THE EXISTENCE OF MULTIPLE SOLUTIONS OF ELLIPTIC DIFFERENTIAL EQUATIONS(1)

BY

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1. **Introduction.** This work is concerned with the problem of the existence and multiplicity of solutions $z(x, y)$ "neighboring" a given initial solution $z_0(x, y)$ of the differential equation

$$F(x, y, z, p, q, r, s, t) = \psi_0(x, y) \tag{1.1}$$

where p, q, r, s, t denote $\partial z/\partial x$, $\partial z/\partial y$, $\partial^2 z/\partial y$, $\partial^2 z/\partial x\partial y$, $\partial^2 z/\partial y^2$ respectively and $z_0(x, y)$ is elliptic relative to F, that is,

$$\left[\frac{\partial F}{\partial s}\right]^2 - 4\left[\frac{\partial F}{\partial r}\right]\left[\frac{\partial F}{\partial t}\right] < 0$$

when z_0 and its derivatives are substituted for z and its derivatives in $\partial F/\partial s$, $\partial F/\partial r$, $\partial F/\partial t$. The solution z_0 is defined on the closure $\overline{K}$ of an open circle K in the xy-plane (that is, $K = [(x, y)/(x-a)^2 + (y-b^2) < r^2]$). If ϕ_0 is the boundary value of z_0, that is, $\phi_0 = z_0/\overline{K} - K$, the function z_0 regarded on $\overline{K} - K$ only, then by a "neighboring" solution of z_0 is meant a function $z_1(x, y)$ whose boundary value ϕ_1 is close to ϕ_0 and which is such that

$$F(x, y, z_1, p_1, q_1, r_1, s_1, t_1) = \psi_1(x, y)$$

where ψ_1 is close to ψ_0 ("close" in the sense of a function space topology to be defined later) and p_1, q_1, r_1, s_1, t_1 denote the derivatives of z_1.

Except for the slight generalization caused by the introduction of the varying $\psi(x, y)$, this is the classical problem. In the earlier work (for a description of this, see [3, pp. 1324–1327(2)) existence theorems were obtained by making the assumption that the Jacobi equation associated with equation (1.1) has only the zero function as solution in $\overline{K}$ if the given boundary value is the zero function. Under this assumption, it was proved by S. Bernstein that equation (1.1) has a solution for each boundary value ϕ sufficiently close to ϕ_0. (The function ψ does not appear in Bernstein's work, that is, $\psi \equiv 0$ throughout his work.)

It was shown by Lichtenstein [9] that if the Jacobi equation has just one linearly independent solution, then for certain quasi-linear elliptic differential equations, there exist two solutions. More extended results concerning the

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Presented to the Society, September 2, 1949; received by the editors May 20, 1949.

(1) This work was done while the writer was under contract to the Office of Naval Research.

(2) Numbers in brackets refer to the bibliography at the end of this work.

number of solutions were obtained by Iglisch ([5] and [6]) for the equation: $\Delta u = f(u)$.

The use of topological techniques to obtain existence theorems was introduced by Schauder [13]. Schauder assumed a uniqueness condition and from this obtained existence theorems for the neighboring solutions of (1.1). Leray and Schauder [8] extended the work in [13] by defining a topological degree for a certain class of mappings in Banach space. One difficulty in applying the topological degree theory is that, in general, the value of the degree for a given mapping cannot be computed. In order to obtain existence theorems by using the degree theory, Leray and Schauder assumed that the associated Jacobi equation has only the zero solution. By making this assumption, they were able to show that the topological degrees of the mappings studied were different from zero and hence to obtain existence theorems.

The object of the present work is to set up and apply a topological technique which may be used to study the solutions of equation (1.1) without making the assumption that the Jacobi equation has only the zero solution. The omission of this assumption leads, in general, to multiple solutions. This is, we obtain solutions whose multiplicity (in a sense to be defined later) is greater than one, or we obtain several distinct solutions. In this work, existence theorems for solutions of equation (1.1) are obtained which are more extensive than the known theorems in that it is not necessary to assume that the Jacobi equation has only the zero solution. Also we demonstrate the existence of several distinct solutions for certain quasi-linear elliptic equations of the type

$$\begin{aligned} A(x, y)\frac{\partial^2 z}{\partial x^2} + B(x, y)\frac{\partial^2 z}{\partial x \partial y} + C(x, y)\frac{\partial^2 z}{\partial y^2} \\ + f\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right) = \psi(x, y). \end{aligned} \tag{1.2}$$

The results are obtained by following, in part, a technique due to Schauder [13] and some of the theory of [2]. We study those solutions of equation (1.1) which are contained in the Banach space $E_{\alpha,3}$ defined in [13], that is, those solutions which have third derivatives satisfying an α-Lipschitz condition.

In §2, we define, following [2], a multiplicity of solutions for a certain class of equations in Banach space. This multiplicity has some of the properties of a topological degree which are useful in studying the existence and number of solutions of a given equation. A method for determining the multiplicity is also given.

In §3, by following the technique of [13], we derive from equation (1.1) an equation in Banach space which is such that the theory of §2 may be applied to it. It is shown that to each solution of this equation, there corresponds a solution of (1.1). Hence the solutions of (1.1) may be investigated by studying the solutions of this equation in Banach space. Existence theorems for solutions of this equation are obtained and these are applied to (1.1). These theorems contain the classical theorem for the case when the Jacobi equation has only the zero solution. We also obtain existence theorems for the case when the Jacobi equation associated with (1.1) has one or more linearly independent solutions.

In §4, we restrict our considerations to equations of the type (1.2). It is shown that when applied to this case, the multiplicity of solutions defined in §2 has all the properties of a topological degree. Actually it is shown that the multiplicity is equal to the Leray-Schauder topological degree of the mapping described by the derived equation in Banach space. By utilizing a particular property of the Leray-Schauder topological degree (Theorem 4.1), we are enabled to show that if the Jacobi equation associated with (1.2) has two or more linearly independent solutions and if certain conditions on the derived equation in Banach space are satisfied, then equation (1.2) has two or more distinct neighboring solutions. That is, for certain pairs ϕ and ψ, there are several distinct solutions z.

Finally, in §5, we indicate how to obtain extensions of the classical theorems concerning the existence of solutions of quasi-linear elliptic differential equations with a parameter, that is, equations of the type:

$$(1.3)\qquad \begin{aligned} A(x, y)\frac{\partial^2 z}{\partial x^2} + B(x, y)\frac{\partial^2 z}{\partial x \partial y} + C(x, y)\frac{\partial^2 z}{\partial y^2} \\ + f\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \lambda\right) = \psi(x, y) \end{aligned}$$

where λ is a real parameter.

2. **Multiplicity of solutions of equations in Banach space.** In this section, a multiplicity of solutions is defined for a certain class of equations in Banach space. This multiplicity is a special case of the multiplicity defined in [2]. For convenience, the definition is repeated in brief form here.

Let $\mathfrak{X}$ be a Banach space. Consider the equation in $\mathfrak{X}$:

$$(2.1)\qquad (I + C + T)x = y \quad \text{or} \quad (I + C)x + T(x) = y$$

where $x, y \in \mathfrak{X}$, I is the identity mapping of $\mathfrak{X}$ into itself, C is a linear, completely continuous(3) mapping of $\mathfrak{X}$ into itself, and T is a mapping of a subset of $\mathfrak{X}$ into itself which satisfies the following conditions:

(P_1) $T(0) = 0$ where 0 is the zero of $\mathfrak{X}$.

(P_2) There exist a neighborhood N of 0 and a positive constant B such that if $x_1, x_2 \in N$, then

(3) A completely continuous mapping is a continuous mapping which has the property that it maps bounded sets into sets which are compact in the space, compact in the sense that every infinite subset has a limit point in the space.

$$\|T(x_1) - T(x_2)\| \leqq B[\|x_1\| + \|x_2\|][\|x_1 - x_2\|].$$

The object of the discussion is to investigate the local solutions of (2.1), that is, the solutions x of (2.1) which are sufficiently close to 0, when y is given sufficiently close to 0. From condition P_1, it follows that $x=y=0$ is an initial solution of equation (2.1). Then the following theorem holds:

THEOREM 2.1. *If the transformation $(I+C)$ is nonsingular, that is, if $(I+C)$ is a* 1-1 *transformation, then equation* (2.1) *has a unique local solution x for each y which is sufficiently close to* 0, *that is, there exist $\epsilon_1>0$, $\epsilon_2>0$ such that if $\|y_0\|<\epsilon_1$, then there exists exactly one x_0 such that $\|x_0\|<\epsilon_2$ and such that (x_0, y_0) satisfies equation* (2.1), *that is,*

$$(I + C)x_0 + T(x_0) = y_0.$$

Proof. The proof of this theorem is a straightforward application of [4, Theorem 2, pp. 134–135]. Condition P_2 makes it possible to apply Theorem 2 of [4].

If the transformation $(I+C)$ is singular, then the solutions x of (2.1) may be studied in the following way. Suppose $\mathfrak{X}_1$ is the null space of $I+C$, that is, $\mathfrak{X}_1$ is the linear space of elements $u\in\mathfrak{X}$ such that $(I+C)u=0$. From the Riesz theory of completely continuous transformations [11], it is known that $\mathfrak{X}_1$ is finite-dimensional and that there exists a complete linear subspace $\mathfrak{X}^1$ of $\mathfrak{X}$ such that

$$\mathfrak{X} = \mathfrak{X}_1 + \mathfrak{X}^1 \text{ (direct sum)},$$

and a nonsingular linear transformation R such that

$$R(I + C)x = x - E_1(x)$$

where E_1 is the projection of $\mathfrak{X}$ onto $\mathfrak{X}_1$. Applying transformation R to equation (2.1), we obtain

$$x - E_1(x) = R(y) - RT(x). \tag{2.2}$$

Multiplying equation (2.2) by E^1, the projection of $\mathfrak{X}$ onto $\mathfrak{X}^1$, we obtain:

$$E^1(x) = E^1R(y) - E^1RT(x). \tag{2.3}$$

Multiplying equation (2.2) by E_1, we obtain:

$$0 = E_1R(y) - E_1RT(x). \tag{2.4}$$

(The symbol 0 is used to designate the zeros of all the linear spaces dealt with in this work.)

