COVERINGS WITH CONNECTED INTERSECTIONS

BY

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If G is a collection of subsets of a set, then a *subintersection* of G is a non-null set which is the common part of the elements of a subcollection of G.

Suppose that a space X is a compact, locally connected, metric continuum. We show that X has a countable basis whose subintersections are connected and uniformly locally connected. In fact, there is a basis for X with the additional property that the collection of closures of elements of this basis is a family of continuous curves such that each subintersection of this family is a continuous curve. This extends a result of Anderson [1](1) showing that there is a sequence $G_1, G_2, \cdots$ such that G_i is a finite $1/i$-collection of continuous curves covering X and the subintersections of $\sum G_i$ are locally connected.

The notion of partitioning [2, 3, 4] will be used in proving these results. A *partitioning* of X is a finite collection of mutually exclusive connected domains whose sum is dense in X. The partitioning U is a *brick partitioning* if each of its elements is uniformly locally connected and equal to the interior of its closure while the interior of the closure of the sum of two adjacent elements of U is connected and uniformly locally connected. If each element of U is of diameter less than ϵ, U is an *ϵ-parititioning*. In general, if each element of a collection is of diameter less than ϵ, the collection is called an *ϵ-collection*.

The brick partitioning V is a *core refinement* of the brick partitioning U if (a) V is a refinement of U, (b) for each pair of adjacent element u', u'' of U there is a pair of adjacent element v', v'' of V in u' and u'' respectively such that $\bar{v}'+\bar{v}''$ is a subset of the interior of $\bar{u}'+\bar{u}''$, and (c) for each element u of U, the elements of V in u may be ordered $v_0, v_1, \cdots, v_n$ such that $\bar{v}_0$ intersects each $\bar{v}_i$ while $\bar{v}_i$ intersects the boundary of u if and only if $i>0$. We call v_0 a *core element* and $v_1, v_2, \cdots, v_n$ *border elements*.

If B is a subset of X and G is a collection of subsets of X, we use $S(B, G)$ to denote the interior of the closure of the sum of the elements of G which have limit points on $\bar{B}$.

We shall use the following result which was proved in [3].

THEOREM 1. *For each brick partitioning U of X and each positive number ϵ, there is a brick ϵ-partitioning V of X which refines U.*

Although the following result is a corollary of Theorem 6, it is given here since its proof is much simpler than that of Theorem 6.

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(1) Numbers in brackets refer to the references cited at the end of the paper.

THEOREM 2. *The space X has a countable basis whose subintersections are connected and uniformly locally connected.*

Proof. Let $U_1, U_2, \cdots$ be a decreasing sequence of brick partitionings of X (U_{i+1} refines U_i and the maximum of the diameters of elements of U_i approaches 0 with $1/i$). The basis is the collection of sets $S(p, U_i)$ for $p \in X$ and $i=1, 2, \cdots$.

If $G=[g]$ is an infinite subcollection of this basis, Πg is either empty or a single point. The conclusion follows in this case. If G is finite, there is a largest integer n such that some $S(p, U_n)$ is an element of G. Any two elements of U_n in Πg are adjacent; moreover, the interior of the closure of their sum is connected, uniformly locally connected, and contained in Πg. Since Πg is the interior of the sum of the closures of adjacent pairs of elements of U_n, it is connected and uniformly locally connected.

THEOREM 3. *For each brick partitioning U of X and each positive number ϵ there is a brick partitioning V of X such that V is a core refinement of U and each border element of V is of diameter less than ϵ.*

Proof. Since U is a brick partitioning of X, there is a positive number ϵ' less than ϵ such that the common boundary of each pair of adjacent elements of U contains a point that is farther than ϵ' from any other element of U. By Theorem 1 there is a brick ϵ'-partitioning U' of X which refines U. For each element u of U, let T_u be a dendron in u which intersects each element of U' in u. There is a brick $(\delta/2)$-partitioning U'' of X which refines U' where δ is less than the distance between any T_u and the corresponding $X-u$. The core element v_0 of V in u is the set which is maximal with respect to being a connected domain containing T_u and being the interior of the closure of the sum of some elements of U'' whose closures lie in u. The border elements of V in u are components of the intersection of elements of U' with $u-\bar{v}_0$.

THEOREM 4. *For each ordering $u_1, u_2, \cdots, u_n$ of the elements of the brick partitioning U of X and each positive number ϵ there is a brick partitioning V of X such that*

(1) *V is a core refinement of U;*

(2) *each border element of V has a diameter of less than ϵ;*

(3) *if v_1, v_2 are adjacent elements of V, one of the sets $S(v_1, U)$, $S(v_2, U)$ contains the other;*

(4) *if v_1, v_2 are adjacent elements of V and the element of U containing v_1 precedes in $u_1, u_2, \cdots, u_n$ the element of U containing v_2, then $S(v_1, U)$ contains $S(v_2, U)$.*

Proof. We first show that there is a brick partitioning V satisfying (1), (2), and (3). Suppose N is a fixed positive integer. Denote by Lemma N the result obtained by replacing in Theorem 4 conditions (3) and (4) by

(3′) *if* v_1, v_2 *are adjacent elements of* V, *one of the sets* $S(v_1, U)$, $S(v_2, U)$ *contains the other if each contains* N *or more elements of* U.

