A GENERAL THEORY OF CONJUGATE NETS IN
PROJECTIVE HYPERSPACE

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Introduction. In a previous paper(1), the author has established a theory of
the projective differential geometry of conjugate nets in a linear space \( S_4 \) of
four dimensions. The purpose of the present paper is to extend the theory to
a general linear space \( S_n \) (\( n \geq 4 \)).

In §1 a completely integrable system of linear homogeneous partial dif-
fferential equations, together with its integrability conditions, is introduced
by a purely geometric method defining a conjugate net \( N_z \) in the space \( S_n \)
except for a projective transformation. A canonical form of the system of
differential equations is obtained in §2 by a geometric determination.

In §3 we deduce the conditions of immovability for a point and a hyper-
plane in the space \( S_n \) relative to an invariant local pyramid of reference asso-
ciated with a point \( x \) of the conjugate net \( N_z \).

§§4, 5, 6 are devoted to proving the following theorems respectively.

Theorem 1. In a linear space \( S_n \) of \( n \geq 3 \) dimensions let \( N_z \) be a conjugate
net and \( \pi \) be a fixed hyperplane; then the points \( M, \bar{M} \) of intersection of the fixed
hyperplane \( \pi \) and the two tangents at a point \( x \) of the net \( N_z \) describe two conju-
gate nets \( N_M, N_{\bar{M}} \) in the hyperplane \( \pi \) respectively, and one of the two nets \( N_M, \)
\( N_{\bar{M}} \) is a Laplace transformed net of the other.

Theorem 2. In a linear space \( S_n \) of \( n \geq 4 \) dimensions let \( N_z \) be a conjugate
net and \( S_{n-2} \) be a fixed linear subspace of \( n - 2 \) dimensions; then the point \( T \) of
intersection of the fixed subspace \( S_{n-2} \) and the tangent plane at a point \( x \) of the
surface sustaining the net \( N_z \) describes a conjugate net \( N_T \) in the subspace \( S_{n-2} \).

Theorem 3. Conjugate nets with equal and nonzero Laplace-Darboux in-
variants in a linear space \( S_n \) of \( n \geq 4 \) dimensions are characterized by the
property that at each point \( x \) of any one of them there exists a proper hyper-
quadric (and therefore \( \propto n(n+3)/2 \) such hyperquadrics) having second order
contact at the Laplace transformed points \( x_{-1}, x_1 \) of the point \( x \) with both
Laplace transformed surfaces \( S_{-1}, S_1 \) of the net \( N_z \), respectively.

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(1) C. C. Hsiung, Projective theory of surfaces and conjugate nets in four-dimensional space,
three-dimensional space has been established in a similar way by E. P. Lane; see his book,
A treatise on projective differential geometry, University of Chicago Press, 1942, Chap. VIII.
1. **Differential equations and integrability conditions.** Let us consider a conjugate net $N_x$ with parameters $u, v$ in a linear space $S_n$ of $n \geq 4$ dimensions so that the homogeneous projective coordinates

$$x^{(1)}, \ldots, x^{(n+1)}$$

of a nonsingular point $x$ on the surface $S$ sustaining the net $N_x$ are given as analytic functions of the two independent variables $u, v$ by equations of the form

$$x = x(u, v).$$

The osculating linear space $S_k(u)$ of $k = 2, \ldots, n-1$ dimensions of the parametric curve $u$ and the osculating linear space $S_{n-k+1}^{(v)}$ of $n-k+1$ dimensions of the parametric curve $v$ at the point $x$ of the net $N_x$ intersect in a line $l_{k-1}$. Let us select on the lines $l_1, \ldots, l_{n-2}$ respectively $n-2$ points $y_1, \ldots, y_{n-2}$, distinct from the point $x$, and suppose that the coordinates $y_i$ of the point $y_i$ ($i = 1, \ldots, n-2$) are functions of $u, v$. Then it can be shown that the coordinates $y_i$ of the corresponding points $y_i$ ($i = 1, \ldots, n-2$) satisfy a system of linear homogeneous partial differential equations of the form