If $E^1(x)$ and $E_1(x)$ are denoted by x^1 and x_1, then equations (2.3) and (2.4) become:

$$x^1 = E^1R(y) - E^1RT(x_1 + x^1), \tag{2.3$'$}$$

$$0 = E_1R(y) - E_1RT(x_1 + x^1). \tag{2.4'}$$

Now if y is given, it is clear that the study of solutions x of equation (2.1) is equivalent to the study of the simultaneous solutions x_1 and x^1 of equations (2.3′) and (2.4′). From conditions P_1 and P_2 on transformation T and the fact that the linear part of (2.3′) in x^1 is nonsingular, it follows that the implicit function theorem of Hildebrandt and Graves [4, Theorem 2, pp. 134–135] may be applied to solve equation (2.3′) for x^1 uniquely in terms of x_1 and y. That is, we have

$$x^1 = F(x_1, y) \tag{2.5}$$

where the function F is uniformly continuous simultaneously in x_1 and y. Substituting from (2.5) into (2.4′), we obtain

$$E_1R(y) - E_1RT[x_1 + F(x_1, y)] = 0. \tag{2.6}$$

Thus the study of the solutions x of equation (2.1) when y is given is reduced to a study of the solutions x_1 of equation (2.6) which is an equation in the finite-dimensional Euclidean space $\mathfrak{X}_1$, the null space of $(I+C)$.

To determine the solutions x_1 of (2.6) when y is fixed, that is, to determine the zeros of the function

$$E_1R(y) - E_1RT[x_1 + F(x_1, y)], \tag{2.7}$$

when y is fixed, is not possible in general. However, it is known that when y is fixed, the function (2.7) is a continuous function of x_1. Hence knowledge of the number of zeros of the function may be obtained by determining the topological degree at 0 of (2.7). In [2], it is proved that the topological degree of (2.7) can be determined in some cases if $\|x_1\|$ is sufficiently small $(<\epsilon)$ and that the topological degree is constant for all sufficiently small y. It follows that there exist $\epsilon_1 > 0$ and $\epsilon_2 > 0$ such that we may investigate the solutions x such that $\|x\| < \epsilon_2$ of equation (2.1) for each y such that $\|y\| < \epsilon_1$. Throughout the remainder of this work, the term local solution will refer to such a solution x.

Combining these considerations with Theorem 2.1, we make the following definition.

DEFINITION 2.1. If the transformation $(I+C)$ is nonsingular, the multiplicity of local solutions x of equation (2.1) for each fixed y_0 such that $\|y_0\| < \epsilon_1$ is one. If $(I+C)$ is singular, the multiplicity of local solutions x of (2.1) for each fixed y_0 such that $\|y_0\| < \epsilon_1$ is the topological degree at 0 of the mapping in $\mathfrak{X}_1$,

$$E_1R(y_0) - E_1RT[x_1 + F(x_1, y_0)]$$

relative to a sphere in $\mathfrak{X}_1$ with center 0 and radius ϵ (where ϵ is determined as described in [2]).

THEOREM 2.2. *The multiplicity of Definition* 2.1 *has the following properties*:

1. *If the multiplicity is different from zero, there is at least one local solution* x *of equation* (2.1).

2. *The multiplicity is constant for all* y *in a sufficiently small neighborhood of* 0.

3. *If the Leray-Schauder topological degree* [8] *is defined for the transformation* $I+C+T$, *then the Leray-Schauder degree at* y_0 *is the same as the multiplicity of solutions at* y_0 *except for at most a factor* -1.

Proof. For the case when $(I+C)$ is singular, the proofs of 1, 2, 3 are contained in [2]. When $(I+C)$ is nonsingular, statements 1 and 2 are clearly true from the definition of the multiplicity and Theorem 2.1. The proof of 3 follows from [8, pp. 55–59]. For a more detailed discussion of some of this work of [8], see [12, Lemma 3].

It should be noted that this multiplicity, defined in terms of a topological degree, gives the same kind of information concerning solutions of equation (2.1) as a topological degree does. For example, if the multiplicity is 5, the only definite statement which can be made is that equation (2.1) has a solution. There may be just one solution whose part in $\mathfrak{X}_1$ has index 5; or there may be 5 solutions each of whose parts in $\mathfrak{X}_1$ has index 1, and so on. (In §4, we shall show how more definite information may be obtained if the Leray-Schauder degree is defined for $I+C+T$.) Moreover, if the multiplicity is zero, we obtain no information concerning the existence of solutions. That is, there may be no solutions, or two solutions whose parts in $\mathfrak{X}_1$ have indices i and $-i$ respectively, and so on.

In order to be able to compute the multiplicity when $I+C$ is singular, we assume that the transformation T satisfies the following additional conditions:

(P_3) There exists a maximal integer $k\geqq 2$ such that if a is a scalar, then $T[ax]=a^kT^1[a, x]$ where T^1 is a continuous mapping of an open subset S of the product space $Z\times\mathfrak{X}$ into $\mathfrak{X}$ where Z denotes the space of scalars. The set S contains the point (0, 0).

(P_4) If the dimension of $\mathfrak{X}_1$ is greater than 1, then the transformation T is split into a term of order k ($k\geqq 2$) and a term of higher order, that is, $T(x) = T^{(k)}(x)+T^{(k+1)}(x)$ where $T^{(k)}$ and $T^{(k+1)}$ are both continuous and:

(1) $T^{(k)}$ is homogeneous of degree k, that is, if m is an arbitrary integer, then

$$T^{(k)}\left(\sum_{i=1}^{m} a_i x_i\right) = \sum_{(b_i)} a_1^{b_1} \cdots a_m^{b_m} T_{b_1\cdots b_m}[x_1, \cdots, x_m]$$

where the summation is over all sets of non-negative integers $[b_1, \cdots, b_m]$ such that $\sum_{i=1}^m b_i=k$, and $T_{b_1\cdots b_m}$ is a continuous mapping from $\mathfrak{X}\times\cdots\times\mathfrak{X}$ (m times) into $\mathfrak{X}$.

$$\text{(2)} \qquad \text{The } \lim_{x \to 0} \frac{\|T^{(k+1)}(x)\|}{\|x\|^k} = 0.$$

(P_5) If the dimension of $\mathfrak{X}_1$ is one, then $E_1RT^1[0, x_1] \neq 0$ where x_1 is a basis for $\mathfrak{X}_1$ such that $\|x_1\|$ is sufficiently small (see [2]).

(P_6) If the dimension of $\mathfrak{X}_1$ is greater than one, then $E_1RT^{(k)}[x] \neq 0$ for all $x \in \mathfrak{X}_1$ such that $x \neq 0$.

If these conditions hold, then the following theorems concerning the value of the multiplicity can be proved. (In Theorems 2.3, 2.4, 2.5, and 2.6, it is assumed that $I+C$ is singular.)

THEOREM 2.3. *The multiplicity of the solutions x of equation* (2.1) *is equal to the topological degree at* 0 *of the following mapping of the n-dimensional Euclidean space $\mathfrak{X}_1$ into itself:*

$$\text{(A)} \qquad \begin{aligned} &\text{(A-1)} \sum_{i=1}^{n} c_{1i} u_i^k + \sum_{(b_{1i})} c_{b_{11} \cdots b_{1n}} u_1^{b_{11}} \cdots u_n^{b_{1n}} = u_1', \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\ &\text{(A-}n\text{)} \sum_{i=1}^{n} c_{ni} u_i^k + \sum_{(b_{ni})} c_{b_{n1} \cdots b_{nn}} u_1^{b_{n1}} \cdots u_n^{b_{nn}} = u_n' \end{aligned}$$

where the summation $\sum_{(b_{ji})}$ $(j=1, \cdots, n)$ is taken over all sets of non-negative integers $[b_{j1}, \cdots, b_{jn}]$ such that $\sum_{i=1}^{n} b_{ji} = k$. The coefficients c_{ji} are the numbers $P_jE_1RT_k[x_i]$ where $x_1, \cdots, x_n$ is a basis for $\mathfrak{X}_1$ and P_j is the projection of $\mathfrak{X}_1$ into the subspace spanned by x_i. The coefficients $c_{b_{j1} \cdots b_{jn}}$ are the numbers $P_jE_1RT_{b_1 \cdots b_m}[x_1, \cdots, x_m]$.

Proof. For the proofs of Theorems 2.3, 2.4, 2.5, and 2.6, see [2].

Thus the multiplicity is equal to the topological degree at $(0, \cdots, 0)$ of the mapping taking $(u_1, \cdots, u_n)$ into $(u_1', \cdots, u_n')$, which is given by n polynomials in n variables (n is the dimension of $\mathfrak{X}_1$), the polynomials homogeneous of degree k (k is the order of T described in conditions P_3 and P_4). The topological degree is always taken relative to the sphere described in Definition 2.1.

In this work, $\mathfrak{X}$ will be a Banach space over the real numbers. Hence $\mathfrak{X}_1$ is a real Euclidean n-space and the coefficients in the mapping (A) are all real numbers. Then the topological degree of mapping (A) can be computed in a number of cases. In particular, the following theorems hold:

THEOREM 2.4. *If $c_{ii} \neq 0$ for $i=1, \cdots, n$ and if the variables $u_1, \cdots, u_{i-1}$ do not appear in equation* (A-i) *for $i=1, \cdots, n$, then the topological degree of mapping* (A) *at* $(0, \cdots, 0)$ *is* $+1$ *or* -1 *if k is odd. The degree is* 0 *if k is even. In particular, if $n=1$, then the topological degree is* $+1$, *or* -1 *if k is odd and is* 0 *if k is even.*

THEOREM 2.5. *If* $n=2n_1$, *and* (A) *is the mapping from real Euclidean n-space into itself which corresponds to the mapping in complex Euclidean* n_1-*space*

$$Z_1^k = Z_1', \cdots, Z_{n_1}^k = Z_{n_1}',$$

then the topological degree of (A) *at* $(0, \cdots, 0)$ *is* $k^{n_1} = k^{n/2}$.

THEOREM 2.6. *If* $n=k=2$ *and if* (A) *is written in the form*

$$\begin{aligned} a_1u_1^2 + c_1u_2^2 &= u_1', \\ a_2u_1^2 + b_2u_1u_2 + c_2u_2^2 &= u_2' \end{aligned}$$

(*mapping* (A) *can always be written in this form by applying a diagonalizing transformation, see* [10, pp. 169–170]), *then if*

$$-1 < \frac{a_1c_2 - a_2c_1}{b_2(-c_1a_1)^{1/2}} < 1, \tag{1}$$

the topological degree of (A) *is* 2 *or* -2. *If* $(-c_1a_1)^{1/2}$ *is complex or*

$$\frac{a_1c_2 - a_2c_1}{b_2(-c_1a_1)^{1/2}} > 1 \quad \text{or} \quad \frac{a_1c_2 - a_2c_1}{b_2(-c_1a_1)^{1/2}} < -1, \tag{2}$$

then the topological degree is 0. (*If* $(a_1c_2 - a_2c_1)/b_2(-c_1a_1)^{1/2} = \pm 1$, *the topological degree is not defined.*)

REMARK FOR §2. In the applications in §§3, 4, and 5 of this multiplicity, it will be seen that conditions P_1, P_2, P_3, and P_4 can be shown to be satisfied in all the cases dealt with. Proving that conditions P_5 and P_6 are satisfied is a different problem. There is no reason why P_5 and P_6 should, in general, hold. Hence it will be necessary to incorporate P_5 or P_6 in the hypotheses of the existence theorems obtained. This is justified, in part, by two facts. First, the special cases of this work that have been considered by other writers (see [5], [6], and [15]) all contain hypotheses of this type. For example, it can be shown that Schmidt's assumption "$L_k \neq 0$" [15, p. 394] is a special case of P_5.

