THE CLASSES L_p AND CONFORMAL MAPPING

BY

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1. **Introduction.** Recently M. Schiffer and the present writer gave new proofs, based upon the notion of a kernel function, of the existence of a number of canonical conformal maps of a multiply-connected domain [6, 7](1). The arguments given were the outcome of investigations of the duality of various extremal problems in the conformal mapping of multiply-connected regions, such as that of Schwarz's lemma to the Szegö kernel function [1, 2, 4, 5, 8, 9, 10]. Much of the reasoning was facilitated by an elementary knowledge of Hilbert space L_2.

Here we shall continue to discuss relations between extremal problems. In this study, a number of existence theorems will be developed, in simplified form, by arguments based upon the uniform convexity of the Banach spaces L_p with $p>1$ [3]. To work thus with minimum functions in L_p is a departure from the classical procedure, often encountered in the calculus of variations, of using minimum functions in Hilbert space L_2 for existence proofs. We shall show that our basic extremal problems in L_p and L_q are related when

$$\frac{1}{p}+\frac{1}{q}=1,$$

thus bringing forward once more the well known duality of these two spaces. Furthermore, in the case of special normalizations in multiply-connected domains, we shall prove that certain problems in the classes L_p are related to those in L_2 and those in the space L_1 of functions of bounded variation and the space L_∞ of bounded functions, and we shall thus extend earlier results [4]. Here it is to be remarked that L_1 and L_∞ correspond to one another in much the same way as do the spaces L_p and L_q with $1/p+1/q=1$.

Our discussions will touch upon relations between problems in L_p, L_q and $L_{pq/(q-p)}$, $L_{q/2}$, $1/p+1/q=1$, which generalize the earlier work on L_1, L_2, L_∞ [4, 7], and we shall investigate the case $p<1$.

2. **The fundamental existence theorem.** Let D be a finite domain of the z-plane bounded by n simple closed curves $C_1, C_2, \cdots, C_n$ which have continuously turning tangents. Denote by C the total boundary $C=\sum_{j=1}^{n} C_j$ of D. Let A denote the class of all functions $\phi(z)$ which are analytic in the closed domain $D+C$, and let B denote the class of all functions $\psi(z)$ of the form

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(1) Numbers in brackets refer to the bibliography at the end of the paper.

$$\psi(z) = \frac{1}{2\pi i} \frac{1}{z - t} + \phi(z), \qquad \phi \in A,$$

where t is a fixed point in D. We shall denote by L_p, $p > 1$, the class of complex-valued functions $f(z)$ defined for $z \in C$ which have a finite norm

$$\oint_C |f(z)|^p ds, \qquad ds = |dz|,$$

and which satisfy the conditions

$$\oint_C f(z)\phi(z)dz = 0, \qquad \phi \in A, \tag{1}$$

$$\oint_C f(z)\psi(z)dz = 1, \qquad \psi \in B. \tag{2}$$

Note that the function $f(z) \equiv 1$ is in this class.

Since the space $\mathcal{L}_p$ of all complex-valued functions $\mu(z)$ defined for $z \in C$ and possessing a finite integral

$$\oint_C |\mu(z)|^p ds$$

is uniformly convex for $p > 1$ [3], that is, since for any pair μ, $\nu \in \mathcal{L}_p$ the conditions

$$\oint_C |\mu|^p ds \leqq 1, \qquad \oint_C |\nu|^p ds \leqq 1, \qquad \oint_C \left|\frac{\mu + \nu}{2}\right|^p ds \geqq 1 - \epsilon$$

imply a relation

$$\oint_C |\mu - \nu|^p ds \leqq \delta(\epsilon), \qquad \lim_{\epsilon \to 0} \delta(\epsilon) = 0,$$

we find readily that there exists in L_p a function $f_0(z)$ which minimizes

$$\oint_C |f(z)|^p ds,$$

that is, a function $f_0(z)$ such that

$$\oint_C |f_0(z)|^p ds \leqq \oint_C |f(z)|^p ds, \qquad f \in L_p. \tag{3}$$

That $f_0(z)$ is unique also follows.

If now $\phi(z) \in A$ and λ is any complex number, we verify easily that the function

$$f(z) = f_0(z)\{1 + \lambda[\phi(z) - \phi(t)]\}$$

is in L_p, since the conditions (1) and (2) are satisfied. Hence we obtain from (3) the inequality

$$\oint_C |f_0(z)|^p ds \leqq \oint_C |f_0(z)|^p |1 + \lambda[\phi(z) - \phi(t)]|^p ds.$$

But a simple calculation yields

$$|1 + \lambda[\phi(z) - \phi(t)]|^p = 1 + p \operatorname{Re}\{\lambda[\phi(z) - \phi(t)]\} + O(|\lambda|^2),$$

and thus

$$\operatorname{Re}\left\{\lambda \oint_C |f_0(z)|^p [\phi(z) - \phi(t)] ds\right\} + O(|\lambda|^2) \geqq 0.$$

Since λ is arbitrary, we deduce that

$$\oint_C |f_0(z)|^p \phi(z) ds = \phi(t) \oint_C |f_0(z)|^p ds, \qquad \phi \in A. \tag{4}$$

Since, by (2),

$$\oint_C f_0(z) \frac{1}{2\pi i} \frac{1}{z - t} dz = 1,$$

we have

$$\oint_C |f_0(z)|^p ds > 0,$$

and we can set

$$\rho(z) = \frac{|f_0(z)|^p}{\oint_C |f_0(z)|^p ds}.$$

From (4) we have

$$\oint_C \rho(z)\phi(z) ds = \phi(t). \tag{5}$$

Now let w and w^* be any pair of points in one component of the exterior of D. We set

$$\phi(z) = \log \frac{z - w^*}{z - w}$$

and find

$$\oint_C \rho(z) \log \frac{z - w^*}{z - w} ds = \log \frac{t - w^*}{t - w} \cdot$$

Taking real parts of both sides of this formula and noticing that $\rho(z)$ is real, we obtain a relation of the form

$$\oint_C \rho(z) \log \frac{1}{|z - w|} ds = \log \frac{1}{|t - w|} - \alpha_j, \tag{6}$$

where α_j is a real constant which depends only upon the component of the exterior of D in which w lies.

For $w \in D$, we set

$$G(w, t) = \log \frac{1}{|w - t|} - \oint_C \rho(z) \log \frac{1}{|z - w|} ds, \tag{7}$$

and we notice that $G(w, t)$ is harmonic for $w \in D$ except for the logarithmic singularity at $w = t$.