$$
\frac{\partial^i x}{\partial u^i} = \alpha_i x + \beta_i x_u + \sum_{j=n-i}^{n-2} \rho_j y_j, \\
\frac{\partial^i x}{\partial v^i} = \delta_i x + \gamma_i x_v + \sum_{k=1}^{i-1} \psi_k y_k \quad (i = 2, \ldots, n-1),
$$

in which subscripts indicate partial differentiation and the coefficients are scalar functions of $u, v$. The first of these equations is merely the Laplace equation for the parametric conjugate net $N_x$.

By using equations (1.2) it is easily seen that the derivatives $y_{iu}$ can be written in the form

$$
\begin{align*}
y_{iu} &= A_1 x + B_1 x_u + E_1 x_v + L_1 y_1, \\
y_{iu} &= A_1 x + B_1 x_u + L_{i-1} y_{i-1} + L_i y_i, \quad (i = 2, \ldots, n-2).
\end{align*}
$$

In particular, by actual calculation one obtains

$$
\begin{align*}
q_1 A_1 &= c_x + a c + b \delta_2 - c \gamma_2 - \beta_2 u, & p_{n-2} A_{n-2} &= \alpha_3 - \alpha_2 \beta_2 - \alpha_2 u, \\
q_1 B_1 &= a_v + a^2 - a \gamma_2 - \delta_2, & p_{n-2} B_{n-2} &= \beta_3 - \alpha_2 - \beta_2 u - \beta_2, \\
q_1 E_1 &= b_v + a b + c - \gamma_2 u, & p_{n-2} E_{n-2} &= \beta_3 - a \gamma_2 - \beta_2, \\
L_1 &= b - (\log q_1)_u; & p_{n-2} L_{n-2} &= \beta_3 - \beta_2 - \beta_2 - p_{n-2}.\end{align*}
$$

Analogous expressions for $y_{iv}$ can be written by making the substitution...
where $H, K$ are the Laplace-Darboux invariants defined by the respective formulas

$$H = c + ab - a_u,$$

$$K = c + ab - b_v,$$

and the $\xi$'s will be defined in §3.

The integrability conditions of equations (1.2) are found by the usual method from the equations

$$\frac{\partial}{\partial u} \left( \frac{\partial^{n-i} x}{\partial u^{n-i}} \right) = \frac{\partial}{\partial v} \left( \frac{\partial^{i+1} x}{\partial v^{i+1}} \right),$$

$$\frac{\partial}{\partial u} \left( \frac{\partial^{i+1} x}{\partial v^{i+1}} \right) = \frac{\partial}{\partial v} \left( \frac{\partial^{n-i} x}{\partial u^{n-i}} \right),$$

$$\left( \begin{array}{c}
(y_{iu})_{u} \\
(y_{iv})_{v}
\end{array} \right) = \left( \begin{array}{c}
(y_{iu})_{v} \\
(y_{iv})_{u}
\end{array} \right)$$

and the fact that the points $x, x_u, x_v, y_1, \ldots, y_{n-2}$ are linearly independent. The result is given by the following equations and the analogous ones obtainable therefrom by the substitution (1.5):

$$A_i = \frac{1}{p_i} \left( \alpha_{n-i+1} - \alpha_{n-i} - \alpha_2 \beta_{n-i} - \sum_{j=i+1}^{n-2} p_j A_j \right)$$

$$B_i = \frac{1}{p_i} \left( \beta_{n-i+1} - \beta_{n-i} - \beta_2 \beta_{n-i} - \sum_{j=i+1}^{n-2} p_j B_j \right)$$

$$L_{i-1} = \frac{p_{i-1}^{n-i+1}}{p_i^{n-i}}, \quad L_i = b - (\log q_{i+1})_u$$

