

ON GENERALIZED JACOBIANS

BY

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INTRODUCTION

1.1. The title of this paper refers to the Jacobians introduced by the writers [2, §1.30; 6, IV.3]⁽¹⁾ and used by them extensively in their work on the theory of the Lebesgue area of surfaces. During the war years Cesari in Italy developed a theory based on similar concepts [3, 4]. Relationships between these two theories have been investigated in some detail in [5] and have been further clarified by Cesari in his invited address on *Area and representation of surfaces*, presented at the annual meeting of the American Mathematical Society in December 1948. This address is to appear in the Bulletin of the American Mathematical Society. In the sequel references to previous literature will be made only in connection with technical matters. The reader interested in historical background is referred to the address of Cesari and to the Colloquium publication [6].

1.2. The principal issue to be studied in this paper may be described as follows. Let \mathfrak{R} be a bounded simply connected Jordan region in a Euclidean uv -plane, and let $x_i(u, v)$, $i = 1, 2, 3$, be continuous functions in \mathfrak{R} . If x_1, x_2, x_3 are Cartesian coordinates in Euclidean three space E_3 , then the equations $x_i = x_i(u, v)$, $(u, v) \in \mathfrak{R}$, for $i = 1, 2, 3$ determine an F -surface S of the type of the 2-cell [6, II.3.44]. If the Lebesgue area of S is finite then there exist generalized Jacobians almost everywhere in the interior \mathfrak{R}^0 of \mathfrak{R} [6, V.2.61]. Now let $\bar{x}_1, \bar{x}_2, \bar{x}_3$ be Cartesian coordinates in E_3 related to x_1, x_2, x_3 by transformation formulas $\bar{x}_i = a_{ij}x_j$ for $i = 1, 2, 3$, where repeated index j indicates a summation over the range of j , and $\|a_{ij}\|$ is a matrix of real numbers which is normal, orthogonal, and has a determinant equal to $+1$. The generalized Jacobians relative to these two systems of coordinates are related to each other almost everywhere in \mathfrak{R}^0 by the linear transformation with the matrix $\|a_{ij}\|$ [10, §4]. However, the exceptional set of points in \mathfrak{R}^0 of measure zero at which no relationship between the Jacobians is available depends upon the frames of reference x_1, x_2, x_3 and $\bar{x}_1, \bar{x}_2, \bar{x}_3$ in E_3 . There arises the question as to whether the transformation formulas for the generalized Jacobians actually hold "uniformly almost everywhere" in \mathfrak{R}^0 in the sense that there is a fixed set of measure zero in \mathfrak{R}^0 such that at every point in \mathfrak{R}^0 not in this set, for every choice of the frames of reference x_1, x_2, x_3 and $\bar{x}_1, \bar{x}_2, \bar{x}_3$, the generalized Jacobians are defined and related by the same law which relates the coordinates

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(1) Numbers in brackets refer to the bibliography at the end of the paper.

in the frames of references. In view of the linear character of the transformation formulas this question may seem trivial, yet we are unable to give a final answer to the question in the form just stated. On the other hand, we have succeeded in answering a series of related questions concerning the "uniformly almost everywhere" validity of various fundamental relationships amongst generalized Jacobians. Our main results are summarized in the theorem in §3.14. The fourth part of this paper contains an application to the ordinary Jacobians.

1.3. While the theory of the Lebesgue area is now practically complete for surfaces of the type of the 2-cell, the extension of this theory to higher dimensions has been initiated only recently by Federer [7]. It appears that various alternative definitions of the fundamental concepts, which are fully interchangeable in the 2-cell case, become essentially different in higher dimensions. The case of the essential multiplicity function κ (6, IV.1) is a significant example. In the 2-cell case there are at least four alternative definitions for κ , which were denoted by ψ , ψ^* , Ψ , Ψ^* in (5, I.1.11). Of these, ψ^* coincides with the original κ , Ψ is the multiplicity function used by Cesari (see §1.1), while ψ and Ψ^* were introduced essentially as auxiliary concepts. It was shown that these four functions differ from each other only on a countable set of points [5, I.4.6], and hence are fully interchangeable. On the other hand, the work of Federer indicates that in higher dimensions, the function defined analogously to Ψ^* may be the most suitable choice. Since $*$ is used in the sequel for a specific purpose, let us use \mathbf{K} as a more convenient notation for Ψ^* . For the purpose of extending the theory to higher dimensions, the following comment may be relevant. In the 2-cell case, essential use is made in (10) not only of the original essential multiplicity function κ , but also of two further functions κ^+ , κ^- [10, §8], which may be regarded as the positive part and the negative part of κ respectively. If we now change to the function \mathbf{K} , we can define in a plausible manner two functions \mathbf{K}^+ , \mathbf{K}^- , corresponding to κ^+ , κ^- respectively. The functions \mathbf{K}^+ , \mathbf{K}^- have properties analogous to those described for κ^+ , κ^- in (10, §§8–16). By methods similar to those in [5] it can be shown that $\mathbf{K}^+ = \kappa^+$, $\mathbf{K}^- = \kappa^-$ excepting possibly at a countable set of points. In other words, the functions κ , κ^+ , κ^- may be replaced by \mathbf{K} , \mathbf{K}^+ , \mathbf{K}^- respectively throughout the entire theory for the 2-cell case. This theory then appears to be formulated in terms of basic concepts which presently appear most suitable for generalization to higher dimensions. The writers plan to extend their work on the transformation of double integrals to the general n -dimensional case by utilizing these concepts.

PRELIMINARIES

2.1. In the sequel, \mathfrak{R} denotes a bounded simply connected Jordan region in Euclidean two-space E_2 , and \mathfrak{R}^0 indicates the interior of \mathfrak{R} . Since \mathfrak{R} is fixed throughout, it will appear in notations only when clarity requires its

presence. In \mathfrak{R}^0 we study real, finite-valued functions f of the following type.

(i) At every point w in \mathfrak{R}^0 where f is defined, its value $f(w)$ is computed by an explicitly given limit process. Conversely, f is defined at every point in \mathfrak{R}^0 where the limit exists. The exceptional set of those points w in \mathfrak{R}^0 where the limit fails to exist is denoted by $e(f)$. Thus f is a real, finite-valued function whose range of definition is precisely $\mathfrak{R}^0 - e(f)$.