We note that if w_1 and w_2 are points on a normal to C, situated at equal distances from C along this normal, and if $z \in C$, then

$$\log \left| \frac{z - w_1}{z - w_2} \right| \to 0 \quad \text{as} \quad |w_2 - w_1| \to 0, \quad \text{uniformly for } z \in C, \tag{8}$$

since C has a continuously turning tangent. Since the logarithm on the right in (6) is uniformly continuous when w ranges over any bounded subset of the exterior of D, it follows readily that the integral

$$\oint_C \rho(z) \log \frac{1}{|z - w|} ds$$

is a continuous function of w across C, and therefore in the finite part of the w-plane. By comparing (6) and (7), we now find

$$\lim_{w \to C_j} G(w, t) = \alpha_j, \qquad w \in D. \tag{9}$$

Hence $G(w, t)$ is harmonic in D, except for a logarithmic singularity at $w = t$, and has constant boundary values α_j on each curve C_j, $j = 1, \cdots, n$.

In the particular case $n = 1$ of a simply-connected domain, we verify that $\alpha_1 = 0$ and that $G(w, t)$ is the Green's function of D. Once in possession of the Green's function, it is not difficult to show by the usual procedure that the simply-connected domain D can be mapped conformally upon the interior of the unit circle. Thus we have presented a simple proof of the Riemann mapping theorem, and, indeed, if we take $p = 2$, even the condition of uniform convexity of $\mathcal{L}_p$ becomes elementary.

3. **Extremal problems in L_p.** We now return to the case of a multiply-

connected domain D, but we make the assumption that the curves $C_1, \cdots, C_n$ are analytic. From (9) and the Schwarz principle of reflection we conclude that $G(w, t)$ is harmonic on C. Since the normal derivative of the potential

$$\oint_C \rho(z) \log \frac{1}{|z-w|} ds$$

has a jump of 2π times the density $\rho(z)$ when w crosses C at $z \in C$, we find that the normal derivative of $G(w, t)$ with respect to the inner normal ν to C is

$$\frac{\partial G(w, t)}{\partial \nu} = 2\pi\rho(w).$$

Thus the extremal function $f_0(z) \in L_p$ has a modulus $|f_0(z)|$ which is proportional to the pth root of the distribution of mass

$$\frac{\partial G(z, t)}{\partial \nu}.$$

We proceed to apply more general variations to the extremal function $f_0(z) \in L_p$. If $\phi \in A$ and λ is a complex number, then $f_0(z) + \lambda[\phi(z) - \phi(t)]$ is in L_p, and hence

$$\oint_C |f_0(z)|^p ds \leqq \oint_C |f_0(z) + \lambda[\phi(z) - \phi(t)]|^p ds.$$

But

$$|f_0(z) + \lambda[\phi(z) - \phi(t)]|^p$$
$$= |f_0(z)|^p + p \operatorname{Re} \left\{ \lambda \frac{|f_0(z)|^p}{f_0(z)} [\phi(z) - \phi(t)] \right\} + O(|\lambda|^2),$$

and therefore

$$\oint_C \frac{|f_0(z)|^p}{f_0(z)} \phi(z) ds = \phi(t) \oint_C \frac{|f_0(z)|^p}{f_0(z)} ds. \tag{10}$$

The reader can verify that $[\phi(z) - \phi(t)]$ can be replaced by $[f_0(z) - 1]$ in this argument, and hence

$$\oint_C \frac{|f_0(z)|^p}{f_0(z)} ds = \oint_C |f_0(z)|^p ds > 0.$$

Thus we can set

$$k(z) = f_0(z) \left[\oint_C |f_0(z)|^p ds \right]^{-1/(p-1)} \tag{11}$$

to obtain

$$\oint_C \frac{|k(z)|^p}{k(z)} \phi(z) ds = \phi(t), \qquad \phi \in A, \tag{12}$$

while at the same time, by (1),

$$\oint_C k(z)\phi(z) dz = 0, \qquad \phi \in A. \tag{13}$$

We introduce the class Ω of functions $g(z)$ defined and integrable on C and satisfying

$$\oint_C g(z)\phi(z) dz = 0, \qquad \phi \in A. \tag{14}$$

Clearly $k(z) \in \Omega$, and by Cauchy's theorem and (12),

$$\oint_C \left\{ \frac{1}{2\pi i} \frac{1}{z - t} - \frac{|k(z)|^p}{k(z)} \bar{z}'(s) \right\} \phi(z) dz = 0, \qquad \phi \in A,$$

where $z(s)$ is the parametric representation of C in terms of arc length, whence

$$m(z) = \frac{1}{2\pi i} \frac{1}{z - t} - \frac{|k(z)|^p}{k(z)} \bar{z}'(s) \tag{15}$$

is also in Ω.

Now for any function $g(z) \in \Omega$, set

$$\Gamma(w, w^*) = \frac{1}{2\pi i} \oint_C g(z) \log \frac{z - w^*}{z - w} dz, \qquad w, w^* \in D. \tag{16}$$

The function $\Gamma(w, w^*)$ is analytic for $w, w^* \in D$, while for w_0, w_0^* in a component of the exterior of D we have

$$\frac{1}{2\pi i} \oint_C g(z) \log \frac{z - w_0^*}{z - w_0} dz = 0, \tag{17}$$

since

$$\log \frac{z - w_0^*}{z - w_0}$$

is in A in this case. We let z_0 and z_0^* denote two points on a component C_j of C, and we choose w and w^* to lie on the normals to C_j at z_0 and z_0^*, respectively, while w_0 and w_0^* are to lie on opposite sides, with respect to C_j, of these same normals, at the same distances as w and w^* from z_0 and z_0^*, respectively. Under these circumstances, we have

$$\log \frac{z - w^*}{z - w} - \log \frac{z - w_0^*}{z - w_0} \to 2\pi i \quad \text{as} \quad |w - w_0| \to 0, \quad |w^* - w_0^*| \to 0,$$

when z lies on one arc of C between z_0 and z_0^*, while for z on the remainder of C

$$\log \frac{z - w^*}{z - w} - \log \frac{z - w_0^*}{z - w_0} \to 0 \quad \text{as} \quad |w - w_0| \to 0, \quad |w^* - w_0^*| \to 0,$$

provided that we choose, once and for all, suitable branches of the logarithm. Thus we find from (16) and (17) that

$$\lim_{w \to z_0, w^* \to z_0^*} \Gamma(w, w^*) = \int_{z_0^*}^{z_0} g(z)dz,$$

and, indeed, the integration is to be extended from z_0^* to z_0 along C_j in the positive sense with regard to D [cf. 7]. Thus the indefinite integral

$$\int^z g(z)dz$$

along C of each function $g \in \Omega$ represents the continuous boundary values on C of a function $\Gamma(w)$ analytic for $w \in D$, but possibly multiple-valued there. Hence both the normalization condition (1), or also (13), and the variational condition (12) lead us to conclude that the expressions involved yield the boundary values of functions analytic in D.