$$p_i^{n-i+1} = p_i^{n-i} + p_i^{n-i} L_i + p_{i+1}^{n-i} L_{i+1}$$

$$p_{n-2}^{n-i+1} = p_{n-2}^{n-i} \beta_{n-i} + p_{n-2}^{n-i} u + p_{n-2}^{n-i} L_{n-2}$$
\[\gamma_{i+1,u} + b\gamma_{i+1} + q_1 E_1 = \frac{\partial^i b}{\partial v^i} + ib \frac{\partial^{i-1} a}{\partial v^{i-1}} + \frac{i}{\partial v^{i-1}} \]

\[+ \sum_{j=1}^{i} C_{i,j} \frac{\partial^{i-j} b}{\partial v^{i-j}} \gamma_{i+1} \]

\[+ \sum_{i=2}^{i} C_{i,j} \left[ \frac{\partial^{i-j} c}{\partial v^{i-j}} \gamma_j + \frac{\partial^{i-j} a}{\partial v^{i-j}} \left( \gamma_{j,u} + b\gamma_j + q_1 E_1 \right) \right] \]

\[(i = 2, \ldots, n - 2),\]

\[q_{ku} + i^{i+1} q_k L_k + q_{k+1} L_{k+1} = \sum_{j=k}^{i} C_{i,j} \frac{\partial^{i-j} b}{\partial v^{i-j}} q_k \]

\[+ \sum_{i=k+1}^{i} C_{i,j} \frac{\partial^{i-j} c}{\partial v^{i-j}} q_k + C_{i,k+2} \frac{\partial^{i-j} k}{\partial v^{i-j}} \left( q_{ku} + q_{k+1} L_{k+1} \right) \]

\[(i = 3, \ldots, n - 2; k = 1, \ldots, i - 2),\]

where \(C_{i,j}\) denotes the number of combinations of \(i\) different things taken \(j\) at a time;

\[A_{i,v} + cB_i + \delta_2 E_1 + D_1 L_{i-1} = D_{i,u} + cC_i + A_1 M_{i-1} + A_2 M_i, \]

\[A_{i,v} + cB_i + D_{i-1} L_{i-1} + D_i L_{i} = D_{i,u} + cC_i + A_1 M_i + A_{i+1} M_{i+1} \]

\[(i = 2, \ldots, n - 3),\]

\[A_{n-2,v} + cB_{n-2} + D_{n-2} L_{n-3} + D_{n-2} L_{n-2} = D_{n-2,u} + \alpha_2 F_{n-2} + \]

\[+ cC_{n-2} + A_{n-2} M_{n-2}, \]

\[B_{i,v} + aB_i = D_i + aC_i + B_1 M_i + B_{i+1} M_{i+1} \]

\[(i = 1, \ldots, n - 3),\]

\[B_{n-2,v} + aB_{n-2} + F_{n-2} L_{n-2} = D_{n-2} + F_{n-2,u} + aB_{n-2} + B_{n-2} M_{n-2}, \]

\[(1.8) A_1 + bB_1 + E_{1,v} + \gamma_2 E_1 + C_1 L_1 = C_{1,u} + bC_1 + E_{1,1}, \]

\[A_{i,v} + bB_i + C_{i-1} L_{i-1} + C_i L_i = C_{i,u} + bC_i \]

\[(i = 2, \ldots, n - 2),\]

\[L_{i-1,v} + L_{i-1} M_{i-1} = L_{i-1} M_i \]

\[(i = 2, \ldots, n - 2),\]

\[L_{1,v} + q_1 E_1 = M_1^1 + L_1 M_{2,1} \]

\[L_{i,v} + L_{i-1} M_i = M_i^i + L_i M_{i+1}^i \]

\[(i = 2, \ldots, n - 3),\]

\[L_{n-2,v} + L_{n-2} M_{n-3} = L_{n-2} M_{n-2}, \]

\[L_{n-2,v} + L_{n-2} M_{n-2} = M_{n-2,u} + q_{n-2} F_{n-2}. \]

