ON FOURIER-STIELTJES INTEGRALS

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1. **Introduction.** A complex valued function $\phi(\tau)$ on $(-\infty, \infty)$ will be said to satisfy condition (A) if for some constant C

$$\text{(A)} \qquad \left| \sum a_n\phi(\tau_n) \right| \leqq C \left\| \sum a_n \exp(i\tau_n s) \right\|$$

for all finite sets of real numbers (τ_n) and complex numbers (a_n), where

$$(1) \qquad \|f(\cdot)\| = \text{LUB}\left[|f(s)| \mid -\infty < s < \infty \right].$$

Bochner [3](1) has shown that a continuous function $\phi(\tau)$ satisfying (A) can be uniquely represented by a Fourier-Stieltjes integral

$$(2) \qquad \phi(\tau) = \int_{-\infty}^{\infty} \exp(i\tau s) dy(s)$$

where $y(s)$ belongs to the class V of functions of bounded variation, continuous on the right with $y(-\infty)=0$. We shall show that this same representation theorem remains valid for almost all τ if $\phi(\tau)$ is any measurable function satisfying (A)(2). The proof of this theorem is sufficiently general to include functions $\phi(\tau)$ with values in a separable Banach space Y [1] provided that the set of elements

$$(3) \qquad \left[\left(\sum a_n\phi(\tau_n)\right) / \left\| \sum a_n \exp(i\tau_n s)\right\| \;\middle|\; \text{all finite sets } (\tau_n) \text{ and } (a_n) \right]$$

is contained in a weakly compact subset of Y.

Any function $\phi(\tau)$ satisfying condition (A) can be used to define a linear bounded functional $L(p)$ on the set of trigonometric polynomials:

$$(4) \qquad L\left[\sum a_n \exp(i\tau_n s)\right] = \sum a_n \phi(\tau_n)$$

where the norm for $p(s)$ is given by (1). It follows that $\|L\| = C^*$, the smallest value of C satisfying (A). Since the trigonometric polynomials are dense in the Banach space X of almost period functions, L can be extended in a unique fashion to be a linear bounded functional on X. Conversely, given such a functional, $\phi(\tau) = L[\exp(i\tau s)]$ will satisfy (A). The functions $\phi(\tau)$ satisfying (A) are therefore in one-to-one correspondence with the functionals $L \in \overline{X}$ and any classification of the functions satisfying (A) is at the same time a

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(1) Numbers in brackets refer to the references cited at the end of the paper.

(2) This result is analogous to that of F. Riesz [9] generalizing the Bochner theorem [2, pp. 74–76] on positive definite functions.

classification of the elements of $\overline{X}$.

2. **Fourier-Stieltjes integral.** In this section we shall extend Bochner's result [3] to measurable functions.

THEOREM 1. *If $\phi(\tau)$ is measurable and satisfies* (A), *then there exists a unique $y(\cdot)\in V$ such that* (2) *holds a.e. and* Var $\{y(\cdot)\}\leqq C^*$. *If $\phi(\tau)$ is continuous, then* (2) *holds for all τ and* Var $\{y(\cdot)\}=C^*$.

Thus if $\phi(\tau)$ is measurable and satisfies (A), it can differ from a continuous function $\phi^*(\tau)$ only on a set of measure zero. Thus $\phi^*(\tau)$ and hence $y(s)$ are uniquely determined (see, for example, [2, p. 67]).

The first step in the proof of this theorem is to show that $\phi(\tau)$ can be used to define a linear bounded functional on the B-space X_0 of continuous functions $f(s)$ which converge to zero as $|s|\to\infty$. The norm in X_0 is defined as in Equation (1). Now the set of continuous functions M which are ultimately zero and have piecewise continuous derivatives form a linear subspace of X_0 dense in X_0. For $f(\cdot)\in M$ we define

$$h(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(s) \exp(-i\tau s)ds. \tag{5}$$

It follows that $h(\tau)$ is bounded, continuous, and absolutely integrable on $(-\infty, \infty)$.