We can set in particular

$$\Gamma_1(w) = \frac{1}{2\pi i} \oint_C k(z) \log \frac{z - w^*}{z - w} dz, \tag{18}$$

$$\Gamma_2(w) = \frac{1}{2\pi i} \oint_C m(z) \log \frac{z - w^*}{z - w} dz. \tag{19}$$

Let γ be a small arc of C, and let γ_0 be an arc inside D which, together with γ, bounds a small simply-connected subdomain Δ of D not including the fixed point $t \in D$. Then by Cauchy's theorem

$$\oint_{\gamma+\gamma_0} \Gamma_1(z)\phi_0(z)dz = 0, \quad \oint_{\gamma+\gamma_0} \Gamma_2(z)\phi_0(z)dz = 0$$

for every function $\phi_0(z)$ analytic in $\Delta+\gamma+\gamma_0$, since $\Gamma_1(z)$ and $\Gamma_2(z)$ are analytic in D and continuous in $D+C$. Since $k(z)$ is integrable of order p over C, and since $m(z)$ is integrable of order $q=p/(p-1)$ over C, we find by application of Hölder's inequality to Cauchy's formula that $\Gamma_1'(z)$ grows at most like $|z-z_0|^{1/q-1}$ and $\Gamma_2'(z)$ grows at most like $|z-z_0|^{1/p-1}$ as $z \to z_0 \in C$ along non-tangential paths. Hence $\Gamma_1'(z)$ and $\Gamma_2'(z)$ are integrable over γ_0 and we can integrate (18) and (19) by parts to obtain

$$\int_{\gamma_0} \Gamma_1'(z)\phi_1(z)dz + \int_{\gamma} k(z)\phi_1(z)dz = 0, \tag{20}$$

$$\int_{\gamma_0} \Gamma_2'(z)\phi_1(z)dz + \int_{\gamma} m(z)\phi_1(z)dz = 0,$$

where now $\phi_1(z)$ can be an arbitrary function analytic in $\Delta+\gamma+\gamma_0$, since $\phi_1'(z)=\phi_0(z)$. The second identity can be rewritten in the form

$$\int_{\gamma_0} \left\{\frac{1}{2\pi i}\frac{1}{z-t} - \Gamma_2'(z)\right\}\phi_1(z)dz + \int_{\gamma} \frac{|k(z)|^p}{k(z)}\bar{z}'(s)\phi_1(z)dz = 0, \tag{21}$$

since

$$\frac{1}{2\pi i}\frac{1}{z-t}$$

is analytic for $z\in\Delta$.

If Δ is sufficiently small, the function $\bar{z}'(s)$ can be continued analytically from γ into Δ, since the curves C are assumed to be analytic. Also, returning to the function $G(w, t)$ of formula (7), the normal derivative $\partial G/\partial\nu$ can be continued analytically throughout $\Delta+\gamma+\gamma_0$. But $|k(z)|$ is proportional to the pth root of $\partial G/\partial\nu$, and hence, in a small region Δ containing none of the finite number of possible zeros of $\partial G/\partial\nu$, the function $|k(z)|$ can be continued analytically. Denote by $T_1(z)$ the continuation of $\bar{z}'(s)$ in $\Delta+\gamma+\gamma_0$, and denote by $T_2(z)$ the continuation of $|k(z)|$ in $\Delta+\gamma+\gamma_0$. We set

$$\phi_1(z) = T_1(z)T_2(z)^{2-p}\phi_2(z)$$

and obtain from (21) the new identity

$$\begin{aligned}\int_{\gamma_0} \left\{\frac{1}{2\pi i}\frac{1}{z-t} - \Gamma_2'(z)\right\}T_1(z)T_2(z)^{2-p}\phi_2(z)dz \\ + \int_{\gamma} \bar{k}(z)(\bar{z}'(s))^2\phi_2(z)dz = 0.\end{aligned} \tag{22}$$

By the reasoning applied to the functions of class Ω, we conclude from (20) and (22) that the functions

$$\Gamma_3(w) = \frac{1}{2\pi i}\int_{\gamma_0}\Gamma_1'(z)\log\frac{z-w^*}{z-w}dz + \frac{1}{2\pi i}\int_{\gamma}k(z)\log\frac{z-w^*}{z-w}dz,$$

$$\begin{aligned}\Gamma_4(w) = \frac{1}{2\pi i}\int_{\gamma_0}\left\{\frac{1}{2\pi i}\frac{1}{z-t} - \Gamma_2'(z)\right\}T_1(z)T_2(z)^{2-p}\log\frac{z-w^*}{z-w}dz \\ + \frac{1}{2\pi i}\int_{\gamma}\bar{k}(z)(\bar{z}'(s))^2\log\frac{z-w^*}{z-w}dz\end{aligned}$$

are, for $w^*\in\Delta$ fixed, analytic functions of w in Δ with boundary values

$$\Gamma_3(w) = \int^w k(z)dz,$$

$$\Gamma_4(w) = \int^w \bar{k}(z)(\bar{z}'(s))^2 dz = \int^w \bar{k}(z)(d\bar{z})$$

on γ. Thus finally

$$\Gamma_3(w) = \bar{\Gamma}_4(w) + \text{const.}$$

on γ.

Our conclusion is that $\Gamma_3(w)+\Gamma_4(w)$ and $i\Gamma_3(w)-i\Gamma_4(w)$ are real on γ and hence may be continued analytically across γ by the Schwarz principle of reflection. Thus $\Gamma_3'(z)=k(z)$ is analytic on γ, and it follows that $k(z)$ is analytic on the entire system of curves C, since γ is an arbitrary arc of C.

An integration by parts can be applied to the derivatives

$$k'(z) = \frac{\partial k(z(s))}{\partial s}\bar{z}'(s), \qquad m'(z) = \frac{\partial m(z(s))}{\partial s}\bar{z}'(s),$$

which we have now shown to exist, in order to prove that they are in the class Ω. It follows that the analytic functions

$$k(w) = \frac{1}{2\pi i}\oint_C \frac{k(z)}{z-w}dz, \qquad m(w) = \frac{1}{2\pi i}\oint_C \frac{m(z)}{z-w}dz$$

in D have boundary values $k(z)$ and $m(z)$, respectively, on C, and are, indeed, analytic in the closed region $D+C$. This follows as well for $m(w)$ as for $k(w)$ by virtue of the relation (15) for $m(z)$ in terms of $k(z)$.