(ii) The exceptional set $e(f)$ has (two-dimensional) measure zero.

(iii) The function $f(w)$, $w \in \mathfrak{R}^0 - e(f)$, is measurable. It will occasionally happen that two different limit processes yield functions f_1 and f_2 of the type described above, such that $f_1(w) = f_2(w)$ for $w \in \mathfrak{R}^0 - [e(f_1) + e(f_2)]$. Then f_1 and f_2 differ only on a set of measure zero and are interchangeable for many purposes. But we shall have to distinguish carefully between the case when f_1 and f_2 are equal on the common part of their ranges of definition but their ranges of definition differ, and the case when f_1 and f_2 are identical—that is, they have exactly the same exceptional sets. For clarity we use the notation $f_1 \equiv f_2$ only when $e(f_1) = e(f_2)$ and $f_1(w) = f_2(w)$ for $w \in \mathfrak{R}^0 - e(f_1) = \mathfrak{R}^0 - e(f_2)$.

2.2. Let f be a function of the type described in 2.1. Then f is said to have an approximate limit α at a point w_0 in \mathfrak{R}^0 , if for every positive number ϵ , the point w_0 is a point of density of the set of points $w \in \mathfrak{R}^0 - e(f)$ where $|f(w) - \alpha| \leq \epsilon$ [1, VII, §3]. The approximate limit α is unique, and is denoted by $f^*(w_0)$ in the sequel. Observe that f^* may be defined at a point where f is not defined, and conversely. The function f^* is regarded as the result of the original limit process which yields f followed by the approximate limit process—that is, the superscript $*$ is thought of as an operator. Since f is approximately continuous almost everywhere in \mathfrak{R}^0 [1, IV, §10], it is evident that f^* is also a function of the type described in 2.1, and that $f = f^*$ almost everywhere in \mathfrak{R}^0 —the set of points in \mathfrak{R}^0 where the relation $f = f^*$ fails to hold is denoted by $e^*(f)$ in the sequel. However, the relation $f \equiv f^*$ does not generally hold, and the sets $e(f)$ and $e(f^*)$ show no relationship to each other; of course, $e^*(f) \supset e(f) + e(f^*)$.

2.3. The following facts for functions of the type described in 2.1 are easy consequences of the properties of approximate limits [1, VII].

(i) $f_1 = f_2$ a.e. in \mathfrak{R}^0 if and only if $f_1^* = f_2^*$ (see 2.1).

(ii) Let e be any set in \mathfrak{R}^0 of measure zero. For any finite set of functions f_i for $i = 1, \dots, m$ and any point w_0 of \mathfrak{R}^0 not in $\sum_{i=1}^m e(f_i^*)$ one has a sequence of distinct points w_n not in $e + \sum_{i=1}^m e(f_i) + \sum_{i=1}^m e(f_i^*)$ such that $f_i(w_n) = f_i^*(w_n)$ and $f_i^*(w_n)$ converges to $f_i^*(w_0)$ for $i = 1, \dots, m$. This fact may be used to prove some of the subsequent statements.

(iii) $f^{**} \equiv f^*$ since $f^* = f$ a.e. in \mathfrak{R}^0 (see (i)).

(iv) If $f_1 \leq f_2$ a.e. in \mathfrak{R}^0 , then $f_1^* \leq f_2^*$ on $\mathfrak{R}^0 - (e(f_1^*) + e(f_2^*))$.

(v) $e(|f|^*) \subset e(f^*)$ and $|f|^* = |f^*|$ on $\mathfrak{R}^0 - e(f^*)$.

(vi) If c_1 and c_2 are real constants, then $e[(c_1 f_1 + c_2 f_2)^*] \subset e(f_1^*) + e(f_2^*)$ and $(c_1 f_1 + c_2 f_2)^* = c_1 f_1^* + c_2 f_2^*$ on $\mathfrak{R}^0 - (e(f_1^*) + e(f_2^*))$.

- (vii) $e[(f_1 f_2)^*] \subset e(f_1^*) + e(f_2^*)$ and $(f_1 f_2)^* = f_1^* f_2^*$ on $\mathfrak{R}^0 - (e(f_1^*) + e(f_2^*))$.
 (viii) $e[(\{f_1^2 + f_2^2 + f_3^2\}^{1/2})^*] \subset e(f_1^*) + e(f_2^*) + e(f_3^*)$ and $(\{f_1^2 + f_2^2 + f_3^2\}^{1/2})^* = \{f_1^{*2} + f_2^{*2} + f_3^{*2}\}^{1/2}$ on $\mathfrak{R}^0 - (e(f_1^*) + e(f_2^*) + e(f_3^*))$.

2.4. In the sequel we have to study situations in which there is associated with each direction ox in Euclidean three-space E_3 a function f_x of the type described in 2.1. A right-handed system of coordinates $ox^1x^2x^3$ constitutes a frame of reference X in E_3 . Let $f_x(w)$ denote the vector with components $f_{x^1}(w), f_{x^2}(w), f_{x^3}(w)$ defined at each point w of \mathfrak{R}^0 not in the set $\sum_{i=1}^3 e(f_{x^i})$. This latter set is of measure zero, and is denoted by $e(f_x)$ in the sequel. At a point of \mathfrak{R}^0 not in $e(f_x)$, $|f_x|$ is the length of the vector f_x . Analogously, the symbol f_x^* is defined at each point of \mathfrak{R}^0 not in the set $e(f_x^*) = \sum_{i=1}^3 e(f_{x^i}^*)$ to be the vector with components $f_{x^1}^*, f_{x^2}^*, f_{x^3}^*$.