If $f(s)$ vanishes outside of the interval $(-l/2, l/2)$, then for $s\in(-l/2, l/2)$

$$f(s) = \sum_{-\infty}^{\infty} a_n \exp(2\pi ins/l) \tag{6}$$

where

$$a_n = (2\pi/l)h(2\pi n/l). \tag{7}$$

The series converges uniformly on $(-\infty, \infty)$; in fact, since $f'(s)$ is piecewise continuous it follows that

$$\begin{aligned}\sum_{-\infty}^{\infty}{}' |a_n| &\leqq [\textstyle\sum' n^{-2}]^{1/2}[\sum |na_n|^2]^{1/2}\\ &\leqq c_1\left[l(2\pi)^{-2}\int_{-\infty}^{\infty} |f'(s)|^2 ds\right]^{1/2} \leqq cl^{1/2},\end{aligned} \tag{8}$$

where c is independent of l.

We now define the functional L_0 on M as

$$L_0(f) = \int_{-\infty}^{\infty} h(\tau)\phi(\tau)d\tau \tag{9}$$

where h is related to f by (5). Since $\phi(\tau)$ is measurable and uniformly bounded by C and since $h(\tau)$ is absolutely integrable, it follows that L_0 is linear on M. It remains to show that L_0 is bounded and, in fact, that $\|L_0\| \leqq C^*$. To this end we consider the auxiliary function

$$F(l, f) = \sum_{-\infty}^{\infty} h\left(\frac{2\pi n}{l}\right)\phi\left(\frac{2\pi n}{l}\right)\left(\frac{2\pi}{l}\right) \tag{10}$$

which roughly approximates the integral on the right side of (9). Making use of the linear bounded functional L defined as in (4) together with the relations (7) and (8), we obtain

$$\begin{aligned} F(l, f) &= \sum_{-\infty}^{\infty} L\left[h\left(\frac{2\pi n}{l}\right)\exp\left(\frac{2\pi ins}{l}\right)\left(\frac{2\pi}{l}\right)\right] \\ &= L\left[\sum_{-\infty}^{\infty} h\left(\frac{2\pi n}{l}\right)\exp\left(\frac{2\pi ins}{l}\right)\left(\frac{2\pi}{l}\right)\right]. \end{aligned} \tag{11}$$

Hence by Equations (6) and (7)

$$|F(l, f)| \leqq \|L\|\cdot\|f\|. \tag{12}$$

We next consider $F(u^{-1}, f) = \sum_{-\infty}^{\infty} h(2\pi nu)\phi(2\pi nu)2\pi u$. As a function of u, $F(u^{-1}, f)$ is the sum of measurable functions and hence measurable. By (7) and (8)

$$\sum_{-\infty}^{\infty} | h(2\pi nu)\phi(2\pi nu)2\pi u | \leqq C^*[cu^{-1/2} + | h(0) | 2\pi u].$$

Hence the following integral exists and we can interchange the order of summation and integration.

$$\begin{aligned} \epsilon^{-1}\int_0^{\epsilon} F(u^{-1}, f)du &= \sum_{-\infty}^{\infty} \epsilon^{-1}\int_0^{\epsilon} h(2\pi nu)\phi(2\pi nu)2\pi u du \\ &= \sum_{n=1}^{\infty} (2\pi n^2\epsilon)^{-1}\int_{-2\pi n\epsilon}^{2\pi n\epsilon} h(\tau)\phi(\tau) | \tau | d\tau + h(0)\phi(0)\pi\epsilon. \end{aligned}$$

If we now define

$$\chi_\epsilon(\tau) = | \tau | (2\pi\epsilon)^{-1}\sum_{k=n}^{\infty} k^{-2}$$

for $2\pi(n-1)\epsilon \leqq |\tau| < 2\pi n\epsilon$, then for $|\tau| < 2\pi\epsilon$

$$\chi_\epsilon(\tau) = | \tau | (2\pi\epsilon)^{-1}(\pi^2/6) \leqq 2$$

whereas for $2\pi(n-1)\epsilon \leqq |\tau| < 2\pi n\epsilon$ $(n \geqq 2)$

$$\begin{aligned} 1/2 \leqq (n-1)/n = 2\pi(n-1)\epsilon(2\pi\epsilon)^{-1}n^{-1} &\leqq \chi_\epsilon(\tau) \\ \leqq 2\pi n\epsilon(2\pi\epsilon)^{-1}(n-1)^{-1} = n/(n-1) &\leqq 2. \end{aligned}$$

