ON THE STRONG DIFFERENTIATION OF THE INDEFINITE INTEGRAL[1]

BY

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1. **Definitions** [1, p. 106][2]. A sequence of intervals[3] $\{I_n\}$ is said to be *regular* if there exists a positive number α such that, with a_n the larger and b_n the smaller side of I_n, we have $b_n/a_n > \alpha$ for every n.

We shall say that the sequence $\{I_n\}$ tends to the point (x, y) if[4] $\delta(I_n) \to 0$ as $n \to \infty$, and the point (x, y) belongs to all the intervals of the sequence.

Given a set function F defined for all intervals, we define $F^*(x, y)$, the *ordinary upper derivate*, as the upper bound of the numbers t such that there exists a regular sequence of intervals $\{I_n\}$ tending to (x, y), for which

$$\lim_n \frac{F(I_n)}{|I_n|} = t\text{[5]}.$$

If we remove the condition of regularity of the sequences of intervals considered, we obtain the definition of *strong upper derivate* $F_s^*(x, y)$. Similarly we define the *lower* derivates $F_*(x, y)$ and $F_{*s}(x, y)$.

If $F^*(x, y) = F_*(x, y)$, we say that $F(I)$ has an *ordinary derivative* $F'(x, y) = F^*(x, y) = F_*(x, y)$. If $F_s^*(x, y) = F_{*s}(x, y)$, we say that $F(I)$ has a *strong derivative* $F_s'(x, y) = F_s^*(x, y) = F_{*s}(x, y)$. Obviously if $F_s'(x, y)$ exists, then also $F'(x, y)$ exists and equals $F_s'(x, y)$.

2. Suppose $f(x, y)$ defined and integrable on a set E of positive measure. We extend the definition of $f(x, y)$ on the whole plane P by putting $f(x, y) = 0$ for $(x, y) \in P - E$. We can therefore, as we shall in the following, consider only functions defined in the whole plane.

We form

$$(1) \qquad W(I) = \iint_I f(x, y)\,dxdy.$$

Then we have the following theorems.

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(1) This problem was first proposed by Professor A. Zygmund and solved under the guidance of Professor A. S. Besicovitch. The author is indebted to both for their many suggestions.

(2) Numbers in brackets refer to the references cited at the end of the paper.

(3) By intervals we shall mean in the following rectangles with sides parallel to coordinate axes.

(4) $\delta(E)$ will mean the diameter of the set E.

(5) The symbol $|E|$ denotes throughout this paper the Lebesgue measure of the plane set E.

THEOREM 1 (Lebesgue). $W^*(x, y) = W_*(x, y) = f(x, y)$ *p.p.*

THEOREM 2. (*See* [2], [3]; *also part* II *of Theorem 6 of this paper.*) *There exist integrable functions for which* $W_s^*(x, y) = +\infty$ *on a set* E *of positive measure. And since* $W_s^*(x, y) \leqq W_*(x, y) = W^*(x, y) = f(x, y) < +\infty$ *p.p., it follows that* $W_s^*(x, y) \neq W_{*s}(x, y)$ *on* E.

THEOREM 3. *If* $f(x, y)$ *is bounded, then* $W_s'(x, y)$ *exists and equals* $f(x, y)$ *p.p.*

The above theorem is an easy generalization of the following:

THEOREM 4(6) (*density theorem, see* [3] *and* [4]). *With* $f(x, y)$ *the characteristic function of a set* E, $W_s'(x, y) = f(x, y)$ *p.p. where* $W(I)$ *is as in* (1).

3. THEOREM 5. *With* $f(x, y)$ *an integrable function, we form*

$$F(I) = \iint_I |f(x, y)|\, dxdy, \qquad W(I) = \iint_I f(x, y)dxdy. \tag{2}$$

If $F(I)$ *is strongly differentiable p.p.,* $W(I)$ *is also strongly differentiable p.p.*

Proof (7). We write $f(x, y) = f_N(x, y) + r_N(x, y)$ where

$$f_N(x, y) = \begin{cases} f(x, y) & \text{for } |f(x, y)| \leqq N, \\ 0 & \text{otherwise.} \end{cases}$$

Obviously $|f(x, y)| = |f_N(x, y)| + |r_N(x, y)|$. $f(x, y)$ is integrable. Hence given $\epsilon > 0$, there exists an N_0 and a plane set M of measure smaller than ϵ, such that $r_N(x, y) = 0$ for $(x, y) \notin M$, and every $N \geqq N_0$.