Secondly, although P_5 and P_6 are difficult to verify in special cases, they are not unreasonable from a theoretical viewpoint. They are analogous to the fundamental assumption used in the definition of topological degree of a mapping in Euclidean n-space, that is, the assumption that the image under the mapping of the boundary of the domain does not contain the point at which the topological degree of the mapping is taken. (See [1, p. 474].) It is not difficult to show that P_5 and P_6 each imply that if $x \neq 0$ and if x is sufficiently small, then $(I+C+T)x \neq 0$.

3. **Application of the multiplicity to elliptic differential equations.** In order that the multiplicity theory of §2 may be applied to the differential equation (1.1), it will be necessary to study an equation in the Banach space

$E_{\alpha,m,K}\times e_{\alpha,n,K}$ which is the topological product of the Banach spaces $E_{\alpha,m,K}$, $e_{\alpha,n,k}$ introduced by Schauder [13.] We recall first the definition given by Schauder of these spaces.

Let K be an open circle in the xy-plane(4). The space $E_{\alpha,m,K}$, where m is an integer and α is a fixed number such that $0<\alpha\leqq 1$, is the linear space of continuous real-valued functions defined on $\overline{K}$, the closure of K, which are m times differentiable at each point of $\overline{K}$ and which are such that the derivatives of order ν $(1\leqq\nu\leqq m)$ are all α-Hölder continuous (are elements of Lip α). That is, if $f(x, y)\in E_{\alpha,m,K}$, then there exists a constant $C_{f,\nu}$ such that

$$\left| f^{(\nu)}(x_1, y_1) - f^{(\nu)}(x_2, y_2)\right| \leqq C_{f,\nu}(r_{12})^{\alpha}$$

where $(r_{12})^2=(x_1-x_2)^2+(y_1-y_2)^2$, $f^{(\nu)}$ is an arbitrary νth order derivative of f, and (x_1, y_1), (x_2, y_2) are an arbitrary pair of points in $\overline{K}$. It is easy to show that for fixed f and ν, there exists a minimum $C_{f,\nu}$ which we denote by $c_{f,\nu}$. The norm of the elements of $E_{\alpha,m,K}$ is defined as follows: first if $g\in E_{\alpha,m,K}$ or g is a νth derivative of an element of $E_{\alpha,m,K}$, we define $\|g\|_{\alpha}$ as:

$$\|g\|_{\alpha} = \max_{(x, y)\in K} \left| g(x, y)\right| + c_{g,0}.$$

The norm of $f(x, y)\in E_{\alpha,m,K}$ is defined as

$$\|f\|_{\alpha,m} = \sum_{i+j=0}^{m} \frac{1}{i!j!}\left\|\frac{\partial^{i+j}f}{\partial x^i\partial y^j}\right\|_{\alpha}.$$

Under this norm, $E_{\alpha,m,K}$ is a Banach space. Also if $f_1, f_2\in E_{\alpha,m,K}$, then $\|f_1\cdot f_2\|_{\alpha,m}\leqq\|f_1\|_{\alpha,m}\cdot\|f_2\|_{\alpha,m}$. (See [13, pp. 667–668].) Hence $E_{\alpha,m,K}$ is a normed ring under pointwise multiplication.

In an entirely similar way, we define the Banach space $e_{\alpha,m,K}$ of continuous functions defined on $\overline{K}-K$, the boundary of K, which are such that the derivatives of order ν $(1\leqq\nu\leqq n)$ are all α-Hölder continuous.

Finally we define the space $E_{\alpha,m,K}\times e_{\alpha,n,K}$, the topological product of $E_{\alpha,m,K}$ and $e_{\alpha,n,K}$. The norm of $E_{\alpha,m,K}\times e_{\alpha,n,K}$ is defined as follows: if $(f, g)\in E_{\alpha,m,K}\times e_{\alpha,n,K}$, that is, $f\in E_{\alpha,m,K}$ and $g\in e_{\alpha,n,K}$, then

$$\|(f, g)\| = \|f\|_{\alpha,m} + \|g\|_{\alpha,n},$$

the sum of the norms of f and g in $E_{\alpha,m,K}$ and $e_{\alpha,n,K}$, respectively. It is clear that $E_{\alpha,m,K}\times e_{\alpha,m,K}$ is a Banach space with this norm. In the work which follows, we shall, for convenience, denote $E_{\alpha,m,K}$ and $e_{\alpha,n,K}$ by $E_{\alpha,m}$ and $e_{\alpha,n}$ respectively.

Now we consider the equation

$$F(x, y, z, p, q, r, s, t) = \psi(x, y) \tag{3.1}$$

(4) From a remark of Schauder [13, footnote 33, p. 690 and III, pp. 702–703] it follows that all the results of this work are valid also if K is a bounded connected open set in the xy-plane whose boundary consists of a finite set of disjoint curves of class Ch.

where F has α-Hölder continuous fifth derivatives in all its variables(5) and $\psi(x, y) \in E_{\alpha,1}$. It is assumed that equation (3.1) has an initial solution in $\overline{K}$ which is elliptic relative to F, that is, we make the following assumption.

ASSUMPTION 3.1. There exist $\psi_0(x, y) \in E_{\alpha,1}$ and $\phi_0(\xi) \in e_{\alpha,3}$ such that if $\psi(x, y) = \psi_0(x, y)$, then equation (3.1) has a solution $z_0(x, y) \in E_{\alpha,3}$ whose boundary value is $\phi_0(\xi)$. Also the solution z_0 is elliptic relative to F, that is,

$$[\partial F/\partial s]^2 - 4\,[\partial F/\partial r]\,[\partial F/\partial t] < 0$$

when z_0 and its derivatives are substituted for z and its derivatives in $\partial F/\partial s$, $\partial F/\partial r$, $\partial F/\partial t$.

The question to be answered is this: if ϵ is sufficiently small, do there exist solutions $z(x, y) \in E_{\alpha,3}$ of equation (3.1) for given $\psi \in E_{\alpha,1}$ and given boundary values $\phi \in e_{\alpha,3}$ such that $\|\psi - \psi_0\|_{\alpha,1} < \epsilon$ and $\|\phi - \phi_0\|_{\alpha,3} < \epsilon$, and what are the multiplicities of these solutions if they exist?

In order to answer this question, we shall derive, following a technique due to Schauder [13, pp. 693–694], an equation in the Banach space $E_{\alpha,1} \times e_{\alpha,3}$ and show that the problem of studying the solutions of equation (3.1) is, in a certain sense, equivalent to the problem of studying the solutions of this abstract equation.

In deriving the equation in $E_{\alpha,1} \times e_{\alpha,3}$, we shall assume that the functions ψ_0 and ϕ_0 of Assumption 3.1 are both identically zero. This will simplify the computations. The same type of derivation may be carried out if it is not assumed that ψ_0 and ϕ_0 are identically zero. (Cf. [13, pp. 693–694].)

We write equation (3.1) in the form:

$$
\begin{aligned}
\psi(x, y) = {} & F(x, y, z, p, q, r_0, s_0, t_0) \\
& + (r - r_0)\frac{\partial F}{\partial r}(x, y, z_0, p_0, q_0, r_0, s_0, t_0) \\
& + (s - s_0)\frac{\partial F}{\partial s}(x, y, z_0, p_0, q_0, r_0, s_0, t_0) \\
& + (t - t_0)\frac{\partial F}{\partial t}(x, y, z_0, p_0, q_0, r_0, s_0, t_0) \\
& + (r - r_0)\int_0^1 \left\{\frac{\partial F}{\partial r}[x, y, z, p, q, r_0 + \lambda(r - r_0), \right. \\
& \qquad\qquad s_0 + \lambda(s - s_0), t_0 + \lambda(t - t_0)] \\
& \qquad\qquad \left. - \frac{\partial F}{\partial r}(x, y, z_0, p_0, q_0, r_0, s_0, t_0)\right\} d\lambda
\end{aligned} \tag{3.2}
$$

(5) We state the hypothesis on F in this way for simplicity. Actually, only certain of the fifth derivatives are needed. An examination of the proofs in this section shows which derivatives of fifth order and which of lower order are needed.

$$
\begin{aligned}
&+ (s - s_0)\int_0^1 \Big\{\frac{\partial F}{\partial s}\,[x, y, z, p, q, r_0 + \lambda(r - r_0), \\
&\qquad\qquad s_0 + \lambda(s - s_0), t_0 + \lambda(t - t_0)] \\
&\qquad\qquad - \frac{\partial F}{\partial s}\,(x, y, z_0, p_0, q_0, r_0, s_0, t_0)\Big\}\, d\lambda \\
&+ (t - t_0)\int_0^1 \Big\{\frac{\partial F}{\partial t}\,[x, y, z, p, q, r_0 + \lambda(r - r_0), \\
&\qquad\qquad s_0 + \lambda(s - s_0), t_0 + \lambda(t - t_0)] \\
&\qquad\qquad - \frac{\partial F}{\partial t}\,(x, y, z_0, p_0, q_0, r_0, s_0, t_0)\Big\}\, d\lambda
\end{aligned} \tag{3.2}
$$

where p_0, q_0, r_0, s_0, t_0 denote the derivatives of z_0.

