A GENERAL THEORY OF CONJUGATE NETS IN PROJECTIVE HYPERSPACE

BY

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Introduction. In a previous paper(1), the author has established a theory of the projective differential geometry of conjugate nets in a linear space S_4 of four dimensions. The purpose of the present paper is to extend the theory to a general linear space S_n ($n \geqq 4$).

In §1 a completely integrable system of linear homogeneous partial differential equations, together with its integrability conditions, is introduced by a purely geometric method defining a conjugate net N_x in the space S_n except for a projective transformation. A canonical form of the system of differential equations is obtained in §2 by a geometric determination.

In §3 we deduce the conditions of immovability for a point and a hyperplane in the space S_n relative to an invariant local pyramid of reference associated with a point x of the conjugate net N_x.

§§4, 5, 6 are devoted to proving the following theorems respectively.

THEOREM 1. *In a linear space S_n of n ($\geqq 3$) dimensions let N_x be a conjugate net and π be a fixed hyperplane; then the points M, $\overline{M}$ of intersection of the fixed hyperplane π and the two tangents at a point x of the net N_x describe two conjugate nets N_M, $N_{\overline{M}}$ in the hyperplane π respectively, and one of the two nets N_M, $N_{\overline{M}}$ is a Laplace transformed net of the other.*

THEOREM 2. *In a linear space S_n of n ($\geqq 4$) dimensions let N_x be a conjugate net and S_{n-2} be a fixed linear subspace of $n-2$ dimensions; then the point T of intersection of the fixed subspace S_{n-2} and the tangent plane at a point x of the surface sustaining the net N_x describes a conjugate net N_T in the subspace S_{n-2}.*

THEOREM 3. *Conjugate nets with equal and nonzero Laplace-Darboux invariants in a linear space S_n of n ($\geqq 4$) dimensions are characterized by the property that at each point x of any one of them there exists a proper hyperquadric (and therefore $\infty^{n(n+3)/2-11}$ such hyperquadrics) having second order contact at the Laplace transformed points x_{-1}, x_1 of the point x with both Laplace transformed surfaces S_{-1}, S_1 of the net N_x, respectively.*

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(1) C. C. Hsiung, *Projective theory of surfaces and conjugate nets in four-dimensional space*, Amer. J. Math. vol. 69 (1947) pp. 607–621. A projective theory of conjugate nets in ordinary three-dimensional space has been established in a similar way by E. P. Lane; see his book, *A treatise on projective differential geometry*, University of Chicago Press, 1942, Chap. VIII.

1. Differential equations and integrability conditions. Let us consider a conjugate net N_x with parameters u, v in a linear space S_n of n ($\geqq 4$) dimensions so that the homogeneous projective coordinates

$$x^{(1)}, \cdots, x^{(n+1)}$$

of a nonsingular point x on the surface S sustaining the net N_x are given as analytic functions of the two independent variables u, v by equations of the form

$$x = x(u, v). \tag{1.1}$$

The osculating linear space $S_k^{(u)}$ of k $(=2, \cdots, n-1)$ dimensions of the parametric curve u and the osculating linear space $S_{n-k+1}^{(v)}$ of $n-k+1$ dimensions of the parametric curve v at the point x of the net N_x intersect in a line l_{k-1}. Let us select on the lines $l_1, \cdots, l_{n-2}$ respectively $n-2$ points $y_1, \cdots, y_{n-2}$, distinct from the point x, and suppose that the coordinates y_i of the point y_i $(i=1, \cdots, n-2)$ are functions of u, v. Then it can be shown that the coordinates y_i of the corresponding points y_i $(i=1, \cdots, n-2)$ satisfy a system of linear homogeneous partial differential equations of the form

$$\begin{aligned} x_{uv} &= cx + ax_u + bx_v, \\ \frac{\partial^i x}{\partial u^i} &= \alpha_i x + \beta_i x_u + \sum_{j=n-i}^{n-2} \overset{i}{p}_j y_j, \\ \frac{\partial^i x}{\partial v^i} &= \delta_i x + \gamma_i x_v + \sum_{k=1}^{i-1} \overset{i}{q}_k y_k \qquad (i = 2, \cdots, n-1), \end{aligned} \tag{1.2}$$

in which subscripts indicate partial differentiation and the coefficients are scalar functions of u, v. The first of these equations is merely the Laplace equation for the parametric conjugate net N_x.