2.5. LEMMA. Assume that with each direction ox in E_3 there is associated a function f_x of the type described in 2.1. Suppose that for each direction ox and each frame of reference X there is a set $e(x, X)$ in \mathfrak{R}^0 of measure zero and depending upon both x and X such that

$$(1) \quad f_x = f_{x^i} \cos x^i ox \quad \text{on } \mathfrak{R}^0 - e(x, X),$$

where the repeated index i indicates a summation over the range of i . Then the set $e(f_x^*)$ is independent of X and is denoted simply by e^* . For any direction oy and any frame of reference Y ,

$$(2) \quad f_y^* = f_{y^j}^* \cos y^j oy \quad \text{on } \mathfrak{R}^0 - e^*.$$

Finally, for any two frames of reference X and Y ,

$$(3) \quad |f_Y^*| = |f_X^*| \quad \text{on } \mathfrak{R}^0 - e^*.$$

Proof. From 2.3 (i), (vi) and relation (1) it follows for any direction ox and any frame of reference X ,

$$(4) \quad e(f_x^*) \subset e(f_x^*),$$

$$(5) \quad f_x^* = f_{x^i}^* \cos x^i ox \quad \text{on } \mathfrak{R}^0 - e(f_x^*).$$

In particular, from (4) one obtains for a frame of reference Y the relation $e(f_Y^*) = \sum_{i=1}^3 e(f_{y^i}^*) \subset e(f_X^*)$, and in view of the symmetry, the fact that $e(f_X^*)$ does not depend upon X is established. From (5) one obtains $f_{y^j}^* = f_{x^i}^* \cos x^i oy^j$ on $\mathfrak{R}^0 - e^*$. Multiplying both sides by $\cos y^j ox^k$ and summing yields $f_{y^j}^* \cos y^j ox^k = f_{x^i}^* \cos x^i ox^k$ on $\mathfrak{R}^0 - e^*$. Then from relation (5)

$$f_y^* = f_{x^i}^* \cos x^i oy = f_{y^j}^* \cos y^j ox^i \cos x^i oy = f_{y^j}^* \cos y^j oy \quad \text{on } \mathfrak{R}^0 - e^*,$$

and relation (2) is established. Relation (3) follows immediately from (2).

2.6. LEMMA. Assume that with each direction ox in E_3 there is associated a

function f_x satisfying all the hypotheses of the lemma in 2.5. Suppose that h is a function of the type described in 2.1 which has the following property: For each frame of reference X there is a set $e(X)$ in \mathfrak{R}^0 of measure zero and depending upon X such that

$$(1) \quad |f_x| \leq h \leq \sum_{i=1}^3 |f_{x^i}| \quad \text{on } \mathfrak{R}^0 - e(X).$$

Put $e^{**} = e(h^*) + e^*$, $e^\# = e^*(h) + e^{**}$ where e^* is the set described in 2.5 (see §2.2). Then e^{**} and $e^\#$ are sets of measure zero in \mathfrak{R}^0 , and for any frame of reference Y ,

$$(2) \quad h^* = |f_Y^*| \quad \text{on } \mathfrak{R}^0 - e^{**},$$

$$(3) \quad h = |f_Y^*| \quad \text{on } \mathfrak{R}^0 - e^\#.$$

Consequently

$$(4) \quad h = |f_Y| \quad \text{a.e. on } \mathfrak{R}^0,$$

where the exceptional set in relation (4) depends upon the frame Y .

Proof. From relation (1) and 2.3 (iv), (v), (vi), (viii) one finds that for every frame X ,

$$(5) \quad |f_x^*| \leq h^* \leq \sum_{i=1}^3 |f_{x^i}^*| \quad \text{on } \mathfrak{R}^0 - e^{**},$$

and from 2.5 (3) one has

$$(6) \quad |f_x^*| = |f_Y^*| \quad \text{on } \mathfrak{R}^0 - e^{**},$$

for any two frames X and Y .

Case (i). Suppose that at a particular point of $\mathfrak{R}^0 - e^{**}$, $|f_Y^*| = 0$. From (5) and (6) it is evident that $h^* = 0$, and thus (2) is established in this case.

Case (ii). Suppose that at a particular point of $\mathfrak{R}^0 - e^{**}$, $|f_Y^*| > 0$. Choose a frame of reference X so that at this point, one has

$$f_{x^1}^* = |f_Y^*|, \quad f_{x^2}^* = 0 = f_{x^3}^*.$$

From relations (5) and (6) it is clear that (2) must be valid in this case also. Relations (3) and (4) follow at once from 2.2.

2.7. When the hypotheses are suitably strengthened, the operator $*$ may be dropped in the results contained in 2.5 and 2.6. By the methods used in the proofs of these results, the reader will easily establish the following facts.

LEMMA. Assume that with each direction ox in E_3 there is associated a func-

tion f_x of the type described in 2.1. Suppose that for a certain frame of reference X there is a set e' in \mathfrak{R}^0 of measure zero and such that for every direction ox

$$(1) \quad e' \supset e(f_x) \quad \text{and} \quad f_x = f_x^i \cos x^i ox \quad \text{on} \quad \mathfrak{R}^0 - e'.$$

Then for any direction oy and any frame of reference Y ,

$$(2) \quad f_y = f_y^j \cos y^j oy \quad \text{on} \quad \mathfrak{R}^0 - e'$$

and

$$(3) \quad |f_Y| = |f_X| \quad \text{on} \quad \mathfrak{R}^0 - e'.$$

Moreover, if h is a function of the type described in 2.1 for which there is a set e'' in \mathfrak{R}^0 of measure zero and containing e' such that for any frame Y ,

$$(4) \quad |f_Y| \leq h \leq \sum_{i=1}^3 |f_{y^i}| \quad \text{on} \quad \mathfrak{R}^0 - e'',$$

then

$$(5) \quad h = |f_Y| \quad \text{on} \quad \mathfrak{R}^0 - e''.$$

2.8. The following result, which the reader may readily verify, contains a sufficient condition in order that the hypotheses of the lemma in 2.7 be fulfilled. In the sequel we assume that a rectangular Cartesian coordinate system is fixed in the w -plane, with reference to which we speak of oriented rectangles R in \mathfrak{R}^0 .

LEMMA. Assume that with each direction ox in E_3 there is associated a function of oriented rectangles $F_x(R)$, $R \subset \mathfrak{R}^0$, which has a derivative $f_x(w)$, $w \in \mathfrak{R}^0 - e(f_x)$, of the type described in 2.1. Suppose that for a certain frame of reference X and for every direction ox ,

$$(1) \quad F_x(R) = F_x^i(R) \cos x^i ox \quad \text{for} \quad R \subset \mathfrak{R}^0.$$

Then f_x satisfies condition 2.7 (1) with $e' = e(f_x)$ (see 2.4). Moreover, if $H(R)$, $R \subset \mathfrak{R}^0$, is a function of oriented rectangles having a derivative $h(w)$, $w \in \mathfrak{R}^0 - e(h)$, of the type described in 2.1 such that for every frame Y ,

$$(2) \quad |F_Y(R)| \leq H(R) \leq \sum_{i=1}^3 |F_{y^i}(R)| \quad \text{for} \quad R \subset \mathfrak{R}^0,$$

where $F_Y(R)$ is the vector with components $F_{y^1}(R)$, $F_{y^2}(R)$, $F_{y^3}(R)$, then condition 2.7 (4) is satisfied by the derivatives f_Y and h when e'' is chosen to be $e' + e(h)$.