From the boundedness of $f_N(x, y)$ and Theorem 3, we conclude that with $F_N(I) = \iint_I |f_N(x, y)|\, dxdy$, $\lim_{\{I_n\} \to (x,y)} F_N(I_n)/|I_n|$ exists and equals $f_N(x, y)$ p.p. But, $\lim_{\{I_n\} \to (x,y)} F(I_n)/|I_n|$ also exists (by assumption), therefore, with $R_N(I) = \iint_I r_N(x, y)dxdy$, $\lim_{\{I_n\} \to (x,y)} R_N(I_n)/|I_n|$ exists p.p. and equals $|r_N(x, y)|$.

Similarly with $W_N(I) = \iint_I f_N(x, y)dxdy$, $\lim_{I_n} W_N(I_n)/|I_n| = f_N(x, y)$ p.p. But

$$\limsup_{I_n} \frac{\left| \iint_{I_n} r_N(x, y) dxdy \right|}{|I_n|} \leqq \limsup_{I_n} \frac{\iint_{I_n} |r_N(x, y)|\, dxdy}{|I_n|} \tag{3}$$
$$= \lim_{I_n} \frac{R_N(I_N)}{|I_N|} = |r_N(x, y)| \text{ p.p.,}$$

(6) This theorem is not true if the intervals I_n (see definition) are replaced by any rectangles (see (3) and (6)).

(7) The following simple proof is due to Professor B. Jessen and was kindly communicated to the author by Professor A. Zygmund.

and since $r_N(x, y)=0$ for $(x, y)\notin M$, it follows that

$$\lim \frac{\iint_{I_n} r_N(x, y)dxdy}{|I_n|} = 0 \quad \text{p.p.} \quad \text{for} \quad (x, y) \notin M.$$

Hence

$$\lim_{I_n} \frac{W(I_n)}{|I_n|} = \lim_{I_n} \frac{W_N(I_n)}{|I_n|} + \lim_{I_n} \frac{\iint_{I_n} r_N(x, y)dxdy}{|I_n|}$$
$$= f_N(x, y) + r_N(x, y) = f(x, y) \quad \text{for} \quad (x, y) \notin M$$

and, since ϵ is arbitrary, Theorem 5 is true.

4. THEOREM 6. *There exist integrable functions $f(x, y)$ for which $W(I)$ is strongly differentiable p.p. but $F_s'(x, y)$ exists nowhere on a set E of positive measure; in fact $F_s^*(x, y) = +\infty$ for every point (x, y) of E.*

Proof. *Part* I (from a construction by H. Bohr on the Vitali covering theorem, see [5, pp. 689–692]). We subdivide the interval

$$I: \begin{array}{l} a < x < a + h, \\ b < y < b + k \end{array}$$

into the intervals

$$I_p^{(1)}: \quad a < x < a + \frac{ph}{N}, \qquad b < y < b + \frac{k}{p}, \qquad 1 \leqq p \leqq N;$$

then for every p we have $|I_p^{(1)}| = hk/N$ and with $V^{(1)} = \sum_{p=1}^{N} I_p^{(1)}$,

$$|V^{(1)}| = \frac{hk}{N}\left(1 + \frac{1}{2} + \cdots + \frac{1}{N}\right) = |I_p^{(1)}|\,\mu(N) \tag{4}$$

where $\mu(N) = 1 + 1/2 + \cdots + 1/N$.

