DENSITY RATIOS AND (ϕ, 1) RECTIFIABILITY IN n-SPACE

BY
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1. **Introduction.** In 1914 Carathéodory defined [1](1) a measure on Euclidean n-space which agrees with total variation on every simple rectifiable arc. Since that time, several other measures have been defined which are, in certain senses, generalizations of length, and which have been called linear measures or 1-dimensional measures. There have been a number of results established, particularly by Besicovitch and Gillis (see the bibliography in [3]) and by Morse and Randolph [4], concerning the relation between the local structure and the rectifiability of sets of finite linear measure in the plane.

Certain of these results have been generalized by Federer [3] to the case of k-dimensional measures defined on Euclidean n-space. Certain other results concerning the relations between density and rectifiability properties which had been shown by Morse and Randolph [4] to hold for 1-dimensional measures in the plane have been proved in the present paper in the case of 1-dimensional measures in n-space.

The main results of this paper depend essentially on Theorem 3.4, which asserts the rectifiability of a set C satisfying certain conditions which are derivable from inequalities involving upper and lower densities. The corresponding theorem in the plane case [4, Theorem 9.1] had been proved in a manner which made essential use of the fact that the boundary of a circle has finite $\mathcal{H}_2^1$ measure. But since if $n \geqq 3$ the boundary of a sphere in Euclidean n-space does not have finite $\mathcal{H}_n^1$ measure, the present proof uses a quite different geometrical construction, making the set C connected by adding to it a sequence of line segments whose total $\mathcal{H}_n^1$ measure is finite.

Certain proofs have been omitted which use routine measure-theoretic methods of little intrinsic interest, or which are only trivially different from the proofs used in the 2-dimensional case [4].

The results obtained are summarized in 4.1–4.3, together with certain results from Federer's paper [3], placed here for the sake of completeness. The results from Federer's paper have been stated in his notation, without redefining all terms here.

These theorems (4.1–4.3) give various equivalent ways of characterizing the (ϕ, 1) rectifiability of a set in terms of its projection properties or of its local measure-theoretic density and restrictedness properties.

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(1) Numbers in brackets refer to the references cited at the end of the paper.

2. **Definitions and preliminary remarks.** Throughout this paper most of the special notations used are such as to conform with those of [3]. Both the empty set and the number 0 are represented by "0," and both numerical and set-theoretic subtraction are indicated by "−." E_n is Euclidean n-space.

2.1. DEFINITION. We say a family F of sets is disjointed if and only if $A \cap B = 0$ whenever A and B are different members of F.

2.2. DEFINITION. The family $\mathfrak{U}_n'$ is defined by $\phi \in \mathfrak{U}_n'$ if and only if ϕ is such a function on the class of all subsets of E_n into $E_t[0 \leqq t \leqq \infty]$ that

$$\phi(0) = 0,$$

$$\phi(B) \leqq \phi(A) \quad \text{whenever} \quad B \subseteq A \subseteq E_n,$$

if F is a countable family of subsets of E_n, then

$$\phi\left(\bigcup_{S \in F} S\right) \leqq \sum_{S \in F} \phi(S),$$

$$\phi(A \cup B) = \phi(A) + \phi(B) \quad \text{whenever distance } (A, B) > 0.$$

2.3. DEFINITION. Let $\mathfrak{H}_n^1$ be the function on the class of all subsets of E_n such that, whenever $A \subseteq E_n$, $\mathfrak{H}_n^1(A)$ is the limit as r approaches zero of the infimum of sums of the form

$$\sum_{i=1}^{\infty} \text{diameter } (B_i)$$

where $B_1, B_2, B_3, \cdots$ is such a sequence of sets that diameter $(B_i) \leqq r$ for all i and

$$A \subseteq \bigcup_{i=1}^{\infty} B_i.$$

2.4. REMARK. $\mathfrak{H}_n^1 \in \mathfrak{U}_n'$, and $\mathfrak{H}_n^1$ is called Hausdorff 1-dimensional measure.

2.5. DEFINITION. For $x \in E_n$, $r > 0$, let

$$K(x, r) = E_y[\,|y - x| < r],$$

$$C(x, r) = E_y[\,|y - x| \leqq r].$$

2.6. DEFINITION. For $\phi \in \mathfrak{U}_n'$, $A \subseteq E_n$, and $x \in E_n$, let

$$\overset{1}{\mathfrak{D}}_n(\phi, A, x) = \limsup_{r \to 0+} \frac{\phi(A \cap K(x, r))}{2r},$$

$$\overset{1}{\mathfrak{Q}}_n(\phi, A, x) = \liminf_{r \to 0+} \frac{\phi(A \cap K(x, r))}{2r},$$

$$\overset{1}{\blacktriangle}_n(\phi, A, x) = \limsup_{r \to 0+, X \in H_{x,r}} \frac{\phi(A \cap X)}{2r},$$

where $H_{x,r} = E_X[x \in X$ and $X = K(y, r)$ for some $y \in E_n]$.