By using equations (1.2) it is easily seen that the derivatives y_{iu} can be written in the form

$$\begin{aligned} y_{1u} &= A_1 x + B_1 x_u + E_1 x_v + \overset{1}{L}_1 y_1, \\ y_{iu} &= A_i x + B_i x_u + \overset{i}{L}_{i-1} y_{i-1} + \overset{i}{L}_i y_i \qquad (i = 2, \cdots, n-2). \end{aligned} \tag{1.3}$$

In particular, by actual calculation one obtains

$$\begin{aligned} \overset{2}{q}_1 A_1 &= c_v + ac + b\delta_2 - c\gamma_2 - \delta_{2u}, & \overset{2}{p}_{n-2} A_{n-2} &= \alpha_3 - \alpha_2\beta_2 - \alpha_{2u}, \\ \overset{2}{q}_1 B_1 &= a_v + a^2 - a\gamma_2 - \delta_2, & \overset{2}{p}_{n-2} B_{n-2} &= \beta_3 - \alpha_2 - \beta_{2u} - \overset{2}{\beta_2}, \\ \overset{2}{q}_1 E_1 &= b_v + ab + c - \gamma_{2u}, & \overset{2}{p}_{n-2} \overset{n-2}{L}_{n-3} &= \overset{3}{p}_{n-3}, \\ \overset{1}{L}_1 &= b - (\log \overset{2}{q}_1)_u; & \overset{2}{p}_{n-2} \overset{n-2}{L}_{n-2} &= \overset{3}{p}_{n-2} - \overset{2}{p}_{n-2}\beta_2 - \overset{2}{p}_{n-2,u}. \end{aligned} \tag{1.4}$$

Analogous expressions for y_{iv} can be written by making the substitution

$$(1.5)\quad \begin{pmatrix} u & \alpha_i & \beta_i & y_i & \overset{k}{p}_i & A_i & B_i & E_1 & \overset{i}{L}_j & H & \xi_1 & \xi_2 & \xi_{i+3} \\ v & \delta_i & \gamma_i & y_{n-(i+1)} & \overset{k}{q}_{n-(i+1)} & D_{n-(i+1)} & C_{n-(i+1)} & F_{n-2} & \overset{n-(i+1)}{M}_{n-(j+1)} & K & \xi_1 & \xi_3 & \xi_{n+2-i} \end{pmatrix},$$

where H, K are the Laplace-Darboux invariants defined by the respective formulas

$$(1.6)\quad \begin{aligned} H &= c + ab - a_u, \\ K &= c + ab - b_v, \end{aligned}$$

and the ξ's will be defined in §3.

The integrability conditions of equations (1.2) are found by the usual method from the equations

$$\frac{\partial}{\partial u}\left(\frac{\partial^{n-i}x}{\partial u^{n-i}}\right) = \frac{\partial^{n-i+1}x}{\partial u^{n-i+1}}, \qquad \frac{\partial}{\partial v}\left(\frac{\partial^{i}x}{\partial v^{i}}\right) = \frac{\partial^{i+1}x}{\partial v^{i+1}} \qquad (i = 2, \cdots, n-2);$$

$$\frac{\partial}{\partial u}\left(\frac{\partial^{i+1}x}{\partial v^{i+1}}\right) = \frac{\partial^{i}x_{uv}}{\partial v^{i}}, \qquad \frac{\partial}{\partial v}\left(\frac{\partial^{n-i}x}{\partial u^{n-i}}\right) = \frac{\partial^{n-(i+1)}x_{uv}}{\partial u^{n-(i+1)}} \qquad (i = 1, \cdots, n-2);$$

$$(y_{iu})_v = (y_{iv})_u \qquad (i = 1, \cdots, n-2);$$

and the fact that the points $x, x_u, x_v, y_1, \cdots, y_{n-2}$ are linearly independent. The result is given by the following equations and the analogous ones obtainable therefrom by the substitution (1.5):

$$(1.7)\quad \begin{aligned} A_i &= \frac{1}{\overset{n-i}{p}_i}\left(\alpha_{n-i+1} - \alpha_{n-i,u} - \alpha_2\beta_{n-i} - \sum_{j=i+1}^{n-2} \overset{n-i}{p}_j A_j\right) \\ &= \frac{1}{\overset{i+1}{q}_i}\left\{\frac{\partial^i c}{\partial v^i} + ic\frac{\partial^{i-1}a}{\partial v^{i-1}} - \delta_{i+1,u} - c\gamma_{i+1} - \sum_{j=1}^{i-1} \overset{i+1}{q}_j A_j \right. \\ &\quad \left. + \sum_{j=2}^{i} C_{i,j}\left[\frac{\partial^{i-j}c}{\partial v^{i-j}}\delta_j + \frac{\partial^{i-j}a}{\partial v^{i-j}}\left(\delta_{ju} + c\gamma_j + \sum_{k=1}^{j-1}\overset{j}{q}_k A_k\right)\right]\right\}, \\ B_i &= \frac{1}{\overset{n-i}{p}_i}\left(\beta_{n-i+1} - \alpha_{n-i} - \beta_{n-i,u} - \beta_2\beta_{n-i} - \sum_{j=i+1}^{n-2}\overset{n-i}{p}_j B_j\right) \\ &= \frac{1}{\overset{i+1}{q}_i}\left[\frac{\partial^i a}{\partial v^i} + ia\frac{\partial^{i-1}a}{\partial v^{i-1}} - a\gamma_{i+1} - \delta_{i+1} - \sum_{j=1}^{i-1}\overset{i+1}{q}_j B_j \right. \\ &\quad \left. + \sum_{j=2}^{i} C_{i,j}\frac{\partial^{i-j}a}{\partial v^{i-j}}\left(\delta_j + a\gamma_j + \sum_{k=1}^{j-1}\overset{j}{q}_k B_k\right)\right] \qquad (i = 2, \cdots, n-2), \end{aligned}$$