Through the vertices of the intervals $I_p^{(1)}$ we draw lines parallel to the y-axis; thus the point-set $I - V^{(1)}$ is subdivided into $(N-1)$ intervals. We perform the above construction on each of these intervals using the same N as before. We so obtain $(N-1)$ point-sets, $V^{(2)}, \cdots, V^{(N)}$, of a staircase form, having no points in common. With

$$\frac{1}{N}\left(1 + \frac{1}{2} + \cdots + \frac{1}{N}\right) = 1 - \theta(N) \tag{5}$$

we have from (4)

$$|V^{(1)}| = (1-\theta(N))\,|I|, \qquad |I - V^{(1)}| = \theta(N)\,|I|\,;$$

reasoning similarly on $V^{(2)}, \cdots, V^{(N)}$ we obtain

$$|I - (V^{(1)} + V^{(2)} + \cdots + V^{(N)})| = \theta(N)\,|I - V^{(1)}| = \theta^2(N)\,|I|.$$

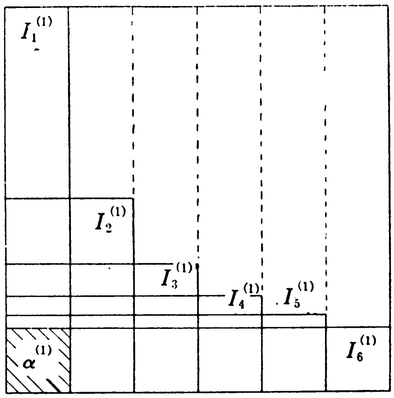

FIG. 1

5. We repeat this process $(k-1)$ times. After the $(k-1)$th iteration we have constructed $(1+(N-1)+\cdots+(N-1)^k)$ sets $V^{(i)}$ of a staircase form; the remaining set $R = I - \sum V^{(i)}$, which we shall call the "rest" set, consists of $(N-1)$: separate intervals of total area,

$$|R| = \theta^k(N)\,|I|. \tag{6}$$

The intervals $I_p^{(i)}$, forming the set $V^{(i)}$, have a nonempty common part: the shaded interval $a^{(i)}$. From our construction it follows that:

$$|a^{(i)}| = \frac{|I_p^{(i)}|}{N} = \frac{|V^{(i)}|}{N\mu(N)}, \tag{7}$$

hence with $A = \sum a^{(i)}$, the sum of all the shaded parts of I,

$$|A| < \frac{|I|}{N\mu(N)}. \tag{8}$$

6. The above construction performed on an interval I will be called in the following the operation B_N^k, N as in §4, k as in §5. We shall mean by $\{i\}$ the set of all the intervals $I_p^{(i)}$ of our construction, and by $\{a\}$ the set of the shaded intervals of I. Then $A = \sum_{a\in\{a\}} a$. If $a\in\{a\}$ is included in an $i\in\{i\}$, that is, if $a = i\cdot A$ (a point-set product), then $|a| = |i|/N$.

Part II. The following construction gives a proof of Theorem 2. It will be used in Part III, with minor modifications, for the proof of Theorem 6; in addition it will familiarize the reader with the method to be followed.

We shall give the construction of an integrable function $f(x, y)$, whose indefinite integral has no finite strong derivative at any point of a set of positive measure.

First step: On the unit square S_1 we perform the operation B_N^k by taking $N=N_1=2^{2^2}$ and k such that $R=R_1<\epsilon/2$. We thus obtain the sets $\{i_1\}$ and $\{a_1\}$. And with $A_1=\sum_{a\in\{a_i\}} a$, $|A_1|<1/N_1\mu(N_1)$, and if $i\in\{i_1\}$, $a=A_1\cdot i$, then

$$(9)\qquad |a| = \frac{|i|}{N_1}, \ \delta(i) < 1.$$

We further define the function

$$(10)\qquad f_1(x, y) = \begin{cases} N_1(\log N_1)^{1/2} = 2^{2^2}\cdot 2 & \text{for } x \in A_1, \\ 0 & \text{elsewhere in } S_1. \end{cases}$$

(In the following, the logarithms will be taken with base 2.) With

$$W_1(I) = \iint_I f_1(x, y)dxdy$$

we get from (8) and (10)

$$W_1(S_1) = |A_1| f_1(x, y) < \frac{N_1(\log N_1)^{1/2}}{N_1\mu(N_1)} = \frac{(\log N_1)^{1/2}}{\mu(N_1)},$$
$$\frac{W_1(i)}{|i|} = \frac{N_1(\log N_1)^{1/2}|A_1\cdot i|}{|i|} = (\log N_1)^{1/2}.$$