Lemma N holds if N is greater than the number of elements in U. We show that it holds for $N=M$ if it holds for $N=M+1$. Induction then establishes Lemma N; Lemma N for $N=1$ is Theorem 4 with condition (4) deleted.

Let U' be a brick partitioning of X satisfying the conditions of Lemma N for $N=M+1$. Define A to be the set of all points p such that $S(p, U)$ contains at least $M+1$ elements of U. We define a brick partitioning U'' of X whose elements are of two types; (a) each element of U' in $W=S(A, U')$ is an element of U''; (b) if u is an element of U, $u-\overline{W}$ is an element of U''. We note that U'' is a refinement of U and a consolidation of U'. However, it may not be a core refinement of U.

There is a positive number δ_1 so small that if B is a subset of $X-W$ of diameter less than δ_1, $S(B, U)$ does not contain $M+1$ elements of U.

Let δ_2 be the minimum of the distances between nonadjacent elements of U'. We note that if B is a subset of X of diameter less than δ_2 and u' is an element of U' with a limit point on B, then $S(u', U)$ contains $S(B, U)$.

We now describe a brick partitioning V which insures that Lemma N holds for $N=M$. Let V' be a brick partitioning of X which is a core refinement of U'' and such that each border element of V' is of diameter less than $\min(\delta_1/2, \delta_2)$. If u'' is an element of U'' of type (b) in an element u of U, the core element of V in u is the interior of the closure of the sum of the elements of V' in u'' whose closures lie in u. The other elements of V are the elements of U' in W and the elements of V' which are not in W and whose closures do not lie in any element of U. We find that V is a core refinement of U.

We now show that the elements of V satisfy conditions (3′). Suppose v_1 and v_2 are two adjacent elements of V such that each of $S(v_1, U)$ and $S(v_2, U)$ contains M or more elements of U. We may suppose that neither v_1 nor v_2 is a core element, for if v_1 is a core element, $S(v_2, U)$ contains $S(v_1, U)$. Hence, if v_i is not in W, it may be supposed to be of diameter less than either $\delta_1/2$ or δ_2.

If both v_1 and v_2 are subsets of W, they are elements of U' and condition (3′) holds for them because each of the sets $S(v_1, U)$, $S(v_2, U)$ contains $M+1$ elements of U.

If v_1 is a subset of W and v_2 is not, then v_1 is an element of U'. Since the diameter of $\bar{v}_2$ is less than δ_2, $S(v_1, U)$ contains $S(v_2, U)$.

If neither v_1 nor v_2 is a subset of W, v_1+v_2 is of diameter less than δ_1. Then $S(v_1+v_2, U)$ does not contain $M+1$ elements of U. Hence $S(v_1, U)=S(v_2, U)$.

By induction we find that Lemma N holds for all values N. Since it holds for $N=1$, there is a sequence $U=V_0, V_1, \cdots, V_n$ of brick partitionings of X such that V_{i+1} satisfies (1) and (3) where U is V_i and the diameters of the border elements of V_{i+1} are less than the distance between any nonadjacent elements of V_i and less than the ϵ mentioned in the statement of Theorem 4.

Consider the core partitioning V of U where the core element of V in u_i

is the interior of the closure of the sum of the elements of V_i whose closures lie in u_i. The border elements of V in u_i are the elements of V_i in u_i which are not in this core.

If v_1 and v_2 are two adjacent elements of V in the same element u_i of V, one of the sets $S(v_1, U)$, $S(v_2, U)$ contains the other because if neither v_1 nor v_2 is a core element, then both are elements of V_i. If v_1 and v_2 are adjacent elements, v_1 is in u_i, v_2 is in u_j, and $j>i$, then $S(v_1, U)$ contains $S(v_2, U)$ because the diameter of v_2 is less than the distance between any two nonadjacent elements of V_i. Hence, V satisfies conditions (1), (2), (3), and (4).

THEOREM 5. *Suppose U is a brick partitioning of X and G is a collection of open sets satisfying the following conditions*:

(a) *each element of G is the interior of the closure of the sum of the elements of a subcollection of U*;

(b) *the subintersections of G are connected and uniformly locally connected*;

(c) *For each subcollection of G, the intersection of the closures of the elements of the subcollection is the closure of the intersection of the elements of the subcollection.*

Then for each positive number ϵ there are a brick partitioning V of X and an ϵ-covering H of X such that $G+H$ satisfies the above conditions (a), (b), *and* (c) *with V substituted for U in condition* (a).