In order to derive the equation in $E_{\alpha,1}\times e_{\alpha,3}$, we shall, roughly speaking, solve the linear part of equation (3.2). For this, we recall the following important lemma:

LEMMA 3.1. *Consider the elliptic differential equation*

$$A(x, y)\frac{\partial^2 w}{\partial x_2} + B(x, y)\frac{\partial^2 w}{\partial x \partial y} + C(x, y)\frac{\partial^2 w}{\partial y^2} = \rho(x, y) \tag{3.3}$$

where $B^2 - 4AC < 0$ *for* $(x, y)\in \overline{K}$, $\rho(x, y)\in E_{\alpha,1}$, *and* $A(x, y)$, $B(x, y)$, $C(x, y)$ $\in E_{\alpha,1}$. *Then for a given boundary value* $\phi \in e_{\alpha,3}$, *equation* (3.3) *has a unique solution* $w(\rho, \phi)\in E_{\alpha,3}$ *in* K *which satisfies the following inequality*:

$$\|w(\rho, \phi)\|_{\alpha,3} \leqq M(\|\rho\|_{\alpha,1} + \|\phi\|_{\alpha,3}) \tag{3.4}$$

where M *is a positive constant. Inequality* (3.4) *implies in particular that*

$$\left.\begin{array}{l} \|w(\rho, \phi)\|_{1,1} \\ \|D_1 w(\rho, \phi)\|_{1,1} \\ \|D_2 w(\rho, \phi)\|_{\alpha,1} \\ \|D_3 w(\rho, \phi)\|_{\alpha} \end{array}\right\} \leqq M(\|\rho\|_{\alpha,1} + \|\phi\|_{\alpha,3})$$

where $D_1 w(\rho, \phi)$, $D_2 w(\rho, \phi)$, $D_3 w(\rho, \phi)$ *denote arbitrary first, second, and third derivatives, respectively, of* $w(\rho, \phi)$ *regarded as a function of* x *and* y.

Moreover, $w(\rho, \phi)$ *regarded as an operator on* $E_{\alpha,1}\times e_{\alpha,3}$ *is linear and* $w(\rho, \phi)$ *is* 1-1 *in* ρ *and* 1-1 *in* ϕ.

Proof. See [14, Satz II′, p. 277] and [13, Hilfssatz 11, p. 687].

The third derivatives of $z_0(x, y)$ are α-Hölder continuous. Hence

$$\mathcal{A}(x, y) = \frac{\partial F}{\partial r}\,(x, y, z_0, p_0, q_0, r_0, s_0, t_0),$$

$$\mathcal{B}(x, y) = \frac{\partial F}{\partial s}(x, y, z_0, p_0, q_0, r_0, s_0, t_0),$$

$$\mathcal{C}(x, y) = \frac{\partial F}{\partial t}(x, y, z_0, p_0, q_0, r_0, s_0, t_0)$$

are elements of $E_{\alpha,1}$. Now we consider the differential equation

$$\mathcal{A}(x, y)\frac{\partial^2 w}{\partial x^2} + \mathcal{B}(x, y)\frac{\partial^2 w}{\partial x \partial y} + \mathcal{C}(x, y)\frac{\partial^2 w}{\partial y^2} = \rho(x, y) \tag{3.5}$$

where $\rho(x, y) \in E_{\alpha,1}$. Applying Lemma 3.1 to equation (3.5), we obtain the solution $w(\rho, \phi)$. Hence equation (3.2) may be written in the form:

$$\begin{aligned}
\psi(x, y) = F\Big[& x, y, z_0 + w(\rho, \phi), p_0 + \frac{\partial}{\partial x} w(\rho, \phi), q_0 + \frac{\partial}{\partial y} w(\rho, \phi), r_0, s_0, t_0\Big] \\
&+ \rho(x, y) + \frac{\partial^2 w(\rho, \phi)}{\partial x^2} \int_0^1 \Big\{\frac{\partial F}{\partial r}\Big[x, y, z_0 + w(\rho, \phi), p_0 \\
&+ \frac{\partial}{\partial x} w(\rho, \phi), q_0 + \frac{\partial}{\partial y} w(\rho, \phi), r_0 + \lambda \frac{\partial^2 w}{\partial x^2}(\rho, \phi), s_0 \\
&+ \lambda \frac{\partial^2 w}{\partial x \partial y}(\rho, \phi), t_0 + \lambda \frac{\partial^2 w}{\partial y^2}(\rho, \phi)\Big] \\
&- \frac{\partial F}{\partial r}[x, y, z_0, p_0, q_0, r_0, s_0, t_0]\Big\} d\lambda \\
&+ \frac{\partial^2 w}{\partial x \partial y} \int_0^1 \Big\{\frac{\partial F}{\partial s}\Big[x, y, z_0 + w, p_0 + \frac{\partial}{\partial x} w, q_0 + \frac{\partial}{\partial y} w, r_0 \\
&+ \lambda \frac{\partial^2}{\partial x^2} w, s_0 + \lambda \frac{\partial^2}{\partial x \partial y} w, t_0 + \lambda \frac{\partial^2}{\partial y^2} w\Big] \\
&- \frac{\partial F}{\partial s}[x, y, z_0, p_0, q_0, r_0, s_0, t_0]\Big\} d\lambda \\
&+ \frac{\partial^2 w}{\partial y^2} \int_0^1 \Big\{\frac{\partial F}{\partial t}\Big[x, y, z_0 + w, p_0 + \frac{\partial}{\partial x} w, q_0 + \frac{\partial}{\partial y} w, r_0 \\
&+ \lambda \frac{\partial^2}{\partial x^2} w, s_0 + \lambda \frac{\partial^2}{\partial x \partial y} w, t_0 + \lambda \frac{\partial^2}{\partial y^2} w\Big] \\
&- \frac{\partial F}{\partial t}[x, y, z_0, p_0, q_0, r_0, s_0, t_0]\Big\} d\lambda
\end{aligned} \tag{3.6}$$

where ϕ is the given boundary value of the sought for solution and $\psi(x, y)$ is also given.

We define the transformation from $E_{\alpha,1}\times e_{\alpha,3}$ into $E_{\alpha,1}$:

$$R^{(1)}(\rho, \phi) = F\left[x, y, z_0 + w, p_0 + \frac{\partial}{\partial x} w, q_0 + \frac{\partial}{\partial y} w, r_0, s_0, t_0\right].$$

Since $F(x, y, z_0, p_0, q_0, r_0, s_0, t_0) \equiv 0$, then by using Taylor's expansion, $R^{(1)}(\rho, \phi)$ may be written in the form:

$$R^{(1)}(\rho, \phi) = R_1^{(1)}(\rho, \phi) + R_2^{(1)}(\rho, \phi)$$

where

$$\begin{aligned} R_1^{(1)}(\rho, \phi) = {} & w(\rho, \phi) \frac{\partial F}{\partial z}(x, y, z_0, p_0, q_0, r_0, s_0, t_0) \\ & + \left[\frac{\partial}{\partial x} w\right]\left[\frac{\partial F}{\partial p}(x, y, z_0, p_0, q_0, r_0, s_0, t_0)\right] \\ & + \left[\frac{\partial}{\partial y} w\right]\left[\frac{\partial F}{\partial q}(x, y, z_0, p_0, q_0, r_0, s_0, t_0)\right] \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} R_2^{(1)}(\rho, \phi) = {} & [w(\rho, \phi)]^2 \int_0^1 \left[\frac{\partial^2 F}{\partial z^2}\left(x, y, z_0 + \tau w, p_0 + \tau \frac{\partial}{\partial x} w, q_0 \right.\right. \\ & \left.\left. + \tau \frac{\partial}{\partial y} w, r_0, s_0, t_0\right)\right][1 - \tau] d\tau \\ & + 2(w)\left(\frac{\partial w}{\partial x}\right)\int_0^1 \left(\frac{\partial^2 F}{\partial z \partial p}\right)(1 - \tau) d\tau \\ & + 2(w)\left(\frac{\partial w}{\partial y}\right)\int_0^1 \left(\frac{\partial^2 F}{\partial z \partial q}\right)(1 - \tau) d\tau \\ & + \left(\frac{\partial w}{\partial x}\right)^2 \int_0^1 \left(\frac{\partial^2 F}{\partial p^2}\right)(1 - \tau) d\tau \\ & + 2\left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial w}{\partial y}\right)\int_0^1 \left(\frac{\partial^2 F}{\partial p \partial q}\right)(1 - \tau) d\tau \\ & + \left(\frac{\partial w}{\partial y}\right)^2 \int_0^1 \left(\frac{\partial^2 F}{\partial q^2}\right)(1 - \tau) d\tau. \end{aligned} \tag{3.8}$$

We define also the transformation

$$
\begin{aligned}
R^{(2)}(\rho, \phi) = \frac{\partial^2 w}{\partial x^2} \int_0^1 & \left\{ \frac{\partial F}{\partial r} \left[x, y, z_0 + w, p_0 + \frac{\partial}{\partial x} w, q_0 + \frac{\partial}{\partial y} w, r_0 \right.\right. \\
& \left. + \lambda \frac{\partial}{\partial x^2} w, s_0 + \lambda \frac{\partial^2}{\partial x \partial y} w, t_0 + \lambda \frac{\partial^2}{\partial y^2} w \right] \\
& \left. - \frac{\partial F}{\partial r} (x, y, z_0, p_0, q_0, r_0, s_0, t_0) \right\} d\lambda \\
& + \frac{\partial^2 w}{\partial x \partial y} \int_0^1 \left\{ \frac{\partial F}{\partial s} \left[x, y, z_0 + w, p_0 + \frac{\partial}{\partial x} w, q_0 + \frac{\partial}{\partial y} w, r_0 \right.\right. \\
& \left. + \lambda \frac{\partial}{\partial x^2} w, s_0 + \lambda \frac{\partial^2}{\partial x \partial y} w, t_0 + \lambda \frac{\partial^2}{\partial y^2} w \right] \\
& \left. - \frac{\partial F}{\partial s} (x, y, z_0, p_0, q_0, r_0, s_0, t_0) \right\} d\lambda \\
& + \frac{\partial^2 w}{\partial y^2} \int_0^1 \left\{ \frac{\partial F}{\partial t} \left[x, y, z_0 + w, p_0 + \frac{\partial}{\partial x} w, q_0 + \frac{\partial}{\partial v} w, r_0 \right.\right. \\
& \left. + \lambda \frac{\partial}{\partial x^2} w, s_0 + \lambda \frac{\partial^2}{\partial x \partial y} w, t_0 + \lambda \frac{\partial^2}{\partial y^2} w \right] \\
& \left. - \frac{\partial F}{\partial t} (x, y, z_0, p_0, q_0, r_0, s_0, t_0) \right\} d\lambda.
\end{aligned} \tag{3.9}
$$

Now equation (3.6) may be written:

$$
\psi = \rho + R_1^{(1)}(\rho, \phi) + R_2^{(1)}(\rho, \phi) + R^{(2)}(\rho, \phi). \tag{3.10}
$$

Instead of studying this equation directly, we shall study the equation in $E_{\alpha,1} \times e_{\alpha,3}$:

$$
(\rho, \phi) + (R_1^{(1)}(\rho, \phi), 0) + (R_2^{(1)}(\rho, \phi), 0) + (R^{(2)}(\rho, \phi), 0) = (\psi, \phi) \tag{3.11}
$$

where 0 is the zero of the space $e_{\alpha,3}$.

