ABELIAN GROUP ALGEBRAS OF FINITE ORDER

BY

SAM PERLIS AND GORDON L. WALKER

Introduction. A group G of finite order n and a field F determine in well known fashion an algebra G_F of order n over F called the group algebra of G over F. One fundamental problem(1) is that of determining all groups H such that H_F is isomorphic to G_F.

It is convenient to recast this problem somewhat: If groups G and H of order n are given, find all fields F such that G_F is isomorphic to H_F (notationally: $G_F \cong H_F$). We present a complete solution of this problem for the case in which G (and thus necessarily H) is abelian and F has characteristic infinity or a prime not dividing n. The result, briefly, is that F shall contain a certain subfield which is determined by the invariants of G and H and the characteristic of F.

1. **Multiplicities.** If G is abelian of order n and F is a field whose characteristic does not divide n, the group algebra G_F has the structure

$$G_F = \sum_{d \mid n} a_d F(\zeta_d) \tag{1}$$

where ζ_d is a primitive dth root of unity, a_d is a non-negative integer, and $a_d F(\zeta_d)$ denotes the direct sum of a_d isomorphic copies of $F(\zeta_d)$. In fact, each irreducible representation S of G_F maps G_F onto a field $F_S \geqq F$ and maps the elements of G on nth roots of unity. The image of G is a subgroup of the group of all nth roots of unity, thus is a cyclic group of some order dividing n. It follows that $F_S = F(\zeta_d)$ where ζ_d is a primitive dth root of unity. Formula (1) expresses the fact that a complete set of irreducible representations of G_F over F include precisely a_d which map G onto a cyclic group of order d. Now if K is the root field over F of $x^n - 1 = 0$ we have

$$G_K = \sum_{d \mid n} n_d K_d \tag{2}$$

where every $K_d = K(\zeta_d)$ is isomorphic to K, $\sum n_d = n$, and each n_d is the number of irreducible representations T of G_K mapping G on a cyclic group of order d.

Lemma 1. *The integer n_d in* (2) *is the number of elements of order d in G.*

There is a one-to-one correspondence between the elements g of G and the

Presented to the Society, April 16, 1948, and October 30, 1948, under the titles *Finite abelian group algebras*, I and II; received by the editors September 22, 1948 and, in revised form, April 29, 1949.

(1) Proposed by R. M. Thrall at the Michigan Algebra Conference in the summer of 1947.

representations $T=T_g$. The formulae(2) for this correspondence make it evident that g has order d if and only if T_g maps a basis of G onto a set of elements, the l.c.m. of whose orders is d. Then some element of G is mapped on an element of order d, all others on elements of order not greater than d. The map of G is thus a cyclic group of order d, and this proves the lemma.

Each irreducible representation S of G_F over F may be extended to a representation of G_K over K, the extension not altering the map of G. If S maps G_F onto $F(\zeta_d)$ where the degree of $F(\zeta_d)/F$ is

$$\deg F(\zeta_d)/F = v_d, \tag{3}$$

then S maps G_K on the direct sum(3)

$$F(\zeta_d)_K = K^{(1)} \oplus \cdots \oplus K^{(v_d)} = v_d K, \tag{4}$$

thus giving rise to v_d irreducible representations T of G_K over K.

Lemma 2. *If S maps G onto a cyclic group of order d, so does each representation T defined above.*

Each element g in G is mapped by S on $g^S = \sum g_i$, g_i in $K^{(i)}$, and the corresponding irreducible representations over K are T_i: $g^{T_i}=g_i$. It may be seen(4) that the g_i are obtainable from one another by automorphisms of $F(\zeta_d)_K$ leaving the elements of K invariant. Hence all the g_i have the same minimum function over K, and all of them are primitive dth roots of unity if g^S is one. Lemma 2 follows immediately, and it follows that the T_i into which the representations S split are the only irreducible representations of G_K mapping G on a cyclic group of order d. The a_d choices of S give rise to $a_d v_d$ representations T, whence $n_d = a_d v_d$.

Theorem 1. *The multiplicity a_d in* (1) *is given*(5) *by $a_d = n_d/v_d$ where n_d is the number of elements of order d in G and v_d is* deg $F(\zeta_d)/F$.