Thus for all τ, $0 \leqq \chi_\epsilon(\tau) \leqq 2$ and for fixed $\tau \neq 0$, $\chi_\epsilon(\tau) \to 1$ as $\epsilon \to 0$. It follows that

$$\epsilon^{-1} \int_0^\epsilon F(u^{-1}, f)du = \int_{-\infty}^{\infty} \phi(\tau)h(\tau)\chi_\epsilon(\tau)d\tau + h(0)\phi(0)\pi\epsilon$$

and, since we have majorized convergence, that

$$\lim_{\epsilon \to 0} \epsilon^{-1} \int_0^\epsilon F(u^{-1}, f)du = \int_{-\infty}^{\infty} \phi(\tau)h(\tau)d\tau.$$

Finally applying (12) to the left-hand side of this equation, we obtain

$$| L_0(f) | = \left| \int_{-\infty}^{\infty} \phi(\tau)h(\tau)d\tau \right| \leqq ||L|| \cdot ||f||$$

and hence

$$||L_0|| \leqq ||L|| = C^*. \tag{13}$$

Thus L_0 is a linear bounded functional on M and can be extended in a unique fashion to be a linear bounded functional on X_0. The remainder of the proof is straightforward and follows directly from a theorem due to I. J. Schoenberg [10]. For purposes of future reference, we shall sketch a different proof, following an argument due to F. Riesz [9].

The general linear bounded functional on X_0 can be uniquely represented by a function $y(\cdot) \in V$ as $L_0(f) = \int_{-\infty}^{\infty} f(s)dy(s)$ for all $f \in X_0$; Var $\{y(\cdot)\} = ||L_0|| \leqq C^*$. In particular, for $f(\cdot) \in M$

$$L_0(f) = \int_{-\infty}^{\infty} f(s)dy(s) = \int_{-\infty}^{\infty} \phi(\tau)h(\tau)d\tau. \tag{14}$$

Let $f_n(s) = \exp\ (its)\,\theta(s/n)$ where

$$\begin{aligned} \theta(s) &= 1 - | s | & &\text{for } | s | < 1 \\ &= 0 & &\text{for } | s | \geqq 1. \end{aligned}$$

Then

$$h_n(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f_n(s) \exp\ (i\tau s)ds = \frac{2}{\pi n} [\{\sin n(t - \tau)/2\}(t - \tau)^{-1}]^2.$$

Finally

$$\lim_{n \to \infty} \int_{-\infty}^{\infty} f_n(s)dy(s) = \int_{-\infty}^{\infty} \exp\ (its)\, dy(s), \tag{15}$$

whereas

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \phi(\tau) h_n(\tau) d\tau = \phi(t) \tag{16}$$

for almost all t by a classical result due to Fejer and Lebesgue [11, p. 49]. Therefore, combining (14), (15), and (16) we obtain

$$\phi(\tau) = \int_{-\infty}^{\infty} \exp(i\tau s) dy(s) \qquad \text{a.e.}$$

If $\phi(\tau)$ is continuous, then this relation must hold for all τ. In this case, let $p(s) = \sum a_n \exp(i\tau_n s)$ be an arbitrary trigonometric polynomial. Then

$$\left| \sum a_n \phi(\tau_n) \right| = \left| \int_{-\infty}^{\infty} p(s) dy(s) \right| \leqq \text{Var} \{y(\cdot)\} \|p\|$$

and hence $\text{Var} \{y(\cdot)\} \geqq C^*$. Combining this with (13), we obtain