With these preliminaries behind us, we define the new domain function

$$l(w) = \frac{1}{2\pi i}\frac{1}{w-t} - m(w),$$

which is analytic in $D+C$, except for the simple pole at $w=t$. Clearly, $l(w)$ has the boundary values

$$\frac{1}{2\pi i}\frac{1}{z-t} - m(z) = \frac{|k(z)|^p}{k(z)}\bar{z}'(s),$$

and thus we find that the pair of analytic functions $l(w)$ and $k(w)$ satisfy the remarkable differential identity

$$l(z) = \frac{|k(z)|^p}{k(z)}\bar{z}'(s), \tag{23}$$

for $z\in C$. It will be our object now to deduce from this identity a number of analytic and extremal properties of the domain functions $l(w)$ and $k(w)$. Finally, we remark that (23) can also be written in the dual form

$$k(z) = \frac{|l(z)|^q}{l(z)} \bar{z}'(s), \tag{24}$$

where

$$q = \frac{p}{p-1}, \qquad \frac{1}{p} + \frac{1}{q} = 1,$$

since $|l(z)|^q = |k(z)|^p$ for $z \in C$. This new form of the identity will already lead the reader to suspect that $l(z)$ will have extremal properties in a class L_q which are analogous to those which $k(z)$ possesses by virtue of its definition in terms of the extremal function $f_0(z) \in L_p$. For the case $p = q = 2$, the relations (23), (24) are already known [4, 6, 7].

Since $l(z)$ has a simple pole of residue $1/2\pi i$ at $z = t$, we find from Cauchy's theorem

$$\begin{aligned} k(t) &= \oint_C l(z)k(z)dz = \oint_C \frac{|k(z)|^p}{k(z)} \bar{z}'(s)k(z)dz \\ &= \oint_C |k(z)|^p ds = \oint_C |l(z)|^q ds, \end{aligned} \tag{25}$$

as is also apparent directly from the variational relation (12). We now prove that among all functions $\phi(z) \in A$ with $\phi(t) = 1$, the function

$$f_0(z) = \frac{k(z)}{k(t)}$$

makes the integral

$$\oint_C |\phi(z)|^p ds$$

a minimum, and that it is the unique function in A with this property. For, by Hölder's inequality

$$\begin{aligned} 1 = \phi(t) &= \oint_C l(z)\phi(z)dz \\ &\leqq \left(\oint_C |l(z)|^q ds\right)^{1/q} \left(\oint_C |\phi(z)|^p ds\right)^{1/p}, \end{aligned}$$

with equality holding if and only if

$$\phi(z)/k(z) = \text{const.}$$

on C, by (23). Hence

$$\min_{\phi\in A,\phi(t)=1}\oint_C |\phi(z)|^p ds = \left(\oint_C |l(z)|^q ds\right)^{-p/q} = k(t)^{1-p}. \tag{26}$$

Of course, it was from this initial extremal property of $f_0(z)$, in the original class L_p, that we arrived at the analytic properties of $k(z)$, and thus (26) holds for a much wider class of analytic functions than A.

More striking, then, is the following inequality. Let $\psi(z)$ be any function of B. Then by the residue theorem and Hölder's inequality

$$1 = \oint_C \frac{k(z)\psi(z)dz}{k(t)} \leqq \frac{1}{k(t)}\left(\oint_C |k(z)|^p ds\right)^{1/p}\left(\oint_C |\psi(z)|^q ds\right)^{1/q}$$
$$= \left(\oint_C |k(z)|^p ds\right)^{-1/q}\left(\oint_C |\psi(z)|^q ds\right)^{1/q},$$

with equality holding for and only for the function $\psi(z)=l(z)$, by (24). Thus $l(z)$ is that function in B which yields the smallest value of the integral

$$\oint_C |\psi(z)|^q ds,$$

or, in other words,

$$\min_{\psi\in B}\oint_C |\psi(z)|^q ds = \oint_C |l(z)|^q ds = k(t). \tag{27}$$

We thus have, in addition to the differential relations (23), (24) between the pair of extremal functions $k(z)$ and $l(z)$, the following identity between the minima for the extremal problems (26) and (27) solved by $k(z)$ and $l(z)$:

$$\left(\min_{\psi\in B}\oint_C |\psi|^q ds\right)^{1/q}\left(\min_{\phi\in A,\phi(t)=1}\oint_C |\phi|^p ds\right)^{1/p} = 1. \tag{28}$$

Let L_q denote the class of functions $g(z)$ defined for $z\in C$, with

$$\oint_C |g(z)|^q ds < \infty,$$

and such that

$$\oint_C g(z)\phi(z)dz = \phi(t), \qquad \phi\in A.$$

The reader will verify with small difficulty that the boundary function $l(z)$, which is in this class L_q, yields the minimum value to the integral

$$\oint_C |g(z)|^q ds, \qquad g \in L_q.$$

With minor changes in technique, we could have arrived at all the results and identities of this section by starting with the class L_q and the extremal problem

$$\oint_C |g(z)|^q ds = \text{minimum}$$

in this class, using the uniform convexity of the corresponding class $\mathcal{L}_q$ of all complex-valued functions integrable of order q over C to obtain the existence of the function $l(z)$. Note that $l(z)$, then, solves the problem (27) in a much wider class of analytic functions than the class B.

The principal point of interest thus far is, therefore, the remarkable relations existing between $k(z)$ and $l(z)$, and the consequent duality between the extremal problems (26) and (27) in the classes L_p and L_q with

$$1/p + 1/q = 1.$$

We proceed to derive, in closing this section, a few more analytic properties of the pair of functions $l(z)$ and $k(z)$.

From (23) we have

$$l(z)k(z)z'(s) = |k(z)|^p \geqq 0 \qquad \text{on } C.$$

Since $l(z)k(z)$ has but one simple pole in D at $z=t$, we conclude from the argument principle that the product $l(z)k(z)$ has at most $n-1$ zeros in D. Thus $l(z)$ and $k(z)$ have together at most $n-1$ zeros. Furthermore, we have on C

$$\frac{2\pi}{k(t)} l(z)k(z)z'(s) = \frac{\partial G(z, t)}{\partial \nu},$$

where $G(z, t)$ is the potential function of the formula (7) with constant boundary values on each curve C_j bounding D. Thus, clearly,

$$-\frac{2\pi i}{k(t)} l(z)k(z) = \frac{\partial G(z, t)}{\partial x} - i\frac{\partial G(z, t)}{\partial y}. \tag{29}$$

Also, by (23),

$$\frac{k(z)^p}{l(z)^q}$$

has unit modulus on C, although this function is in general multiple-valued in D. Thus

$$\log k(z)^p - \log l(z)^q$$

has imaginary boundary values on C. This fact, together with (29), can be used to determine $k(z)$ and $l(z)$ in terms of the solutions of various Dirichlet problems, and thus we can relate our new problems (26) and (27) to classical potential theory [cf. 4, 8, 9].