Now let G and H be abelian of common order $n = p_1^{e_1} \cdots p_k^{e_k}$ for distinct primes p_i, so there are unique expressions $G = G_1 \times \cdots \times G_k$ and $H = H_1 \times \cdots \times H_k$ for G and H as direct products of groups G_i and H_i of order $n_i = p_i^{e_i}$. Then:

Corollary 1. *$G_F \cong H_F$ if and only if $G_{iF} \cong H_{iF}$ for $i = 1, \cdots, k$.*

By hypothesis and Theorem 1

(2) A. Speiser, *Die Theorie der Gruppen von endlicher Ordnung*, New York, 1945, p. 179.

(3) A. A. Albert, *Structure of algebras*, Amer. Math. Soc. Colloquium Publications, vol. 24, New York, 1939, p. 31.

(4) Ibid.

(5) The authors are indebted to the referees for the simple approach to Theorem 1 which has been presented here.

$$G_F = \sum_{d|n} m_d/v_d F(\zeta_d) \cong H_F,$$

$$G_{iF} = \sum_{d|n_i} g_{id}/v_d F(\zeta_d), \qquad H_{iF} = \sum_{d|n_i} h_{id}/v_d F(\zeta_d)$$

where the number of elements of order d in G_i is g_{id}, in H_i is h_{id}, and in G or H is m_d. But if $d \mid n_i$, the elements of G having order d lie in G_i, so $m_d = g_{id}$ and likewise $m_d = h_{id}$ so $g_{id} = h_{id}$, whence $G_{iF} \cong H_{iF}$. The converse is trivial.

In the remaining sections only the prime-power case is considered.

2. **Cyclotomic fields.** When $n = p^\alpha$ for a prime p the notation in (1) will be changed to

$$G_F = \sum_{i=0}^{\alpha} a_i F(\zeta_i) \tag{5}$$

where ζ_i and a_i are new symbols for ζ_d and a_d, $d = p^i$. This section explores conditions under which $F(\zeta_i) \cong F(\zeta_j)$. Taking $i \leqq j$ we may and shall assume that $F(\zeta_i) \leqq F(\zeta_j)$, so the question now is concerned with the equality of these fields. Let P always denote the prime subfield of F.

Lemma 3. *Let i and j be positive integers such that $i < j$. Then $F(\zeta_i) = F(\zeta_j)$ if and only if F has a subfield $F_0 \leqq P(\zeta_j)$ such that $F_0(\zeta_i) = F_0(\zeta_j)$.*

Proof. If $F_0(\zeta_i) = F_0(\zeta_j)$, the field $F(\zeta_i)$ must contain ζ_j. Conversely, suppose $F(\zeta_i) = F(\zeta_j)$. The minimum function $f(x)$ of ζ_j over F has degree s equal to that of ζ_i, and is a factor of the minimum function $m(x)$ of ζ_j over P. The coefficients of $f(x)$ then must lie in the root field $P(\zeta_j)$ of $m(x)$ over P, and hence generate a subfield F_0 of $P(\zeta_j)$ such that $F_0 \leqq F$. Then $F_0(\zeta_j) \geqq F_0(\zeta_i)$, and

$$\deg F_0(\zeta_j)/F_0 = s \geqq \deg F_0(\zeta_i)/F_0 = r \geqq \deg F(\zeta_i)/F = s,$$

whence $r = s$, $F_0(\zeta_i) = F_0(\zeta_j)$.

It is necessary now to make a brief detour because of some peculiarities arising if P is finite. Suppose that

$$P \leqq P(\zeta_1) = \cdots = P(\zeta_e) < P(\zeta_{e+1}) \qquad (e \geqq 1) \tag{6}$$

if p is odd, and

$$P \leqq P(\zeta_2) = \cdots = P(\zeta_e) < P(\zeta_{e+1}) \qquad (e \geqq 2) \tag{7}$$

if $p = 2$. These equalities never occur if $P = R$ but do occur if P is a finite prime field whose characteristic is appropriately related to p (see Lemma 5).