2.9. LEMMA. Assume that with each direction ox in E_3 there is associated a function of oriented rectangles $F_x(R)$, $R \subset \mathfrak{R}^0$, which has a derivative $f_x(w)$, $w \in \mathfrak{R}^0 - e(f_x)$, of the type described in 2.1. Suppose $G(R)$, $R \subset \mathfrak{R}^0$, is a function of oriented rectangles having a derivative $g(w)$, $w \in \mathfrak{R}^0 - e(g)$, of the type described

in 2.1 such that for any pair of directions ox and oy ,

$$(1) \quad |F_x(R) - F_y(R)| \leq \rho(xoy)G(R) \quad \text{for } R \subset \mathfrak{R}^0,$$

where $\rho(xoy)$ is a function of the angle xoy which converges to zero with that angle. Then there is a set e''' in \mathfrak{R}^0 of measure zero such that for any direction ox ,

$$(2) \quad e''' \supset e(f_x),$$

and for any pair of directions ox and oy ,

$$(3) \quad |f_x(w) - f_y(w)| \leq \rho(xoy)g(w) \quad \text{for } w \in \mathfrak{R}^0 - e'''.$$

Proof. Choose a dense countable set \mathfrak{B} of directions oz in E_3 , and put $e''' = e(g) + \sum e(f_z)$ for oz in \mathfrak{B} . Evidently e''' is a set of measure zero in \mathfrak{R}^0 . Let w be any point in $\mathfrak{R}^0 - e'''$, and ox any direction in E_3 . Denote by Δ and δ the upper and lower derivates respectively of F_x at w . From (1), it follows that for any direction oz in \mathfrak{B} , $|\Delta - f_z(w)| \leq \rho(xoz)g(w)$ and $|\delta - f_z(w)| \leq \rho(xoz)g(w)$, hence $|\Delta - \delta| \leq 2\rho(xoz)g(w)$. Since the directions oz in \mathfrak{B} are dense, and $\rho(xoz)$ tends to zero with xoz , it follows that $\Delta = \delta = f_x(w)$. Thus relation (2) is established, and relation (3) follows at once from (1), (2) and the fact that e''' contains $e(g)$.

2.10. LEMMA. Assume that with each direction ox in E_3 there are associated functions of oriented rectangles $F_{ix}(R)$, $R \subset \mathfrak{R}^0$, $i=1, 2$, which have derivatives $f_{ix}(w)$, $w \in \mathfrak{R}^0 - e(f_{ix})$, of the type described in 2.1. Suppose $G_i(R)$, $R \subset \mathfrak{R}^0$, $i=1, 2$, are functions of oriented rectangles having derivatives $g_i(w)$, $w \in \mathfrak{R}^0 - e(g_i)$, of the type described in 2.1 such that for any pair of directions ox and oy ,

$$(1) \quad |F_{ix}(R) - F_{iy}(R)| \leq \rho_i(xoy)G_i(R) \quad \text{for } R \subset \mathfrak{R}^0, \quad i = 1, 2,$$

where the $\rho_i(xoy)$ for $i=1, 2$ are functions of the angle xoy which converge to zero with that angle. Finally, assume that for each direction ox there is a set $e(x)$ in \mathfrak{R}^0 of measure zero such that $f_{1x} = f_{2x}$ on $\mathfrak{R}^0 - e(x)$. Then there is a set e'''' in \mathfrak{R}^0 of measure zero such that for any direction ox ,

$$(2) \quad e'''' \supset e(f_{ix}) \quad \text{for } i = 1, 2$$

and

$$(3) \quad f_{1x} = f_{2x} \quad \text{on } \mathfrak{R}^0 - e''''.$$

Proof. Let e_i''' for $i=1, 2$, be the set described in 2.9 which corresponds to the functions F_{ix} and G_i for $i=1, 2$. Choose a dense countable set \mathfrak{B} of directions oz in E_3 and put $e'''' = e_1''' + e_2''' + \sum e(z)$ for oz in \mathfrak{B} . Evidently e'''' is a set of measure zero in \mathfrak{R}^0 , and relation (2) holds. Let w be a point in $\mathfrak{R}^0 - e''''$, and ox any direction in E_3 . From 2.9(3) and the assumptions, it follows for any direction oz in \mathfrak{B} that

$$|f_{1x}(w) - f_{2x}(w)| \leq \rho_1(xoz)g_1(w) + \rho_2(xoz)g_2(w).$$

Since the directions oz in \mathfrak{R} are dense and $\rho_i(xoz)$ tends to zero with xoz , it follows that $f_{1x}(w) = f_{2x}(w)$, and (3) is established.

APPLICATIONS

3.1. Again \mathfrak{R} is a bounded simply connected Jordan region in E_2 (see §2.1) and a rectangular Cartesian coordinate system is fixed in the plane of \mathfrak{R} as in §2.8. Let $T: \mathfrak{R} \rightarrow E_3$ be a continuous mapping from \mathfrak{R} into Euclidean three-space E_3 . Since T and \mathfrak{R} are kept fixed in this paper, they are displayed in subsequent notations only when clarity requires it. Now $T: \mathfrak{R} \rightarrow E_3$ represents an F -surface of the type of the 2-cell [6; II.3.44]. It is assumed in the sequel that this surface has finite Lebesgue area or, equivalently, that T is an eBV representation [6; V.1.15, 2.61, 2.65]. Presently some well known facts are listed to which the results in the preceding chapter are applied.