$$\overset{i}{L}_{i-1} = \overset{n-i+1}{p}_{i-1}/\overset{n-i}{p}_i, \quad \overset{i}{L}_i = b - (\log \overset{i+1}{q}_i)_u \qquad (i = 2, \cdots, n-2),$$

$$\overset{n-i+1}{p}_j = \overset{n-i}{p}_{ju} + \overset{n-i}{p}_j\overset{j}{L}_i + \overset{n-i}{p}_{j+1}\overset{j+1}{L}_i \quad (i = 2, \cdots, n-3;\ i \leqq j \leqq n-3),$$

$$\overset{n-i+1}{p}_{n-2} = \overset{2}{p}_{n-2}\beta_{n-i} + \overset{n-i}{p}_{n-2,u} + \overset{n-i}{p}_{n-2}\overset{n-2}{L}_{n-2} \qquad (i = 2, \cdots, n-2),$$

$$\gamma_{i+1,u} + b\gamma_{i+1} + \overset{i+1}{q_1} E_1 = \frac{\partial^i b}{\partial v^i} + ib\frac{\partial^{i-1}a}{\partial v^{i-1}} + i\frac{\partial^{i-1}c}{\partial v^{i-1}}$$
$$+ \sum_{j=1}^{i} C_{i,j}\frac{\partial^{i-j}b}{\partial v^{i-j}}\gamma_{j+1}$$
$$+ \sum_{j=2}^{i} C_{i,j}\left[\frac{\partial^{i-j}c}{\partial v^{i-j}}\gamma_j + \frac{\partial^{i-j}a}{\partial v^{i-j}}(\gamma_{ju} + b\gamma_j + \overset{j}{q_1}E_1)\right]$$
$$(i = 2, \cdots, n-2),$$

(1.7) $$\overset{i+1}{q_{ku}} + \overset{i+1}{q_k}\overset{k}{L_k} + \overset{i+1}{q_{k+1}}\overset{k+1}{L_k} = \sum_{j=k}^{i} C_{i,j}\frac{\partial^{i-j}b}{\partial v^{i-j}}\overset{j+1}{q_k}$$
$$+ \sum_{j=k+1}^{i} C_{i,j}\frac{\partial^{i-j}c}{\partial v^{i-j}}\overset{j}{q_k} + C_{i,k+2}\frac{\partial^{i-k+1}a}{\partial v^{i-k+1}}(\overset{k+1}{q_{ku}} + \overset{k+1}{q_k}\overset{k}{L_k})$$
$$+ \sum_{j=k+2}^{i} C_{i,j}\frac{\partial^{i-j}a}{\partial v^{i-j}}(\overset{j}{q_{ku}} + \overset{j}{q_k}\overset{k}{L_k} + \overset{j}{q_{k+1}}\overset{k+1}{L_k})$$
$$(i = 3, \cdots, n-2;\ k = 1, \cdots, i-2),$$

where $C_{i,j}$ denotes the number of combinations of i different things taken j at a time;

$$A_{1v} + cB_1 + \delta_2 E_1 + D_1\overset{1}{L_1} = D_{1u} + cC_1 + A_1\overset{1}{M_1} + A_2\overset{1}{M_2},$$
$$A_{iv} + cB_i + D_{i-1}\overset{i}{L_{i-1}} + D_i\overset{i}{L_i} = D_{iu} + cC_i + A_i\overset{i}{M_i} + A_{i+1}\overset{i}{M_{i+1}}$$
$$(i = 2, \cdots, n-3),$$
$$A_{n-2,v} + cB_{n-2} + D_{n-3}\overset{n-2}{L_{n-3}} + D_{n-2}\overset{n-2}{L_{n-2}} = D_{n-2,u} + \alpha_2 F_{n-2}$$
$$+ cC_{n-2} + A_{n-2}\overset{n-2}{M_{n-2}},$$
$$B_{iv} + aB_i = D_i + aC_i + B_i\overset{i}{M_i} + B_{i+1}\overset{i}{M_{i+1}} \qquad (i = 1, \cdots, n-3),$$
$$B_{n-2,v} + aB_{n-2} + F_{n-2}\overset{n-2}{L_{n-2}} = D_{n-2} + F_{n-2,u} + \beta_2 F_{n-2}$$
$$+ aC_{n-2} + B_{n-2}\overset{n-2}{M_{n-2}},$$

