ON THE L-HOMOMORPHISMS OF FINITE GROUPS

BY

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Let G be a finite group. We shall denote by $L(G)$ the lattice formed by all subgroups of G. A homomorphic mapping from $L(G)$ onto a lattice L is called an L-homomorphism from G onto L.

In his previous paper (Suzuki [5](1)), dealing with L-isomorphisms of finite groups, the author determined the structure of groups, L-isomorphic to a p-group, and proved that groups L-isomorphic to a solvable or a perfect group are also solvable or perfect respectively. In this paper we shall generalize these results to the case of L-homomorphisms and study the relations between L-homomorphisms and L-isomorphisms. In particular, we shall determine all L-homomorphisms from a perfect group, and as an application, we shall also determine the neutral elements of $L(G)$.

L-homomorphisms of finite groups were first considered by P. Whitman [6], who dealt with the case when L is the subgroup lattice of a cyclic group. His result will be sharpened to Theorem 1 in §1 which will play a fundamental rôle in our study.

1. Some remarks on L-homomorphisms

Let G be a group and ϕ be an L-homomorphism from G onto a lattice L. A set of elements of $L(G)$, which is mapped onto a fixed element of L, forms a convex sublattice(2) of $L(G)$, and in particular elements mapped to the least (greatest) element 0 (I)(3) of L, form a (dual) ideal of $L(G)$. The greatest (least) element of such a (dual) ideal is called the "lower (upper) kernel," or shortly "l- (u-) kernel" of ϕ in G.

First we shall prove the following lemma.

Lemma 1. [Cf. 6]. *Let G be a group and ϕ be an L-homomorphism from G onto a chain C_n of dimension n. Then there are two subgroups N and G_0 of G and a prime number p with the following properties*:

(1) *N is a Sylow p-complement*(4) *of G*,

(2) *a p-Sylow subgroup S_p contains G_0 and is cyclic or a generalized quaternion group (g. q. group)*,

Received by the editors April 8, 1950.

(1) Numbers in brackets refer to the bibliography at the end of the paper.

(2) For general lattice theory, see Birkhoff [2].

(3) In the following we always denote by 0 (I) the least (greatest) elements of various lattices and do not mention it particularly, if there is no risk of misunderstanding.

(4) Sylow p-complements of a group of order $p^n g$, $(p, g)=1$, are subgroups of index p^n. Cf. Suzuki [5, footnote 8].

(3) *If the order of* G_0 *is* p^m, *we have* $m \geqq n$, *and*

(4) *If* S_p *is a g. q. group, the order of* G_0 *is* 2.

Conversely if there are normal subgroups N *and* G_0 *of* G *and a prime number* p *with the properties* (1)–(4), *then* $L(G)$ *is homomorphic to a chain* C_n *of dimension* n.

Proof. Denote by G_0 the u-kernel of ϕ. G_0 has only one maximal subgroup and hence G_0 is a cyclic group of prime power order. Let p^m be this order. Take a Sylow subgroup S_p of G containing G_0. If there were a noncyclic subgroup V of S_p covering G_0, V would be L-homomorphic to C_n. Since the factor group $V/\Phi(V)$[5] is a P-group, there would exist at least two maximal subgroups M_1 and M_2 of V, different from G_0. Both $\phi(M_1)$ and $\phi(M_2)$ would be maximal elements of C_n, and we should, therefore, have $\phi(V)=\phi(M_1 \cup M_2) = \phi(M_1) \cup \phi(M_2) \neq I$, which is clearly a contradiction. Hence all subgroups of S_p covering G_0 are cyclic and S_p has only one subgroup of order p. S_p must be cyclic or a g. q. group [cf. 7, p. 112].

Take a q-Sylow subgroup S_q, where q is any prime factor of the order of G other than p. We have $\phi(S_q) \cap \phi(S_p)=0$ because $S_q \cap S_p = e$. This implies that $\phi(S_q)=0$. Put $N=\bigcup_{q \neq p} S_q$, where q runs through all prime factors of the order of G except p. Then N is clearly self-conjugate. Take a normalizer N_q of S_q in G, then we have $N_q \cdot N = G$. Hence N_q contains a p-Sylow subgroup of G. Choosing a suitable q-Sylow subgroup S_q we may assume that $N_q \supseteqq S_p \supseteqq G_0$. We shall prove that G_0 is self-conjugate in $H=G_0 \cdot S_q$, using induction on the dimension of the interval H/G_0. We take a maximal subgroup M of H containing G_0, then $M \cap S_q$ is self-conjugate in H. $H/M \cap S_q$ is L-homomorphic to C_n because $\phi(M \cap S_q)=0$ and $\phi(H)=I$. Hence we have only to prove our assertion in the case where G_0 is maximal. If G_0 were not self-conjugate in such a case, there would be at least two subgroups G_1 and G_2 of H, conjugate to and different from G_0. We should then have $\phi(G_1) = \phi(G_2) \neq I$, which gives the contradiction that $\phi(H)=\phi(G_1) \cup \phi(G_2) \neq I$. Hence G_0 is self-conjugate in H. Since q is an arbitrary prime factor other than p, this implies that G_0 is self-conjugate in G and that G_0 is elementwise permutable with N. By the definition of N this implies that $N \cap G_0 = e$ and $N \cdot S_p = G$. The former part of our lemma now follows immediately.

Conversely, suppose G to have such a structure. Then G is proved to be L-homomorphic to a chain as follows.

When S_p is a g. q. group, the mapping ϕ from $L(G)$ onto the two-element lattice C_2 defined by

$$\phi(V) = \begin{cases} I & \text{if the order of } V \text{ is even,} \\ 0 & \text{if the order of } V \text{ is odd,} \end{cases}$$

[5] We mean by $\Phi(V)$ the Φ-subgroup of V, which is defined to be the intersection of all maximal subgroups of V. Cf. Zassenhaus [7, p. 44].

is an L-homomorphism from G onto C_2. For subgroups of even order contain G_0 and those of odd order are contained in N.