2.7. Remark. The correspondence between this notation and that of Morse and Randolph [4, Definition 2.11] is as follows:

$$\mathfrak{D}_{\phi}^{\vartriangle}(A, x) = \overline{\mathfrak{S}}_2^1(\phi, A, x),$$

$$\mathfrak{D}_{\phi}^{\triangledown}(A, x) = \underline{\mathfrak{S}}_2^1(\phi, A, x),$$

$$\mathfrak{D}_{\phi}^{\blacktriangle}(A, x) = \blacktriangle_2^1(\phi, A, x).$$

2.8. Remark. If $\phi \in \mathfrak{U}_n'$, $A \subseteq E_n$, and $x \in E_n$, then

$$\underline{\mathfrak{S}}_n^1(\phi, A, x) \leqq \overline{\mathfrak{S}}_n^1(\phi, A, x) \leqq \blacktriangle_n^1(\phi, A, x) \leqq 2\overline{\mathfrak{S}}_n^1(\phi, A, x).$$

Proof. It suffices to note that if $x \in E_n$, $r > 0$, then $K(x, r) \in H_{x,r}$, and $Y \in H_{x,r}$ implies $Y \subset K(x, 2r)$.

2.9. Remark. If $\phi \in \mathfrak{U}_n'$, $A \subseteq E_n$, and $\delta > 0$, then

$$\overline{\mathfrak{S}}_n^1(\phi, A, x), \qquad \underline{\mathfrak{S}}_n^1(\phi, A, x), \qquad \blacktriangle_n^1(\phi, A, x),$$

$$\inf_{0<r<\delta} [\phi(A \cap K(x, r))/2r], \ \sup_{0<r<\delta} [\phi(A \cap K(x, r))/2r],$$

and

$$\sup_{0<r<\delta}[\sup_{Y \in H_{x,r}} [\phi(A \cap Y)/2r]]$$

are Borel measurable functions of x.

2.10. Theorem. *If $\phi \in \mathfrak{U}_n'$, $B \subseteq A \subseteq E_n$, $\phi(A - B) < \infty$, B is a Borel subset of E_n, $\overline{\mathfrak{S}}_n^1(\phi, A - B, x) < \infty$ for ϕ almost all x in B, and $F = B \cap E_x[\overline{\mathfrak{S}}_n^1(\phi, A - B, x) = 0]$, then F is a Borel subset of E_n, $\phi(B - F) = 0$, and the following three propositions hold for each $x \in F$:*

(1) $$\overline{\mathfrak{S}}_n^1(\phi, F, x) = \overline{\mathfrak{S}}_n^1(\phi, A, x),$$

(2) $$\underline{\mathfrak{S}}_n^1(\phi, F, x) = \underline{\mathfrak{S}}_n^1(\phi, A, x),$$

(3) $$\blacktriangle_n^1(\phi, F, x) = \blacktriangle_n^1(\phi, A, x).$$

A proof is as in [3, Remark 3.9].

2.11. Definition. A set $A \subseteq E_n$ is said to be 1 rectifiable if and only if there exists a function f such that domain f is a bounded subset of the real numbers, range $f \supseteq A$, and f satisfies the Lipschitz condition

$$|f(x) - f(y)| \leqq |x - y| \quad \text{whenever } x, y \in \text{domain } f.$$

2.12. Definition. A set $A \subseteq E_n$ is said to be $(\phi, 1)$ rectifiable if and only if $\phi \in \mathfrak{U}_n'$ and for each $\epsilon > 0$ there is a 1 rectifiable subset B of A for which

$\phi(A-B)<\epsilon$.

2.13. REMARK. If $\phi\in\mathfrak{U}_n'$, $\phi(E_n)<\infty$, and if every Borel set with positive ϕ measure contains a rectifiable subset with positive ϕ measure, then E_n is $(\phi, 1)$ rectifiable.

A proof is given by Morse and Randolph [4, 7.15].

2.14. THEOREM. *If A is a compact connected subset of E_n, and $\mathfrak{H}_n^1(A)<\infty$, then A is* 1 *rectifiable.*

A proof is given by Eilenberg and Harrold [2, Theorem 2].