(1.8) $$A_1 + bB_1 + E_{1v} + \gamma_2 E_1 + C_1\overset{1}{L_1} = C_{1u} + bC_1 + E_1\overset{1}{M_1},$$
$$A_i + bB_i + C_{i-1}\overset{i}{L_{i-1}} + C_i\overset{i}{L_i} = C_{iu} + bC_i \qquad (i = 2, \cdots, n-2),$$
$$\overset{i}{L_{i-1,v}} + \overset{i}{L_{i-1}}\overset{i-1}{M_{i-1}} = \overset{i}{L_{i-1}}\overset{i}{M_i} \qquad (i = 2, \cdots, n-2),$$
$$\overset{1}{L_{1v}} + \overset{2}{q_1}E_1 = \overset{1}{M_{1u}} + \overset{2}{L_1}\overset{1}{M_2},$$
$$\overset{i}{L_{iv}} + \overset{i}{L_{i-1}}\overset{i-1}{M_i} = \overset{i}{M_{iu}} + \overset{i+1}{L_i}\overset{i}{M_{i+1}} \qquad (i = 2, \cdots, n-3),$$
$$\overset{n-2}{L_{n-3,v}} + \overset{n-2}{L_{n-3}}\overset{n-3}{M_{n-3}} = \overset{n-2}{L_{n-3}}\overset{n-2}{M_{n-2}},$$
$$\overset{i}{L_i}\overset{i}{M_{i+1}} = \overset{i}{M_{i+1,u}} + \overset{i+1}{L_{i+1}}\overset{i}{M_{i+1}} \qquad (i = 1, \cdots, n-3),$$
$$\overset{n-2}{L_{n-2,v}} + \overset{n-2}{L_{n-3}}\overset{n-3}{M_{n-2}} = \overset{n-2}{M_{n-2,u}} + \overset{2}{p_{n-2}}F_{n-2}.$$

Making use of the third of equations (1.4), the ninth, the tenth, and the thirteenth of equations (1.8), and the substitution (1.5) we obtain

$$(1.9)\qquad \left(a + \gamma_2 + \sum_{i=1}^{n-2} \overset{i}{M_i}\right)_u = \left(b + \beta_2 + \sum_{i=1}^{n-2} \overset{i}{L_i}\right)_v .$$

It follows that there exists a function θ of u, v which is defined, except for an arbitrary additive constant, as a solution of the differential equations

$$(1.10)\qquad \theta_u = b + \beta_2 + \sum_{i=1}^{n-2} \overset{i}{L_i}, \qquad \theta_v = a + \gamma_2 + \sum_{i=1}^{n-2} \overset{i}{M_i}.$$

Accordingly, the following formula is valid:

$$(1.11)\qquad (x, x_u, x_v, y_1, \cdots, y_{n-2}) = e^\theta,$$

where a determinant is indicated by writing only a typical row within parentheses.

2. **Canonical form of the differential equations.** We now proceed to choose for the points $y_1, \cdots, y_{n-2}$ $n-2$ particular covariant points on the lines $l_1, \cdots, l_{n-2}$ respectively. To this end, at first we observe that the point X_1 defined by $X_1 = y_1 + kx$, where k is a scalar function of u, v, is on the line l_1. When the point x varies along a curve C_λ of the family represented by the differential equation

$$(2.1)\qquad dv - \lambda du = 0,$$

λ being a function of u, v, on the surface S, the point X_1 generates a curve C_{X_1} whose tangent at X_1 is determined by X_1 and the point X_1' given by

$$X_1' = y_{1u} + y_{1v}\lambda + k(x_u + x_v\lambda) + k'x \qquad (X_1' = dX_1/du, \cdots).$$

Expressing X_1' as a linear combination of x, x_u, x_v, y_1, y_2 by means of the first of equations (1.3) and the substitution (1.5), and equating to zero the coefficients of x_u, x_v therein, we obtain two conditions on the functions k and λ which are necessary and sufficient that the tangent to the curve C_{X_1} at the point X_1 lies in the plane l_1l_2, namely,

$$k + B_1 = 0, \qquad E_1 + (C_1 + k)\lambda = 0.$$

Similarly, we can also determine a unique point X_{n-2} on the line l_{n-2} and a unique curve of the family (2.1) such that as the point x varies along the curve the tangent to the locus of the point X_{n-2} at the point lies in the plane $l_{n-3}l_{n-2}$. If we choose these two points respectively for the points y_1 and y_{n-2},

$$(2.2)\qquad B_1 = 0, \qquad C_{n-2} = 0,$$

and the differential equations of the two curves become

$$E_1 du + C_1 dv = 0, \qquad B_{n-2}du + F_{n-2}dv = 0. \tag{2.3}$$

Finally, we can determine a unique point X_h on the line l_h $(h=2, \cdots, n-3)$ such that as the point x varies along any curve, except the v-curve, on the surface S the tangent to the locus of the point X_h at the point is in the four-dimensional space determined by the lines l_{h-1}, l_h, l_{h+1} and the v-tangent. If we choose this point to be the point y_h, then

$$B_h = 0 \qquad (h = 2, \cdots, n-3). \tag{2.4}$$

Hereafter it will be supposed that the differential equations (1.2) are in *the canonical form* for which the conditions (2.2), (2.4) are satisfied.