Finally, let $\zeta(z)$ be a conformal transformation of the domain D in the z-plane onto a new domain D^* in the ζ-plane, and let $l^*(\zeta)$ and $k^*(\zeta)$ be the functions in D^* with extremal properties corresponding to those of $l(z)$ and $k(z)$ in D. If $\zeta'(t)=1$, then we should expect to have

$$l(z)dz^{1/q} = l^*(\zeta)d\zeta^{1/q}, \tag{30}$$

$$k(z)dz^{1/p} = k^*(\zeta)d\zeta^{1/p}. \tag{31}$$

That is, we should expect k and l to be differentials of orders $1/p$ and $1/q$, respectively. However, for $p\neq 2\neq q$ this follows only when the transformation $\zeta(z)$ does not reverse the sense of rotation, taken to be positive with respect to the domain, of the curves C_j bounding D, for only under this restriction are the functions

$$\left(\frac{d\zeta}{dz}\right)^{1/q}, \quad \left(\frac{d\zeta}{dz}\right)^{1/p}$$

single-valued in D. Thus (30) and (31) hold for and only for transformations $\zeta(z)$ carrying the outer boundary of D into the outer boundary of D^*. These remarks follow from the fact that when orientation of a boundary curve C_j is preserved under the correspondence $\zeta(z)$, then

$$\Delta_{C_j} \arg \zeta'(z) = 0,$$

while when orientation of C_j is not preserved,

$$\Delta_{C_j} \arg \zeta'(z) = \pm 2.$$

Thus for $p, q\neq 2$, it must be borne in mind that our extremal problems in L_p and L_q have only a restricted, although significant, degree of conformal invariance.

4. **Further relations among extremal problems.** It has been shown in the papers [4, 7] that for $p=q=2$, the function $l(z)$ has no zeros, while the function $k(z)$ has $n-1$ zeros interior to D. Now since $k(z)$ and $l(z)$ are unique, it is not hard to show that they depend continuously upon p and q. Thus for values of p and q sufficiently near 2, the zeros of the corresponding functions $l(z)$ and $k(z)$ are all given to $k(z)$. This must be so, for example, in some interval $2\leqq q\leqq 2+\epsilon$, $p=q/(q-1)\leqq 2$. For these values of q we can set

$$L(z) = 2\pi i l(z)^2, \qquad K(z) = \frac{(2\pi)^{q/2-1}}{i}\,\frac{k(z)}{l(z)},$$

and we obtain in $L(z)$ a function analytic in $D+C$, except for a double pole at $z=t$ with leading coefficient $1/2\pi i$, while in $K(z)$ we obtain a function regular throughout $D+C$ with $K(t)=0$, $K'(t)=(2\pi)^{q/2}k(t)>0$. Now we have

$$\begin{aligned} L(z)K(z)z'(s) &= (2\pi)^{q/2}l(z)^2 \frac{k(z)}{l(z)} z'(s) \\ &= (2\pi)^{q/2}l(z)k(z)z'(s) \geqq 0 \qquad \text{on } C, \end{aligned}$$

and furthermore

$$L(z) = \frac{|K(z)|^{pq/(q-p)}}{K(z)} \bar{z}'(s), \qquad \text{on } C, \tag{32}$$

since

$$\begin{aligned} 2\pi i l(z)^2 &= \frac{(2\pi)^{q/2}}{(2\pi)^{q/2-1}/i} l(z) \frac{|k(z)|^p}{k(z)} \bar{z}'(s) \\ &= (2\pi)^{q/2} \frac{|k(z)|^p}{K(z)} \bar{z}'(s) = (2\pi)^{q/2} \frac{|k(z)|^{((q-p))/q(pq/(q-p))}}{K(z)} \bar{z}'(s). \end{aligned}$$

From (32) we obtain alternately

$$K(z) = \frac{|L(z)|^{q/2}}{L(z)} \bar{z}'(s). \tag{33}$$

Since

$$\frac{1}{q/2} + \frac{1}{pq/(q-p)} = 1,$$

we see immediately that the differential identities (32) and (33) are altogether analogous to (23) and (24). We shall proceed to deduce extremal properties of $L(z)$ and $K(z)$ from (32) and (33) which are altogether analogous to those of $l(z)$ and $k(z)$. For $p=q=2$ this has already been done in the paper [4].

Let $L_{pq/(q-p)}$ be the class of functions $\phi(z)$ analytic in $D+C$ with $\phi(t)=0$, $\phi'(t)=1$. Let $L_{q/2}$ be the class of functions $\psi(z)$ analytic in $D+C$ except for a pole

$$\psi(z) = \frac{1}{2\pi i} \frac{1}{(z-t)^2} + \frac{a_{-1}}{z-t} + a_0 + a_1(z-t) + \cdots$$

of order two at $z=t$. From a strict pedagogical point of view, the reader can define, alternately, the classes $L_{pq/(q-p)}$ and $L_{q/2}$ to consist merely of the boundary values of the functions described. Clearly

$$\frac{K(z)}{K'(t)} \in L_{pq/(q-p)}, \qquad L(z) \in L_{q/2}.$$

We maintain that

$$\oint_C |\phi(z)|^{pq/(q-p)}ds \geqq K'(t)^{-2/(q-2)} = (2\pi)^{-q/(q-2)}k(t)^{-2/(q-2)}, \tag{34}$$

for $\phi \in L_{pq/(q-p)}$, with equality holding for and only for

$$\phi(z) = \frac{K(z)}{K'(t)} \cdot$$

Indeed, by Cauchy's theorem and Hölder's inequality, we have

$$\begin{aligned} 1 &= \oint_C L(z)\phi(z)dz \\ &\leqq \left(\oint_C |L(z)|^{q/2}ds\right)^{2/q}\left(\oint_C |\phi(z)|^{pq/(q-p)}ds\right)^{(q-p)/pq}, \qquad \phi \in L_{pq/(q-p)}, \end{aligned}$$

and by the identity (32), equality can hold for and only for the function $\phi(z) = K(z)/K'(t)$. The inequality (34) follows when we remark that

$$\begin{aligned} \left(\oint_C |L(z)|^{q/2}ds\right)^{(2/q)(pq/(q-p))} &= \left(\oint_C L(z)K(z)dz\right)^{2p/(q-p)} \\ &= K'(t)^{2p/(q-p)} = K'(t)^{2/(q-2)}. \end{aligned}$$

Likewise, we obtain the inequality

$$\oint_C |\psi(z)|^{q/2}ds \geqq K'(t) = (2\pi)^{q/2}k(t), \qquad \psi \in L_{q/2}, \tag{35}$$

with equality holding for and only for the function $\psi(z) = L(z)$. Indeed, this follows from Hölder's inequality and (33), since we can write

$$\begin{aligned} K'(t) &= \oint_C \psi(z)K(z)dz \\ &\leqq \left(\oint_C |\psi(z)|^{q/2}ds\right)^{2/q}\left(\oint_C |K(z)^{pq/(q-p)}ds|\right)^{(q-p)/pq}, \qquad \psi \in L_{q/2}. \end{aligned}$$

We note that by the procedure of §3, the inequalities (34) and (35) can be obtained with far more general classes of functions $L_{pq/(q-p)}$ and $L_{q/2}$ than we have used here.