2.15. DEFINITION. Whenever p, $q\in E_n$, let $p^*q=K(p, |p-q|)\cap K(q, |p-q|)$.

2.16. DEFINITION. Whenever p, $q\in E_n$, let $M(p, q)=E_x[x=tp+(1-t)q$ for some t such that $0\leqq t\leqq 1]$.

2.17. REMARK. If p, $q\in E_n$, then $\mathfrak{H}_n^1(M(p, q))=|p-q|$.

Letting f be an isometry which maps $A=E_y[0\leqq y\leqq |p-q|\,]$ onto $M(p, q)$, and applying [3, Theorem 5.2] to f and f^{-1}, we have $\mathfrak{H}_n^1(M(p, q))=\mathfrak{H}_1^1(A)$, but by [3, Remark 5.6], $\mathfrak{H}_1^1(A)=|p-q|$.

2.18. DEFINITION. Whenever $1/2<t<1$ and p, $q\in E_n$, let $L(t, p, q)=C(p, t|p-q|)\cap C(q, t|p-q|)$.

3. **Density ratios and rectfiability.**

3.1. THEOREM. *If $\psi\in\mathfrak{U}_n'$, $P=(0, 0, \cdots, 0)$, $Q=(1, 0, \cdots, 0)$, $A\subseteq B\subseteq E_n$, and if also*

(i) $P, Q\in A$,

(ii) $1\leqq\psi(K(x, r))/2r<1.01$ *whenever* $x\in B$ *and* $0<r\leqq 2$,

(iii) $1\leqq\psi(B\cap K(x, r))/2r<1.01$ *whenever* $x\in A$ *and* $0<r\leqq 2$,

then $\psi\ (L(.999, P, Q))>.012$.

Proof. First note that, by a proof quite analogous to the two-dimensional case [4, Theorem 8.1], we have

$$\psi(P^*Q)>.06.$$

Then observe that

$$\begin{aligned}\psi(L(.999, P, Q)) &= \psi(C(P, .999)\cap C(Q, .999)) \geqq \psi\ (K(P, .998)\cap K(Q, .998))\\ &= \psi(P^*Q) - \psi(K(P, 1)\cap (K(Q, 1) - K(Q, .998)))\\ &\quad - \psi(K(Q, .998)\cap (K(P, 1) - K(P, .998)))\\ &> .06 - \psi(K(Q, 1) - K(Q, .998)) - \psi(K(P, 1) - K(P, .998))\\ &= .06 - \psi(K(Q, 1)) + \psi(K(Q, .998))\\ &\quad - \psi(K(P, 1)) + \psi(K(P, .998))\\ &\geqq .06 - 2.02 + 1.996 - 2.02 + 1.996 = .012.\end{aligned}$$

3.2. THEOREM. *If* $\phi\in\mathfrak{U}_n'$, $0<\phi(E_n)<\infty$, *and* $\mathfrak{S}_n^1(\phi, E_n, x)<1.01\mathfrak{Q}_n^1$

$\cdot(\phi, E_n, x)$ *for* ϕ *almost all* x, *then there exist compact sets* B, C, *and positive numbers* α, t, δ, *and* β, *such that* $C \subseteq B \subseteq E_n$, $\phi(C) > 0$, diameter $(C) < \delta/2$, $1/2 < t < 1$, *and*

$$(1) \qquad \alpha \leqq \phi(K(x, r))/2r \leqq 2\alpha \textit{ whenever } x \in B \textit{ and } 0 < r < \delta,$$

$$(2) \qquad \phi(B \cap L(t, p, q)) \geqq \beta |p - q| \textit{ whenever } p, q \in C.$$

Proof. By a straightforward uniformization procedure (using 2.9, 2.10, and measure-theoretic arguments) choose positive numbers α and δ and compact sets C, E, and B such that $C \subseteq E \subseteq B \subseteq E_n$, $\phi(C) > 0$, diameter $(C) < \delta/2$, and

$$\alpha \leqq \phi(K(x, r))/2r < 1.01\alpha \text{ whenever } x \in B \text{ and } 0 < r < \delta,$$
$$\alpha \leqq \phi(B \cap K(x, r))/2r < 1.01\alpha \text{ whenever } x \in E \text{ and } 0 < r < \delta,$$
$$\alpha \leqq \phi(E \cap K(x, r))/2r < 1.01\alpha \text{ whenever } x \in C \text{ and } 0 < r < \delta.$$

Then let $\beta = .012\alpha$, and $t = .999$.