It should be noted that the above choice of the points $y_2, \cdots, y_{n-3}$ is not symmetric with respect to the parameters u, v. However if the dimension of the space S_n is even and equal to $n=2m$, we may determine the points $y_2, \cdots, y_{2m-3}$ by the conditions

$$B_h = 0 \qquad (h = 2, \cdots, m-1), \tag{2.5}$$

and the analogous ones

$$C_i = 0 \qquad (i = m, \cdots, 2m-3). \tag{2.6}$$

3. **Conditions of immovability.** If the points $x, y_1, \cdots, y_{n-2}$ and the Laplace transformed points x_{-1}, x_1 at the point x of the net N_x, given by equations

$$x_{-1} = x_u - bx, \qquad x_1 = x_v - ax, \tag{3.1}$$

are used as the vertices of the pyramid of reference with unit point suitably chosen, then any point P in the space given by an expression of the form

$$P \equiv \xi_1 x + \xi_2 x_{-1} + \xi_3 x_1 + \sum_{i=1}^{n-2} \xi_{i+3} y_i \tag{3.2}$$

has local coordinates proportional to $\xi_1, \cdots, \xi_{n+1}$. Differentiating the expression (3.2) and making use of the relations $P_u=0$, $P_v=0$, we can easily obtain the following conditions of immovability and the analogous ones obtainable therefrom by the substitution (1.5):

$$\begin{aligned}
\xi_{1u} &= -b\xi_1 - H\xi_3 - (A_1 + aE_1)\xi_4 - \sum_{i=5}^{n+1} (A_{i-3} + bB_{i-3})\xi_i, \\
\xi_{2u} &= -\xi_1 + (b - \beta_2)\xi_2 - \sum_{i=5}^{n+1} B_{i-3}\xi_i, \\
\xi_{3u} &= -b\xi_3 - E_1\xi_4, \\
\xi_{iu} &= -\overset{i-3}{L}_{i-3}\xi_i - \overset{i-2}{L}_{i-3}\xi_{i+1} \qquad (i = 4, \cdots, n), \\
\xi_{n+1,u} &= -\overset{2}{p}_{n-2}\xi_2 - \overset{n-2}{L}_{n-2}\xi_{n+1}.
\end{aligned} \tag{3.3}$$

Let π be a fixed hyperplane in the space S_n, which does not pass through the point x and has the equation

$$\pi \equiv \xi_1 + \sum_{i=2}^{n+1} \lambda_i \xi_i = 0, \tag{3.4}$$

where $\lambda_2, \cdots, \lambda_{n+1}$ are functions of u, v. In order that the hyperplane π be fixed in the space S_n, it is necessary and sufficient that there be two functions k_1, k_2 of u, v such that

$$\pi_u = k_1 \pi, \qquad \pi_v = k_2 \pi, \tag{3.5}$$

provided that the derivatives of $\xi_1, \cdots, \xi_{n+1}$ in equations (3.5) be substituted from equations (3.3) and the analogous ones. Comparison of the coefficients of the corresponding terms in the first of equations (3.5) thus derived and elimination of k_1, k_2 yield[4]

$$\begin{aligned}
\lambda_{2u} &= (\beta_2 - 2b)\lambda_2 - \lambda_2^2 + p^{2}_{n-2}\lambda_{n+1}, \\
\lambda_{3u} &= H - \lambda_2\lambda_3, \\
\lambda_{4u} &= A_1 + aE_1 + E_1\lambda_3 + (L^{1}_{1} - b)\lambda_4 - \lambda_2\lambda_4, \\
\lambda_{iu} &= A_{i-3} + bB_{i-3} + B_{i-3}\lambda_2 + L^{i-3}_{i-4}\lambda_{i-1} + (L^{i-3}_{i-3} - b)\lambda_i - \lambda_2\lambda_i \\
&\qquad\qquad (i = 5, \cdots, n+1).
\end{aligned} \tag{3.6}$$

An analogous set of equations can be obtained from the second of equations (3.5) or by the substitution (1.5).