3.2. Let R be an oriented rectangle in \mathfrak{R} (see §2.8). Then $T: R \rightarrow E_3$ determines a F -surface whose Lebesgue area is denoted by $A(R)$ for R in \mathfrak{R} . The function of oriented rectangles $A(R)$, $R \subset \mathfrak{R}$, possesses a derivative $D(w, A)$ almost everywhere in \mathfrak{R}^0 which is summable in \mathfrak{R}^0 [6; V.2.18], and hence of the type described in §2.1. The exceptional set $e(D(w, A))$ is denoted by e^1 . It is important to observe that the function $A(R)$, $R \subset \mathfrak{R}$, and hence its derivative $D(w, A)$, $w \in \mathfrak{R}^0 - e^1$, are independent of the choice of a frame of reference in E_3 [6; V.2.8].

3.3. Let ox be any direction in E_3 . Then $T_x: \mathfrak{R} \rightarrow E_2$ will denote the continuous plane transformation obtained by going from \mathfrak{R} into E_3 under T and then projecting orthogonally into the plane π_x through o with normal ox . Since T is eBV by assumption, it follows that T_x is eBV in \mathfrak{R}^0 [6; V.1.15]. Consequently T_x gives rise to a number of important functions which depend essentially upon the direction ox . A description of the ones needed in the sequel follows. Close inspection of their definitions shows that they are independent of the choice of a frame of reference in π_x . The symbol \mathfrak{z} stands for a generic point in the plane π_x .

3.4. Let R be an oriented rectangle in \mathfrak{R} , and ox any direction in E_3 . Then the mapping $T_x: R^0 \rightarrow E_2$ gives rise to the essential multiplicity $\kappa(\mathfrak{z}, x, R^0)$ of each point \mathfrak{z} in the plane π_x in the image of R^0 under T_x [6; IV.1.43, 1.47]. Since T_x is eBV in R^0 [6; IV.4.1], $\kappa(\mathfrak{z}, x, R^0)$ is summable in π_x ; denote its integral over π_x by $G(R, x, \kappa)$ —this is the function G_4^* of [6; IV.3.8]. Now $G(R, x, \kappa)$, $R \subset \mathfrak{R}$, has a derivative $D(w, x, \kappa)$ almost everywhere in \mathfrak{R}^0 which is summable in \mathfrak{R}^0 [6; IV.3.13], and hence of the type described in §2.1. The exceptional set $e(D(w, x, \kappa))$ is denoted by $e^2(x)$ in the sequel. For a frame of reference X in E_3 given by coordinates $ox^1x^2x^3$, the set $\sum_{i=1}^3 e^2(x^i)$ is denoted by $e^2(X)$. At each point w of \mathfrak{R}^0 not in $e^2(X)$, $D(w, X, \kappa)$ is the vector with components $D(w, x^1, \kappa)$, $D(w, x^2, \kappa)$, $D(w, x^3, \kappa)$.

3.5. Retain the assumptions made in §3.4. Let $g(R, x, \kappa)$ for $R \subset \mathfrak{R}$ be the integral over the plane π_x of the characteristic function $c(\mathfrak{z}, x, R^0)$ of the set

of points in π_x where $\kappa(\mathfrak{r}, x, R^0)$ is not zero—this is the function G_4 of [6; IV.3.7]. Now $g(R, x, \kappa)$, $R \subset \mathfrak{R}$, has a derivative $d(w, x, \kappa)$ almost everywhere in \mathfrak{R}^0 which is summable in \mathfrak{R}^0 [6; IV.3.13], and hence of the type described in §2.1. The exceptional set $e(d(w, x, \kappa))$ is denoted by $e^3(x)$ in the sequel. Moreover [6; IV.3.13], $d(w, x, \kappa) = D(w, x, \kappa)$ almost everywhere in \mathfrak{R}^0 .

3.6. Retain the working hypotheses of §3.5. At each point w of $\mathfrak{R}^0 - e^3(x)$ the essential generalized Jacobian $\mathfrak{J}_e(w, x)$ is defined [6, IV.3.21]. Evidently $\mathfrak{J}_e(w, x)$, $w \in \mathfrak{R}^0 - e^3(x)$, is again a function of the type described in §2.1. For a frame of reference X in E_3 given by coordinates $ox^1x^2x^3$, the set $\sum_{i=1}^3 e^3(x^i)$ is denoted by $e^3(X)$ (see §2.4). At each point w of \mathfrak{R}^0 not in $e^3(X)$, $\mathfrak{J}_e(w, X)$ is the vector with components $\mathfrak{J}_e(w, x^1)$, $\mathfrak{J}_e(w, x^2)$, $\mathfrak{J}_e(w, x^3)$.

3.7. Continue with the assumptions of §3.4. The mapping $T_x: R^0 \rightarrow E_2$ gives rise to non-negative multiplicity functions $\kappa^+(\mathfrak{r}, x, R^0)$, $\kappa^-(\mathfrak{r}, x, R^0)$ whose sum is $\kappa(\mathfrak{r}, x, R^0)$ for $R \subset \mathfrak{R}$ [10; §8]. Both κ^+ and κ^- are summable in π_x ; let $G(R, x)$ denote the integral of $\kappa^+(\mathfrak{r}, x, R^0) - \kappa^-(\mathfrak{r}, x, R^0)$ over π_x —this is the function G° of [10; §23]. Then $G(R, x)$ has a derivative $D(w, x)$ almost everywhere in \mathfrak{R}^0 which is summable in \mathfrak{R}^0 [10; §23], and hence of the type described in §2.1. The exceptional set $e(D(w, x))$ is denoted by $e^4(x)$ in the sequel. For a frame of reference X in E_3 , $e^4(X)$ is the set $\sum_{i=1}^3 e^4(x^i)$, and at each point w of \mathfrak{R}^0 not in $e^4(X)$, $D(w, X)$ is the vector with components $D(w, x^1)$, $D(w, x^2)$, $D(w, x^3)$.