When S_p is cyclic, the mapping ϕ from $L(G)$ onto the chain C_m of dimension m defined by

$$\phi(V) = a_\nu \qquad (\nu = \min(m, \lambda),\ p^\lambda \| (V:e))$$

is an L-homomorphism from G onto C_m, where a_ν is the element of C_m with dimension ν, and λ is the exact power of p dividing the order of V. For $G_0 \cup N$ is L-decomposable, and subgroups of order $p^\mu g$ with $\mu \geqq m$ $((p, g)=1)$ contain G_0. Hence G is clearly L-homomorphic to a chain C_n with $n \leqq m$. Note that the mapping ϕ defined above is equivalent to the mapping $U \rightarrow G_0 \cap U$ from $L(G)$ onto a chain $L(G_0)$.

By this lemma we can easily generalize Whitman's theorem as follows.

THEOREM 1. *A group G is L-homomorphic to a cyclic group G' of order $\prod_{i=1}^n q_i^{e_i}$ if and only if there exist prime numbers p_i $(i=1, 2, \cdots, n)$ and two normal subgroups G_0 and N with the following properties*:

(1) $p_i \neq p_j$ $(i \neq j)$,

(2) *the order of G is* $\prod_{i=1}^n p_i^{f_i} \cdot g$, $(p_i, g)=1$ $(i=1, 2, \cdots, n)$,

(3) *the order of G_0 is* $\prod_{i=1}^n p_i^{a_i}$ *with* $f_i \geqq a_i$ $(i=1, 2, \cdots, n)$,

(4) *N is of order g and the factor group G/N is a nilpotent group whose p-Sylow subgroups are cyclic, or a g. q. group, and*

(5) *if $p_i=2$ and if a 2-Sylow subgroup is a g. q. group, then* $a_i=e_i=1$.

Proof. The subgroup lattice $L(G')$ of a cyclic group G' is a direct product of chains, so that there are natural homomorphisms ψ_i $(i=1, 2, \cdots, n)$ from $L(G')$ onto its direct components. Let ϕ be the homomorphism from $L(G)$ onto $L(G')$. Then $\psi_i\phi$ is clearly a homomorphism from $L(G)$ onto a chain. Hence G has a prime factor p_i and two normal subgroups G_i and N_i with the properties given in Lemma 1. Now we have clearly $p_i \neq p_j$ $(i \neq j)$. Put $G_0 = \bigcup G_i$ and $N = \bigcap N_i$, then G_0 and N satisfy the properties of Theorem 1.

Conversely, suppose that G has prime factors p_i $(i=1, 2, \cdots, n)$ and two normal subgroups with the above properties. Then G has the Sylow p_i-complement N_i and G_0 is nilpotent. Let G_i be a p_i-Sylow subgroup of G_0. Then both N_i and G_i are self-conjugate in G. By Lemma 1, G is L-homomorphic to $L(G_i)$. We shall denote by ϕ_i this L-homomorphism from G onto $L(G_i)$. We have then

$$(*) \qquad \phi_i(G_j) = 0 \qquad (i \neq j).$$

Let ϕ_0 be a mapping from $L(G)$ into a direct product $L = L(G_1) \times \cdots \times L(G_n)$ defined by

$$\phi_0(V) = (\phi_1(V), \cdots, \phi_n(V)).$$

ϕ_0 is clearly an L-homomorphism from G into L, and in virtue of (*) it is surely onto L. As is easily proved, there exists a homomorphism ψ from L

onto $L(G')$ of a cyclic group G' of order $\prod q_i^{e_i}$. $\psi\phi_0$ is clearly an L-homomorphism from G onto $L(G')$. q.e.d.

REMARK. The l-kernel and the u-kernel of ϕ are both self-conjugate, if L is a chain.

We obtain now the following two theorems.

THEOREM 2. *Let G be a group, and ϕ be an L-homomorphism from G onto a lattice L. Then the l-kernel of ϕ is self-conjugate in G.*

Proof. The greatest element of L is represented as a join of elements l_i such that the intervals $l_i/0$ are chains. Let $l_1, \cdots, l_n$ be all such elements of L. Take a subgroup V_i of G such that $\phi(V_i)=l_i$ $(i=1, 2, \cdots, n)$ and let V_i be maximal under this condition. Then we have $\bigcup_{i=1}^n V_i=G$. Let E be the l-kernel of ϕ. Then we have $\phi(V_i\cup E)=\phi(V_i)\cup\phi(E)=\phi(V_i)=l_i$, which implies that $V_i\cup E=V_i$ or $V_i\supseteq E$. Hence E is self-conjugate in V_i, as the l-kernel of ϕ(6) between V_i and $l_i/0$. E is, therefore, self-conjugate in G.

THEOREM 3. *Under the same assumptions as in Theorem 2, the u-kernel G_0 of ϕ is also self-conjugate in G.*

Proof. We shall prove our theorem by induction on the dimension of L. Since the greatest element of the interval G/G_0 is represented as a join of join-irreducible (that is, covering only one element) elements, we may assume that G has only one maximal subgroup containing G_0. If no other maximal subgroup exists, G is cyclic and our theorem is obvious. If there exists another maximal subgroup M, $\phi(M)$ must be a dual atom of L. By the hypothesis of induction, the u-kernel M_0 of ϕ in M is self-conjugate in M. Since $\phi(M\cap G_0)=\phi(G_0)\cap\phi(M)=\phi(M)$, we have $M\cap G_0\supseteq M_0$. Take any element a of M, then $a\cdot G_0\cdot a^{-1}\cup M=G$. Hence we have $\phi(a\cdot G_0\cdot a^{-1})\cup\phi(M)=I$. On the other hand, we have $\phi(a\cdot G_0\cdot a^{-1})\supseteq\phi(a\cdot M_0\cdot a^{-1})=\phi(M_0)=\phi(M)$. Hence we have $I=\phi(a\cdot G_0\cdot a^{-1})$ which implies that $a\cdot G_0\cdot a^{-1}\supseteq G_0$ and hence $a\cdot G_0\cdot a^{-1}=G_0$. G_0 is therefore self-conjugate in G. q.e.d.