4. **Laplace transformed nets in a fixed hyperplane derived from the net N_x.** Now we consider the points M, $\overline{M}$ where the fixed hyperplane π cuts the u-, v-tangents of the net N_x at the point x respectively. By means of equations (1.2), (1.3), (3.1), (3.4), (3.6), and the substitution (1.5), a simple calculation gives the following equations

$$\begin{aligned}
M &= -(b + \lambda_2)x + x_u, \\
M_u &= [\alpha_2 - b_u + (2b - \beta_2)\lambda_2 + \lambda_2^2 - p^{2}_{n-2}\lambda_{n+1}]x \\
&\quad - (b - \beta_2 + \lambda_2)x_u + p^{2}_{n-2}y_{n-2}, \\
M_v &= (-ab + \lambda_2\lambda_3)x + ax_u - \lambda_2 x_v, \\
M_{uv} &= [a\alpha_2 - a_u b - ab_u + (ab - a_u)\lambda_2 + (\beta_2 - 2b)\lambda_2\lambda_3 \\
&\quad + p^{2}_{n-2}\lambda_3\lambda_{n+1} - 2\lambda_2^2\lambda_3]x + (a\beta_2 - ab + a_u - a\lambda_2 + \lambda_2\lambda_3)x_u \\
&\quad + [(b - \beta_2)\lambda_2 + \lambda_2^2 - p^{2}_{n-2}\lambda_{n+1}]x_v + a p^{2}_{n-2}y_{n-2},
\end{aligned} \tag{4.1}$$

from which it is easily seen that the coordinates of the point M satisfy the equation of Laplace

$$M_{uv} = \mathcal{C}M + \mathcal{A}M_u + \mathcal{B}M_v, \tag{4.2}$$

where

$$\mathcal{A} = a,$$
$$(4.3)\qquad \mathcal{B} = \beta_2 - b - \lambda_2 + p^2_{n-2}\lambda_{n+1}/\lambda_2,$$
$$\mathcal{C} = a(b - \beta_2) + a_u + a\lambda_2 + \lambda_2\lambda_3 - ap^2_{n-2}\lambda_{n+1}/\lambda_2.$$

Thus the point M describes a conjugate net N_M in the fixed hyperplane π. The Laplace transformed points M_1, M_{-1} and the Laplace-Darboux invariants $\mathcal{H}$, $\mathcal{K}$ at the point M of this net N_M are given by the equations

$$(4.4)\qquad \begin{aligned} M_1 &= -\lambda_2\overline{M},\\ M_{-1} &= bp^2_{n-2}\frac{\lambda_{n+1}}{\lambda_2}x - p^2_{n-2}\frac{\lambda_{n+1}}{\lambda_2}x_u + p^2_{n-2}y_{n-2},\\ \mathcal{H} &= \lambda_2\lambda_3,\\ \mathcal{K} &= p^2_{n-2}K\frac{\lambda_{n+1}}{\lambda_2^2} - \frac{p^2_{n-2}}{\lambda_2}(D_{n-2} + bF_{n-2}). \end{aligned}$$

By means of the substitution (1.5) we can write out immediately the similar equations for the point $\overline{M}$. Combining the above results and the similar one for a conjugate net in an ordinary space(2) we arrive at Theorem 1.

From equations (3.4), (4.4), it is easily seen that if every point x_{-1} lies in the fixed hyperplane π, then the net N_M coincides with the net N_{-1}, described by the point x_{-1}, which reduces to a u-curve. Similarly, if every point x_1 lies in the fixed hyperplane π, then $H=0$, $\mathcal{H}=0$, and therefore the first Laplace transformed net of the net N_M coincides with the net N_1, described by the point x_1, which reduces to a v-curve. In each of these two special cases, the fixed hyperplane π is uniquely determined for the net N_x.

Finally, it should be noted that the net N_M has equal and nonzero Laplace-Darboux invariants $\mathcal{H}$, $\mathcal{K}$ if and only if

$$(4.5)\qquad \lambda_2^3\lambda_3 = p^2_{n-2}K\lambda_{n+1} - p^2_{n-2}(D_{n-2} + bF_{n-2})\lambda_2.$$

5. **A conjugate net associated with the net N_x in a fixed linear space S_{n-2} of $n-2$ dimensions.** In this section we consider in the space S_n a fixed linear subspace S_{n-2} of $n-2$ dimensions determined by two fixed hyperplanes given respectively by equations (3.4) and

$$(5.1)\qquad \xi_1 + \sum_{i=2}^{n+1}\mu_i\xi_i = 0,$$

where $\mu_2, \cdots, \mu_{n+1}$ are functions of u, v. In order that the second hyperplane (5.1) be fixed in the space S_n it is necessary and sufficient that $\mu_2, \cdots, \mu_{n+1}$

(2) C. C. Hsiung, *Conjugate nets in three- and four-dimensional spaces*, to appear in Duke Math. J.

satisfy a system of equations similar to (3.6) and the analogous ones obtainable by the substitution (1.5). From these two systems of equations it is easily seen that for a general net $N_x \lambda_2 \neq \mu_2$, as otherwise the two hyperplanes (3.4), (5.1) would be coincident. Similarly, $\lambda_3 \neq \mu_3$.