To check that (2) holds, let $p, q \in C$, and note that (2) is trivially satisfied if $p = q$. Assume $p \neq q$, let P, Q be as in 3.1, and let T be such a function that $T(p) = P$, $T(q) = Q$, and T is an isometry superimposed on a magnification. Then let $A = T(C)$, $D = T(E)$, and let ψ be the function such that

$$\psi(X) = \frac{\phi(B \cap T^{-1}(X))}{\alpha |p - q|} \quad \text{whenever } X \subseteq E_n.$$

Then noting that A, D, and ψ satisfy the hypotheses of 3.1, we conclude

$$\frac{\phi(B \cap T^{-1}(L(t, P, Q)))}{\alpha |p - q|} = \psi(L(t, P, Q)) > .012.$$

However, since $T^{-1}(L(t, P, Q)) = L(t, p, q)$, we have $\phi(B \cap L(t, p, q)) > .012\alpha|p - q| = \beta|p - q|$.

3.3. THEOREM. *If* $\phi \in \mathfrak{U}_n'$, $0 < \phi(E_n) < \infty$, *and* $3\blacktriangle_n^1(\phi, E_n, x) < 4\circledcirc_n^1(\phi, E_n, x)$ *for* ϕ *almost all* x, *then there exist compact sets* B, C, *and positive real numbers* α, t, δ, *and* β, *such that* $C \subseteq B \subseteq E_n$, $\phi(C) > 0$, diameter $(C) < \delta/2$, $1/2 < t < 1$, *and*

$$(1) \qquad \alpha \leqq \phi(K(x, r))/2r \leqq 2\alpha \quad \textit{whenever } x \in B \textit{ and } 0 < r < \delta,$$

$$(2) \qquad \phi(B \cap L(t, p, q)) \geqq \beta |p - q| \quad \textit{whenever } p, q \in C.$$

Proof. Choose positive numbers α, δ, and ζ, and compact sets C and B such that $C \subseteq B \subseteq E_n$, $\phi(C) > 0$, diameter $(C) < \delta/2$, $1 < \zeta < 4/3$, and

$$(3) \qquad \alpha \leqq \phi(K(x, r))/2r, \qquad \phi(Y)/2r \leqq \alpha\zeta$$

whenever $x \in B$, $0 < r < \delta$, and $Y \in H_{x,r}$, and furthermore

$$(4) \qquad \alpha \leqq \phi(B \cap K(x, r))/2r, \qquad \phi(B \cap Y)/2r \leqq \alpha\zeta$$

whenever $x \in C$, $0 < r < \delta$, and $Y \in H_{x,r}$.

Then let $s = (4-\zeta)/(8-4\zeta)$, and note that $3/4 < s < 1$. Also let $\beta = \alpha(4s - (1+2s)\zeta)$. Then $\beta > 0$ since $4s-(1+s)\zeta > 0$. Let $t = (1+s)/2$. To show (2), note that (2) is trivially true if $p = q$, hence assume $|p-q| > 0$. Then let $p' = (p+q)/2$. Let $w = (s+1/2)|p-q|$. Note that

$$K(p', w) \supseteq K(p, s|p-q|) \cup K(q, s|p-q|),$$

hence

$$\phi(B \cap K(p', w)) \geqq \phi(B \cap K(p, s\,|\,p-q\,|\,)) + \phi(B \cap K(q, s\,|\,p-q\,|\,)) - \phi(B \cap K(p, s\,|\,p-q\,|\,) \cap K(q, s\,|\,p-q\,|\,)). \tag{5}$$

Then since $s|p-q| < w < (3/2)|p-q| < 2$ diameter $(C) < \delta$, and since $K(p', w) \in H_{p,w}$, it follows from (4) and (5) that $(2s+1)|p-q|\alpha\zeta \geqq 4s|p-q|\alpha - \phi(B \cap K(p, s|p-q|) \cap K(q, s|p-q|))$. Hence $\phi(B \cap L(t, p, q)) \geqq \phi(B \cap K(p, s|p-q|) \cap K(q, s|p-q|)) \geqq \alpha(4s-(1+2s)\zeta)|p-q| = \beta|p-q|$.

3.4. Theorem. *If* $\phi \in \mathfrak{U}_n'$, $\phi(E_n) < \infty$, *C and B are compact sets, and* α, t, δ, *and* β *are positive real numbers, such that* $C \subseteq B \subseteq E_n$, $\phi(C) > 0$, diameter $(C) < \delta/2$, $1/2 < t < 1$, *and*

$$\alpha \leqq \phi(K(x, r))/2r \leqq 2\alpha \text{ whenever } x \in B \text{ and } 0 < r < \delta, \tag{1}$$

$$\phi(B \cap L(t, p, q)) \geqq \beta\,|\,p-q\,|\, \text{ whenever } p, q \in C, \tag{2}$$

then C is 1 *rectifiable.*

Let m be the least positive integer satisfying $m > (8n^{1/2})/(1-t)$. Let $\epsilon = \min\,(\beta(1-t)/8\alpha(m^{2n}+1), (1-t)/2)$. Then $1/4 > \epsilon > 0$. Choose a sequence $V_1, V_2, V_3, \cdots$ such that for each k, $V_k = (p_k, q_k, s_k)$, $p_k, q_k, s_k \in E_n$, and the conditions inductively stated below are fulfilled.