3.8. Recall the conditions set forth in §3.3. If the mapping $T_x: \mathfrak{R} \rightarrow E_2$ possesses an ordinary Jacobian $J(w, x)$ at a point w in \mathfrak{R}^0 [6, IV.3.21], it is well known that neither its value nor its existence is dependent upon the choice of a frame of reference in π_x . Let $E(x)$ denote the set of points in \mathfrak{R}^0 where T_x fails to possess an ordinary Jacobian. Then $E(x)$ is not generally of measure zero when T_x is eBV in \mathfrak{R}^0 —in fact, $E(x)$ may be identical with \mathfrak{R}^0 . Thus the function $J(w, x)$, $w \in \mathfrak{R}^0 - E(x)$, is not generally of the type described in §2.1. For a frame of reference X in E_3 , let $E(X)$ be the set $\sum_{i=1}^3 E(x^i)$ and at each point w of \mathfrak{R}^0 not in $E(X)$, let $J(w, X)$ be the vector with components $J(w, x^1)$, $J(w, x^2)$, $J(w, x^3)$. It is well known that $E(X)$ does not depend upon the vector X , hence it is denoted simply by E^5 . For any direction oy , any frame of reference Y , one has

$$(1) \quad J(w, y) = J(w, y^i) \cos y^i oy \quad \text{on } \mathfrak{R}^0 - E^5.$$

3.9. Given a frame of reference X and a direction ox in E_3 , the function of rectangles $G(R, x)$, $R \subset \mathfrak{R}$, introduced in §3.7 satisfies the relation $G(R, x) = G(R, x^i) \cos x^i ox$ for $R \subset \mathfrak{R}$ [10; §§5, 29]. From the lemmas in §2.8 and §2.7 follows the existence of a set $e^6 \equiv e^4(X)$ in \mathfrak{R}^0 of measure zero and such that for any direction oy and any frame of reference Y , $e^6 \supset e^4(y)$ and

$$(1) \quad \begin{aligned} D(w, y) &= D(w, y^i) \cos y^i oy \quad \text{on } \mathfrak{R}^0 - e^6, \\ |D(w, Y)| &= |D(w, X)| \quad \text{on } \mathfrak{R}^0 - e^6. \end{aligned}$$

3.10. For any direction ox it is known that $\mathfrak{J}_e(w, x) = D(w, x)$ almost everywhere in R^0 [10; §§5, 23]. In view of §3.9 (1) it follows that for any frame of reference X in E_3 , one has $\mathfrak{J}_e(u, x) = \mathfrak{J}_e(u, x^i) \cos x^i ox$ almost everywhere in \mathfrak{R}^0 , where the exceptional set now depends upon both x and X . From the lemma in §2.5 follows the existence of a set e^7 in \mathfrak{R}^0 of measure zero and such that for any direction oy and any frame of reference Y , $e^7 \supset e(\mathfrak{J}_e^*(w, y)) + e^6$ and

$$\begin{aligned} \mathfrak{J}_e^*(w, y) &= \mathfrak{J}_e^*(w, y^i) \cos y^i oy \quad \text{on } \mathfrak{R}^0 - e^7, \\ (1) \quad |\mathfrak{J}_e^*(w, Y)| &= |\mathfrak{J}_e^*(w, X)| \quad \text{on } \mathfrak{R}^0 - e^7, \\ \mathfrak{J}_e^*(w, y) &= D(w, y) \quad \text{on } \mathfrak{R}^0 - e^7. \end{aligned}$$

3.11. For any direction ox it is known that $|D(w, x)| = D(w, x, \kappa)$ almost everywhere in \mathfrak{R}^0 [6; IV.3.25, 3.13], where the exceptional set depends upon the direction ox . However, it will now be shown that there is a set e^8 in \mathfrak{R}^0 of measure zero and such that for any direction oy , $e^8 \supset e^2(y) + e^4(y)$ and

$$(1) \quad |D(w, y)| = D(w, y, \kappa) \quad \text{on } \mathfrak{R}^0 - e^8.$$

In view of the lemma in §2.10, this fact is established by the result in the next section.

3.12. LEMMA. *The functions of oriented rectangles $A(R), G(R, x, \kappa), G(R, x), R \subset \mathfrak{R}$, introduced in §§3.2, 3.4, 3.7 respectively, satisfy the following conditions. For any pair of directions ox and oy ,*

$$(1) \quad \begin{aligned} |G(R, x, \kappa) - G(R, y, \kappa)| &\leq 2 \sin xoy/2 \cdot A(R) \quad \text{for } R \subset \mathfrak{R}, \\ |G(R, x) - G(R, y)| &\leq 2 \sin xoy/2 \cdot A(R) \quad \text{for } R \subset \mathfrak{R}. \end{aligned}$$

Proof. Case (i). Assume that the mapping $T: \mathfrak{R} \rightarrow E_3$ is eAC [6; V.1.15]; it is known that this property is independent of the choice of the frame of reference in E_3 [10; §30]. Then for any direction ox in E_3 , T_x is eAC in \mathfrak{R}^0 , and consequently [6; IV.4.7; 10; §24]

$$\begin{aligned} G(R, x, \kappa) &= \iint_R D(w, x, \kappa) = \iint_R |D(w, x)|, \\ G(R, x) &= \iint_R D(w, x). \end{aligned}$$

For any frame of reference X , it follows from §3.10 (1) and §3.11 that

$$|D(w, X)| = |\mathfrak{J}_e(w, X)|$$

almost everywhere in \mathfrak{R}^0 , and consequently [6; V.2.61]

$$A(R) = \int \int_R |D(w, X)|.$$

Thus the inequalities in (1) are established if one proves that

$$\int \int_R |D(w, x) - D(w, y)| \leq 2 \sin \alpha \sin \beta / 2 \cdot \int \int_R |D(w, X)|.$$

But this inequality follows immediately from §3.9 (1) and the inequality of Schwarz.

Case (ii). Assume that T is merely eBV (see §3.1). Then there is a sequence of continuous mappings $T_n: R \rightarrow E_3$ each of which is eAC—in fact, each may be chosen to represent a polyhedron [6; V.2.5]—and such that the T_n converge uniformly to T on R and their Lebesgue areas $A_n(R)$ converge to $A(R)$. Let $G_n(R, x, \kappa)$ and $G_n(R, x)$ be the functions of oriented rectangles generated by T_n which correspond to $G(R, x, \kappa)$ and $G(R, x)$ respectively for T (see §§3.4, 3.7). From case (i); [9, §3]; [10, §26] one obtains the following relations:

$$\begin{aligned} |G_n(R, x, \kappa) - G_n(R, y, \kappa)| &\leq 2 \sin \alpha \sin \beta / 2 \cdot A_n(R) && \text{for } n = 1, 2, \dots, \\ |G_n(R, x) - G_n(R, y)| &\leq 2 \sin \alpha \sin \beta / 2 \cdot A_n(R) && \text{for } n = 1, 2, \dots, \\ \lim_{n \rightarrow \infty} G_n(R, x, \kappa) &= G(R, x, \kappa), && \lim_{n \rightarrow \infty} G_n(R, y, \kappa) = G(R, y, \kappa), \\ \lim_{n \rightarrow \infty} G_n(R, x) &= G(R, x), && \lim_{n \rightarrow \infty} G_n(R, y) = G(R, y). \end{aligned}$$

Clearly the inequalities in (1) follow.