2. GROUPS WHICH ADMIT PROPER L-HOMOMORPHISMS

An L-homomorphism is called proper if it is neither an L-isomorphism nor a trivial L-homomorphism. Otherwise we call it improper. We shall say that a group G admits a proper L-homomorphism when there exists a lattice L and an L-homomorphism from G onto L which is proper. In this section we shall consider the structure of groups which admit proper L-homomorphisms. First we shall prove the following lemma.

LEMMA 2. *If a p-group G admits a proper L-homomorphism, G is either a cyclic group or a g. q. group.*

(6) Strictly speaking, it is a contraction of ϕ onto U. We shall, in this paper, not distinguish a contraction of ϕ from ϕ, as long as no confusion arises.

Proof. Let ϕ be a proper L-homomorphism from G onto a lattice L. If the u-kernel G_0 of ϕ differs from G, we can prove our lemma in a similar way as in the proof of Lemma 1. In the following we shall assume that $G_0 = G$, and prove our lemma by induction on the order of G. Since G is a p-group, L satisfies the Jordan-Dedekind chain condition. Since ϕ is a proper L-homomorphism, the dimension of L is different from that of $L(G)$. Hence every maximal subgroup of G admits a proper L-homomorphism, that is, that induced by ϕ. By the hypothesis of induction, every maximal subgroup of G contains only one subgroup of order p. Hence G is either a P-group of order p^2, or one of the types stated in Lemma 2. On the other hand, P-groups admit no proper L-homomorphism. Hence we have our lemma.

Let ϕ be again a proper L-homomorphism from G onto L. We shall denote by E the l-kernel and by G_0 the u-kernel of ϕ and put $E_0 = G_0 \cap E$ and $G_1 = G_0 \cup E$. Then these four subgroups E, G_0, E_0, and G_1 are all self-conjugate. Hence we may consider the factor group $\overline{G}_1 = G_1/E_0$ which is clearly a direct product of $\overline{G}_0 = G_0/E_0$ and $\overline{E} = E/E_0$. These notations will be fixed throughout this section.

We shall prove the following propositions.

(a) The groups $\overline{G}_0$ and $\overline{E}$ have mutually prime orders.

Proof. If the orders of $\overline{G}_0$ and $\overline{E}$ had a common prime factor p, there would exist two subgroups V_1 and V_2 of $\overline{G}_0$ and $\overline{E}$ respectively whose orders are p. Hence $V_1 \cup V_2$ would contain another subgroup V such that $\overline{G}_0 \cap V = e$ and $\overline{E} \cap V = e$. The first equality implies that $\phi(V) = 0$ and $V \subseteq \overline{E}$, but the second equality implies that $\overline{E} \not\supseteq V$. This is a contradiction. q.e.d.

(b) $\Phi(G_0)$ contains E_0.

Proof. Take any maximal subgroup M of G_0. $\phi(M)$ must be a dual atom of L. We have $\phi(M \cup E_0) = \phi(M) \cup \phi(E_0) = \phi(M) \cup 0 = \phi(M)$ and hence $M \cup E_0 = M$. This implies that $M \supseteq E_0$ and that $\phi(G_0) \supseteq E_0$. q.e.d.

(b′) (Cf. [5, Lemma 4].) E_0 is nilpotent, and if a prime number p divides the order of E_0, p divides also that of $\overline{G}_0$.

(c) G_1 is a direct product of G_0 and another group N. N is isomorphic to $\overline{E}$ and its order is relatively prime to that of G_0.

Proof. By (b′) and (a) the order of E_0 is relatively prime to that of E/E_0. Hence by a theorem of Schur (cf. [7, p. 125]) there exists a subgroup N of E such that $N \cup E_0 = E$ and $N \cap E_0 = e$. Take the normalizer N^* of N in G. Then we have $N^* \cup E = G$, since E_0 is nilpotent by (b′) (cf. [7, p. 125]). Hence we have $I = \phi(G) = \phi(N^* \cup E) = \phi(N^*) \cup \phi(E) = \phi(N^*)$. This implies that $N^* \supseteq G_0$. Hence $N^* \supseteq G_0 \cup N = G_0 \cup E_0 \cup N = G_0 \cup E = G_1$. It follows then that N is a normal subgroup of G. G_1 is clearly a direct product of G_0 and N, and N is isomorphic to $\overline{E}$.

(d) If a prime number p divides the order of G/G_1, then p divides that of G_1/E. Hence the groups G/N and N have mutually prime orders.

Proof. Take any prime factor p of the order of G/G_1. If p did not divide the order of G_1/E, a p-Sylow subgroup $\overline{S}$ of G/E would satisfy the condition $\overline{S}\cap G_1/E=e$. We mean by S a subgroup of G corresponding to $\overline{S}$ by the natural homomorphism from G onto G/E. Then we should have $S\cap G_1=E$ and $\phi(S)=\phi(S\cap G_1)=\phi(E)=0$. This implies that $S\subseteq E$, which gives a contradiction. Hence p divides the order of G_1/E. q.e.d.

Hence again by Schur's theorem, G contains a subgroup H such that $G=H\cdot N$, $H\cap N=e$ and $H\supseteq G_0$. Now we have, in a similar way as for (b),

(e) $\Phi(H)$ contains E_0.

Next we shall prove the following proposition.

(f) If ϕ induces an improper L-homomorphism of every Sylow subgroup of G into L, then H is mapped isomorphically onto L by ϕ and we have $G=G_0\times E$.

Proof. By the assumption of this proposition and by propositions (b′) and (d), we have $E_0=e$ and $H=G_0$. Our proposition follows then immediately.

By means of proposition (f) we shall deal with a Sylow subgroup in which ϕ induces a proper L-homomorphism. We shall prove the following propositions.

(g) If a g. q. group Q is mapped by ϕ onto a chain of dimension two, H is a direct product of its 2-Sylow subgroup S_2 and its Sylow 2-complement K In this case, L is also a direct product of $\phi(S_2)$ and $\phi(K)$.