Denoting $(R_1^{(1)}(\rho, \phi), 0)$ by $C(\rho, \phi)$ and denoting $(R_2^{(1)}(\rho, \phi) + R^{(2)}(\rho, \phi), 0)$ by $T(\rho, \phi)$ where C and T are then transformations from $E_{\alpha,1} \times e_{\alpha,3}$ into itself, we may write equation (3.11) as:

$$
I(\rho, \phi) + C(\rho, \phi) + T(\rho, \phi) = (\psi, \phi) \tag{3.12}
$$

where I is the identity transformation.

In the remainder of this section, we investigate the solutions (ρ, ϕ) of equation (3.12) when (ψ, ϕ) is given. The investigation is made by applying the theory of §2 to equation (3.12). The information obtained concerning the solutions of equation (3.12) yields, in turn, results concerning the existence of solutions of equation (3.1).

We show first the relationship between the solutions $z(x, y)$ of equation (3.1) for given ψ and ϕ and the solutions of equation (3.12) for given (ψ, ϕ). If for given $(\psi_1, \phi_1) \in E_{\alpha,1} \times e_{\alpha,3}$ there exists a solution $(\rho_1, \phi_1) \in E_{\alpha,1} \times e_{\alpha,3}$ of equation (3.12), then there exists a solution $z_1(x, y) \in E_{\alpha,3}$ of equation (3.1) when $\psi = \psi_1$. The boundary value of z_1 is ϕ_1 and $z_1 = z_0 + w(\rho_1, \phi_1)$ where z_0 is the initial solution of Assumption 3.1. The existence and properties of z_1 follow from the derivation of equation (3.12). Conversely, if there exists a solution z_1 of equation (3.1) when $\psi = \psi_1$ such that z_1 has boundary value ϕ_1, then there exists a solution (ρ_1, ϕ_1) of (3.12) when $(\psi, \phi) = (\psi_1, \phi_1)$ and

$$\rho_1(x, y) = \mathcal{A}(x, y) \frac{\partial^2}{\partial x^2} [z_1 - z_0] + \mathcal{B}(x, y) \frac{\partial^2}{\partial x \partial y} [z_1 - z_0] + \mathcal{C}(x, y) \frac{\partial^2}{\partial y^2} [z_1 - z_0].$$

The fact that (ρ_1, ϕ_1) is a solution of equation (3.12) follows from the derivation of equation (3.12).

Since $w(\rho, \phi)$ is 1-1 in ρ and ϕ (Lemma 3.1), it is clear that corresponding to n distinct solutions of (3.1), there are exactly n distinct solutions of (3.12) and vice versa.

However, by using the theory of §2, we shall be able to investigate only the local solutions of equation (3.12). If it is shown that equation (3.12) has a local solution, that is, for each (ψ, ϕ) in a neighborhood N_1 of $(0, 0)$, there is a solution (ρ, ϕ) of (3.12) contained in a neighborhood N_2 of $(0, 0)$, this will imply only that equation (3.1) has, for given ψ and ϕ, a solution in the set $z_0 + w(H_2)$ where H_2 is the set of solutions of (3.12) which correspond to $(\psi, \phi) \in N_1$ and which are contained in N_2. (The operator w was defined in Lemma 3.1. It will be shown later (Lemma 3.3) that w is completely continuous. Hence since H_2 is bounded, $w(H_2)$ is compact.) There is, however, no reason to exclude the possibility that equation (3.1) may have a solution outside the set $z_0 + w(H_2)$. So the number of local solutions of equation (3.12) is a lower bound for the number of solutions of equation (3.1).

We summarize this discussion in the following theorem.

THEOREM 3.1. *The number of distinct local solutions of equation* (3.12) *for given* (ψ, ϕ) *is a lower bound for the number of distinct solutions of equation* (3.1) *for given* ψ *and* ϕ. *The number of local solutions of equation* (3.12) *for given* (ψ, ϕ) *is equal to the number of solutions of equation* (3.1) *in the compact set* $z_0 + w[H_2]$ *defined above.*

Theorem 3.1 and the discussion preceding it justify the following definition.

DEFINITION 3.1. The multiplicity of solutions of equation (3.1) is the multiplicity of solutions (in the sense of Definition 2.1) of equation (3.12). (In Theorem 3.2, it will be shown that Definition 2.1 can be applied to equation (3.12).)

The relationship between equations (3.1) and (3.12) is further revealed by the following lemma which shows that the linear parts of equations (3.1) and (3.12) are essentially the same.

LEMMA 3.2. *The dimension of the null space of the operator* $I(\rho, \phi)+C(\rho, \phi)$ *is equal to the number of linearly independent solutions of the Jacobi equation associated with equation* (3.1).

Proof. The Jacobi equation associated with equation (3.1) is the linear differential equation

$$\begin{aligned}\text{(3.13)}\qquad &\left[\frac{\partial F}{\partial r}\right]_0 \frac{\partial^2 u}{\partial x^2}+\left[\frac{\partial F}{\partial s}\right]_0 \frac{\partial^2 u}{\partial x \partial y}+\left[\frac{\partial F}{\partial t}\right]_0 \frac{\partial^2 u}{\partial y^2} \\ &\qquad +\left[\frac{\partial F}{\partial p}\right]_0 \frac{\partial u}{\partial x}+\left[\frac{\partial F}{\partial q}\right]_0 \frac{\partial u}{\partial y}+\left[\frac{\partial F}{\partial z}\right]_0 u=0\end{aligned}$$

where the notation $[\ \ ]_0$ means that the expression in the brackets has substituted in it z_0 and its derivatives for z and its derivatives.

By a solution of the Jacobi equation, we mean a function $u(x, y)$ which is an element of $E_{\alpha,3}$, which satisfies equation (3.13), and which is zero on the boundary of the set K.

Suppose (ρ_1, ϕ_1) is an element of the null space of $I(\rho, \phi)+C(\rho, \phi)$, that is, (ρ_1, ϕ_1) is a solution of the equation

$$\text{(3.14)}\qquad I(\rho, \phi)+C(\rho, \phi)=(0,0).$$

By definition of $C(\rho, \phi)$, this implies that $\phi_1 \equiv 0$ and

$$\text{(3.15)}\qquad \rho_1+w(\rho_1, \phi_1)\left[\frac{\partial F}{\partial z}\right]_0+\frac{\partial}{\partial x} w(\rho_1, \phi_1)\left[\frac{\partial F}{\partial p}\right]_0+\frac{\partial}{\partial y} w(\rho_1, \phi_1)\left[\frac{\partial F}{\partial q}\right]_0=0.$$

But $w(\rho_1, \phi_1)$ is a solution of (3.5) when ρ_1 and $\phi_1 \equiv 0$ are given; that is,

$$\rho_1=\left[\frac{\partial F}{\partial r}\right]_0 \frac{\partial^2 w(\rho_1, \phi_1)}{\partial x^2}+\left[\frac{\partial F}{\partial s}\right]_0 \frac{\partial^2 w(\rho_1, \phi_1)}{\partial x \partial y}+\left[\frac{\partial F}{\partial t}\right]_0 \frac{\partial^2 w(\rho_1, \phi_1)}{\partial y^2}.$$

Hence equation (3.15) may be written as:

$$\begin{aligned}&\left[\frac{\partial F}{\partial r}\right]_0 \frac{\partial^2 w(\rho_1, \phi_1)}{\partial x^2}+\left[\frac{\partial F}{\partial s}\right]_0 \frac{\partial^2 w(\rho_1, \phi_1)}{\partial x \partial y}+\left[\frac{\partial F}{\partial t}\right]_0 \frac{\partial^2 w(\rho_1, \phi_1)}{\partial y^2} \\ &+w(\rho_1, \phi_1)\left[\frac{\partial F}{\partial z}\right]_0+\left[\frac{\partial F}{\partial p}\right]_0 \frac{\partial w(\rho_1, \phi_1)}{\partial x}+\left[\frac{\partial F}{\partial q}\right]_0 \frac{\partial w(\rho_1, \phi_1)}{\partial y}=0.\end{aligned}$$

Hence since $\phi_1 \equiv 0$, $w(\rho, \phi_1)=w(\rho_1, 0)$ is a solution of the Jacobi equation.

Since $w(\rho, \phi)$ is linear and 1-1 in (ρ, ϕ) (Lemma 3.1), it is clear that if $(\rho_1, 0)$ and $(\rho_2, 0)$ are linearly independent solutions of (3.14), then $w(\rho_1, 0)$ and $w(\rho_2, 0)$ are linearly independent solutions of the Jacobi equation.

By following a similar procedure, it can be shown that corresponding to two linearly independent solutions of the Jacobi equation, there are two linearly independent solutions $(\rho_1, 0)$ and $(\rho_2, 0)$ of equation (3.14). This completes the proof of the lemma.

We show now that equation (3.12) is an example of equation (2.1) and hence that the multiplicity theory of §2 can be applied to equation (3.12).

THEOREM 3.2. *Equation* (3.12) *is an example of equation* (2.1). *That is, the transformation* $C(\rho, \phi)$ *is linear and completely continuous and the transformation* T *satisfies conditions* P_1, P_2, P_3, P_4 *of* §2.

Proof. The proof will be broken up into lemmas.

LEMMA 3.3. *The transformation* $C(\rho, \phi)$ *is a linear, completely continuous transformation of* $E_{\alpha,1} \times e_{\alpha,3}$ *into itself.*

Proof. The linearity follows from the definition of $C(\rho, \phi)$ and the linearity of $w(\rho, \phi)$ (Lemma 3.1). From the definition of $C(\rho, \phi)$, it is clear that in order to prove that $C(\rho, \phi)$ is completely continuous, it is sufficient to prove that $w(\rho, \phi)$, $\partial w(\rho, \phi)/\partial x$, and $\partial w(\rho, \phi)/\partial y$ are completely continuous in $E_{\alpha,1}$ because $[\partial F/\partial p]_0$ and $[\partial F/\partial q]_0$ are bounded in $\bar{K}$. The proof that $w(\rho, \phi)$, $\partial w(\rho, \phi)/\partial x$, and $\partial w(\rho, \phi)/\partial y$ are completely continuous is a direct consequence of [13, Hilfssatz 12, p. 691]. Hilfssatz 12 shows that $w(\rho, \phi)$ is "vollstetig" in the sense of [13, Definition 5, p. 674]. But "vollstetig" implies complete continuity in the usual sense.

LEMMA 3.4. *The transformation* $T(\rho, \phi)$ *satisfies the conditions* P_1, P_2, P_3, *and* P_4 *described in* §2.