Second step: We divide S_1 into 4 congruent squares S_2 and perform in each of them the operation B_N^k by taking $N=N_2=2^{2^{2\cdot 2}}$ and k such that, with R_2 the sum of all the "rests" (§5) in the 4 squares S_2, we have $|R_2|<\epsilon/2^2$. By $\{i_2\}$ we now mean the set of intervals i included in all 4 squares S_2 and by $\{a_2\}$ the set of the shaded intervals whose sum we call A_2. Again

$$(11)\qquad \begin{gathered} |A_2| < \frac{1}{N_2\mu(N_2)} \text{ and if } i\in\{i_2\},\ a=A_2\cdot i, \text{ then} \\ |a| = \frac{|i|}{N_2}, \ \delta(i) < \frac{1}{2}. \end{gathered}$$

We define:

$$(12)\qquad f_2(x, y) = \begin{cases} N_2(\log N_2)^{1/2} = 2^{2^2}\cdot 2^2 & \text{for } x \in A_2, \\ 0 & \text{elsewhere in } S_1, \end{cases}$$

and with

$$W_2(I) = \iint_I f_2(x, y)dxdy$$

we get from (11) and (12)

$$W_2(S_1) = |A_2| f_2(x, y) < \frac{(\log N_2)^{1/2}}{\mu(N_2)},$$

$$\frac{W_2^{(i)}}{|i|} = (\log N_2)^{1/2}.$$

Similarly we continue.

nth step: We divide S_1 into 4^{n-1} congruent squares S_n and perform in each of them operation B_N^k by taking

$$N = N_n = 2^{2^{2n}} \quad \text{and } k \text{ such that} \quad |R_n| < \frac{\epsilon}{2^n} \cdot \tag{13}$$

As before:

$$\begin{aligned} &|A_n| < \frac{1}{N_n\mu(N_n)}, \quad \text{and if} \quad i \in \{i_n\}, \qquad a = A_n \cdot i, \quad \text{then} \\ &|a| = \frac{|i|}{N_n}, \qquad \delta(i) < \frac{1}{2^{n-1}}, \end{aligned} \tag{13a}$$

$$f_n(x, y) = \begin{cases} N_n(\log N_n)^{1/2} = 2^{2^{2n}} \cdot 2^n & \text{for } x \in A_n, \\ 0 & \text{elsewhere in } S_1, \end{cases} \tag{14}$$

$$W_n(I) = \iint_I f_n(x, y)dxdy,$$

$$W_n(S_1) = |A_n| f_n(x) < \frac{(\log N_n)^{1/2}}{\mu(N_n)}, \tag{15}$$

$$\frac{W_n(i)}{|i|} = (\log N_n)^{1/2} = 2^{2^{n}}$$

and so forth.

7. Form $f(x, y) = \sum_{n=1}^{\infty} f_n(x, y)$. We shall show that $f(x, y)$ satisfies our requirements. Indeed, from (15) we have

$$\sum_{n=1}^{\infty} W_n(S_1) < \sum_{n=1}^{\infty} \frac{2^n}{\mu(2^{2^{2n}})}$$

but

$$\mu(N) = 1 + 1/2 + \cdots + \frac{1}{N} \sim \log N$$

hence

$$\mu(N) > \frac{\log N}{2} \quad \text{for} \quad N > N_0$$

or $\mu(2^{2^{2n}})2^{2n}/2$ for $n > n_0$; hence

$$\sum_{n=n_0}^{\infty} W_n(S_1) < 2 \sum_{n=n_0}^{\infty} \frac{2^n}{2^{2n}} = \frac{4}{2^{n_0}} \cdot \tag{16}$$

Therefore $\sum_{n=1}^{\infty} W_n(S_1)$ converges and

$$W(I) = \iint_I f(x, y) dxdy < + \infty. \tag{17}$$

We shall now show that $W_s^*(x) = +\infty$ for every $x \in E = S_1 - \sum_{n=1}^{\infty} R_n$ where $|E| > 1 - \epsilon$ because of (13).

x is contained in one of the intervals of $\{i_n\}$ for every n. Call i_n the interval of $\{i_n\}$ containing x; then $i_1, i_2, \cdots, i_n, \cdots$ is a sequence of intervals containing x and with diameter tending to zero (13a), and since $W(i_n)/|i_n| > W_n(i_n)/|i_n| = (\log N_n)^{1/2}$ (see (14)) and since $(\log N_n)^{1/2} \to \infty$ with $n \to \infty$, we conclude that $W_s^*(x) = +\infty$.