For the case $p = q = 2$, the inequalities analogous to (34) and (35) have been obtained in a previous paper [4]. In that work, the Hölder inequality was not required, and the estimates were therefore all the more elementary. The results obtained there relate an extremal problem in a class L_∞ of bounded functions and an extremal problem in a class L_1 of functions of bounded variation to our problems in L_2. The problem referred to in the class L_∞ is nothing more than the extension of Schwarz's lemma to multiply-connected regions, while the problem in L_1 is one of distortion of length in conformal mapping. Of course, the class L_2 to which these classes L_1 and L_∞ are related leads to the Szegö kernel function [4], and the identities (23), (24) and (32), (33) may be viewed as extensions to the Banach spaces L_p of the boundary relations usually arrived at in the theory of kernel functions in Hilbert space L_2.

Combining (34) and (35), we have

$$\left(\min_{\psi\in L_{q/2}} \oint_C |\psi|^{q/2}ds\right)^{2/q}\left(\min_{\phi\in L_{pq/(q-p)}} \oint_C |\phi|^{pq/(q-p)}ds\right)^{(q-p)/pq} = 1, \tag{36}$$

in analogy with (28). We have here also, however, additional identities such as

$$\left[\left(\min_{\psi\in B} \oint_C |\psi|^{q}ds\right)^{1/q}\right]^2\left(\min_{\phi\in L_{pq/(q-p)}} \oint_C |\phi|^{pq/(q-p)}ds\right)^{(q-p)/pq} = \frac{1}{2\pi}. \tag{37}$$

Thus we can relate extremal problems in the four classes L_p, L_q, $L_{pq/(q-p)}$, $L_{q/2}$, if $1/p+1/q=1$, $2\leqq q\leqq 2+\epsilon$. Furthermore, it is evident that if ϵ can be chosen large enough so that for integral $m=2+\epsilon\geqq q\geqq 2$ the extremal functions $l(z)$ corresponding to these values of q continue to have no zeros, then we can relate problems in classes $L_{pq/(q-\mu p)}$, $L_{q/(\mu+1)}$ to our original problems, where μ runs over all integers between 1 and $m-1$, inclusive. Indeed, we merely divide the differential

$$l(z)k(z)dz$$

into new component factors defined by the formula

$$\left\{l(z)^{\mu+1}\right\}\left\{\frac{k(z)}{l(z)^\mu}\right\}dz = l(z)k(z)dz$$

to arrive at this conclusion, and these factors yield our new extremal functions.

It is worth remarking that in doubly-connected domains the above case where ϵ can exceed 1 may occur, and that, indeed, a situation can arise which leads to arbitrarily large values of ϵ. To see this, we return to the study of the pair of domain functions $l(z)$ and $k(z)$ through Dirichlet problems mentioned at the end of §3.

Let D be the annulus $1<|z|<\rho$, let

$$\omega(z) = \frac{\log |z|}{\log \rho},$$

and suppose $1<t<\rho$. Then the functions $l(z)$, $k(z)$, and consequently the functions

$$q \log l(z) - p \log k(z), \qquad l(z)k(z),$$

can exist in D with $l(z) \neq 0$ there if and only if there is a point ζ with $-\rho < \zeta < -1$ and with

$$q\omega(t) + p\omega(\zeta) = q. \tag{38}$$

This can be verified easily directly, or by the methods in [4], for one notices that the point $z=\zeta$ is merely the zero of $k(z)$ in D, so that Green's theorem yields (38). From (38) we find that q must be given by the formula

$$\frac{q}{p} = q - 1 = \frac{\omega(\zeta)}{1 - \omega(t)};$$

thus, by suitable choice of ζ, q can have any value >1, but not greater than

$$1 + \frac{1}{1 - \omega(t)} = 2 + \frac{\omega(t)}{1 - \omega(t)}\cdot$$

Hence when the point of normalization $z=t$ is sufficiently close to the outer boundary, $|z| = \rho$, of D, the upper estimate on q exceeds any prescribed integral value m, and all the dualities displayed above can occur. Notice the particular role played here by the outer boundary of D, a result of the incomplete conformal invariance of our extremal problem when $q \neq 2$.

Thus we see that the fundamental properties of our extremal functions $l(z)$, $k(z)$ follow from application of the basic differential identity (23), together with Hölder's inequality and the general method of contour integration. When a new identity of this form, such as (32), can be obtained, even by mere algebraic manipulation, the new functions involved, such as $L(z)$ and $K(z)$, are immediately shown to possess extremal properties to correspond with the identity. Thus various unexpected relations are obtained between these several extremal problems in the multiply-connected domain D.

5. **Normalization at infinity.** With our normalization so far in the finite domain D we have had to require that q be not too large in order to obtain the results (34), (35). We shall now assume that D is infinite and that the point $z=t$ of normalization is the point at infinity. The symmetry obtained in the new normalization will to some degree offset the lack of complete conformal invariance of our problems in L_p, $p \neq 2$, and will lead to a new relation amongst extremal problems in all classes L_q with no restriction on q of this indefinite nature.

Transforming $z=t$ to $z=\infty$, taking $q=p=2$ in (23), and multiplying by a suitable factor of $2\pi i$, we find that there exist in D two functions $l(z)$ and $k(z)$ with expansions

$$l(z) = 1 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots,$$

$$k(z) = \frac{1}{i}\frac{\alpha}{z} + \frac{b_2}{z^2} + \frac{b_3}{z^3} + \cdots, \qquad \alpha > 0,$$

about the point at infinity, which satisfy on C the identity

$$l(z) = \frac{|k(z)|^2}{k(z)}\bar{z}'(s) = \bar{k}(z)\bar{z}'(s). \tag{39}$$

The function $l(z)$ is found to have no zeros in $D+C$ and no changes of argument over the curves C_j [4], so that $l(z)^\mu$ is defined and analytic throughout $D+C$ for all μ. Of course, $k(z)$ has in D n zeros, counting the zero at infinity. The two functions $l(z)$, $k(z)$ are Szegö kernel functions [4].