For each integer k, assume V_j has been chosen for all integers $j < k$, and let A_k be the set of all ordered triples of the form (p, q, s) such that $p, q \in C$, and each of the following conditions are satisfied:

(i) $C \cap (p^*q) = 0,$

(ii) $s \in ((B \cap L(t, p, q)) - \bigcup_{j=1}^{k-1} K(s_j, \epsilon\,|\,p_j - q_j\,|\,)),$

(iii) for all $j < k$, either $|\,p_j - p\,| \geqq |\,p - q\,|$ or $|\,q_j - q\,| \geqq |\,p - q\,|$.

Let f be the function on A_k such that $f(p, q, s) = |p-q|$; note that in the cartesian product topology A_k is compact and f is continuous. Then whenever A_k is not empty, choose $V_k = (p_k, q_k, s_k)$ as some point for which f is maximal. Then the maximal choice for each k, together with the fact that $A_{k+1} \subseteq A_k$ for each k, insures that the sequence $|p_1 - q_1|, |p_2 - q_2|, |p_3 - q_3|, \cdots$ is non-increasing.

If for some k we have A_k is empty, let k_0 be the least such k, and terminate the construction by choosing some $x \in C$ and letting $p_j = q_j = s_j = x$ for all $j \geqq k_0$.

Regardless of whether the sequence was so terminated, we define

$$S = C \cup \bigcup_{j=1}^{\infty} M(p_j, q_j).$$

The remainder of the proof is divided into five parts.

Part I. $\mathcal{H}_n^1(C) < \infty$.

Proof. For any r such that $0 < r < \delta$, let N be the set of all integers j such that there exists $Y_j = \{x_1\} \cup \{x_2\} \cup \cdots \cup \{x_j\}$ such that $Y_j \subseteq C$ and $\{K(x_1, r/4)\} \cup \{K(x_2, r/4)\} \cup \cdots \cup \{K(x_j, r/4)\}$ is a disjointed family of spheres. Then for such Y_j,

$$\phi(E_n) \geqq \sum_{i=1}^{j} \phi(K(x_i, r/4)) \geqq \alpha j r/2.$$

Hence if $j \in N$, $j \leqq 2\phi(E_n)/\alpha r < \infty$.

Then since N is bounded, let k be the largest member of N. Choose Y_k satisfying the above conditions. Let $Z = K(x_1, r/2) \cup \cdots \cup K(x_k, r/2)$. Then $C \subseteq Z$ by the maximality of k. Then

$$\sum_{i=1}^{k} \text{diameter } (K(x_i, r/2)) = rk \leqq 2\phi(E_n)/\alpha.$$

Hence by 2.3,

$$\mathcal{H}_n^1(C) \leqq 2\phi(E_n)/\alpha < \infty.$$

Part II. $\mathcal{H}_n^1(S) < \infty$.

Proof. By 2.17, $\mathcal{H}_n^1(M(p_j, q_j)) = |p_j - q_j| \leqq \phi(K(s_j, \epsilon|p_j - q_j|/2))/\epsilon\alpha$, but from (ii) and the fact that $|p_1 - q_1|, |p_2 - q_2|, \cdots$ is a non-increasing sequence we conclude that

$$\bigcup_{j=1}^{\infty} \{K(s_j, \epsilon|p_j - q_j|/2)\}$$

is a disjointed family of spheres, hence

$$\sum_{j=1}^{\infty} \mathfrak{H}_n^1(M(p_j, q_j)) \leqq \sum_{j=1}^{\infty} \frac{\phi(K(s_j, \epsilon|p_j - q_j|/2))}{\epsilon\alpha} \leqq \frac{\phi(E_n)}{\epsilon\alpha} < \infty. \tag{3}$$

Thus $\mathcal{H}_n^1(S) \leqq \mathfrak{H}_n^1(C) + \phi(E_n)/\epsilon\alpha < \infty$.

Part III. S is compact.