Proof. By Lemma 3.1, $w(\rho, \phi)$ is linear in (ρ, ϕ). Hence $w(0, 0) \equiv 0$. From this and from the definition of $T(\rho, \phi)$, it follows that $T(0, 0) = 0$, hence that $T(\rho, \phi)$ satisfies P_1.

In order to prove that $T(\rho, \phi)$ satisfies P_2, we recall that $T(\rho, \phi) = (R_2^{(1)}(\rho, \phi), 0) + (R^{(2)}(\rho, \phi), 0)$. That $R^{(2)}(\rho, \phi)$ satisfies condition P_2 is proved in [13, pp. 697–701]. Hence to show that $T(\rho, \phi)$ satisfies condition P_2, it is only necessary to show that $R_2^{(1)}(\rho, \phi)$ satisfies condition P_2. We prove that P_2 holds for

$$\begin{aligned}(3.16)\qquad [w(\rho, \phi)]^2 \int_0^1 \Big[\frac{\partial^2 F}{\partial z^2}\Big(x, y, z_0 + \tau w(\rho, \phi), p_0 + \tau \frac{\partial}{\partial x} w(\rho, \phi),& \\ q_0 + \tau \frac{\partial}{\partial x} w(\rho, \phi), r_0, s_0, t_0\Big)\Big][1 - \tau]d\tau.&\end{aligned}$$

The same type of proof holds for the other five terms in $R_2^{(1)}(\rho, \phi)$.

For convenience, we introduce, following [13, p. 695], the notations X, Z_0, R_0, $W(\rho, \phi)$, $V(\rho, \phi)$ for (x, y), (z_0, p_0, q_0), (r_0, s_0, t_0), $(w(\rho, \phi), \partial w(\rho, \phi)/\partial x, \partial w(\rho, \phi)/\partial y)$, and $(\partial^2 w(\rho, \phi)/\partial x^2, \partial^2 w(\rho, \phi)/\partial x\partial y, \partial^2 w(\rho, \phi)/\partial y^2)$ respectively. Then (3.16) becomes

$$[w(\rho, \phi)]^2 \int_0^1 \left[\frac{\partial^2 F}{\partial z^2}(X, Z_0 + \tau W(\rho, \phi), R_0)\right][1 - \tau]d\tau.$$

By definition,

$$\begin{aligned}
&\left\| \left([w(\rho_1, \phi_1)]^2 \int_0^1 \left[\frac{\partial^2 F}{\partial z^2}(X, Z_0 + \tau W(\rho_1, \phi_1), R_0)\right][1 - \tau]d\tau, 0\right)\right. \\
&\qquad \left. - \left([w(\rho_2, \phi_2)]^2 \int_0^1 \left[\frac{\partial F}{\partial z}(X, Z_0 + \tau W(\rho_2, \phi_2), R_0)\right][1 - \tau]d\tau, 0\right)\right\| \\
&= \left\| (w_1)^2 \int_0^1 \left[\frac{\partial^2 F}{\partial z^2}(X, Z_0 + \tau W(\rho_1, \phi_1), R_0)\right][1 - \tau]d\tau \right. \\
&\qquad \left. - (w_2)^2 \int_0^1 \left[\frac{\partial^2 F}{\partial z^2}(X, Z_0 + \tau W(\rho_2, \phi_2), R_0)\right][1 - \tau]d\tau \right\|_{\alpha,1} \\
&\qquad\qquad \text{(where } w_i \text{ denotes } w(\rho_i, \phi_i) \text{ for } i = 1, 2) \\
&\leqq \left\| (w_1)^2 \int_0^1 \left[\frac{\partial^2 F}{\partial z^2}(X, Z_0 + \tau W(\rho_1, \phi_1), R_0)\right.\right. \\
&\qquad \left.\left. - \frac{\partial^2 F}{\partial z^2}(X, Z_0 + \tau W(\rho_2, \phi_2), R_0)\right][1 - \tau]d\tau \right\|_{\alpha,1} \\
&\qquad + \left\| (w_1^2 - w_2^2) \int_0^1 \left[\frac{\partial^2 F}{\partial z^2}(X, Z_0 + \tau W(\rho_2, \phi_2), R_0)\right][1 - \tau]d\tau \right\|_{\alpha,1}.
\end{aligned} \tag{3.17}$$

The expression

$$\int_0^1 \left[\frac{\partial^2 F}{\partial z^2}(X, Z_0 + \tau W(\rho_2, \phi_2), R_0)\right][1 - \tau]d\tau = G(x, y)$$

is a function defined on $\overline{K}$. Since F has α-Hölder continuous third derivatives, there exists $M_1 > 0$ such that

$$\|G(x, y)\|_{\alpha,1} < M_1.$$

Hence

$$\begin{aligned}
\|(w_1^2 - w_2^2)G(x, y)\|_{\alpha,1} &< M_1 \|w_1^2 - w_2^2\|_{\alpha,1} \\
&\leqq M_1[\|w_1 - w_2\|_{\alpha,1}][\|w_1 + w_2\|_{\alpha,1}]
\end{aligned}$$

since $E_{\alpha,1}$ is a normed ring. But

$$\begin{aligned}
&M_1[\|w_1 - w_2\|_{\alpha,1}][\|w_1 + w_2\|_{\alpha,1}] \\
&\qquad = M_1[\|w(\rho_1, \phi_1) - w(\rho_2, \phi_2)\|_{\alpha,1}][\|w(\rho_1, \phi_1) + w(\rho_2, \phi_2)\|_{\alpha,1}] \\
&\qquad \leqq M_1[\|w(\rho_1 - \rho_2, \phi_1 - \phi_2)\|_{\alpha,3}][\|w(\rho_1 + \rho_2, \phi_1 + \phi_2)\|_{\alpha,3}] \\
&\qquad \leqq M_1M^2[\|\rho_1 - \rho_2\|_{\alpha,1} + \|\phi_1 - \phi_2\|_{\alpha,3}][\|\rho_1 + \rho_2\|_{\alpha,1} + \|\phi_1 + \phi_2\|_{\alpha,3}] \\
&\qquad = M_1M^2[\|(\rho_1 - \rho_2, \phi_1 - \phi_2)\|][\|(\rho_1 + \rho_2, \phi_1 + \phi_2)\|] \\
&\qquad = M_1M^2[\|(\rho_1, \phi_1) - (\rho_2, \phi_2)\|][\|(\rho_1, \phi_1) + (\rho_2, \phi_2)\|].
\end{aligned} \tag{3.18}$$

Now we deal with the term

$$\left\| w_1^2 \int_0^1 \left\{ \frac{\partial^2 F}{\partial z^2}(X, Z_0 + \tau W(\rho_1, \phi_1), R_0) - \frac{\partial^2 F}{\partial z^2}(X, Z_0 + \tau W(\rho_2, \phi_2), R_0) \right] [1 - \tau] d\tau \right\|_{\alpha\, 1}.$$

By Taylor's expansion, this may be written as:

$$\begin{aligned}
\Bigg\| w_1^2 \int_0^1 \Bigg[&\tau [W(\rho_1, \phi_1) - W(\rho_2, \phi_2)] \frac{\partial^3 F}{\partial Z \partial z^2}(X, Z_0 + \tau W(\rho_2, \phi_2), R_0) \\
&+ \tau^2 [W(\rho_2, \phi_2) - W(\rho_1, \phi_1)]^2 \int_0^1 \left[\frac{\partial^4 F}{\partial Z^2 \partial z^2} [X, \lambda \tau [W(\rho_1, \phi_1) \right. \\
&\left. - W(\rho_2, \phi_2)], R_0] \right] d\lambda \Bigg\} \{1 - \tau\} d\tau \Bigg\|_{\alpha,1}.
\end{aligned}$$

(We use in this expression the same type of abbreviation as is used in [13, p. 695. See especially footnote 37].) It readily follows that this expression satisfies condition P_2. This completes the proof that $R_2^{(1)}(\rho, \phi)$ satisfies condition P_2 and hence that $T(\rho, \phi)$ satisfies condition P_2.

Now we prove that $T(\rho, \phi)$ satisfies condition P_4, regardless of the dimension of the null space of $I(\rho, \phi) + C(\rho, \phi)$. We prove first that $R_2^{(1)}(\rho, \phi)$ satisfies condition P_4. It is sufficient to show that P_4 is satisfied by the expression

$$[w(\rho, \phi)]^2 \int_0^1 \left[\frac{\partial F}{\partial z}(X, Z_0 + \tau W(\rho, \phi), R_0) \right] [1 - \tau] d\tau. \tag{3.19}$$

The same type of proof holds for the other terms in $R_2^{(1)}(\rho, \phi)$.

Expression (3.19) may be written as:

$$\begin{aligned}
&[w(\rho, \phi)]^2 \int_0^1 \left[\frac{\partial F}{\partial z}(X, Z_0, R_0) \right] [1 - \tau] d\tau \\
&\quad + [w(\rho, \phi)]^2 \int_0^1 \left[\frac{\partial F}{\partial z}(X, Z_0 + \tau W(\rho, \phi), R_0) - \frac{\partial F}{\partial z}(X, Z_0, R_0) \right] [1 - \tau] d\tau.
\end{aligned}$$

It is clear that the expression $[w(\rho, \phi)]^2\int_0^1[(\partial F/\partial z)(X, Z_0, R_0)][1-\tau]d\tau$ regarded as an operator on (ρ, ϕ) satisfies the conditions of part (1) of P_4. Also it is easy to show that the expression

$$[w(\rho, \phi)]^2\int_0^1\left[\frac{\partial F}{\partial z}(X, Z_0+\tau W(\rho, \phi), R_0)-\frac{\partial F}{\partial z}(X, Z_0, R_0)\right][1-\tau]d\tau$$

regarded as an operator on (ρ, ϕ) satisfies the condition of part (2) of P_4. Hence $R_2^{(1)}(\rho, \phi)$ satisfies condition P_4.