The tangent plane of the net N_x at the point x intersects the fixed subspace S_{n-2} in a point T whose coordinates are given by

$$
\begin{aligned}
T = {} & [a(\lambda_2 - \mu_2) - b(\lambda_3 - \mu_3) + \lambda_2\mu_3 - \lambda_3\mu_2]x \\
& + (\lambda_3 - \mu_3)x_u - (\lambda_2 - \mu_2)x_v.
\end{aligned}
\tag{5.2}
$$

Differentiating the expression (5.2) and making use of certain equations obtained in §3, we may show that the coordinates of the point T satisfy the equation of Laplace

$$
T_{uv} = \mathcal{C}^*T + \mathcal{A}^*T_u + \mathcal{B}^*T_v,
\tag{5.3}
$$

where we have placed

$$
\begin{aligned}
\mathcal{A}^* &= \gamma_2 - a - (\lambda_3 + \mu_3) + \frac{\overset{2}{q}_1(\lambda_4 - \mu_4)}{\lambda_3 - \mu_3}, \\
\mathcal{B}^* &= \beta_2 - b - (\lambda_2 + \mu_2) + \frac{\overset{2}{p}_{n-2}(\lambda_{n+1} - \mu_{n+1})}{\lambda_2 - \mu_2}, \\
\mathcal{C}^* &= a_u + b_v - c - 2ab + a\beta_2 + b\gamma_2 - \beta_2\gamma_2 + (\gamma_2 - a)(\lambda_2 + \mu_2) \\
&\quad + (\beta_2 - b)(\lambda_3 + \mu_3) - \lambda_2\mu_3 - \lambda_3\mu_2 \\
&\quad + \frac{\overset{2}{q}_1(\lambda_4 - \mu_4)}{\lambda_3 - \mu_3}(b - \beta_2 + \lambda_2 + \mu_2) \\
&\quad + \frac{\overset{2}{p}_{n-2}(\lambda_{n+1} - \mu_{n+1})}{\lambda_2 - \mu_2}(a - \gamma_2 + \lambda_3 + \mu_3) \\
&\quad - \overset{2}{p}_{n-2}\overset{2}{q}_1 \frac{(\lambda_4 - \mu_4)(\lambda_{n+1} - \mu_{n+1})}{(\lambda_2 - \mu_2)(\lambda_3 - \mu_3)}.
\end{aligned}
\tag{5.4}
$$

Thus we obtain Theorem 2.

The Laplace-Darboux invariants $\mathcal{H}^*$, $\mathcal{K}^*$ of the net N_T at the point T are given by the equations

$$
\begin{aligned}
\mathcal{H}^* &= \frac{\overset{2}{q}_1}{(\lambda_3 - \mu_3)^2}(\lambda_2 - \mu_2)(\lambda_3\mu_4 - \lambda_4\mu_3), \\
\mathcal{K}^* &= \frac{\overset{2}{p}_{n-2}}{(\lambda_2 - \mu_2)^2}(\lambda_3 - \mu_3)(\lambda_2\mu_{n+1} - \lambda_{n+1}\mu_2).
\end{aligned}
\tag{5.5}
$$

It is obvious that $\mathcal{H}^* = 0$ if, and only if, the line x_1y_1 corresponding to each point x of the net N_x intersects the fixed subspace S_{n-2}. We can easily show that in this case the termination of the Laplace sequence determined by the net N_T in the fixed subspace S_{n-2} is that of Laplace, that is, its first Laplace transformed net reduces to a v-curve. Similarly, the minus-first Laplace transformed net of the net N_T reduces to a u-curve in case the line $x_{-1}y_{n-2}$ corresponding to each point x of the net N_x intersects the fixed subspace S_{n-2}. Moreover, the Laplace sequence determined by the net N_T in the fixed subspace S_{n-2} terminates in both directions after one transformation of Laplace according to the case of Laplace if, and only if, the lines x_1y_1, $x_{-1}y_{n-2}$ corresponding to each point x of the net N_x both intersect the fixed subspace S_{n-2}.

Finally, from equations (5.5) it follows immediately that the net N_T has equal and nonzero Laplace-Darboux invariants $\mathcal{H}^*$, $\mathcal{K}^*$ in case

$$\overset{2}{p}_{n-2}(\lambda_3 - \mu_3)^{3}(\lambda_2\mu_{n+1} - \lambda_{n+1}\mu_2) = \overset{2}{q}_1(\lambda_2 - \mu_2)^{3}(\lambda_3\mu_4 - \lambda_4\mu_3). \tag{5.6}$$

6. Conjugate nets with equal and nonzero Laplace-Darboux invariants. It is known that as u, v vary the Laplace transformed points x_{-1}, X_1 given by equations (3.1) at the point x of the conjugate net N_x in the space S_n generate two surfaces S_{-1}, S_1, on which the parametric curves also form two conjugate nets N_{-1}, N_1. As usual, we call the surfaces S_{-1}, S_1 and the nets N_{-1}, N_1 respectively, the minus-first and first Laplace transformed surfaces and nets of N_{-x}. In this section we shall first find the power series expansions of the surfaces S_{-1}, S_1 at the points x_{-1}, x_1.