$$\text{Var} \{y(\cdot)\} = C^*.$$

3. **Banach space valued functions.** The proof of Theorem 1 can be reinterpreted so as to give a representation theorem for Banach space valued function $\phi(\tau)$ which in addition to satisfying condition (A) also satisfy a certain compactness condition defined below. A function $\phi(\tau)$ on $(-\infty, \infty)$ to a B-space Y satisfying condition (A) corresponds as in (4) to a linear bounded transformation T on the set of trigonometric polynomials which can be extended in a unique way to be linear and bounded on the B-space of almost periodic functions X to Y. Conversely any such transformation defines a function $\phi(\tau) = T[\exp(i\tau s)]$ on $(-\infty, \infty)$ to Y which satisfies (A). The compactness condition which we shall impose on $\phi(\tau)$ is equivalent to the condition that the corresponding transformation T be weakly (or strongly) compact and separable valued.

Definition. A function $\phi(\tau)$on $(-\infty, \infty)$ to Y will be said to satisfy condition (C_w) [or (C_s)] if the set of elements

$$\left[\left(\sum a_n \phi(\tau_n) \right) / \left\| \sum a_n \exp(i\tau_n s) \right\| \;\middle|\; \text{all finite sets } (\tau_n) \text{ and } (a_n) \right] \tag{17}$$

is contained in a weakly (or strongly) compact and separable subset of Y.

Since the set of elements (17) is the T transform of the set of all polynomials of norm one, it is clear that T weakly (or strongly) compact and separable valued implies that this set is weakly (or strongly) conditionally compact and separable valued. Conversely, since the closed convex extension of conditionally compact sets is compact ([7] and [8]), it follows that if (C_w) [or (C_s)] is satisfied then T takes bounded subsets of X into weakly (or strongly) conditionally compact and separable subsets of Y. If Y is separable and reflexive, then (A) itself implies (C_w), whereas if Y is finite-dimensional,

(A) implies (C_s). Without any loss of generality we shall restrict ourselves in the remainder of this paper to separable B-spaces Y.

Before giving the proof of the representation theorem, we shall introduce some appropriate concepts. For functions defined on $(-\infty, \infty)$ to a separable Y, weak measurability is equivalent to strong measurability [6, p. 37]. Integrals involving $\phi(\tau)$ will be interpreted as Bochner integrals [6, p. 37], whereas those involving $y(s)$ will be interpreted as a generalized Riemann-Stieltjes integral [6, p. 51]. We shall also be concerned with the following function classes(3):

DEFINITION. $V_w(Y)$ [or $V_s(Y)$] is the set of all functions $y(s)$ on $(-\infty, \infty)$ to separable Y such that $g[y(s)] \in V$ for all $g \in \overline{Y}$ and the set

$$\left[\sum [y(s_i) - y(s_i')] \mid \text{all finite sets of disjoint intervals } (s_i, s_i')\right]$$

is weakly (or strongly) conditionally compact.

For $y(\cdot) \in V(Y)$ we define the norm

$$\|y(\cdot)\| = \text{LUB } [\text{Var } \{g[y(s)]\} \mid g \in \overline{Y}, \|g\| = 1].$$

It is clear that $V_w(Y) \supset V_s(Y)$.

LEMMA 1. *Let $\{y_n\}$ be a weakly (or strongly) conditionally compact subset of Y and let $[g_\alpha \mid \alpha \in A] \subset \overline{Y}$ be total on Y. If* $\lim_{n\to\infty} g_\alpha(y_n)$ *exists for each $\alpha \in A$, then there exists a unique $y_0 \in Y$ such that y_n converges weakly (strongly) to y_0.*

Suppose the contrary, then since $\{y_n\}$ is weakly (strongly) conditionally compact, we can find two weakly convergent subsequences $y_{n_k'} \to y_0$ and $y_{n_k''} \to y_1$, where $y_1 \neq y_0$. However this implies

$$g_\alpha(y_0) = \lim_{k\to\infty} g_\alpha(y_{n_k'}) = \lim_{k\to\infty} g_\alpha(y_{n_k''}) = g_\alpha(y_1)$$

and hence that $y_1 = y_0$.