Remarks. (1) The sequence $\{i_n\}$ is not regular.

(2) $f(x, y) \cdot \log |f(x, y)|$ is not finitely integrable, in agreement with the theorem of Jessen, Marcinkiewicz, Zygmund.

Part III. We shall make the following modifications in the construction of Part II. The operation B_N^k of the first step will remain the same. With m_1 the smallest side of the shaded intervals $\{a_1\}$, we divide S_1 in the second step, not into 4 squares, but into such a number of congruent squares $\{S_2\}$ that their side is smaller than m_1/N_1^2. With m_2 the smallest side of the intervals $\{a_2\}$, we divide S_1 in the third step into such a number of congruent squares $\{S_3\}$ that their side is smaller than m_2/N_2^2; and so we continue.

From the way the squares S_n have been constructed it follows that:

$$|a_n| > |S_{n+1}| N_n^4. \tag{18}$$

As before (13) and (13a) hold.

We now subdivide each rectangle of the set $\{a_n\}$ into N_n^4 congruent rectangles, half of them we call *black* and the remaining *white* (as on a chessboard). We thus obtain the point-set B_n of the black rectangles of A_n, and the point-set W_n of the white rectangles of A_n. With W one of the rectangles of the set W_n, we obtain from (18)

$$|W| = \frac{|a_n|}{N_n^4} > |a_{n+1}| \tag{19}$$

because a_{n+1} is included in one of the squares of $\{S_{n+1}\}$ whose side is smaller than $1/N_n^2$ times the smaller side of a_n.

We define

$$f_n^{(+)}(x, y) = \begin{cases} N_n(\log N_n)^{1/2} & \text{for } x \in W_n, \\ 0 & \text{elsewhere,} \end{cases} \qquad f^{(+)}(x, y) = \sum_{n=1}^{\infty} f_n^{(+)}(x, y), \tag{20}$$

$$f_n^{(-)}(x, y) = \begin{cases} N_n(\log N_n)^{1/2} & \text{for } x \in B_n, \\ 0 & \text{elsewhere,} \end{cases} \qquad f^{(-)}(x, y) = \sum_{n=1}^{\infty} f_n^{(-)}(x, y), \tag{21}$$

$$f_n(x, y) = f_n^{(+)}(x, y) - f_n^{(-)}(x, y), \qquad f(x, y) = f^{(+)}(x, y) - f^{(-)}(x, y).$$

With similar reasoning as in Part II, we conclude that $f^{(+)}(x, y), f^{(-)}(x, y)$, and $f(x, y)$ are integrable, with $F^{(+)}(I)$, $F^{(-)}(I)$, and $W(I) = F^{(+)}(I) - F^{(-)}(I)$ their respective integrals over I. We put

$$F(I) = \int\int_I |f(x, y)|\, dxdy. \tag{22}$$

8. Consider the set ΔA_n consisting of all rectangles similar to a_n, with the same center and with sides twice the sides of a_n. Clearly from (13a) we have

$$|\Delta A_n| = 4|A_n| < \frac{4}{N_n\mu(N_n)} < \frac{4}{16^n} \tag{23}$$

hence

$$\sum_{n=1}^{\infty} |\Delta A_n| < \frac{4}{15}$$

and with

$$E = S_1 - \sum_{n=1}^{\infty} \Delta A_n - \sum_{n=1}^{\infty} R_n$$

we obtain from (13) and (23)

$$|E| > 1 - \frac{4}{15} - \epsilon. \tag{24}$$

Consider the rectangle $a \in \{a_n\}$; it is subdivided into $N_n^4/2$ black and $N_n^4/2$ white rectangles of area $|a|/N_n^4$. With L any rectangle, we can easily see that the difference D of the areas of the black and white parts of a, which are included in L, is not greater than the area of one of the N_n^4 rectangles into which a has been subidivded; that is