Definition. Let p be a prime and let P be a prime field of characteristic not equal to p. Then the integer e defined by (6) and (7) is called the cyclotomic number of P relative to p (or cyclotomic p-number of P).

Lemma 4. *Let P be a finite prime field of characteristic π, n be an integer not* *divisible by* π, *and* $P(\zeta)$ *be the root field over* P *of* x^n-1. *Then* $\deg P(\zeta)/P=\epsilon$ *where* ϵ *is defined as the exponent to which* π *belongs modulo* n.

Let P_f be a field of degree f over P so its nonzero quantities are roots of $x^\nu-1=0$, $\nu=\pi^f-1$. Then P_f contains the nth roots of unity if n divides ν. Conversely, if P_f contains a primitive nth root of unity, ζ, the equation $\nu=qn+r$ $(0\leqq r<n)$ leads to $\zeta^\nu=1=\zeta^r$ so $r=0$, and n divides ν. The smallest value of $\nu=\pi^f-1$ obeying this condition is given by $f=\epsilon$. On the other hand the smallest value surely belongs to $P_f=P(\zeta)$.

Now let $n=p^i$, where p is a prime not equal to π, and denote the corresponding integer ϵ of Lemma 4 by ϵ_i. Then the cyclotomic p-number of P is the integer e determined by the conditions $\epsilon_1=\epsilon_2=\cdots=\epsilon_e<\epsilon_{e+1}$ (p odd), $\epsilon_2=\epsilon_3=\cdots=\epsilon_e<\epsilon_{e+1}$ $(p=2)$. Hence:

LEMMA 5. *The cyclotomic p-number of P is the maximum integer e such that p^e divides $\pi^\epsilon-1$ where ϵ is the exponent to which π belongs modulo p if p is odd, or modulo* 4 *if* $p=2$.

The fact that $P(\zeta_i)<P(\zeta_{i+1})$ for every $i\geqq e$ is a consequence of the following result.

LEMMA 6. *The extension* $P(\zeta_{e+i})/P(\zeta_e)$ *has degree* $\delta_i=p^i$ $(i=1, 2, \cdots)$.

Writing $\epsilon_e=\epsilon$ we have $\delta_i=\epsilon_{e+i}/\epsilon$ and know(6) that $\delta_i=p^j$, $j\leqq i$, $\epsilon_{e+i}=p^j\epsilon$. By Lemma 5, $\pi^\epsilon=1+ap^e$ where a is not divisible by p. A trivial induction shows that

$$\pi^{p^i\epsilon}=1+a_ip^{e+i},\qquad (a_i, p)=1,$$

for $i=0, 1, 2, \cdots$. This proves that $\epsilon_{e+i}=p^i\epsilon$.

LEMMA 7. *If p is an odd prime and P is any prime field of characteristic not p, $P(\zeta_q)$ has the structure*

$$P(\zeta_q)=P(\zeta_1)\times L_q,\qquad \deg L_q/P=\text{power of } p, \tag{8}$$

where L_q is unique. Moreover, $L_q=P$ if q does not exceed the cyclotomic p-number of P.

The proof of this result is similar to the known(7) proof for the case $P=R$.

LEMMA 8. *Let p be odd and $q>1$. Then the following conditions are equivalent*:

(i) $F(\zeta_q)=F(\zeta_i)$, $1\leqq i<q$.
(ii) $F(\zeta_q)=F(\zeta_{q-1})=\cdots=F(\zeta_1)$.
(iii) *F contains the field L_q defined by Lemma* 7.

(6) A. A. Albert, *Modern higher algebra*, Chicago, 1937, p. 188, Theorem 21. The desired result is obtained by repeated application of this reference theorem.

(7) Robert Fricke, *Lehrbuch der Algebra*, vol. 3, Braunschweig, 1928, p. 205.

The condition (iii) implies that $F(\zeta_1)$ contains $L_q(\zeta_1)=P(\zeta_q)$, $F(\zeta_1)=F(\zeta_q)$, so (ii) follows. That (ii) implies (i) is obvious. Now we assume (i) and use Lemma 3 to reduce considerations to the case $F\leqq P(\zeta_q)=F(\zeta_q)$. If $q\leqq e$ where e is the cyclotomic p-number of P, $L_q=P\leqq F$ so (iii) is valid. Now let q be greater than e.