Proof. First, using induction on the order of G, we prove that G has a self-conjugate Sylow 2-complement. By Lemma 2, 2-Sylow subgroups of G are g.q. groups. Take any proper subgroup V of G. If its 2-Sylow subgroup is cyclic. V has a self-conjugate Sylow 2-complement by a theorem of Burnside (cf. [7, p. 131]). The same holds from the hypothesis of induction if its 2-Sylow subgroup is a g.q. group. Hence every proper subgroup of G has a self-conjugate Sylow 2-complement. By a theorem of Ito(7), G has also a self-conjugate Sylow 2-complement, or all proper subgroups of G are nilpotent. In the latter case, if its Sylow 2-complement were not self-conjugate, G would be of order $p^\alpha 2^\beta$ (p is a prime greater than 2). The structure of such a group has been completely determined by Iwasawa(8). We can prove by direct examinations that our assumption does not hold in this case. Hence G has a self-conjugate Sylow 2-complement.

Next using again induction on the order of H, we prove that H is a direct product of its 2-Sylow subgroup and the Sylow 2-complement. We shall denote by K the Sylow 2-complement of H and assume for a while that the l-kernel of ϕ coincides with e. Considering normalizers of Sylow subgroups

(7) Cf. N. Ito, Zenkoku Sizyô Sûgaku-Danwa-Kai 2-93 (1948) (In Japanese). His theorem asserts that if all proper subgroups of a finite group G have the self-conjugate Sylow p-complement, then G has also a self-conjugate Sylow p-complement except when all proper subgroups are nilpotent. His proof is a slight modification of the proof given in K. Iwasawa, Proc. of P-M. Soc. of Japan, 3-23 (1941).

(8) Cf. A paper of Iwasawa quoted in footnote 7.

of K, we can assume K to be a p-group ($p>2$). If K is cyclic, the centralizer of K contains the center Z of a 2-Sylow subgroup S_2. Since $\phi(K\cup Z)=\phi(K)\cup\phi(Z)=\phi(K)\cup\phi(S_2)=\phi(H)$, KZ contains the u-kernel of ϕ and it is a direct product of K and Z. Hence we have $L=(\phi(K)/0)\times(\phi(Z)/0)$. Let ψ be the natural homomorphism from L onto $\phi(K)/0$. Then $\psi\phi$ is an L-homomorphism from H onto $\phi(K)/0$ and S_2 is the l-kernel of $\psi\phi$, since we assumed the l-kernel of ϕ to coincide with e. Hence by Theorem 2, S_2 is self-conjugate in H and we have $H=K\times S_2$.

If K is not cyclic, ϕ induces an L-isomorphism from K into L by Lemma 2. We can, therefore, assume also that S_2 is maximal. If the center Z of S_2 is self-conjugate in H, S_2 is self-conjugate in the same way as above. If Z were not self-conjugate in H, Z would be conjugate to another group Z_1. Z_1 would be the center of a 2-Sylow subgroup Q and $Q\neq S_2$. Then we should have $\phi(Z\cup Z_1)=\phi(Z)\cup\phi(Z_1)=\phi(S_2)\cup\phi(Q)=\phi(H)$ and hence $Z\cup Z_1\supseteq K$. This implies that K would be cyclic, which gives a contradiction.

If the l-kernel E_0 of ϕ in H differs from e, the l-kernel of ϕ in H/E_0 coincides with e. Hence the 2-Sylow subgroup $\overline{V}$ of H/E_0 is self-conjugate. Let V be a subgroup of H corresponding to $\overline{V}$ by the natural homomorphism from H onto H/E_0. Then V is self-conjugate in H. Take the normalizer N_2 of a 2-Sylow subgroup S_2 of H. Then we have $N_2V=H$ because $S_2\subseteq V$. On the other hand, we have $N_2V=N_2\cup S_2\cup E_0=N_2\cup E_0$. Hence we have $N_2E_0=H$, which implies that $H=N_2$ by (e)([9]). Hence S_2 is self-conjugate and we have $H=S_2\times K$. q.e.d.

(h) If ϕ induces a proper L-homomorphism from a cyclic p-Sylow subgroup S into L, then G has a self-conjugate Sylow p-complement.

Proof. We shall prove that S is contained in the center of its normalizer. If this is done, our proposition follows from a theorem of Burnside (cf. [7, p. 131]). Choosing a suitable subgroup of G, we may assume S to be self-conjugate. We shall then prove that G is a direct product of S and the Sylow p-complement K. Using induction on the order of G we have only to prove our assertion assuming K to be a cyclic group of prime power order, that is, $K=\{b\}$. Put $S=\{a\}$, then we have $b\cdot a\cdot b^{-1}=a^r$. If $r\not\equiv 1$ (mod the order of a), G would admit no proper L-homomorphism, against our assumption. Hence we have $r\equiv 1$ and $G=K\times S$. q.e.d.

By propositions (g) and (h), we get the following propositions.

(i) The factor group H/G_0 is a nilpotent group each of whose Sylow subgroups is either cyclic or a dihedral group.

(j) If ϕ induces a proper L-homomorphism of G_0/E_0, G_0 contains a normal subgroup G_2 of G such that the factor group G_0/G_2 is cyclic and ϕ induces an L-isomorphism of G_2/E_0. Moreover the order of G_0/G_2 is relatively prime to that of G_2/E_0.

(9) Let Φ be the Φ-subgroup of G, then $\Phi H=G$ implies $H=G$ for any subgroup H of G. Cf. Zassenhaus [7, p. 45].

REMARK. If the center Z of a g.q. group Q is mapped onto 0 by ϕ, and if $\phi(Q)\neq 0$, Z is clearly self-conjugate by (b′), since $Z\subseteq E_0$. Hence Z is contained in the center of G. Conversely if a 2-Sylow subgroup Q of G is a g.q. group and if the center Z of Q is self-conjugate in G, then the natural homomorphism from G onto G/Z induces an L-homomorphism from G onto G/Z (see Lemma 4 below).

From (b′), (h) and the remark given above we obtain:

(k) E_0 is a cyclic group contained in the center of G.