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Making use of the third of equations (1.4), the ninth, the tenth, and the thirteenth of equations (1.8), and the substitution (1.5) we obtain

\[
(a + \gamma_2 + \sum_{i=1}^{n-2} M_i^i, \quad (b + \beta_2 + \sum_{i=1}^{n-2} L_i^i).
\]

It follows that there exists a function \( \theta \) of \( u, v \) which is defined, except for an arbitrary additive constant, as a solution of the differential equations

\[
\theta_u = b + \beta_2 + \sum_{i=1}^{n-2} L_i^i, \quad \theta_v = a + \gamma_2 + \sum_{i=1}^{n-2} M_i^i.
\]

Accordingly, the following formula is valid:

\[
(x, x_u, x_v, y_1, \cdots, y_{n-2}) = e^{\theta},
\]

where a determinant is indicated by writing only a typical row within parentheses.

2. Canonical form of the differential equations. We now proceed to choose for the points \( y_1, \cdots, y_{n-2} \) particular covariant points on the lines \( l_1, \cdots, l_{n-2} \) respectively. To this end, at first we observe that the point \( X_1 \) defined by \( Y_1 = y_1 + kx, \) where \( k \) is a scalar function of \( u, v, \) is on the line \( l_1. \) When the point \( x \) varies along a curve \( C_1 \) of the family represented by the differential equation

\[
dv - \lambda du = 0,
\]

\( \lambda \) being a function of \( u, v, \) on the surface \( S, \) the point \( X_1 \) generates a curve \( C_{X_1} \) whose tangent at \( X_1 \) is determined by \( X_1 \) and the point \( X_1' \) given by

\[
X_1' = y_1u + y_1\lambda + k(x_u + x_\lambda) + k'x \quad (X_1' = dX_1/du, \cdots).
\]

Expressing \( X_1' \) as a linear combination of \( x, x_u, x_\lambda, y_1, y_2 \) by means of the first of equations (1.3) and the substitution (1.5), and equating to zero the coefficients of \( x_u, x_\lambda \) therein, we obtain two conditions on the functions \( k \) and \( \lambda \) which are necessary and sufficient that the tangent to the curve \( C_{X_1} \) at the point \( X_1 \) lies in the plane \( l_1l_2, \) namely,

\[
k + B_1 = 0, \quad E_1 + (C_1 + k)\lambda = 0.
\]

Similarly, we can also determine a unique point \( X_{n-2} \) on the line \( l_{n-2} \) and a unique curve of the family (2.1) such that as the point \( x \) varies along the curve the tangent to the locus of the point \( X_{n-2} \) at the point lies in the plane \( l_{n-2}l_{n-2}. \) If we choose these two points respectively for the points \( y_1 \) and \( y_{n-2}, \)

\[
B_1 = 0, \quad C_{n-2} = 0,
\]

and the differential equations of the two curves become
Finally, we can determine a unique point $X_h$ on the line $l_h$ ($h = 2, \cdots, n-3$) such that as the point $x$ varies along any curve, except the $v$-curve, on the surface $S$ the tangent to the locus of the point $X_h$ at the point is in the four-dimensional space determined by the lines $l_{h-1}, l_h, l_{h+1}$ and the $v$-tangent. If we choose this point to be the point $y_h$, then

\[ B_h = 0 \quad (h = 2, \cdots, n-3). \]

Hereafter it will be supposed that the differential equations (1.2) are in the canonical form for which the conditions (2.2), (2.4) are satisfied.