(25) $$D < \frac{|a|}{N_n^4}.$$

If therefore L is such that $|L| > |a|$, then

(26) $$D < \frac{|L|}{N_n^4}.$$

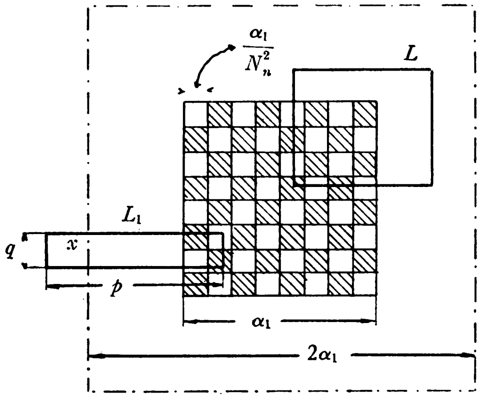

FIG. 2

Suppose now that L is any rectangle with sides p and q, containing a point x of E. Clearly (see Fig. 2)

$$p > \frac{\alpha_1}{2} \quad \text{and} \quad D < \frac{\alpha_1 q}{N_n^2},$$

hence

(27) $$D < \frac{2pq}{N_n^2} = \frac{2|L|}{N_n}.$$

9. We shall first show(7) that $W(I)$ has a strong derivative p.p. on E. We write

$$W(I) = W_m(I) + R_m(I) \quad \text{where} \quad W_m(I) = \sum_{n=1}^{m} \int\int_I f_n(x, y)dxdy$$

with $x \in E$, consider an arbitrary interval L containing the point x; we have

$$\frac{R_n(I)}{L} < \frac{\sum_{n=m}^{\infty} \left| \int\int_I f_n(x, y)dxdy \right|}{L},$$

(7) The proof of this paragraph is based on a suggestion by Professor A. Zygmund. Originally, it was proved by using Ward's theorem.

but $|\int\int_I f_n(x, y)dxdy|$ equals $N_n(\log N_n)^{1/2}$ times the difference D_n of the areas of the black and white parts of A_n which are included in L. Hence from (27) we have

$$\frac{R_m(I)}{|L|} < \sum_{n=m}^{\infty} \frac{D_n}{|L|} N_n(\log N_n)^{1/2} < 2\sum_{n=m}^{\infty} \frac{(\log N_n)^{1/2}}{N_n}$$

which tends to zero with $m\rightarrow\infty$ (see 16)), and since $W_m(I)$ has a strong derivative p.p. $(W_m)_s'$ (Theorem 3), we conclude that given $\epsilon>0$ we can find m_0 such that

$$(W_m)_s'(x, y) - \epsilon < W_{*s}(x, y) \leqq W_s^*(x, y) < (W_{m0})_s' + \epsilon;$$

hence with $\epsilon\rightarrow 0$ we must have $W_{*s}(x, y) = W_s^*(x, y)$, that is, $W(I)$ is strongly differentiable p.p.

10. It remains to show that $F_s^*(x, y) = +\infty$ for $x\in E$.

We first see by an easy induction that with $\rho(n) = N_n(\log N_n)^{1/2} = 2^{2^{2n}}\cdot 2^n$,

$$\rho(n-1) + \rho(n-2) + \cdots + \rho(1) < \rho(n)/2 \cdot \tag{28}$$

Hence for every $(x, y)\in W_n$ we have (see (20))

$$\sum_{k=1}^{n-1} |f_k(x, y)| < \frac{f_n^{(+)}(x, y)}{2} = \frac{f_n(x, y)}{2}$$

(since $f_k(x, y) = 0$ or $\rho(k)$ for $k\leqq n-1$), therefore

$$\sum_{k=1}^{n} f_k(x, y) > f_n(x, y) - \sum_{k=1}^{n-1} |f_k(x, y)| > \frac{f_n(x, y)}{2} = \frac{f_n^{(+)}(x, y)}{2} \geqq 0. \tag{29}$$