Proof. By (b′) and Lemma 2, E_0 is cyclic. Let T be a p-Sylow subgroup of E_0, and S be a p-Sylow subgroup of G. S is then cyclic or a g.q. group. If it is a g.q. group, T is contained in the center of G as remarked above. If S is cyclic, ϕ induces a proper L-homomorphism of S. Hence by (h), G has a self-conjugate Sylow p-complement K. As T is self-conjugate by (b′), $K\cup T$ is a direct product of K and T, which implies that T is contained in the center of G. This proves proposition (k).

These propositions may be summarized as follows.

THEOREM 4. *If G admits a proper L-homomorphism ϕ, then G contains a normal subgroup N and a subgroup H such that*

(1) *$NH=G$ and $N\cap H=e$,*

(2) *The orders of N and H are relatively prime,*

(3) *H contains the u-kernel G_0 of ϕ, and*

(4) *N is contained in the l-kernel E of ϕ.*

Moreover putting $E_0=E\cap G_0$ we have

(5) *E_0 is a cyclic group, contained in the center of G.*

The factor group H/G_0 is a nilpotent group, each of whose Sylow subgroups is either cyclic or a dihedral group. If H/G_0 contains a dihedral group, H is a direct product of its 2-Sylow subgroup and the Sylow 2-complement. If, moreover, ϕ induces a proper L-homomorphism of $\overline{G}_0=G_0/E_0$, $\overline{G}_0$ contains a normal subgroup $\overline{G}_2$ such that

(6) *$\overline{G}_0/\overline{G}_2$ is cyclic,*

(7) *the order of $\overline{G}_0/\overline{G}_2$ is relatively prime to that of $\overline{G}_2$, and*

(8) *ϕ induces an L-isomorphism from $\overline{G}_2$ into L.*

As special cases of this theorem we obtain the following theorem.

THEOREM 5. *If none of the Sylow complements of a group G is self-conjugate, any L-homomorphism from G onto a lattice L is either one of the natural homomorphisms from $L(G)$ onto its direct components, or the L-homomorphism from G onto G/Z, where Z is the center of a 2-Sylow subgroup, which is a g.q. group, or combinations of these L-homomorphisms. Hence L is isomorphic to the subgroup lattice of some group.*

Since a group L-isomorphic to a perfect group is also perfect (cf. [5, Theorem 12]) we obtain the following theorem.

THEOREM 6. *Let G be a perfect group. If G is L-homomorphic to the subgroup lattice $L(H)$ of a group H, then H is perfect.*

3. GROUPS L-HOMOMORPHIC TO A NILPOTENT GROUP

In the following two sections we shall consider a homomorphism from the subgroup lattice $L(G)$ of a group G onto $L(G')$ of another group G'. We shall call this homomorphism the L-homomorphism from G onto G'. In this section we assume in particular G' to be nilpotent, then we can obtain more precise results than those of the preceding section.

Let G be a group and ϕ be an L-homomorphism from G onto a lattice L. Then by Theorem 4, G has a normal subgroup N and a subgroup H with properties (1)–(4) of Theorem 4, and if we denote by E or G_0 the l-kernel or the u-kernel of ϕ respectively, these groups are self-conjugate in G. Put $E_0 = E \cap G_0$. These notations will be fixed throughout this section.

LEMMA 3. *$L(H)$ is directly decomposable if and only if L is directly decomposable.*

Proof. If $L(H)$ is directly decomposable, L is clearly decomposable. Assume conversely that L is directly decomposable: $L = L_1 \times L_2$. Then there is a natural homomorphism ψ_i from L onto L_i ($i=1, 2$). $\psi_i\phi$ is clearly an L-homomorphism from G onto L_i. We shall denote the l-kernel of $\psi_i\phi$ by E_i. By Theorem 2, E_i is self-conjugate in G. We have clearly $E_1 \cap E_2 = E$ and $E_1 \cup E_2 = G$. When we regard $\psi_1\phi$ as an L-homomorphism from G/E onto L_1, the u-kernel of $\psi_1\phi$ is contained in E_2/E, and therefore the order of E_1/E is relatively prime to that of E_2/E by Theorem 4. Hence $L(G/E)$ is directly decomposable. Since $G/E \cong H/E_0$ and since $E_0 \subseteq \Phi(H)$ by proposition (e) of §2, $L(H)$ is also directly decomposable (cf. [5, Lemma 5]). q.e.d.

In the following we shall assume that L is the subgroup lattice of a nilpotent group G' and determine the structure of the group H. In virtue of Lemma 3, we can assume G' to be a p-group.

THEOREM 7. *Let G be a group, and ϕ be an L-homomorphism from G onto a p-group G'. If G' is neither cyclic nor a P-group, H is also a p-group and coincides with G_0. G is therefore a direct product of N and G_0. If G' is a P-group, H is either a p-group or an upper semi-modular group of order $p^m q^n$*[10]*, where q is a prime number and $p>q$, and G_0 is its maximal self-conjugate M-group.*

Proof. We shall assume that G' is not cyclic. Since $L(G')$ has no irreducible interval, H/G_0 is cyclic by Theorem 4 and Lemma 3. If ϕ induces a proper L-homomorphism from G_0/E_0, G_0 has a normal subgroup G_2 and ϕ

[10] Such a group G has been completely determined by Sato [4]. According to him, a group of order $p^m q^n$ ($p>q$) is an upper semimodular group if and only if its p-Sylow subgroup P is a P-group, a q-Sylow subgroup Q is cyclic, $Q=\{b\}$, and for any element a of P, $bab^{-1}=a^x$, $x^{q^t}\equiv 1 \pmod p$.

induces an L-isomorphism from G_2/E_0 into G'. Hence by Theorem 3 of Suzuki [5], G_2/E_0 is a p-group or a P-group. If G_2/E_0 were a nonabelian P-group, ϕ would induce an L-isomorphism from a group V/E_0, where V is a subgroup of G_0, covering G_2. Since the order of V/E_0 would be divisible by three distinct primes, this is a contradiction. Hence by proposition (b′) and (d) of §2, we see that H is a p-group or a group of order $p^m q^n$ $(p>q)$. If H is a p-group, by Lemma 2 we have $H=G_0$. We have now only to prove that if the order of H is $p^m \cdot q^n$, H is an upper semi-modular group, and G is a P-group.