We remark that the lemma itself is valid for weakly (strongly) conditionally compact sets $[y_\pi \mid \pi \in \text{directed set } \Pi]$ such that $\lim_\pi g_\alpha(y_\pi)$ exists for all $\alpha \in A$ (4).

LEMMA 2. *If $y(\cdot) \in V_w(Y)$, then $y(s)$ is right weakly continuous for all s, weakly continuous at all but a denumerable set, weak* $\lim_{s\to-\infty} y(s) = 0$, *and weak* $\lim_{s\to\infty} y(s)$ *exists. If $y(\cdot) \in V_s(Y)$ all of these limits are limits in the norm.*

(3) Gelfand [4] has carried through a development similar to ours for the function class $V_s(Y)$ on the interval [0, 1].

(4) The sets $F_\pi \equiv$ weak (strong) closure of $[y_{\pi'} \mid \pi' \geqq \pi]$ have the finite intersection property and are contined in a weakly (strongly) compact subset of Y. Hence there exists a y_0 common to all F_π. Clearly $\lim_\pi g_\alpha(y_\pi) = g_\alpha(y_0)$ for all $\alpha \in A$. On the other hand if y_π does not converge to y_0 in the weak (strong) topology, then there will exist a subset Π' of Π cofinal with Π, an $\epsilon > 0$, and a $g \in \overline{Y}$ such that $|g(y_{\pi'}) - g(y_0)| > \epsilon$ (or in the strong case $\|y_{\pi'} - y_0\| > \epsilon$) for all $\pi' \in \Pi'$. Reiterating the above argument with Π' instead of Π leads to a y_1 such that $|g(y_1) - g(y_0)| \geqq \epsilon$ (or $\|y_1 - y_0\| \geqq \epsilon$) and $g_\alpha(y_0) = \lim_\pi g_\alpha(y_\pi) = \lim_{\pi'} g_\alpha(y_{\pi'}) = g_\alpha(y_1)$. Since the $[g_\alpha]$ are total this is impossible.

If $y(\cdot)\in V_w(Y)$ (or $V_s(Y)$), then the set $[y(s)|\ s\in(-\infty,\ \infty)]$ is weakly (strongly) conditionally compact. Since $\lim_{s\to-\infty} g[y(s)]=0$ for all $g\in\overline{Y}$, we have by Lemma 1 that weak (strong) $\lim_{s\to-\infty} y(s)=0$. Likewise, weak (strong) $\lim_{s\to\infty} y(s)$ exists. For a separable Y, there exists a denumerable set g_k total on Y. For each k, $g_k[y(s)]$ is right continuous for all s and continuous at all but a denumerable subset. The desired result therefore follows directly from Lemma 1.

LEMMA 3. *If* $y(\cdot)\in V_w(Y)$ (*or* $V_s(Y)$), *then*

$$K\equiv\left[\sum_{i=1}^{n} a_i[y(s_i)-y(s_{i-1})]\ \Big|\ |a_i|\leqq 1,\ \text{all subdivisions } s_0<s_1<s_2<\cdots<s_n\right]$$

is weakly (*strongly*) *conditionally compact.*

If R_+, R_-, I_+, I_- are defined as K above for non-negative a's, non-positive a's, imaginary non-negative a's, and imaginary non-positive a's respectively, then $K\subset R_++R_-+I_++I_-$ and if each of these sets are weakly (strongly) conditionally compact then so is K. We shall therefore limit the a's to be non-negative, and show that R_+ is contained in the convex extension of the set

$$[\textstyle\sum'[y(s_i)-y(s_{i-1})]\ |\ \text{all finite partial sums}]. \tag{18}$$

For a given set of a's, choose $0=b_0<b_1<\cdots<b_k=1$ so that the b's contain all of the a's. Let σ_j be the set of i's such that $a_i=b_j$. We can then write the identity

$$\sum_{i=1}^{n} a_i[y(s_i)-y(s_{i-1})]=\sum_{m=1}^{k}(b_m-b_{m-1})\left[\sum_{j=m}^{k}\left(\sum_{\sigma_j}[y(s_i)-y(s_{i-1})]\right)\right].$$

Since $(b_m-b_{m-1})\geqq 0$ and $\sum_{m=1}^{k}(b_m-b_{m-1})=1$, it follows that R_+ is contained in the convex extension of the set (18). By hypothesis, the set (18) is weakly (strongly) conditionally compact, and, by a theorem due to Krein [7] for the weak case and Mazur [8] for the strong case, so is its convex extension.