Proof. Note that diameter (S) = diameter $(C) < \delta/2$, and to show S is closed, assume the contrary, and let $x \in$ closure $(S) - S$. Since C is closed and $M(p_j, q_j)$ is closed for each integer j, we select distinct integers $m_1, m_2, m_3, \cdots$ such that $\lim_{i\to\infty}$ distance $(\{x\}, M(p_{m_i}, q_{m_i}))=0$. But we infer from (3) that $\lim_{i\to\infty}$ diameter $(M(p_{m_i}, q_{m_i}))=0$. Thus since $p_{m_i}\in C$ for each i, it follows that x is a limit point of the closed set C, hence $x\in C\subseteq S$, contradicting our assumption.

Part IV. S is connected.

Proof. Assume the contrary, and let S' be a proper subset open and closed in S. Then, using part III, $X=C\cap S'$ and $Y=C-S'$ are disjoint compact non-empty sets, such that $X\cup Y=C$. Then distance $(X, Y)=r>0$. Choose points $x\in X$, $y\in Y$ such that $|x-y|=r$. Since the sequence $|p_1-q_1|$, $|p_2-q_2|, \cdots$ decreases to zero, we can choose the smallest integer i such that $|p_i-q_i|<r$. Then $(x, y, s)\notin A_i$ for any s, for if $(x, y, s)\in A_i$, this would contradict the maximality of $|p_i-q_i|$. Hence either (i), (ii), or (iii) must fail to hold for (x, y, s), regardless of the choice of s. Each of these assumptions will be shown to lead to a contradiction.

Case I. (i) fails. If $C\cap(x^*y)$ is not empty, let $z\in C\cap(x^*y)$. Then either $z\in X$ and $|z-y|<r$ or $z\in Y$ and $|z-x|<r$, contradicting the definition of r in either case.

Case II. (ii) fails. If

$$((B\cap L(t, x, y)) - \bigcup_{j=1}^{i-1} K(s_j, \epsilon|p_j-q_j|)) = 0,$$

then by (2),

$$\sum_{j\in E}\phi(K(s_j, \epsilon|p_j-q_j|)\cap(B\cap L(t, x, y))) \geqq \beta|x-y| = \quad . \tag{4}$$

where the summation is taken over the set E of all those integers j such that $j<i$ and $K(s_j, \epsilon|p_j-q_j|)\cap L(t, x, y)\neq 0$. For $j\in E$, choose $z_j\in K(s_j, \epsilon|p_j-q_j|)\cap L(t, x, y)$. Then $|s_j-z_j|<\epsilon|p_j-q_j|\leqq((1-t)/2)|p_j-q_j|$. Since $x\notin(p_j{}^*q_j)$, $|x-q_j|\geqq|p_j-q_j|$, and since $s_j\in L(t, p_j, q_j)$, $|s_j-q_j|\leqq t|p_j-q_j|$, hence $|x-s_j|\geqq|x-q_j|-|q_j-s_j|\geqq(1-t)|p_j-q_j|$. Thus $(1-t)|p_j-q_j|\leqq|x-z_j|+|z_j-s_j|\leqq r+((1-t)/2)|p_j-q_j|$, hence $((1-t)/2)|p_j-q_j|\leqq r$, which gives $|p_j-q_j|\leqq 2r/(1-t)$ whenever $j\in E$.

Then since $\epsilon|p_j-q_j|<\epsilon(\delta/2)<\delta$, we apply (1) and (4) to conclude

$$\sum_{j\in E}\frac{8\alpha\epsilon}{1-t}\geqq\sum_{j\in E}\frac{4\alpha\epsilon}{r}|p_j-q_j|\geqq\frac{1}{r}\sum_{j\in E}\phi(K(s_j, \epsilon|p_j-q_j|))\geqq\beta,$$

and since $m^{2n}+1\leqq\beta(1-t)/8\alpha\epsilon$, the set E has at least $m^{2n}+1$ members.

Let P be a cube of side length $8r/(1-t)$ and center x. Then if $j\in E$, $|p_j-x|\leqq|p_j-z_j|+|z_j-x|\leqq|p_j-q_j|+r\leqq 2r/(1-t)+2r/(1-t)=4r/(1-t)$, hence $p_j\in P$. Similarly $j\in E$ implies $q_j\in P$.