G_0/E_0 is a group of order $p^\alpha q^\beta$ and its p-Sylow subgroup $\overline{S}$ is self-conjugate by Theorem 3 of Suzuki [5] and our Theorem 4. ϕ induces an L-isomorphism from $\overline{S}$ into G'. Take a subgroup $\overline{T}$ of G_0/E_0 covering $\overline{S}'$, then ϕ induces also an L-isomorphism in $\overline{T}$. Hence T is a P-group. Next take a q-Sylow subgroup $\overline{Q}$ of G_0/E_0 and a subgroup $\overline{V}$ covering $\overline{Q}$; then $\overline{Q}$ is cyclic. Since G' is a p-group, $\phi(\overline{V})\cap\phi(\overline{S})$ is of prime order. Hence $\overline{V}\cap\overline{S}$ is a normal subgroup of G_0/E_0 of order p. By direct examination we see that $\phi(\overline{V})$ is a P-group. This implies that $G'=\phi(\overline{T})$ and $G_0/E_0=\overline{T}$. Hence we see that G_0/E_0 and G' are both P-groups.

Since Sylow p-complements of H are not self-conjugate, the orders of H/G_0 and E_0 are both powers of q by proposition (h) of §2. The p-Sylow subgroup S of H is clearly self-conjugate in H and ϕ induces an L-isomorphism from S into G'. Take any subgroup V of order p and any q-Sylow subgroup Q of H. Then $\phi(V\cup Q)$ is a P-group of order p^2. Hence $(V\cup Q)\cap S$ is of prime order and hence coincides with V; $(V\cup Q)\cap S= V$. This implies that V is a normal subgroup of H. Put $Q=\{b\}$; then for any element a of S we have

$$b\cdot a\cdot b^{-1} = a^x, \qquad x \not\equiv 1, \qquad x^{q^t} \equiv 1 \pmod{p}.$$

Hence H is an upper semi-modular group and G_0 is its maximal self-conjugate M-group. q.e.d.

In order to prove the converse of this theorem we shall first prove the following lemma.

LEMMA 4. *Let Z be a cyclic subgroup of prime power order contained in the center of a group G. If Sylow subgroups containing Z are cyclic or g.q. groups, the natural homomorphism from G onto G/Z induces an L-homomorphism.*

Proof. We can assume that Z is of prime order. We have only to prove $(U\cap V)\cup Z=(U\cup Z)\cap(V\cup Z)$ for any two subgroups U and V of G. If $U\supseteq Z$ and $V\supseteq Z$, we have clearly this equality. If $U\not\supseteq Z$, the order of U is prime to p. Hence we have $L(U\cup Z)=L(Z)\times L(U)$ (cf. [3]). If moreover $V\supseteq Z$, we have $(U\cup Z)\cap V=Z\cup(((U\cup Z)\cap V)\cap U)=Z\cup(U\cap V)$. If $V\not\supseteq Z$, $(U\cup Z)\cap(V\cup Z)=Z\cup W$ for some subgroup W. We have then $U\cap V\supseteq W$. Hence we have $(U\cup Z)\cap(V\cup Z)\subseteq(U\cap V)\cup Z$. On the other hand, we have clearly $(U\cap V)\cup Z\subseteq(U\cup Z)\cap(V\cup Z)$. Hence $(U\cap V)\cup Z=(U\cup Z)\cap(V\cup Z)$. q.e.d.

If a group G is a direct product of two groups G_0 and N (having relatively prime orders), and if G_0 is a p-group, G is clearly L-homomorphic to G_0. If H is an upper semi-modular group and G_0 is its maximal self-conjugate M-group, G is L-homomorphic to a P-group as follows. First the mapping $U\rightarrow U\cup E_0$ from $L(G)$ onto $L(G/E_0)$ is surely an L-homomorphism by Lemma 4. Hence we may assume that $E_0=e$. As H is an upper semi-modular group, the mapping $U\rightarrow U\cap G_0$ from $L(H)$ onto $L(G_0)$ is an L-homomorphism. We shall prove that the mapping $U\rightarrow U\cap G_0(U\subseteq G)$ is also an L-homomorphism from G onto G_0. First we shall show that $(U\cap G_0)\cup N=(U\cup N)\cap(G_0\cup N)$ for any subgroup U of G. Suppose that the order of U is $p^\alpha q^\beta g$, $(pq, g)=1$. If $\beta=0$, U is contained in $S\cup N$, where S is a p-Sylow subgroup of G. Since $L(S\cup N)=L(S)\times L(N)$, we have easily $(U\cap G_0)\cup N=(U\cup N)\cap(G_0\cup N)$. If $\beta\neq 0$, the index $[(U\cap G_0)\cup N:N]$ is equal to $p^\alpha q$, and $[(U\cup N)\cap(G_0\cup N):N]$ is also equal to $p^\alpha q$. On the other hand, we have $(U\cap G_0)\cup N\subseteq(U\cup N)\cap(G_0\cup N)$. Hence we have $(U\cap G_0)\cup N=(U\cup N)\cap(G_0\cup N)$.

Now we shall show that $(U\cup V)\cap G_0=(U\cap G_0)\cup(V\cap G_0)$. In fact, we have

$$N\cup((U\cup V)\cap G_0)=(U\cup V\cup N)\cap(G_0\cup N).$$

On the other hand, as G/N is an upper semi-modular group,

$$\begin{aligned}((U\cup N)\cup(V\cup N))\cap(G_0\cup N)&\\ &=((U\cup N)\cap(G_0\cup N))\cup((V\cup N)\cap(G_0\cup N))\\ &=((U\cap G_0)\cup N)\cup((V\cap G_0)\cup N)\\ &=((U\cap G_0)\cup(V\cap G_0))\cup N.\end{aligned}$$

Since $G_0\cap N=e$, we have

$$\begin{aligned}(U\cup V)\cap G_0 &\cong N\cup((U\cup V)\cap G_0)/N\\ &\cong((U\cap G_0)\cup(V\cap G_0))\cup N/N\cong(U\cap G_0)\cup(V\cap G_0).\end{aligned}$$

Hence we have $(U\cup V)\cap G_0=(U\cap G_0)\cup(V\cap G_0)$. The mapping $U\rightarrow U\cap G_0$ is thus an L-homomorphism from G onto a P-group G_0.