Now for any bounded continuous function $f(s)$ defined on $(-\infty,\ \infty)$, $\int_a^b f(s)dy(s)$ exists and is clearly contained in $\|f\|\cdot\overline{K}$(5). Further for each $g\in\overline{Y}$,

$$\lim_{a\to-\infty,b\to\infty}\int_a^b f(s)dg[y(s)]=\lim_{a\to-\infty,b\to\infty} g\left[\int_a^b f(s)dy(s)\right]$$

exists. Hence by Lemma 1, we may define

$$\int_{-\infty}^{\infty} f(s)dy(s)=\lim_{a\to-\infty,b\to\infty}\int_a^b f(s)dy(s)\ \text{[weak (strong) limit]}.$$

Further

(5) $\overline{K}$ designates the weak (strong) closure of K and is likewise weakly (strongly) compact.

$$\left| g\left[\int_{-\infty}^{\infty} f(s)dy(s) \right| = \left| \int_{-\infty}^{\infty} f(s)dgy(s) \right| \leqq \|f\| \cdot \text{Var } \{g[y(s)]\},$$

and taking the least upper bound over all $g \in \overline{Y}$ of norm one we have

$$\left\| \int_{-\infty}^{\infty} f(s)dy(s) \right\| \leqq \|f\| \cdot \|y(\cdot)\|.$$

If we now define

$$T(f) = \int_{-\infty}^{\infty} f(s)dy(s)$$

for all bounded continuous $f(s)$ on $(-\infty, \infty)$, then T is a bounded linear transformation with $\|T\| \leqq \|y(\cdot)\|$. For $\|f\| \leqq 1$, $T(f) \in \overline{K}$, so that T is a weakly (strongly) compact linear bounded transformation. Finally even if we limit f in X_0 (that is, to continuous functions such that $f(s) \rightarrow 0$ as $|s| \rightarrow \infty$),

$$\|T\| = \underset{\|g\|=1}{\text{LUB}} \underset{\|f\|=1}{\text{LUB}} |g[T(f)]| = \underset{\|g\|=1}{\text{LUB}} \text{Var } \{g[y(\cdot)]\} = \|y(\cdot)\|.$$

The converse is likewise true. In fact:

LEMMA 4. *If T_0 is a weakly (strongly) compact linear transformation on X_0 to Y, then there exists a unique $y(\cdot) \in V_w(Y)$ (or $V_s(Y)$) such that*

$$T_0(f) = \int_{-\infty}^{\infty} f(s)dy(s)$$

and $\|T_0\| = \|y_1(\cdot)\|$.

We have shown above that $\|T_0\| = \|y(\cdot)\|$. The uniqueness follows from the fact that the difference of two $y(\cdot)$'s would again belong to $V_w(Y)$, represent the zero transformation, and so have norm zero.

In order to obtain a $y(\cdot) \in V_w(Y)$ (or $V_s(Y)$), we first transform the interval $(-\infty, \infty)$ into $(0, 1)$ by, say, the transformation $u = (1/\pi) \cot^{-1}(-s) = \omega(s)$. Then $U(f) = h(u) = f[\omega^{-1}(u)]$ maps X_0 isometrically onto the space X_1 of continuous functions on $[0, 1]$ which vanish at the end points. It is therefore clearly sufficient to prove the lemma for X_1 and for this the methods of I. Gelfand [4, §8] will suffice. For the sake of completeness, we sketch an independent proof. In this we make use of the Bernstein polynomials and a device due to Hildebrandt and Schoenberg [5]. We set $W = T_0 U^{-1}$ so that $\|W\| = \|T_0\|$. Now