The field $F(\zeta_i)$ is the composite $F\cup P(\zeta_i)$. Denoting the intersection $F\cap P(\zeta_i)$ by F_i, we have

$$\deg F/F_i = \deg F(\zeta_i)/P(\zeta_i) = \deg P(\zeta_q)/P(\zeta_i). \tag{9}$$

Also, deg $P(\zeta_k)/P=p^{\epsilon_k}u$, deg $F/P=p^a v$ for suitable integers ϵ_k, a, $u=\deg (P(\zeta_1)/P$, and v a divisor of u. To complete preparations for substituting in (9) note that $P(\zeta_q)/P$ is cyclic, hence possesses a unique subfield of any given degree dividing $p^{\epsilon_q}u$. Thus: deg $F_i/P=gcd[p^a v,\ p^{\epsilon_i}u]=p^\mu v$ where $\mu=\min\ [a,\ \epsilon_i]$. From (9), $p^{a-\mu}=p^c$ where $c=\epsilon_q-\epsilon_i=a-\mu$. Since $q>e$, we have $\epsilon_q-\epsilon_i>0$, $\mu<a$, $\mu=\epsilon_i$, so $a=\epsilon_q$, deg $F/P=p^{\epsilon_q}v$. Every such subfield F of $P(\zeta_q)$ must contain the subfield L_q of degree p^{ϵ_q}.

For the case $p=2$ similar results are obtainable. The extension $P(\zeta_q)/P$ is cyclic of degree a power of 2 if P is finite, and for this case we define

$$L_q = P \quad \text{if} \quad q \leqq e, \qquad L_q = P(\zeta_q) \quad \text{if} \quad q > e, \tag{10}$$

where e is the cyclotomic number of P relative to $p=2$. For $P=R$ we have $P(\zeta_q)=P(\zeta_2)\times L_q$ where L_q is arbitrarily one of the fields

$$L_q = P(\zeta_q + \zeta_q^{-1}), \qquad L_q = P(\zeta_q - \zeta_q^{-1}) \tag{11}$$

and deg $L_q/P=2^{q-2}$. We then state without proof:

LEMMA 9. *Let $p=2$ and $q>2$. Then the following conditions are equivalent:*
(i) $F(\zeta_q)=F(\zeta_i)$, $2\leqq i<q$.
(ii) $F(\zeta_q)=F(\zeta_{q-1})=\cdots=F(\zeta_2)$.
(iii) *F contains one of the fields L_q above.*

3. **Determination of the fields.** Let G and H be abelian groups of common prime-power order p^α and let F be any field of characteristic not p. In this section all fields F are determined such that $G_F\cong H_F$.

As in (5) we have

$$G_F = \sum_{i=0}^{\alpha} a_i F(\zeta_i), \qquad H_F = \sum_{i=0}^{\alpha} b_i F(\zeta_i), \tag{12}$$

so there is a unique integer $q=q(G,\ H)$ defined as the maximum integer i such that $a_i\neq b_i$. From Theorem 1 this integer is the maximum i such that $m_i\neq n_i$ where m_i and n_i are the numbers of elements of order p^i in G and H, respectively. Thus q is independent of F. Since $m_0=n_0=1$, q is never less than 2, but it may happen that q does not exist, that is, every $m_i=n_i$. In this case we define $q=0$.

THEOREM 2. *The group algebras G_F and H_F are isomorphic if and only if* (α) *holds when p is odd, and* (β) *or* (γ) *holds when $p=2$:*

(α) *$F \geqq L_q$ defined by Lemma* 7.

(β) *G and H have the same number of invariants and F contains one of the fields L_q defined by Lemma* 9.