From Lemmas 1 and 3, Theorem 7, and the remark given above we obtain:

THEOREM 8. *Let G be a group. There exists an L-homomorphism ϕ from G onto a nilpotent group $G'=\prod_{i=1}^{t} S_i$, where S_i is a p_i-Sylow subgroup of G', if and only if G has a normal subgroup N and a subgroup H with the following properties*:

(1) *$NH=G$ and $N\cap H=e$.*

(2) *the order of N is relatively prime to that of H,*

(3) *H is a direct product of groups H_i $(i=1, 2, \cdots, t)$ having mutually prime orders*: $H=\prod_{i=1}^{t} H_i$,

(4) $\phi(H_i)=S_i$ $(i=1, 2, \cdots, t)$,

(5) *if S_j is cyclic, H_j is a cyclic group of prime power order or a g. q. group, and H_j contains a normal subgroup K_j of G such that $\phi(K_j)=S_j$*,

(6) *if S_k is a P-group of order p_k^{n+1} $(n\geqq 1)$, H_k is either isomorphic to S_k, or a quaternion group $(n=1, p_k=2)$, or an upper semi-modular group of order $p_k^n q^m$ (q is a prime and $p_k>q$), and its maximal self-conjugate M-group is a normal subgroup of G*,

(7) *if S_l is neither cyclic nor a P-group, H_l is also a p_l-group and self-conjugate in G. In this case if H_l is not L-isomorphic to S_l, H_l is a g.q. group and S_l is isomorphic to the factor group H_l/Z_l of H_l modulo its center Z_l.*

We shall omit the proof of this theorem, since it runs along similar lines as the proof of Theorem 1.

4. The L-homomorphic image of a solvable group

In this section we shall prove the following theorem.

Theorem 9. *Let G be a solvable group, and ϕ be an L-homomorphism from G onto another group G'. Then G' is also solvable.*

Denote by E or G_0 the l-kernel or the u-kernel of ϕ respectively. Then by Theorems 2 and 3, E and G_0 are self-conjugate. Put $E_0=E\cap G_0$. ϕ induces an L-homomorphism $\bar{\phi}$ from G_0/E_0 onto G'. If ϕ is an L-isomorphism, our theorem follows from a theorem on the L-isomorphism which asserts that groups L-isomorphic to a solvable group are also solvable (cf. [5, Theorem 12]). If $\bar{\phi}$ is a proper L-homomorphism, G_0/E_0 contains a normal subgroup G_2/E_0 such that G_0/G_2 is cyclic and $\bar{\phi}$ induces an L-isomorphism from G_2/E_0 into G'. Hence in order to prove our Theorem 9, it is sufficient to prove the following theorem.

Theorem 10. *Assume L to be a lattice of subgroups of a group G'. Then under the same notations as in Theorem* 4, *$\phi(G_2)$ is self-conjugate in G'.*

Proof. In changing the notations, we shall assume that the u-kernel of ϕ coincides with G and that the l-kernel of ϕ coincides with e. Take a p-Sylow subgroup S of G in which ϕ induces a proper L-homomorphism. By Lemma 3 and proposition (g) of §2, S must be cyclic, and by proposition (h) of §2, G has a Sylow p-complement N. We shall first prove that $\phi(S)$ is also a Sylow subgroup of G.

Since $\phi(S)$ is a cyclic group of prime power order, it is contained in some Sylow subgroup S' of G'. Take the greatest subgroup U of G such that $\phi(U)=S'$. Then U clearly contains S. If S' were a P-group, $\phi(S)$ would be of prime order. On the other hand, taking the maximal subgroup M of S, we have $\phi(M)\neq\phi(S)$, as the u-kernel of ϕ coincides with G. Hence we would have $\phi(M)=e$, that is, M would be contained in the l-kernel of ϕ and by our assumption $M=e$. Hence S is mapped L-isomorphically onto $\phi(S)$, contrary to our assumption. Hence S' is not a P-group and U is also of prime power order by Theorem 8. Hence U must coincide with S, that is, $S'=\phi(S)$.

Next we shall prove that $S'=\phi(S)$ is contained in the center of its normalizer. Take a subgroup V' of G' such that S' is self-conjugate in V', and V'/S' is of prime power order, say of order q^n (q is a prime number). Take a subgroup V of G such that $\phi(V)=V'$; then $\phi(V\cap N)$ is a q-Sylow subgroup Q' of V'. If $V\cap N$ is cyclic and not L-isomorphic to Q', S is self-conjugate in V by proposition (h) of §2, and hence V and also V' are directly decomposable.

We can then assume $V\cap N$ to be L-isomorphic to Q'(11). Since the l-kernel of ϕ coincides with e, a subgroup T of V, covering $N\cap V$, is L-isomorphic to $\phi(T)=T'$, and ϕ induces an L-isomorphism from T onto T'. By our assumption, $T'\cap S'$ is self-conjugate in T'. If $T\cap S$ were not self-conjugate in T, T would be a P-group (cf. [5, Theorems 13 and 14]) which would imply that Q' has prime order. Hence $V\cap N$ would also be of prime order. Since $\phi(S)$ is self-conjugate in V', V' is a P-group, which leads us to the same contradiction as above. Hence $T\cap S$ is self-conjugate in T and so T is a direct product of $N\cap V$ and $T\cap S$. This implies that $T\cap S$ is self-conjugate in V. If S were not self-conjugate in V, there would be another p-Sylow subgroup S^* of V. S^* would also contain $T\cap S$. Hence we would have $\phi(S^*)\cap S'\neq e$. Since $\phi(S^*)$ is a cyclic group of prime power order, this gives a contradiction. Hence we have $V=(N\cap V)\times S$ and $V'=Q'\times S'$. S' is thus contained in the center of its normalizer and G' contains a normal subgroup N' such that $N'S'=G$ and $N'\cap S'=e$(12).