We can set

$$L(z) = l(z)^{1+\mu} = 1 + \cdots,$$

$$K(z) = \frac{k(z)}{l(z)^\mu} = \frac{1}{i}\frac{\alpha}{z} + \cdots,$$

and $L(z)$, $K(z)$ are found to be analytic throughout D. If we set

$$p = \frac{2}{1-\mu}, \qquad q = \frac{2}{1+\mu}, \qquad 1 < p, q < \infty,$$

we find immediately from (39) the identity

$$L(z) = \frac{|K(z)|^p}{K(z)}\bar{z}'(s), \qquad \text{on } C, \tag{40}$$

or

$$K(z) = \frac{|L(z)|^q}{L(z)}\bar{z}'(s), \qquad \text{on } C, \tag{41}$$

and also

$$\frac{1}{p} + \frac{1}{q} = \frac{1-\mu+1+\mu}{2} = 1.$$

From (40) we find that if

$$\psi(z) = 1 + \cdots, \qquad |z| \text{ large},$$

is analytic in $D+C$, then

$$2\pi\alpha = \oint_C \psi(z)K(z)dz \leqq \left(\oint_C |\psi(z)|^q ds\right)^{1/q}\left(\oint_C |K(z)|^p ds\right)^{1/p},$$

with equality holding for and only for $\psi(z) = L(z)$. Also, from (41), if

$$\phi(z) = \frac{1}{i}\frac{\alpha}{z} + \cdots, \qquad |z| \text{ large},$$

is analytic in $D+C$, then

$$2\pi\alpha = \oint_C L(z)\phi(z)dz \leqq \left(\oint_C |L(z)|^q ds\right)^{1/q}\left(\oint_C |\phi(z)|^p ds\right)^{1/p},$$

with equality for and only for $\phi(z) = K(z)$. Thus, by our usual methods, we find

$$\min_{\psi} \oint_C |\psi|^q ds = \oint_C |L|^q ds = 2\pi\alpha, \tag{42}$$

$$\min_{\phi} \oint_C |\phi|^p ds = \oint_C |K|^p ds = 2\pi\alpha, \tag{43}$$

and similar inequalities [4] hold in L_1 and L_∞ for the extremal functions

$$\frac{k(z)}{l(z)}, \quad l(z)^2 \quad \text{and} \quad 1, \quad l(z)k(z),$$

corresponding to the limiting values $\mu = 1$, $\mu = -1$. Thus far we restrict ourselves to the cases $-1 < \mu < 1$.

In this way we get for all our extremal problems the same extremal functions, except for exponents. Hence our problems in L_p and L_q lead back here to the kernel functions in L_2 already studied [4, 7] and to Schwarz's lemma in L_∞ and distortion of length in L_1.

6. **The cases** $p<1$, $q<1$. The discussions in §5 lead us to ask whether it is possible to solve problems such as (26), (27) or (42), (43) when $p<1$ or $q<1$. The cases $p=1$ and $q=1$ are already known to yield to our present method [6, 7]. We shall show here that all our problems have solutions for positive $p<1$, $q<1$, and that identities of the form (23), (24) are satisfied by the extremal functions, but we shall not be able to obtain the uniqueness of the extremal functions in this case, nor a duality between L_p and L_q with $1/p+1/q=1$. Indeed, for $q=1$ examples are easily found where the extremal functions are not unique, for example, the extremal function $l(z)k(z)$ of §5 is not unique in L_1.

It will be necessary to change our method in order to discuss cases where $p<1$. We shall return to ideas developed previously [1, 4], and since the work is no longer so elegant or new as that of the foregoing sections, we shall be content to sketch proofs, giving reference to the earlier papers for detailed exposition of method. A point of exceptional interest will be, however, the fact that §5 can be extended, including a uniqueness proof, for sufficiently large $q<1$.

Suppose, then, that we try, for example, to find an extremal function $f_0(z)$

for the problem (26), with $p<1$. We can certainly pick a minimal sequence $f_m(z)$ with

$$\oint_C |f_m(z)|^p ds \to \inf_{\phi\in A,\phi(t)=1} \oint_C |\phi(z)|^p ds,$$

where A is the class of functions analytic in $D+C$. If $f_m(z)$ has zeros $z_1, \cdots, z_\sigma$ in D, then we can write

$$\log |f_m(z)| = -\sum_{\mu=1}^{\sigma} g(z; z_\mu) + \log |F_m(z)|,$$

where $g(z; \zeta)$ is the Green's function of D, and where $F_m(z)$ is analytic in D. We can vary the zeros $z_1, \cdots, z_\sigma$ to new positions $z_1^*, \cdots, z_\sigma^*$ and thus define a new function $f_m^*(z)$ by the formula

$$\log |f_m^*(z)| = -\sum_{\mu=1}^{\sigma} g(z; z_\mu^*) + \log |F_m(z)|,$$

provided that

$$\sum_{\mu=1}^{\sigma} \oint_{C_j} \left\{\frac{\partial g(z; z_\mu)}{\partial \nu} - \frac{\partial g(z; z_\mu^*)}{\partial \nu}\right\} ds = 0, \qquad j = 1, \cdots, n,$$

$$\sum_{\mu=1}^{\sigma} \{g(t; z_\mu) - g(t; z_\mu^*)\} = \delta > 0,$$

so that the new function $f_m^*(z)$ is single-valued and $f_m^*(t)=e^\delta>1$, while on C

$$|f_m^*(z)| = |f_m(z)|.$$

Now for sufficiently small $\delta>0$ all these equations can be satisfied by making suitable choice of the points z_μ^*, provided that $z_1, \cdots, z_\sigma$ are not all critical points of a harmonic function

$$h(z) = g(z; t) + \sum_{j=1}^{n-1} \lambda_j \oint_{C_j} \frac{\partial g(\zeta; z)}{\partial \nu} ds.$$

This follows, indeed, by application of the implicit function theorem [1, 4]. Since, by the argument principle, $h(z)$ can have at most $n-1$ critical points in D, we conclude in particular that if $\sigma>n-1$, then we can replace the competing function $f_m(z)$ in our minimal sequence by the new function $e^{-\delta}f_m^*(z)$, and this function will yield for the norm

$$\oint_C |f(z)|^p ds$$

a value less than that obtained with $f_m(z)$. If we increase δ in this process until it becomes as large as possible, then clearly the function $e^{-\delta}f_m^*(z)$ will have at most $n-1$ zeros. Thus we can assume in the first place that our minimal sequence $f_m(z)$ consists of functions with at most $n-1$ zeros. We note, in passing, that the above function $h(z)$ plays a role analogous to that of the function $G(z, t)$ of formula (7).

It follows now that the zeros of the functions of our minimal sequence can be assumed to have at most $n-1$ limit positions ζ_j in D. We can, indeed, choose, with no loss of generality, only competing functions $\phi(z)$ of the form

$$\phi(z) = \phi_1(z)F(z),$$

where $F(z)$ is analytic in $D+C$ with $F(t)=1$, $F(\zeta_j)=0$, and where $\phi_1(z)\neq 0$ in D. In this discussion, if we count a zero of order greater than one according to its multiplicity, all statements remain valid [4].