3.13. Given T as in §3.1, the functions of rectangles $A(R)$, $G(R, x, \kappa)$, $R \subset \mathfrak{R}$, introduced in §§3.2, 3.4 respectively satisfy the following condition [6; V.1.2, 1.3, 1.16, 2.60, 2.65]. For every frame of reference Y ,

$$|G(R, Y, \kappa)| \leq A(R) \leq \sum_{i=1}^3 |G(R, y^i, \kappa)| \text{ for } R \subset \mathfrak{R}.$$

In view of §3.11 (1), it follows from this relation that

$$(1) \quad |D(w, Y)| \leq D(w, A) \leq \sum_{i=1}^3 |D(w, y^i)| \text{ on } \mathfrak{R}^0 - (e^1 + e^8).$$

From §3.9 (1) and §3.11 (1) follows by the lemma in §2.7 the existence of a set e^9 in \mathfrak{R}^0 of measure zero and such that $e^9 \supset e^1 + e^8 + e^8$ and for every frame of reference Y ,

$$(2) \quad D(w, A) = |D(w, Y)| \text{ on } \mathfrak{R}^0 - e^9.$$

3.14. The information derived in the preceding sections on relations which hold almost everywhere uniformly with respect to the frame of reference is

summarized in the following statement.

THEOREM. *Given a continuous mapping $T: \mathfrak{R} \rightarrow E_3$ which is eBV (see §3.1), there exists a set $e(T)$ in \mathfrak{R}^0 of measure zero such that the following statements hold at each point w in $\mathfrak{R}^0 - e(T)$ for every direction oy and every frame of reference Y .*

- (i) *The functions $D(w, A)$, $D(w, y, \kappa)$, $D(w, y)$, $\mathfrak{J}_e^*(w, y)$ exist;*
- (ii) $D(w, y) = \mathfrak{J}_e^*(w, y)$;
- (iii) $D(w, y, \kappa) = |D(w, y)| = |\mathfrak{J}_e^*(w, y)|$;
- (iv) $D(w, A) = |D(w, Y)| = |\mathfrak{J}_e^*(w, Y)| = |D(w, Y, \kappa)|$;
- (v) $D(w, y) = D(w, y^i) \cos y^i o y$;
- (vi) $\mathfrak{J}_e^*(w, y) = \mathfrak{J}_e^*(w, y^i) \cos y^i o y$.

A proof follows at once from §§3.9–3.13 with $e(T) = e^7 + e^9$.

3.15. It will be noticed that the operator $*$ plays a role in this theorem only on the essential generalized Jacobians $\mathfrak{J}_e(w, y)$. The question as to whether this $*$ may be removed seems to involve certain topological problems which we have not solved thus far. However, for a fixed frame of reference X , we have (see §2.2) $D(w, A) = \mathfrak{J}_e(w, X)$ almost everywhere in \mathfrak{R}^0 , where the exceptional set now depends upon the frame X . This result answers a question raised in [6; V.2.18]—namely, that the derivative of the Lebesgue area function $A(R)$, $R \subset \mathfrak{R}$, of a surface having finite area exists almost everywhere in \mathfrak{R}^0 and equals almost everywhere the square root of the sum of the squares of the three essential generalized Jacobians associated with a given frame of reference.

A THEOREM ON ORDINARY JACOBIANS

4.1. In the theorem of §3.14, no reference to the ordinary Jacobians appears, since these may not exist at a single point. It is interesting to observe that if one adds to the assumptions made in §3.14 the hypothesis that the set E^5 of §3.8 is of measure zero, one obtains the following result.

THEOREM. *Given a continuous mapping $T: \mathfrak{R} \rightarrow E_3$ which is eBV (see §3.1), assume that the set E^5 described in §3.8 is of measure zero. There exists a set $E(T)$ in \mathfrak{R}^0 of measure zero such that $E(T) \supset e(T)$ and the following statements hold at each point w in $\mathfrak{R}^0 - E(T)$ for every direction oy and every frame of reference Y .*

- (i) *The functions $D(w, A)$, $D(w, y)$, $\mathfrak{J}_e^*(w, y)$, $J(w, y)$ exist;*
- (ii) $D(w, y) = \mathfrak{J}_e^*(w, y) = J(w, y)$;
- (iii) $D(w, A) = |D(w, Y)| = |\mathfrak{J}_e^*(w, Y)| = |J(w, Y)|$.

Proof. In view of relations §3.8 (1) and §3.9 (1) it is evident from the theorem in §3.13 that this theorem is established if one shows that for any direction ox ,

$$(1) \quad D(w, x) = J(w, x) \quad \text{a.e. in } \mathfrak{R}^0.$$

Now the set E^5 of points w in \mathfrak{R}^0 where $J(w, x)$ fails to exist is of measure zero by assumption. Put $E^4 = e^4(x) + e^3(x) + e^2(x)$ (see §§3.4–3.7); then E^4 is a set in \mathfrak{R}^0 of measure zero. Let E^3 be the set of points w in \mathfrak{R}^0 not in E^4 where $d(w, x, \kappa) \neq D(w, x, \kappa)$; then E^3 is a set of measure zero (see §3.5). A set E^2 in \mathfrak{R}^0 is defined by saying that a point w_0 not in E^5 is not in E^2 if the following conditions are fulfilled. Suppose the transformation $T_x: \mathfrak{R} \rightarrow E_2$ is given by the relation:

$$T_x: x^k = x^k(w) = x^k(u, v), \quad (u, v) = w \in \mathfrak{R}, \quad k = 1, 2.$$

Associate with $w_0 = (u_0, v_0)$ the affine transformation

$$T_*: x^k = x^k(w_0) + x^k_u(w_0)(u - u_0) + x^k_v(w_0)(v - v_0), \quad k = 1, 2.$$

Then there is a sequence of oriented squares s_n with center w_0 whose side lengths converge to zero and for which the constants $m(s_n)$ associated with T_* [6; IV. 3.31] converge to zero as n tends to infinity. The set E^2 is of measure zero [6; I.3.14; IV.3.42]. Put $E^1 = E^2 + E^3 + E^4 + E^5$; then E^1 is a set of measure zero in \mathfrak{R}^0 . For w_0 not in E^1 it is presently shown that $D(w_0, x) = J(w_0, x)$. It is convenient to consider two cases.