We shall now prove that $\phi(N)=N'$. Take all p-Sylow subgroups $S=S_1$, $S_2, \cdots, S_t$ of G. Then ϕ induces a proper L-homomorphism in every S_i. Hence the $\phi(S_i)$ are Sylow subgroups of G' and are contained in centers of their normalizers, as proved above. G then has Sylow complements $N'=N_1'$, $N_2', \cdots, N_r'$. Put $D'=\bigcap_{i=1}^{r}N_i'$. Take a subgroup D of G such that $\phi(D)=D'$. Since $D'\cap\phi(S_i)=e$ $(i=1, 2, \cdots, t)$, we have $D\cap S_i=e$ $(i=1, 2, \cdots, t)$, which implies that the order of D is prime to p, or $D\subseteq N$. Since $\phi(N)\supseteq D'$, $\phi(N)\cap\phi(S)=e$ and $\phi(N)\cup\phi(S)=G'$, we have $\phi(N)=N'$. This proves our theorem.

5. Neutral elements of $L(G)$

An element l of a lattice L is called neutral if every triple $\{l, x, y\}$ of elements of L generates a distributive sublattice of L. An element l of L is neutral if and only if the mappings $x\rightarrow x\cup l$ and $x\rightarrow x\cap l$ are homomorphisms, and $x\cup l=y\cup l$ and $x\cap l=y\cap l$ imply $x=y$ for any two elements x, y of L

(11) Cf. Theorem 7.

(12) By Burnside's theorem, cf. Zassenhaus [7, p. 131].

(Birkhoff [1]). If L is directly decomposable, an element is neutral if and only if all its components are neutral.

In this section we shall determine the neutral elements of a subgroup lattice $L(G)$ of a group G. Because of the above remark we may assume $L(G)$ to be irreducible.

Let K be a neutral element of $L(G)$. Then the mapping ϕ: $U\rightarrow U\cup K$ is an L-homomorphism from G onto an interval G/K. As K is the l-kernel of ϕ, it is self-conjugate in G by Theorem 2. Denote the u-kernel of ϕ by G_0; then we have $G_0\cup K=G$. By proposition (c) of §2, we have either $G_0\supseteq K$ or $L(G)$ is directly decomposable. Hence from our assumptions we have $G_0\supseteq K$, so $G_0=G$. By Theorem 4, K is a cyclic group contained in the center of G. On the other hand, the mapping $U\rightarrow U\cap K$ is also an L-homomorphism from G onto K. Since K is cyclic, the structure of G is determined by Theorem 1. Let $K=\prod_{i=1}^{t} K_i$ be the decomposition of K into a direct product of its Sylow subgroups K_i. Then G has a normal subgroup N and a subgroup H with the following properties:

(1) $NH=G$, $N\cap H=e$, and $H\supseteq K$,

(2) the order of N is prime to that of H,

(3) H is a direct product $\prod_{i=1}^{t} H_i$ of its Sylow subgroups H_i, and

(4) H_i is either cyclic or a g.q. group.

Conversely suppose that a subgroup K of a group G is contained in the center of G and G has a normal subgroup N and a subgroup H with the properties (1)–(4) given above. Then K is a neutral element of $L(G)$.

Proof. By (4), K is cyclic. Let K_i be a p_i-Sylow subgroup of K. We shall show that K_i is neutral. By Lemma 4, the mapping $U\rightarrow U\cup K_i$ is an L-homomorphism from G onto G/K_i. By Lemma 1, the mapping $U\rightarrow U\cap K_i$ is also an L-homomorphism from G onto K_i. We have only to prove that $U\cup K_i = V\cup K_i$ and $U\cap K_i= V\cap K_i$ imply $U= V$ for any two subgroups U, V of G. G has a Sylow p_i-complement N_i. We have $U\supseteq K_i$, or $U\subseteq K_iN_i$ for any subgroup U of G. Suppose now that $U\cup K_i= V\cup K_i$ and $U\cap K_i= V\cap K_i$. If $U\supseteq K_i$, we have $U\cap K_i=K_i$. Hence we have $V\cap K_i=K_i$, or $V\supseteq K_i$. We have, therefore, $U= U\cup K_i= V$. If $U\not\supseteq K_i$, we have also $V\not\supseteq K_i$, that is, N_iK_i contains both U and V. On the other hand, N_iK_i is a direct product of N_i and K_i, and we have $L(N_i, K_i) =L(N_i)\times L(K_i)$. Hence we have clearly

$$\begin{aligned} U &= (U \cap K_i) \cup (U \cap N_i) = (U \cap K_i) \cup ((U \cup K_i) \cap N_i) \\ &= (V \cap K_i) \cup ((V \cup K_i) \cap N_i) = V. \end{aligned}$$

Since the join of neutral elements is also neutral, $K=\bigcup_{i=1}^{t} K_i$ is neutral. Thus we obtain the following theorem, which gives an answer to a problem of Birkhoff(13).

THEOREM 11. *Assume that the subgroup lattice $L(G)$ of a group G is ir-*

(13) Problem 35, described in the revised edition of his book *Lattice theory*.

reducible. A subgroup K of G is a neutral element of $L(G)$ if and only if K is contained in the center of G, and G has a normal subgroup N and subgroup H with the properties (1)–(4) *given above.*

Added in proof. After writing this paper, the author learned that G. Zappa has obtained some theorems concerning L-homomorphisms of finite groups, in particular Theorem 1 of this paper: Cf. G. Zappa, *Determinazione dei gruppi finiti in omomorfismo strutturale con un gruppo ciclico*, Rendiconti del seminario Matematico, Univ. di Padova (1949) pp. 140–162, and *Sulla condizione perche un omomorfismo ordinario sia anche un omomorfismo strutturale*, Giornale di Matematiche vol. 78 (1949) pp. 182–192.

For the detailed proof of a theorem of N. Ito, cited in footnote 7 of this paper, see his forthcoming paper: *Note on (LM)-groups of finite orders*, Kôdai Mathematical Seminar Reports.

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