With $(x, y)\in E$, we denote by i_n the interval of the set $\{i_n\}$, which contains the point (x, y); then since $f^{(+)}(x, y)\geqq 0$, we have

$$\int\int_{i_n} f^{(+)}(x, y)dxdy \geqq \int\int_{i_n W_n} f^{(+)}(x, y)dxdy.$$

Denote by W one of the white rectangles of the set W_n, and by D_{n+1} the remaining part of $W\cdot B_{n+1}$, after subtracting from it a set of area equal to the area of $W\cdot W_{n+1}$; thus

$$|W\cdot B_{n+1}| \leqq |D_{n+1}| + |W\cdot W_{n+1}|. \tag{30}$$

From (19) we conclude that $|W|$ is greater than the area of any one of the rectangles of the set $\{a_{n+1}\}$, hence (see (26)) we must have

$$|D_{n+1}| < \frac{|W|}{N_{n+1}^4} \cdot \tag{31}$$

But $f_{n+1}(x, y)$ increases $\sum_{k=1}^{n} f_k(x, y)$ (which is positive as we can see from (29) in all points of $(W-D_{n+1})W_{n+1})$ by $N_{n+1}(\log N_{n+1})^{1/2}$, and it decreases it by the same amount in all points of $(W-D_{n+1})B_{n+1}$, and

$$|(W - D_{n+1})B_{n+1}| \leqq |(W - D_{n+1})W_{n+1}|$$

(see (30). Therefore

$$\iint_W \left|\sum_{k=1}^{n+1} f_k(x, y)\right| dxdy \geqq \iint_{W-D_{n+1}} \left|\sum_{k=1}^{n+1} f_k(x, y)\right| dxdy,$$

but

$$\iint_{W-D_{n+1}} \left|\sum_{k=1}^{n+1} f_k(x, y)\right| dxdy \geqq \iint_{W-D_{n+1}} \left|\sum_{k=1}^{n} f_k(x, y)\right| dxdy;$$

hence

$$\begin{aligned}\iint_W \left|\sum_{k=1}^{n+1} f_k(x, y)\right| dxdy &\geqq \iint_{W-D_{n+1}} \left|\sum_{k=1}^{n} f_k(x, y)\right| dxdy \\ &\geqq \frac{1}{2}\iint_{W-D_{n+1}} |f_n(x, y)|\, dxdy \\ &= \frac{1}{2}|W - D_{n+1}|\, N_n(\log N_n)^{1/2} \\ &\geqq \frac{1}{2}|W|\left(1 - \frac{1}{N_{n+1}^4}\right)N_n(\log N_n)^{1/2}\end{aligned} \tag{32}$$

(because of (31)). Similarly we obtain

$$\iint_W \left|\sum_{k=1}^{n+r} f_k(x, y)\right| dxdy > \frac{1}{2}|W|\left(1 - \frac{1}{N_{n+1}^4}\right) \cdots \left(1 - \frac{1}{N_{n+r}^4}\right)N_n(\log N_n)^{1/2};$$

but

$$\sum_{r=1}^{\infty} \frac{1}{N_{n+r}^4} < 10^{-4},$$

hence with $r \to \infty$ we obtain

$$\iint_W \left|\sum_{k=1}^{\infty} f_k(x, y)\right| dxdy > \frac{1}{2}|W|\,(1 - 10^{-4})N_n(\log N_n)^{1/2}$$

or

$$\iint_W |f(x, y)| \, dxdy > \frac{1}{2} |W| (1 - 10^{-4}) N_n (\log N_n)^{1/2}. \tag{33}$$

We now sum over all the W's which are included in the set i_n. From (13a) we have $|i_n W_n| = |i_n A_n|/2 = |i_n|/2N_n$. Therefore

$$\iint_{i_n W_n} |f(x, y)| \, dxdy > \frac{|i_n|}{4N_n} (1 - 10^{-4}) N_n (\log N_n)^{1/2},$$

hence

$$F(i_n) = \iint_{i_n} |f(x, y)| \, dxdy > \frac{|i_n|}{4} (1 - 10^{-4})(\log N_n)^{1/2}$$

or

$$\frac{F(i_n)}{|i_n|} > \frac{1}{4} (1 - 10^{-4})(\log N_n)^{1/2} = \frac{2^n}{4} (1 - 10^{-4}) \to \infty$$

with n; and since (see (13a)) $\delta i_n \to 0$, we conclude that $F_s^*(x, y) = +\infty$ and the theorem is proved.

REMARK. Even if we assume that not only $f(x, y)$, but also $\sigma(|f|)|f| \log |f|$, is integrable where $\sigma(t)$ is an arbitrary function such that $\liminf \sigma(t) = 0$ with $t \to \infty$, Theorem 6 is still true (see [2]).

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