Now let τ be an integer so large that $\tau p>1$. The periods of the logarithms of our competing functions $f_m(z)$, now assumed to have zeros only at the points ζ_j, are of the form

$$2\pi i(\mu_j\tau + \nu_j), \qquad 0 \leqq \nu_j < \tau, \qquad \text{on } C.$$

For a suitable subsequence $f_{m_\mu}(z)$ of the minimal sequence $f_m(z)$ the integers ν_j all tend to limits, $j=1, \cdots, n$. Thus we may equally well assume in the first place that the ν_j have their limit values. Hence, finally, we see that there is no loss of generality in our extremal problem (26) if we assume that our competing sequence $f_m(z)$ consists of functions of the form

$$f_m(z) = \phi_m(z)^\tau F(z),$$

where $F(z)$ is fixed, with zeros at the points ζ_j and with fixed changes of argument about the C_j.

Thus our problem reduces to that of finding

$$\min_{\phi\in A,\phi(t)=1} \oint_C |F(z)|^p|\phi(z)|^{\tau p}ds.$$

Here $|F(z)|^p$ merely plays the role of a positive weight function to be taken into consideration in our problem, that is, we replace ds by $|F(z)|^p\, ds$ as a measure along C. Since $\tau p>1$, the new problem has a unique extremal function $f_F(z)$, as can easily be verified by application of the methods of §§2 and 3. Indeed, the weight function $|F(z)|^p$ does not add serious difficulties to our work. Thus the original problem possesses an extremal function

$$f_0(z) = f_F(z)^\tau F(z),$$

and since $F(z)$ may be chosen to be analytic on C, and $f_F(z)$ can be proved to be analytic on C by the methods of §3, it follows that the extremal function $f_0(z)$ is regular on C as well as in D.

Once in possession of the analytic extremal function $f_0(z)$, it is not hard to make the variations of §3 and to prove that an identity of the form (23) holds with

$$k(z) = f_0(z)\left[\oint_C |f_0(z)|^p ds\right]^{1/(1-p)}.$$

Thus the problem (26) always possesses at least one extremal function $f_0(z)$ which satisfies an identity (23) in terms of the usual associated pair of functions $l(z)$, $k(z)$ when $p>0$. No uniqueness can be deduced in general, since Hölder's inequality does not hold with $p<1$. But $f_0(z)$ is the unique extremal function of the form $\phi(z)^\tau F(z)$, for fixed $F(z)$, since the minimum problem with weight function $|F(z)|^p$ and exponent $\tau p>1$ has a unique solution.

Thus the more important deductions of §3 carry over to the case $p<1$, and we have the less obvious result that our extremal problems in L_p have here nonvanishing solutions, even though no inequality is available from elementary processes.

We proceed now to study the problem (42) for $q<1$. A similar study can be made generalizing the work of §4, carried out for $q>2$ near 2, to the case of values of $q<2$ and sufficiently near 2, but we leave the results there for the reader to complete.

By the considerations of this section, we see that the problem (42) always possesses minimum functions $L_q(z)$, even when $q<1$, and that each of these solutions satisfies an identity

$$K_p(z) = \frac{|L_q(z)|^q}{L_q(z)}\bar{z}'(s), \qquad p = \frac{q}{q-1} < 0,$$

for suitable $K_p(z)$ associated with $L_q(z)$. In the notation of §5, we shall prove that for q sufficiently near 1, $L_q(z)$ is unique and is given by the formula

$$L_q(z) = l(z)^{1+\mu}, \qquad \mu = 2/q - 1 > 1. \tag{44}$$

This will complete the results of §5.

First, we remark that in the paper [4] it was shown that $L_1(z) = l(z)^2$ is unique. It follows therefore from the relation

$$\lim_{q\to 1}\oint_C |L_q|^q ds = \oint_C |L_1|\, ds$$

that

$$\lim_{q\to 1} L_q(z) = l(z)^2,$$

since $L_q(z)$ has at most $n-1$ zeros and since $|L_q(z)|^q$ is on C of the form

$$|L_q(z)|^q = L_q(z)K_p(z)z'(s) \geqq 0,$$

and this quantity is proportional to the normal derivative of a generalized Green's function such as that given in formula (7). Thus $L_q(z)$ has no zeros in $D+C$ and no changes of argument on the C_j for q sufficiently near 1, $q<1$. Thus we can define

$$l(z) = L_q(z)^{q/2},$$

and we discover that this function has all the properties described for it in §5, when q is near 1. Hence, by the uniqueness of $l(z)$, proved in §5, we find that $L_q(z)$ is unique and is given by (44).

Let $q_0<1$ be a number for which (44) holds. Then we can show by considering the relation

$$\lim_{q\to q_0} L_q(z) = L_{q_0}(z)$$

that for $q<q_0$ sufficiently near q_0 the formula (44) holds. The argument here is, indeed, that just given for $q_0=1$. Hence there is no smallest q_0 for which (44) holds.

We close by obtaining an inequality directly which is related to the problem (42) in the infinite domain D. Similar inequalities can be derived for our other extremal problems.

Let $g(z, \infty)$ be the Green's function of D with a positive logarithmic singularity at $z=\infty$, and let $\psi(z)=1+\phi(z)$ be any function of the class used in (42), so that $\phi(z)$ is regular in $D+C$ and vanishes at infinity. Then by Green's theorem we have

$$\begin{aligned}\oint_C \frac{\partial g}{\partial \nu}|\psi|^p ds - \oint_C \frac{\partial g}{\partial \nu} ds &= \oint_C \frac{\partial g}{\partial \nu}\{|1+\phi|^p - 1\}ds \\ &\quad - \oint_C g\frac{\partial}{\partial \nu}\{|1+\phi|^p\}ds \\ &= \iint_D g\Delta\{|1+\phi|^p\}dxdy \\ &= p^2\iint_D g|(1+\phi)^{p-2}\phi'^2|\,dxdy \geqq 0,\end{aligned}$$

with equality holding for and only for $\psi(z)\equiv 1$. Hence

$$\oint_C \frac{\partial g}{\partial \nu}|\psi|^p ds \geqq \oint_C \frac{\partial g}{\partial \nu} ds = 2\pi, \tag{45}$$

and we have, finally, the inequality

$$\oint_C |\psi(z)|^p ds \geqq \frac{2\pi}{\max\limits_C \dfrac{\partial g}{\partial \nu}}. \tag{46}$$

Also, we have, from (45) and Hölder's inequality, for all $p>0$ the inequality

$$\oint_C |\psi(z)|^p ds \geqq (2\pi)^Q \left\{ \oint_C \left(\frac{\partial g}{\partial \nu}\right)^{Q/(Q-1)} ds \right\}^{1-Q}, \qquad Q > 1. \tag{47}$$

We remark that if the function $G(w, t)$ of (7) ,with $t=\infty$, should coincide with the Green's function $g(w, \infty)$ of D, then (45) can be reformulated in such a way as to yield a sharp inequality for the problem (42). If this should be the case, it can be shown that (42) has the solution (44) for all $q>0$. Finally, we point out that when the infinite domain D has rotational symmetry about the origin, the functions $G(z, \infty)$ and $g(z, \infty)$ do coincide, except for an inessential additive constant.

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