$$h(u) = \lim_{n \rightarrow \infty} \sum_{k=0}^{n} C_k^n h\left(\frac{k}{n}\right) u^k (1-u)^{n-k}$$

uniformly on $(0, 1)$ and hence

$$W(h) = \lim_{n\to\infty} \sum_{k=0}^{n} h\left(\frac{k}{n}\right) W[C_k^n u^k (1-u)^{n-k}] = \lim_{n\to\infty} W_n(h)$$

where $W_n(h) = \int_0^1 h(u) dy_n(u)$ and $y_n(u)$ is defined to be zero at $u=0$, constant elsewhere except at the points k/n $(0<k<n)$ where it has a jump of $W[C_k^n u^k(1-u)^{n-k}]$. Since

$$\|W_n(h)\| = \left\| W\left[\sum C_k^n h\left(\frac{k}{n}\right) u^k (1-u)^{n-k} \right] \right\| \leqq \|W\| \cdot \|h\|,$$

we have $\|y_n(\cdot)\| \leqq \|W\|$. The set of all sums

$$H \equiv [\sum [y_n(u_i) - y_n(u_i')] \mid n = 1, 2, \cdots, \text{ all finite sets of disjoint intervals } (u_i, u_i')]$$

consists of elements of the type

$$W\left[\sum_{k/n \in \cup (u_i, u_i')} C_k^n u^k (1-u)^{n-k} \right].$$

Hence H is contained in the transform of the unit sphere of X_1 and is therefore a weakly (strongly) conditionally compact subset of Y. Let $\{g_k\}$ be a denumerable set of functionals of norm one total on Y and such that

$$\operatorname{LUB}_k |g_k(y)| = \|y\|$$

for every $y \in Y$. By Helly's theorem there exists for each k a subsequence of n's such that $g_k[y_{n_i}(u)]$ converges pointwise to a function of bounded variation with variation not greater than $\|W\|$. By the diagonal process there exists a subsequence of n's (which we renumber) such that $\lim_n g_k[y_n(u)]$ exists for all u and k. By Lemma 1, the weak (strong) limit of $y_n(u)$ exists for all u. Set $z(u)$ equal to this limit. Then the values of $z(u)$ are contained in $\overline{H}$ and $\operatorname{Var}\{g[z(u)]\} \leqq \limsup \operatorname{Var}\{g[y_n(u)]\} \leqq \|W\|$ for all $g \in \overline{Y}$ of norm one. Since all of the $g_k[z(u)]$ are continuous except perhaps at a denumerable set of points and even at these points the right and left limits exist, it follows from Lemma 1 that $z(u)$ is weakly (strongly) continuous except at this denumerable set and can be redefined at these points to be right weakly (strongly) continuous. We now define $y(u)$ to be the so redefined $z(u)$ less $z(0+)$. Since $\operatorname{Var}\{g[y(u)]\} \leqq \operatorname{Var}\{g[z(u)]\}$, we have $\|y(\cdot)\| \leqq \|W\|$. Finally, the set $[\sum [y(u_i) - y(u_i')] \mid$ all finite sets of disjoint intervals $(u_i, u_i')]$ is contained in the weak (strong) closure of H. Therefore $y[\omega(s)] \in V_w(Y)$ (or $V_s(Y)$). For $h \in X_1$,

$$g[W(h)] = \lim_n g[W_n(h)] = \lim_n \int_0^1 h(u) dg[y_n(u)] = \int_0^1 h(u) dg[y(u)];$$

so that

$$W(h) = \int_0^1 h(u)dy(u)$$

and

$$T_0(f) = \int_{-\infty}^{\infty} f(s)dy[\omega(s)].$$

We come now to the main theorem of this section.