From the system (1.2), equations (1.3), (1.4), (3.1), and the substitution (1.5) by differentiation and substitution, any derivative of x_{-1} can be expressed as a linear combination of x, x_u, x_v, y_1, $\cdots$, y_{n-2}. In particular, one obtains

$$\begin{aligned} x_{-1u} &= (\alpha_2 - b_u)x + (\beta_2 - b)x_u + \overset{2}{p}_{n-2}y_{n-2},\\ x_{-1v} &= (c - b_v)x + ax_u,\\ x_{-1uu} &= (\alpha_3 - b_{uu} - b\alpha_2)x + (\beta_3 - 2b_u - b\beta_2)x_u\\ &\quad + \overset{3}{p}_{n-3}y_{n-3} + (\overset{3}{p}_{n-2} - b\overset{2}{p}_{n-2})y_{n-2},\\ x_{-1uv} &= (c_u - b_{uv} + ac)x + (a_u - b_v + c + a\beta_2)x_u + a\overset{2}{p}_{n-2}y_{n-2},\\ x_{-1vv} &= (c_v - b_{vv} + ac)x + (a_v + a^2)x_u + Kx_v. \end{aligned} \tag{6.1}$$

The coordinates X, where

$$X = x_{-1}(u + \Delta u, v + \Delta v),$$

of any point X near the point x_{-1} on the surface S_{-1} can be represented by the Taylor's expansion as power series in the increments Δu, Δv corresponding to displacement on the surface S_{-1} from the point x_{-1} to the point X:

$$X = x_{-1} + x_{-1u}\Delta u + x_{-1v}\Delta_v$$
$$(6.2)\qquad + \frac{1}{2}(x_{-1uu}\Delta u^2 + 2x_{-1uv}\Delta u\Delta v + x_{-1vv}\Delta v^2) + \cdots.$$

Substituting the expressions (6.1) in equation (6.2) we obtain the local coordinates $\xi_1, \cdots, \xi_{n+1}$ of the point X:

$$(6.3)\qquad \begin{aligned} \xi_1 &= K\Delta v + \cdots, \\ \xi_2 &= 1 + (\beta_2 - b)\Delta u + a\Delta v + \cdots, \\ \xi_3 &= \frac{1}{2} K\overset{2}{\Delta v} + \cdots, \\ &\cdots\cdots\cdots\cdots\cdots\cdots, \\ \xi_n &= \frac{1}{2} \overset{3}{p}_{n-3} \overset{2}{\Delta_u} + \cdots, \\ \xi_{n+1} &= \overset{2}{p}_{n-2}\Delta u + \frac{1}{2}(\overset{3}{p}_{n-2} - b\overset{2}{p}_{n-2})\overset{2}{\Delta u} + a\overset{2}{p}_{n-2}\Delta u\Delta v + \cdots, \end{aligned}$$

where the unwritten coordinates are of at least third degree in Δu, Δv. By means of (6.3) it follows that a hyperquadric with the general equation

$$(6.4)\qquad \sum_{i,k=1}^{n+1} a_{ik}\xi_i\xi_k = 0 \qquad (a_{ik} = a_{ki})$$

has second order contact with the surface S_{-1} at the point x_{-1} in case

$$(6.5)\qquad \begin{gathered} a_{12} = a_{1,n+1} = a_{22} = a_{2,n+1} = 0, \qquad a_{23} = -Ka_{11}, \\ a_{n+1,n+1} = -\frac{\overset{3}{p}_{n-3}}{(\overset{2}{p}_{n-2})^2} a_{2n}. \end{gathered}$$

Similarly, the hyperquadric (6.4) has second order contact with the surface S_1 at the point x_1 if, and only if,

$$(6.6)\qquad \begin{gathered} a_{13} = a_{14} = a_{33} = a_{34} = 0, \qquad a_{23} = -Ha_{11}, \\ a_{44} = -\frac{\overset{3}{q}_2}{(\overset{2}{q}_1)^2} a_{35}. \end{gathered}$$

Combining the conditions (6.5), (6.6) we thus reach Theorem 3[3].

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(3) This theorem was formerly obtained by the author for the case $n=4$, loc. cit. (see footnote 1). However, it is not true for a general conjugate net with equal and nonzero Laplace-Darboux invariants in ordinary space; see the author's paper, *New geometrical characterizations of some special conjugate nets*, Duke Math. J. vol. 12 (1945) p. 252.