(γ) *G and H have unequal numbers, γ and η, of invariants and F contains $P(\zeta_q)$ where P is the prime subfield of F.*

If $q=0$ the theorem is trivial, so we assume $q>0$, hence $q \geqq 2$. Note that $G_F \cong H_F$ if and only if $A \cong B$ where

$$A = \sum_{i=0}^{q} a_i F(\zeta_i), \qquad B = \sum_{i=0}^{q} b_i F(\zeta_i). \tag{13}$$

Suppose (α) holds. Then (Lemma 8) both A and B becomes $F \oplus mF(\zeta_1)$ for a suitable integer m, so $A \cong B$. If $p=2$, $F(\zeta_1)=F$, $a_1=2^\gamma-1$ so

$$A = 2^\gamma F \oplus \sum_{i=2}^{q} a_i F(\zeta_i), \qquad B = 2^\eta F \oplus \sum_{i=2}^{q} b_i F(\zeta_i) \tag{14}$$

whence (β) implies that $A=2^\gamma F \oplus mF(\zeta_2) \cong B$. If ($\gamma$) holds, A and B are diagonal over F and of the same order, hence isomorphic. Conversely, suppose $A \cong B$ and first let p be odd. The assumption that $F(\zeta_q)$ is not isomorphic to $F(\zeta_i)$ for $i<q$ implies that A has precisely a_q components $F(\zeta_q)$ and B has precisely b_q such components. But then the fact that $a_q \neq b_q$ conflicts with the isomorphism of A and B. Hence $F(\zeta_q)=F(\zeta_i)$ for $i<q$ so $F \geqq L_q$. The proofs for $p=2$ are obtained in similar fashion.

The case in which F is a prime field is interesting.

THEOREM 3. *Let G and H be abelian groups of order p^α. If R is the rational number field, $G_R \cong H_R$ if and only if $G \cong H$. If P is a finite prime field of characteristic $\pi \neq p$, $G_P \cong H_P$ if and only if $q \leqq e$ (where e is the cyclotomic p-number of P) unless $p=2$ and G and H have different numbers of invariants. In the latter case $G_P \cong H_P$ if and only if $q \leqq e$ and $\pi \equiv 1$* (mod 4).

For $F=R$ the decompositions (12) are unique. Hence the condition $G_R \cong H_R$ implies that $q=0$, and for each integer $k=p^h$ dividing p^α, G and H have the same number of elements of order k. This number is $N_k(G)\phi(k)$ where ϕ denotes the Euler ϕ-function and $N_k(G)=N_k$ the number of cyclic subgroups of order k in G. The numbers N_k have been determined([8]) by formulae which show that the group invariants are determined when the N_k

([8]) G. A. Miller, *Number of the sub-groups of any abelian group*, Proc. Nat. Acad. Sci. U. S. A. vol. 25 (1939) pp. 256–262; see also Yenchien Yeh, *On prime power abelian groups*, Bull. Amer. Math. Soc. vol. 54 (1948) pp. 323–327.

are specified. Thus $G \cong H$. The remaining parts of the theorem follow from Theorem 2 and our lemmas.

To compute the "q-number" directly from the invariants of G and H, denote the latter by p^{e_i} $(i=1, \cdots, \gamma)$ and p^{f_i} $(i=1, \cdots, \eta)$, respectively, numbered in descending order of magnitude.

THEOREM 4. *Define* λ *as the minimum integer* i *such that* $e_i \neq f_i$. *Then* $q = \max [e_\lambda, f_\lambda]$.

For proof, note that $G = K \times \overline{G}$, $H = K \times \overline{H}$ where K has invariants p^{e_i}, $i=1, \cdots, \lambda-1$, and those of $\overline{G}$ and $\overline{H}$ are evident. Let the common order of $\overline{G}$ and $\overline{H}$ be $\bar{n}$ and let the numbers of elements of order p^i in G, H, and K, respectively, be m_i, n_i, and k_i. Then $i > e_\lambda$ implies $m_i = \bar{n}k_i$ and $i > f_\lambda$ implies $n_i = \bar{n}k_i$. For definiteness take $e_\lambda > f_\lambda$, so $i > e_\lambda$ implies $m_i = n_i$, $q \leqq e_\lambda$. For $i = e_\lambda > f_\lambda$, however, $n_i = \bar{n}k_i$, $m_i > n_i$. This proves that $q = e_\lambda$.

PURDUE UNIVERSITY,
LAFAYETTE, IND.