THEOREM 2. *If $\phi(\tau)$ is measurable on $(-\infty, \infty)$ to Y and satisfies* (A) *and* (C_w) (*or* (C_s)), *then there exists a unique $y(\cdot) \in V_w(Y)$ (or $V_s(Y)$) such that* (2) *holds a.e. and $\|y(\cdot)\| \leqq C^*$. If $\phi(\tau)$ is weakly continuous, then* (2) *holds for all τ and $\|y(\cdot)\| = C^*$.*

The proof proceeds exactly as in Theorem 1. One shows that

$$T_0(f) = \int_{-\infty}^{\infty} h(\tau)\phi(\tau)d\tau$$

is linear and bounded on M as before. It is, however, also necessary to show that T_0 takes the unit sphere in M into a weakly (or strongly) conditionally compact subset of Y. By (10) and (12), $F(l, f)$ is a linear bounded transformation on those functions in M which vanish outside of $(-l/2, l/2)$. For such functions, $F(l, f)$ is precisely equal to the T transform of f repeated to be of period l. Since T is weakly (strongly) compact, it follows that

$$[F(l, f) \mid f \in M, \|f\| \leqq 1, l \text{ sufficiently large}]$$

is contained in a weakly (strongly) conditionally compact subset of Y and that

$$\left[\epsilon^{-1}\int_0^{\epsilon} F(u^{-1}, f)du \mid f \in M, \|f\| \leqq 1, \epsilon \text{ sufficiently small}\right], \tag{19}$$

contained in the convex extension of this set, is likewise weakly (strongly) conditionally compact. Since as before

$$\begin{aligned}\lim_{\epsilon\to 0} g\left[\epsilon^{-1}\int_0^{\epsilon} F(u^{-1}, f)du\right] &= \lim_{\epsilon\to 0} \epsilon^{-1}\int_0^{\epsilon} g[F(u^{-1}, f)]du \\ &= \int_{-\infty}^{\infty} g[\phi(\tau)]h(\tau)d\tau \\ &= g\left[\int_{-\infty}^{\infty} \phi(\tau)h(\tau)d\tau\right],\end{aligned}$$

the set

$$\left[\int_{-\infty}^{\infty}\phi(\tau)h(\tau)d\tau \mid f\in M, \|f\|\leqq 1\right]$$

is in the weak closure of the set (19) and hence is weakly (strongly) conditionally compact. As in (12),

$$|g[F(l,f)]|\leqq \|g\|\cdot\|T\|\cdot\|f\|,$$

so that

$$\left\|\int_{-\infty}^{\infty}\phi(\tau)h(\tau)d\tau\right\|\leqq \|T\|\cdot\|f\|.$$

Thus T_0 is a weakly (strongly) compact linear bounded transformation.

With the aid of Lemma 4, the proof then proceeds as in Theorem 1. Thus for all $f\in M$ we have, corresponding to (14),

$$T_0(f)=\int_{-\infty}^{\infty}f(s)dy(s)=\int_{-\infty}^{\infty}\phi(\tau)h(\tau)d\tau,$$

with $\|y(\cdot)\|=\|T_0\|\leqq C^*$. Let $\{g_k\}$ be a denumerable set of functionals total on Y. Then as in (15) and (16), for each k we have

$$\lim_n \int_{-\infty}^{\infty}f_n(s)dg_k[y(s)]=\int_{-\infty}^{\infty}\exp(its)dg_k[y(s)]=g_k\left[\int_{-\infty}^{\infty}\exp(its)dy(s)\right]$$

for all t and

$$\lim_n\int_{-\infty}^{\infty}g_k[\phi(\tau)]h_n(\tau)d\tau=g_k[\phi(t)]$$

for all $t\notin E_k$, where E_k is of measure zero. If $E_0=\bigcup E_k$, then

$$g_k\left[\int_{-\infty}^{\infty}\exp(its)dy(s)\right]=g_k[\phi(t)]$$

and hence

$$\int_{-\infty}^{\infty}\exp(i\tau s)dy(s)=\phi(\tau) \tag{20}$$

for $\tau\notin E_0$, that is, a.e. for $\phi(\tau)$ weakly continuous this relation holds for all τ since the left side of (20) is clearly weakly continuous in any case.

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