Proof. Let U' be a brick $(\epsilon/2)$-partitioning of X that refines U. We note the G satisfies condition (a) with U' substituted for U.

Since U' has only a finite number of elements, G has only a finite number. Let $u_1, u_2, \cdots, u_n$ be an ordering of the elements of U' such that if $i<j$, u_j intersects as many elements of G as u_i does. It follows from condition (c) that if $\bar{u}_i$ intersects $\bar{u}_j$, then each element of G containing u_i also contains u_j.

Let δ be a positive number so small that if B is a subset of X of diameter less than δ, then there exists a point p of X such that $S(p, U')$ contains $S(B, U')$. Suppose V is a core refinement of U' such that V satisfies conditions (3) and (4) of Theorem 4 and the border elements of V are of diameter less than $\delta/2$. Let $v_1, v_2, \cdots, v_m$ be an ordering of the elements of V such that v_i precedes v_j provided either (1) v_i lies in an element of U' which precedes the element of U' containing v_j in the ordering $u_1, u_2, \cdots, u_n$ or (2) v_i and v_j lie in the same element of U' and $S(v_i, U')$ contains more elements of U' than $S(v_j, U')$ does . We note that if $\bar{v}_i$ intersects $\bar{v}_j$ and $i<j$, then each element of G containing v_i also contains v_j.

For each point p, define $h(p)$ to be the interior of the closure of the sum of all elements of V whose closures lies in $S(p, U')$. Let H be the collection of all such sets $h(p)$. We prove that H is an ϵ-covering of X and that $G+H$ satisfies conditions (a), (b), and (c) with V substituted for U in condition (a).

To prove that H is a covering, consider a point q. Since each border element of V is of diameter less than $\delta/2$, there exists a point p such that $S(p, U')$

contains $S[S(q, V), U']$. Then $h(p)$ contains q. As the elements of U' are of diameter less than $\epsilon/2$, each $S(p, U')$ is of diameter less than ϵ and the elements of H are of diameter less than ϵ.

We next verify condition (b). Let J be a subcollection of $G+H$ and π be the intersection of the elements of J. Since π is the interior of the closure of the sum of the elements of a subcollection of V, it is uniformly locally connected. If J contains no element of H, then π is connected by hypothesis. Suppose J contains an element $h(p)$ of H. Let v_i, v_j be elements of V contained in π and u_r, u_s be the elements of U' containing v_i and v_j respectively. Then $\bar{u}_r \cdot \bar{u}_s$ is not null. Hence π contains v_i, v_j, the core elements of V in u_r and u_s, and the border elements of V whose boundaries intersect the closure of no elements of U' except u_r and u_s. Then π is connected.

Finally, we check condition (c). If $\bar{v}_i$ intersects $\bar{v}_j$ and $i<j$, $S(v_i, U')$ contains $S(v_j, U')$. Hence $\bar{v}_j$ lies in $S(v_i, U')$ and if $h(p)$ contains v_i, it also contains v_j. Hence each element of $G+H$ containing v_i also contains v_j. If p is a point of the intersection of the closures of the elements of J, the last element of $v_1, v_2, \cdots, v_m$ having p on its closure is in each element of J. Hence $\bar{\pi}$ contains p and is the intersection of the closures of the elements of J.

THEOREM 6. *The space X has a countable basis G whose subintersections are connected and uniformly locally connected and such that if G' is a subcollection of G, then the intersection of the closures of the elements of G' is the closure of the intersection of the elements of G'.*

Proof. Let G_0 be the covering of X whose only element is X itself, and U_0 be the brick partitioning of X whose only element is X itself. Repeated applications of Theorem 5 give a sequence $G_0, G_1, \cdots$ of coverings of X such that G_i $(i \geqq 1)$ is a $(1/i)$-covering and such that $G_1+G_2+ \cdots +G_i$ satisfies conditions (b) and (c). Define G to be $\sum G_i$ and the theorem follows. The following result is a consequence of Theorem 6.

THEOREM 7. *For each positive integer i, X is the sum of a finite $(1/i)$-collection G_i of continuous curves such that each subintersection of $\sum G_i$ is a continuous curve.*

REFERENCES

1. R. D. Anderson, *Concerning upper semi-continuous collections of continua*, Trans. Amer. Math. Soc. vol. 67 (1949) pp. 451–460.

2. R. H. Bing, *Partitioning a set*, Bull. Amer. Math. Soc. vol. 55 (1949) pp. 1101–1110.

3. ———, *Complements of continuous curves*, Fund. Math. vol. 36 (1950) pp. 303–318.

4. E. E. Moise, *Grille decomposition and convexification theorems for compact locally connected continua*, Bull. Amer. Math. Soc. vol. 55 (1949) pp. 1111–1121.

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