Case (i). Assume that $J(w_0, x)$ is not zero. Since w_0 is not in E^2 there follows the existence of a sequence of oriented squares s_n in \mathfrak{R}^0 with center w_0 , with side lengths l_n converging to zero, and with values $m(s_n)$ less than $k/2$ and converging to zero as n tends to infinity [6; IV.3.28, 3.31]. Let s'_n, s''_n be oriented squares with center w_0 and side lengths l'_n, l''_n respectively given by

$$l'_n = (1 - 2m(s_n)/k)l_n, \quad l''_n = (1 + 2m(s_n)/k)l_n.$$

Let p'_n, p_n, p''_n denote the parallelograms $T_*(s'_n), T_*(s_n), T_*(s''_n)$ respectively, and let q_n be the doubly connected region composed of points in p'_n but not in p''_n . From [6; IV 3.35] it follows that

$$\mu(\xi, s_n) = \begin{cases} 0 & \text{for } \xi \text{ not in } p''_n; \\ \text{sgn } J(w_0, x) & \text{for } \xi \text{ in } p'_n. \end{cases}$$

By [6; IV.3.38] it is true that the image of the boundary of s_n under T_x is contained in the interior of q_n . Hence from [2; §2.20] one sees that for almost all points ξ not in q_n , $\mu(\xi, s_n)$ is equal to the sum of the generalized local indices of the essential maximal model continua for ξ under T_x in s_n^0 . But from [5; I.1.11, 1.4.6] it is clear that, excepting a set of points ξ in π_x of measure zero, $\mu(\xi, s_n) = \kappa^+(\xi, x, s_n^0) - \kappa^-(\xi, x, s_n^0)$ for ξ not in q_n (see §3.7). Consequently

$$\kappa^+(\xi, x, s_n^0) - \kappa^-(\xi, x, s_n^0) = \begin{cases} 0 & \text{for a.e. } \xi \text{ not in } p''_n; \\ \text{sgn } J(w_0, x) & \text{for a.e. } \xi \text{ in } p'_n. \end{cases}$$

From this fact it follows that

$$\begin{aligned}
 (2) \quad G(s_n, x) &= J(w_0, x) \cdot l_n'^2 + \iint_{q_n} [\kappa^+(\xi, x, s_n^0) - \kappa^-(\xi, x, s_n^0)], \\
 &\left| \iint_{q_n} [\kappa^+(\xi, x, s_n^0) - \kappa^-(\xi, x, s_n^0)] \right| \\
 (3) \quad &\leq \iint_{q_n} \kappa(\xi, x, s_n^0) \leq \iint_{q_n} c(\xi, x, s_n^0) + \iint [\kappa(\xi, x, s_n^0) - c(\xi, x, s_n^0)] \\
 &\leq |J(w_0, x)| \cdot 8m(s_n)l_n^2/k + G(s_n, x, \kappa) - g(s_n, x, \kappa).
 \end{aligned}$$

Since w_0 is not in E^3 it is true that $D(w_0, x, \kappa) = d(w_0, x, \kappa)$. Because $m(s_n)$ converges to zero as n tends to infinity, it follows from (3) that

$$\lim_{n \rightarrow \infty} \iint_{q_n} [\kappa^+(\xi, x, s_n^0) - \kappa^-(\xi, x, s_n^0)]/l_n^2 = 0.$$

Since w_0 is not in E^4 , one now obtains from (2) the relation $D(w_0, x) = J(w_0, x)$, as asserted.

Case (ii). Assume that $J(w_0, x)$ is zero. There exists a sequence of oriented squares s_n as described in case (i). Let r_n be the rectangle in π_x associated with s_n in the manner described in [6; IV.3.40]; its side lengths are $2^{1/2}[K + m(s_n)]l_n$ and $2^{1/2}m(s_n)l_n$ respectively, and it contains the image of the boundary of s_n under T_x in its interior. Now $\mu(\xi, s_n) = 0$ for ξ not in r_n , and it follows that $\kappa^+(\xi, x, s_n^0) - \kappa^-(\xi, x, s_n^0) = 0$ for almost every ξ not in r_n by the reasoning used in case (i). Thus

$$\begin{aligned}
 |G(s_n, x)| &= \iint [\kappa^+(\xi, x, s_n^0) - \kappa^-(\xi, x, s_n^0)] \\
 &\leq \iint_{r_n} \kappa(\xi, x, s_n^0) \\
 &\leq \iint_{r_n} c(\xi, x, s_n^0) + \iint [\kappa(\xi, x, s_n^0) - c(\xi, x, s_n^0)] \\
 &\leq 2[K + m(s_n)]m(s_n)l_n^2 + G(s_n, x, \kappa) - g(s_n, x, \kappa).
 \end{aligned}$$

Reasoning as in case (i), one concludes that $D(w_0, x) = 0 = J(w_0, x)$. Thus relation (1) is established.

4.2. From the preceding theorem one easily obtains the following result.

COROLLARY. *Let $t: \mathfrak{R} \rightarrow E_2$ be a continuous plane transformation which is eBV in \mathfrak{R}^0 [6; IV.4.1]. Assume that the ordinary Jacobian $J(w, t)$ exists almost everywhere in \mathfrak{R}^0 . Then $J(w, t) = \mathfrak{J}_e(w, t)$ almost everywhere in \mathfrak{R}^0 .*

This constitutes an improvement of the theorem in [6; IV.3.42], where a stronger assumption is placed upon the transformation t . In that theorem, however, the domain of definition for t is taken to be a bounded connected open set instead of a bounded simply connected Jordan region \mathfrak{R} . The methods used in the proof of the preceding theorem apply at once to this case.

4.3. Under the assumption of the above corollary [5; II.1.2; 4, §7] Cesari has shown that the ordinary Jacobian equals his generalized Jacobian almost everywhere. However, he has not discussed uniformity features such as those described for our generalized Jacobian in the theorem in 4.1.

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