Subdivide P into m^n cubes, each of side length $8r/(1-t)m<r/n^{1/2}$. Since E has at least $m^{2n}+1$ members, we can choose Q as one of these cubes such that

the set

$$F = \underset{j}{E}\,[p_j \in Q \text{ and } j \in E]$$

has more than m^n members. Then we can choose R as one of the cubes such that the set

$$G = \underset{j}{E}\,[q_j \in R \text{ and } j \in F]$$

has at least two members. Then choose j, $k \in G$, such that $j<k$. But since diameter (Q) = diameter $(R) < r$, we have $|p_j - p_k| < r$ and $|q_j - q_k| < r$. But since $k \in G$, $k<i$, hence $|p_k - q_k| \geqq r$, hence $(p_k, q_k, s_k) \notin A_k$, since (p_k, q_k, s_k) fails to satisfy (iii). But $(p_k, q_k, s_k) \notin A_k$ contradicts the inductive definition of (p_k, q_k, s_k).

Case III. (iii) fails. Suppose $|p_j - x| < r$ and $|q_j - y| < r$ for some $j<i$. Then since $p_j, q_j \in C$, and using the definition of r, we have $p_j \in X$ and $q_j \in Y$. Hence $p_j \in S'$ and $q_j \in (S - S')$. But $M(p_j, q_j)$ is a connected subset of S, hence p_j and q_j belong to the same component of S, a contradiction.

Part V. C is a 1 rectifiable set.

Proof. By parts II, III, IV, and Theorem 2.14, S is 1 rectifiable. Hence, by 2.11, C is 1 rectifiable.

3.5. Theorem. *If* $\psi \in \mathfrak{U}_n'$, $\psi(E_n) < \infty$, $\overline{\odot}_n^1(\psi, E_n, x) < 1.01\,\underline{\odot}_n^1(\psi, E_n, x)$ *for* ψ *almost all* x, *then* E_n *is* $(\psi, 1)$ *rectifiable.*

Proof. Applying 2.13, let B be any Borel subset of E_n with positive ψ measure. Then by 2.10 there exists a Borel subset F of E_n with $\psi(F) > 0$ such that $\overline{\odot}_n^1(\psi, F, x) < 1.01\,\underline{\odot}_n^1(\psi, F, x)$ for ψ almost all x in F. Then letting ϕ be the function such that $\phi(Y) = \psi(F \cap Y)$ for all $Y \subseteq E_n$, we have $\phi \in \mathfrak{U}_n'$, $0 < \phi(E_n) < \infty$, and $\overline{\odot}_n^1(\phi, E_n, x) < 1.01\,\underline{\odot}_n^1(\phi, E_n, x)$ for ϕ almost all x in E_n, hence by 3.2 and 3.4 there is a compact rectifiable set C with $\phi(C) > 0$. Hence $B \cap C$ is a rectifiable set of positive ψ measure.

3.6. Theorem. *If* $\psi \in \mathfrak{U}_n'$, $\psi(E_n) < \infty$, *and* $3\blacktriangle_n^1(\psi, E_n, x) < 4\,\underline{\odot}_n^1(\psi, E_n, x)$ *for* ψ *almost all* x, *then* E_n *is* $(\psi, 1)$ *rectifiable.*

A proof parallels that of 3.5, making use of 3.3 instead of 3.2. Use is also made of 2.8 in checking that $\overline{\odot}_n^1(\psi, E_n - B, x) < \infty$ for ψ almost all x in B.

3.7. Theorem. *If* $\phi \in \mathfrak{U}_n'$, $A \subseteq E_n$, $\phi(A) < \infty$, *and* $\overline{\odot}_n^1(\phi, A, x) < 1.01\,\underline{\odot}_n^1(\phi, A, x)$ *for* ϕ *almost all* x *in* A, *then* A *is* $(\phi, 1)$ *rectifiable.*

3.8. Theorem. *If* $\phi \in \mathfrak{U}_n'$, $A \subseteq E_n$, $\phi(A) < \infty$, *and* $3\blacktriangle_n^1(\phi, A, x) < 4\,\underline{\odot}_n^1(\phi, A, x)$ *for* ϕ *almost all* x *in* A, *then* A *is* $(\phi, 1)$ *rectifiable.*

3.9. Theorem. *If* $\phi \in \mathfrak{U}_n'$, $A \subseteq E_n$, $\phi(A) < \infty$, $\underline{\odot}_n^1(\phi, A, x) < \infty$ *for* ϕ *almost* *all x in A, and A is* (ϕ, 1) *rectifiable, then* $0<\underline{\circ}_n^1(\phi, A, x)=\blacktriangle_n^1(\phi, A, x)<\infty$ *for* ϕ *almost all x in A.*

3.10. Corollary. *If the hypotheses of* 3.9 *hold, then* $0<\underline{\circ}_n^1(\phi, A, x) = \overline{\circ}_n^1(\phi, A, x)<\infty$ *for* ϕ *almost all x in A.*

The proof is by 2.8.