Now we show that $R^{(2)}(\rho, \phi)$ satisfies P_4. From the definition of $R^{(2)}(\rho, \phi)$, it is clear that it is sufficient to prove the statement for the expression:

$$\frac{\partial^2 w}{\partial x^2}(\rho, \phi)\int_0^1\left\{\frac{\partial F}{\partial r}(X, Z_0+W(\rho, \phi), R_0+\lambda V(\rho, \phi)) - \frac{\partial F}{\partial r}(X, Z_0, R_0)\right\}d\lambda. \tag{3.20}$$

By Taylor's expansion, the integrand in (3.20) may be written in the form:

$$\begin{aligned}
&w(\rho, \phi)\frac{\partial^2 F}{\partial z\partial r}+\frac{\partial w(\rho, \phi)}{\partial x}\left[\frac{\partial^2 F}{\partial p\partial r}\right]_0+\frac{\partial w(\rho, \phi)}{\partial y}\left[\frac{\partial^2 F}{\partial q\partial r}\right]_0\\
&+\lambda\frac{\partial^2 w(\rho, \phi)}{\partial x^2}\left[\frac{\partial^2 F}{\partial r^2}\right]_0+\lambda\frac{\partial^2 w(\rho, \phi)}{\partial x\partial y}\left[\frac{\partial^2 F}{\partial s\partial r}\right]_0\\
&+\lambda\frac{\partial^2 w(\rho, \phi)}{\partial y^2}\left[\frac{\partial^2 F}{\partial t\partial r}\right]_0+[w(\rho, \phi)]^2\int_0^1\left[\frac{\partial^3 F}{\partial^2 z\partial r}\right]_\tau(1-\tau)d\tau\\
&+[w]\left[\frac{\partial w}{\partial x}\right]\int_0^1\left[\frac{\partial^3 F}{\partial z\partial p\partial r}\right]_\tau[1-\tau]d\tau\\
&+[w]\left[\frac{\partial w}{\partial y}\right]\int_0^1\left[\frac{\partial^3 F}{\partial z\partial q\partial r}\right]_\tau[1-\tau]d\tau+\cdots\\
&+\lambda^2\left[\frac{\partial^2 w}{\partial x^2}\right]\left[\frac{\partial^2 w}{\partial y^2}\right]\int_0^1\left[\frac{\partial^3 F}{\partial t\partial r^2}\right]_\tau[1-\tau]d\tau\\
&+\lambda^2\left[\frac{\partial^2 w}{\partial x\partial y}\right]^2\int_0^1\left[\frac{\partial^3 F}{\partial s^2\partial r}\right]_\tau[1-\tau]d\tau\\
&+\lambda^2\left[\frac{\partial^2 w}{\partial x\partial y}\right]\left[\frac{\partial^2 w}{\partial y^2}\right]\int_0^1\left[\frac{\partial^3 F}{\partial s\partial t\partial r}\right]_\tau[1-\tau]d\tau\\
&+\lambda^2\left[\frac{\partial^2 w}{\partial y^2}\right]^2\int_0^1\left[\frac{\partial^3 F}{\partial s^2\partial r}\right]_\tau[1-\tau]d\tau
\end{aligned}\tag{3.21}$$

where

$$\left[\frac{\partial^2 F}{\partial z \partial r}\right]_0 = \frac{\partial^2 F}{\partial z \partial r}(X, Z_0, R_0),$$

$$\left[\frac{\partial^3 F}{\partial z^2 \partial r}\right]_\tau = \frac{\partial^3 F}{\partial z^2 \partial r}(X, Z_0 + \tau W(\rho, \phi), R_0 + \tau\lambda V(\rho, \phi))$$

and the other symbols have similar meanings.

When the integrand of (3.20) is written in this form, it is clear that (3.20) satisfies P_4.

Thus we have shown that $(R_2^{(1)}(\rho, \phi), 0)+(R^{(2)}(\rho, \phi), 0)$ is of the form $T^{(2)}+T^{(3)}$. If the second order terms, that is, the terms in $T^{(2)}$, vanish, then by further application of the mean value theorem, it may be shown that the transformation is of the form $T^{(3)}+T^{(4)}$ or, in general, $T^{(k)}+T^{(k+1)}$ where $k \geqq 3$. In order to apply Taylor's expansion in these cases, it may be necessary to assume that the function F has derivatives of order higher than 5.

Since we have shown that condition P_4 is satisfied, regardless of the dimension of the null space of $I+C$, condition P_3 is a consequence of P_4. This completes the proof of the lemma and hence the proof of Theorem 3.2.

Theorem 3.2 shows that the theory of §2 may be applied to investigate the local solutions of equation (3.12) and hence, by Theorem 3.1, to investigate the solutions of equation (3.1). We prove first a version of the classical existence theorem.

THEOREM 3.3. *If the Jacobi equation* (3.13) *associated with equation* (3.1) *has only the zero solution, then equation* (3.1) *has a neighboring solution in the following sense: there exists* $\epsilon_1>0$ *and* $\epsilon_2>0$ *such that for each pair* $\psi \in E_{\alpha,1}$ *and* $\phi \in e_{\alpha,3}$ *satisfying the conditions*

$$\|\psi_0 - \psi\|_{\alpha,1} < \epsilon_1, \qquad \|\phi_0 - \phi\|_{\alpha,3} < \epsilon_1,$$

there is at least one function $z(x, y) \in E_{\alpha,3}$ *such that*:

(1) $\|z_0 - z\|_{\alpha,3} < \epsilon_2$.

(2) *The boundary value of* $z(x, y)$ *is* ϕ.

(3) *In* $\overline{K}$, $z(x, y)$ *satisfies the differential equation*

$$F(x, y, z, p, q, r, s, t) = \psi(x, y).$$

Proof. By Lemma 3.2, the hypothesis implies that $I(\rho, \phi)+C(\rho, \phi)$ is a nonsingular, 1-1 transformation. Hence by Theorem 2.1, equation (3.12) has a unique local solution. The theorem then follows from Theorem 3.1.

Theorem 3.3 is contained in the work of Leray and Schauder (see [8, pp. 74–78]) except for the generalization introduced by varying ψ. The proof given here is new. Theorem 3.3 is more general than the classical result of S. Bernstein (see [3, p. 1325]) in that solutions in $E_{\alpha,3}$ rather than just analytic solutions are considered, the varying ψ is introduced and a bound, $\|z_0\|_{\alpha,3}+\epsilon_2$, is given for the solutions z.

Now we consider the case when the Jacobi equation associated with equation (3.1) has a nonzero solution. This means, by Lemma 3.2, that the operator $I+C$ of equation (3.12) has a null space different from zero. We have first:

THEOREM 3.4. *If the multiplicity of solutions of equation* (3.1) (*Definition* 3.1) *is different from zero, there exist neighboring solutions for equation* (3.1) *in the following sense: there exist* $\epsilon_1>0$ *and* $\epsilon_2>0$ *such that for each pair* $\psi\in E_{\alpha,1}$ *and* $\phi\in e_{\alpha,3}$ *satisfying the conditions*:

$$\|\psi_0 - \psi\|_{\alpha,1} < \epsilon_1, \qquad \|\phi_0 - \phi\|_{\alpha,3} < \epsilon_1,$$

there exists at least one function $z\in E_{\alpha,3}$ *such that*:

(1) $\|z_0-z\|_{\alpha,3}<\epsilon_2$.

(2) *The boundary value of* z *is* ϕ.

(3) *In* $\overline{K}$, z *is a solution of the differential equation*:

$$F(x, y, z, p, q, r, s, t) = \psi(x, y).$$

Proof. This follows from Theorem 2.2 and Definition 3.1.

Combining the results concerning the value of the multiplicity that are given by Theorems 2.4, 2.5, and 2.6 with Theorem 3.6, we obtain more concrete existence theorems. In particular, by applying Theorems 2.4, we obtain:

THEOREM 3.5. *If the Jacobi equation associated with equation* (3.1) *has just one linearly independent solution, if the transformation* T *in the corresponding equation* (3.8) *is of the form* $T^{(k)}+T^{(k+1)}$ *where* k *is odd* (*see condition* P_4), *and if condition* P_5 *is satisfied, then equation* (3.1) *has at least one neighboring solution in the sense described in Theorem* 3.4.

Proof. This follows from Theorem 3.4 and Theorem 2.4.

The significance of the hypothesis $T=T^{(k)}+T^{(k+1)}$ where k is odd may be seen by considering an example: $k=3$. By definition,

$$T(\rho, \phi) = (R_2^{(1)}(\rho, \phi), 0) + (R^{(2)}(\rho, \phi), 0)$$

where $R_2^{(1)}(\rho, \phi)$ and $R^{(2)}(\rho, \phi)$ are defined by equations (3.8) and (3.9) respectively. An examination of (3.8) and the treatment of $R^{(2)}(\rho, \phi)$ in the proof of Lemma 3.4, in particular equations (3.20) and (3.21), shows that if the terms $[\partial^2F/\partial u\partial v]_0$, where $u, v=z, p, q$, and the terms $[\partial^2F/\partial u\partial r]_0$, where $u=z, p, q, r, s, t$, are identically zero, then $T=T^{(3)}+T^{(4)}$.

By making similar application of Theorems 2.4, 2.5, and 2.6, other existence theorems analogous to Theorem 3.5 may be obtained.

An example of these results is the work of R. Iglisch ([5] and [6]). Iglisch considers the equation

$$\Delta u = F(u) \tag{3.22}$$

where $F(u)$ is a real, analytic function. By using Green's function, an integral equation is derived from (3.22). The problem of solving (3.22) is equivalent to the problem of solving the integral equation. The study of the integral equation runs parallel to the study of equation (3.12). Iglisch obtains a theorem [5, p. 101] which is a special case of Theorem 3.3. He also obtains a theorem which is, in a certain sense, a special case of Theorem 3.5. It is not strictly a special case of Theorem 3.5 because while it is a special case of equation (2.1), it is not a special case of equation (3.12). Iglisch assumes that the Jacobi equation associated with (3.22) has just one linearly independent solution. His hypothesis that $L_k \neq 0$ [5, p. 110] is a special case of the hypotheses that $T = T^{(k)} + T^{(k+1)}$ and that P_5 is satisfied. Hence this hypothesis is exactly analogous to the application of the hypotheses $T = T^{(k)} + T^{(k+1)}$ and P_5 in the statement of Theorem 3.5.

REMARKS FOR §3.

1. It can be seen that the crucial statement in §3 is Lemma 3.1. This lemma is applied repeatedly in the work. It is because of the hypotheses of this lemma that we are restricted to studying solutions of the differential equation (3.1) that are in the space $E_{\alpha,3}$. The possibility of generalizing the theory of §3 to a study of solutions of (3.1) in $E_{\alpha,2}$ depends entirely upon the possibility of generalizing Lemma 3.1 in an analogous fashion. According to a footnote in [13] (footnote 43a, p. 703), Giraud asserted in a letter to Schauder that Lemma 3.1 can be generalized in this way.

2. In this and the following sections, we consider, for simplicity, the case of two independent variables. All the discussion can be carried through for the case of n variables. The only change which has to be made is that Lemma 3.1 has to be generalized to a statement in n variables. The statement in n variables of Lemma 3.1 can be obtained from [14].