4. **Summary.** Theorem 4.1 can be proved by using 3.7, 3.8, 3.9, 3.10, and 2.8. Theorems 4.2 and 4.3 also use the results of, and employ the notation of, Federer [3, Theorems 9.1, 9.2]. A proof of 4.3 also involves 2.10.

4.1. Theorem. *If* $n\geqq 1$, $\phi\in\mathfrak{U}_n'$, $A\subseteq E_n$, *and* $\phi(A)<\infty$, *then the following five propositions are equivalent*:

(1) $\underline{\circ}_n^1(\phi, A, x)<\infty$ *for* ϕ *almost all x in A, and A is* (ϕ, 1) *rectifiable,*

(2) $\overline{\circ}_n^1(\phi, A, x)<1.01\,\underline{\circ}_n^1(\phi, A, x)$ *for* ϕ *almost all x in A,*

(3) $3\blacktriangle_n^1(\phi, A, x)<4\underline{\circ}_n^1(\phi, A, x)$ *for* ϕ *almost all x in A,*

(4) $0<\overline{\circ}_n^1(\phi, A, x)=\underline{\circ}_n^1(\phi, A, x)<\infty$ *for* ϕ *almost all x in A,*

(5) $0<\blacktriangle_n^1(\phi, A, x)=\underline{\circ}_n^1(\phi, A, x)<\infty$ *for* ϕ *almost all x in A.*

4.2. Theorem. *If* $n\geqq 2$, $\phi\in\mathfrak{U}_n'$, $A\subseteq E_n$, $\phi(A)<\infty$, *and* $\overline{\circ}_n^1(\phi, A, x)<\infty$ *for* ϕ *almost all x in A, then the following eight propositions are equivalent*:

(1) *A is* (ϕ, 1) *rectifiable,*

(2) *A is* (ϕ, 1) *restricted at* ϕ *almost all of its points,*

(3) *corresponding to* ϕ *almost all x in A there is an* $R\in G_n$ *such that*

$$\varnothing_n^1(\phi, A, R, x) = 0 < \overline{\circ}_n^1(\phi, A, x),$$

(4) *corresponding to* ϕ *almost all x in A we can find R and* η *such that* $R\in G_n$, $0<\eta<1$, *and* $\lim\sup_{r\to 0+}\nabla_n^1(\phi, A, R, \eta, r, x)<.005\,\overline{\circ}_n^1(\phi, A, x)$,

(5) $\overline{\circ}_n^1(\phi, A, x)<1.01\,\underline{\circ}_n^1(\phi, A, x)$ *for* ϕ *almost all x in A,*

(6) $3\blacktriangle_n^1(\phi, A, x)<4\underline{\circ}_n^1(\phi, A, x)$ *for* ϕ *almost all x in A,*

(7) $0<\overline{\circ}_n^1(\phi, A, x)=\underline{\circ}_n^1(\phi, A, x)$ *for* ϕ *almost all x in A,*

(8) $0<\blacktriangle_n^1(\phi, A, x)=\underline{\circ}_n^1(\phi, A, x)$ *for* ϕ *almost all x in A.*

4.3. Theorem. *If* $n\geqq 2$, $\phi\in\mathfrak{U}_n'$, *A is a Borel subset of* E_n, $0<\phi(A)<\infty$, $\overline{\circ}_n^1(\phi, A, x)>0$ *for all x in A, and* $\overline{\circ}_n^1(\phi, A, x)<\infty$ *for* ϕ *almost all x in A, then the following six propositions are equivalent*:

(1) *A is positively* (ϕ, 1) *unrectifiable,*

(2) *A has a subset B for which* $\phi(A-B)=0$ *and* $\lim\sup_{r\to 0+}\nabla_n^1(\phi, A, R, \eta, r, x)\geqq .005\,\overline{\circ}_n^1(\phi, A, x)$ *whenever* $x\in B$, $R\in G_n$, *and* $0<\eta<1$,

(3) *A has a subset B for which* $\phi(A-B)=0$ *and* $x\in B$ *implies*

$$\varnothing_n^1(\phi, A, R, x)>0 \textit{ for } \phi_n \textit{ almost all } R \textit{ in } G_n,$$

(4) $\mathcal{L}_1[P_R^1{}^*(A)]=0$ *for* ϕ_n *almost all R in* G_n,

(5) $\overline{\circ}_n^1(\phi, A, x)\geqq 1.01\,\underline{\circ}_n^1(\phi, A, x)$ *for* ϕ *almost all x in A,*

(6) $3\blacktriangle_n^1(\phi, A, x)\geqq 4\underline{\circ}_n^1(\phi, A, x)$ *for* ϕ *almost all x in A.*

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