4. **The relation between the multiplicity of solutions and the number of distinct solutions.** The multiplicity of Definition 2.1 gives no criterion for determining if there is more than one solution of equation (2.1). That is, the multiplicity may be m such that $|m| > 1$, but there may be just one local solution x of the equation. From Definition 2.1 it can be seen that this inability of the multiplicity to distinguish whether or not there are several distinct solutions stems directly from the corresponding inability of the topological degree of a mapping in Euclidean space. That is, if the topological degree at point p of a mapping M which takes Euclidean n-space E^n into itself is m such that $|m| > 1$, then the equation $M(x) = p$ has at least one but not necessarily more than one solution. A simple example of this is the equation in E^2: $Z^m = 0$ where $m > 1$. This equation is said to have m roots and the mapping $Z \rightarrow Z^m$ has topological degree m at 0. But there is just one point, the point 0, which is taken into 0 by this mapping.

Hence the applications of the multiplicity in §3 yield no information concerning the number of distinct solutions of equation (3.1). However, by restricting our considerations to a class of quasi-linear elliptic differential equations, we show in this section how the theory of §2 may be used to demonstrate the existence of several distinct local solutions for certain quasi-linear elliptic equations.

The class of differential equations considered will be restricted so that the abstract equation (equation (3.12)) associated with the differential equation has the form

$$I(\rho, \phi) + D(\rho, \phi) = (\rho, \phi)$$

where D is a completely continuous transformation. For a mapping of the form $I+D$, the Leray-Schauder degree (hereafter to be designated as the LS degree) [8] is defined. Moreover the LS degree and the multiplicity of Definition 2.1 differ at most by a factor -1 (Theorem 2.2). Hence for this class of differential equations, the multiplicity has the properties of the LS degree. In particular, the property of the LS degree that is described in Theorem 4.1 is used to demonstrate the existence of several distinct solutions for certain equations.

We consider the following quasi-linear elliptic differential equation:

$$\begin{aligned} F(x, y, z, p, q, r, s, t) &= A(x, y)\frac{\partial^2 Z}{\partial x^2} + B(x, y)\frac{\partial^2 Z}{\partial x \partial y} \\ &\quad + C(x, y)\frac{\partial^2 Z}{\partial y^2} + f(x, y, z, p, q) = \psi(x, y) \end{aligned} \tag{4.1}$$

where $A(x, y)$, $B(x, y)$, $C(x, y) \in E_{\alpha,1}$, f has α-Hölder continuous fifth derivatives in all its variables, and $\psi(x, y) \in E_{\alpha,1}$. We also make the following assumption.

ASSUMPTION 4.1. There exists $\psi_0(x, y) \in E_{\alpha,1}$ and $\phi_0 \in e_{\alpha,3}$ such that if $\psi = \psi_0$, equation (4.1) has a solution $z_0(x, y) \in E_{\alpha,3}$ whose boundary value is ϕ_0.

The theory of §3 may be applied to equation (4.1) and we obtain as before equation (3.11) in $E_{\alpha,1} \times e_{\alpha,3}$. However, in this case, $\partial F/\partial r$, $\partial F/\partial s$, $\partial F/\partial t$ are $A(x, y)$, $B(x, y)$, and $C(x, y)$ respectively. Hence from the definition of $R^{(2)}(\rho, \phi)$ (see equation (3.9)) it follows that, in this case, $R^{(2)} = 0$. Hence the abstract equation associated with equation (4.1) is

$$I(\rho, \phi) + (R^{(1)}(\rho, \phi), 0) + (R_1^{(2)}(\rho, \phi), 0) = (\psi, \phi). \tag{4.2}$$

Denoting $(R_1^{(1)}(\rho, \phi), 0)$ by $C(\rho, \phi)$ and $(R_2^{(1)}(\rho, \phi), 0)$ by $T(\rho, \phi)$, we may write equation (4.2) as

$$I(\rho, \phi) + C(\rho, \phi) + T(\rho, \phi) = (\psi, \phi). \tag{4.3}$$

LEMMA 4.1. *The LS degree is defined for the transformation* $I+C+T$ *in equation* (4.3).

Proof. The LS degree is defined for transformations of the type $I+D$ where D is completely continuous [8]. In [13, pp. 694–697] it is proved that $R_1^{(1)}(\rho, \phi)+R_1^{(2)}(\rho, \phi)$ is completely continuous. Hence $C+T$ is completely continuous. This completes the proof of the lemma.

Now by Theorem 2.2, the LS degree of $I+C+T$ and the multiplicity of solutions of equation (4.2) are the same except for at most a factor -1. That means that in this case, the multiplicity of solutions has all the properties of the LS degree besides the multiplicity properties described in §2.

We want to determine when equation (4.3) (and consequently equation (4.1)) has more than one solution. We suppose that equation (4.3) has a multiplicity of solutions m such that $|m|>1$. If (4.3) has more than one distinct solution, then we are all through. However, the fact that the multiplicity is m does not imply that (4.3) has more than one solution. All that is known is that the sum of the topological indices of the solutions is m. (See [8, pp. 54–55].) There might be just one solution with topological index m.

However, we shall show that in this case, if ϕ and ψ are varied, no matter how slightly, then equation (4.3), and hence equation (4.1), will have at least two distinct solutions. In order to show this, we prove the following theorem concerning the Leray-Schauder topological index.

THEOREM 4.1. *Suppose $I+D$ is a mapping of a subset of Banach space $\mathfrak{X}$ into $\mathfrak{X}$ for which the LS degree is defined. Suppose $(I+D)p=p_1$ and that the LS topological index of p is m such that $|m|>1$. Then if U is an arbitrary neighborhood of p_1, there exists $q\in U$ and a neighborhood V of p such that V contains at least two distinct points which map under $I+D$ into q.*

Proof. We prove first:

LEMMA 4.2. *Let $I+D$ be a* 1-1 *mapping. If Q is a closed bounded subset in the domain of $I+D$, then $I+D$ is a homeomorphism of Q onto $(I+D)Q$.*

Proof. We show that $I+D$, regarded as a mapping on Q, is an open mapping. Let N be an open set in the induced topology on Q. Then $Q-N$ is closed in the topology of Q and the topology of X. But $(I+D)(Q-N)$ is closed because D is completely continuous. Since $I+D$ is 1-1,

$$(I + D)N = (I + D)Q - (I + D)[Q - N]$$

is an open set in the induced topology on $(I+D)Q$. This completes the proof of the lemma.

LEMMA 4.3. *If $I+D$ is a homeomorphism of $\overline{O}$, the closure of a bounded, connected, open set, onto $(I+D)\overline{O}$, and if $q\in(I+D)O$, then the LS degree of $I+D$ at q is* 1 *or* -1.

Proof. The proof of this lemma is given in [7].

Now the theorem can be proved. We suppose that there exists a neighborhood U_1 of p_1 such that each point of U_1 is the image of just one point under the mapping $I+D$. Let V_1 be a connected open neighborhood of $(I+D)^{-1}p_1$ such that $\overline{V}_1 \subset (I+D)^{-1}U_1$. Then since $I+D$ is 1-1 on $\overline{V}_1$, $I+D$ is a homeomorphism of $\overline{V}_1$ onto $(I+D)\overline{V}_1$, by Lemma 4.2. Hence by Lemma 4.3, the LS degree at p_1 relative to $\overline{V}_1$ of $I+D$ is 1 or -1. This is in contradiction to the hypothesis. This completes the proof of the theorem.

Theorem 4.1 makes it possible to demonstrate the existence of at least two distinct solutions for certain elliptic equations. From Theorem 3.1, Lemma 4.1, and Theorem 4.1, we obtain:

THEOREM 4.2. *If the multiplicity of solutions of equation* (4.1) (*Definition* 3.1) *is m such that $|m|>1$, then equation* (4.1) *admits at least two distinct solutions in the following sense: if $\epsilon>0$, there exists $\psi_1 \in E_{\alpha,1}$ and $\phi_1 \in e_{\alpha,3}$ such that*

$$\|\psi_1 - \psi_0\|_{\alpha,1} < \epsilon, \qquad \|\phi_1 - \phi_0\|_{\alpha,3} < \epsilon,$$

and such that there exist l $(\geqq 2)$ distinct functions $z_{11}, \cdots, z_{1l} \in E_{\alpha,3}$ which have ϕ_1 as their boundary value and which satisfy the equation:

$$A(x, y)\frac{\partial^2 z}{\partial x^2} + B(x, y)\frac{\partial^2 z}{\partial x \partial y} + C(x, y)\frac{\partial^2 z}{\partial y^2} + f(x, y, z, p, q) = \psi_1(x, y).$$

Criteria for determining when the absolute value of the multiplicity is greater than one are given by Theorems 2.5 and 2.6. These theorems together with Theorem 4.2 yield the desired existence theorems. A typical such theorem is:

THEOREM 4.3. *If the Jacobi equation associated with equation* (4.1) *has two linearly independent solutions* (*then by Lemma* 3.2, *the null space of $I+C$ in equation* (4.3) *has dimension two*); *if in equation* (4.3), $T=T^{(2)}+T^{(3)}$ (*cf. condition* P_4); *if condition* (1) *of Theorem* 2.6 *is satisfied, then equation* (4.1) *has at least two distinct solutions in the sense described in Theorem* 4.2.

Proof. This is an immediate consequence of Theorems 2.6 and 4.2.

5. **Quasi-linear differential equations with a parameter.** The techniques of §§3 and 4 may also be used to study the existence of multiple solutions of quasi-linear equations with a parameter. For this work, it is necessary to know how the multiplicity of solutions of the differential equation changes when the parameter is varied. Now it is known [8] that the LS degree is invariant under homotopy. Hence if we restrict ourselves as in §4 to quasi-linear equations, that is, to equations for which the multiplicity is equal to the LS degree of the associated mapping given by equation (3.12) (equal except possibly for sign), then it will be certain that the multiplicity remains constant (except possibly for a change of sign) when the parameter is varied.

By using the same methods as in §3, theorems which are generalizations and extensions of the classical existence theorems for differential equations with a parameter may be obtained for the quasi-linear elliptic differential equation:

$$A(x, y)\frac{\partial^2 z}{\partial x^2} + B(x, y)\frac{\partial^2 z}{\partial x \partial y} + C(x, y)\frac{\partial^2 z}{\partial y^2} + f(x, y, z, p, q, \lambda) = \psi(x, y) \tag{5.1}$$

where $A(x, y)$, $B(x, y)$, $C(x, y) \in E_{\alpha,1}$, λ is a real parameter such that $0 \leqq \lambda < 1$, f has α-Hölder continuous fifth derivatives in all its variables, and $\psi(x, y) \in E_{\alpha,1}$.

Just as in §4, it is also possible to demonstrate the existence of several distinct solutions if the multiplicity of solutions of equation (5.1) is greater than one.

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