It should be noted that the above choice of the points $y_2, \cdots, y_{n-3}$ is not symmetric with respect to the parameters $u, v$. However if the dimension of the space $S_n$ is even and equal to $n = 2m$, we may determine the points $y_2, \cdots, y_{2m-3}$ by the conditions

\[ B_h = 0 \quad (h = 2, \cdots, m-1), \]

and the analogous ones

\[ C_i = 0 \quad (i = m, \cdots, 2m-3). \]

3. Conditions of immovability. If the points $x, y_1, \cdots, y_{n-2}$ and the Laplace transformed points $x_{-1}, x_1$ at the point $x$ of the net $N_x$, given by equations

\[ x_{-1} = x_u - bx, \quad x_1 = x_v - ax, \]

are used as the vertices of the pyramid of reference with unit point suitably chosen, then any point $P$ in the space given by an expression of the form

\[ P = \xi_1 x + \xi_2 x_{-1} + \xi_3 x_1 + \sum_{i=1}^{n-2} \xi_{i+3} y_i \]

has local coordinates proportional to $\xi_1, \cdots, \xi_{n+1}$. Differentiating the expression (3.2) and making use of the relations $P_u = 0, P_v = 0$, we can easily obtain the following conditions of immovability and the analogous ones obtainable therefrom by the substitution (1.5):

\[ \xi_{1u} = -b\xi_1 - H\xi_3 - (A_1 + aE_1)\xi_4 - \sum_{i=5}^{n+1} (A_{i-3} + bB_{i-3})\xi_i, \]

\[ \xi_{2u} = -\xi_1 + (b - \beta_2)\xi_3 - \sum_{i=5}^{n+1} B_{i-3}\xi_i, \]

\[ \xi_{3u} = -b\xi_3 - E_1\xi_4, \]

\[ \xi_{iu} = -L_{i-3}\xi_i - L_{i-3}\xi_{i+1} \quad (i = 4, \cdots, n), \]

\[ \xi_{n+1,u} = -p_{n-3}\xi_2 - L_{n-3}\xi_{n+1}. \]
Let $\pi$ be a fixed hyperplane in the space $S_n$, which does not pass through the point $x$ and has the equation

$$
(3.4) \quad \pi = \xi_1 + \sum_{i=2}^{n+1} \lambda_i \xi_i = 0,
$$

where $\lambda_2, \ldots, \lambda_{n+1}$ are functions of $u, v$. In order that the hyperplane $\pi$ be fixed in the space $S_n$, it is necessary and sufficient that there be two functions $k_1, k_2$ of $u, v$ such that

$$
(3.5) \quad \pi_u = k_1 \pi, \quad \pi_v = k_2 \pi,
$$

provided that the derivatives of $\xi_1, \ldots, \xi_{n+1}$ in equations (3.5) be substituted from equations (3.3) and the analogous ones. Comparison of the coefficients of the corresponding terms in the first of equations (3.5) thus derived and elimination of $k_1, k_2$ yield

$$
(3.6) \quad \begin{align*}
\lambda_{2u} &= (\beta_2 - 2b)\lambda_2 - \lambda_2^2 + \delta_{n-2}\lambda_{n+1}, \\
\lambda_{3u} &= H - \lambda_2 \lambda_3, \\
\lambda_{4u} &= A_1 + aE_1 + E_1 \lambda_3 + (L_1 - b)\lambda_4 - \lambda_2 \lambda_4, \\
\lambda_{iu} &= A_i - 3 + bB_{i-3} + B_{i-3} \lambda_2 + L_{i-3} \lambda_{i-1} + (L_{i-3} - b)\lambda_i - \lambda_2 \lambda_i
\end{align*}
$$

$(i = 5, \ldots, n + 1)$.

An analogous set of equations can be obtained from the second of equations (3.5) or by the substitution (1.5).

4. Laplace transformed nets in a fixed hyperplane derived from the net $N_x$. Now we consider the points $M, M$ where the fixed hyperplane $\pi$ cuts the $u$, $v$-tangents of the net $N_x$ at the point $x$ respectively. By means of equations (1.2), (1.3), (3.1), (3.4), (3.6), and the substitution (1.5), a simple calculation gives the following equations

$$
(4.1) \quad \begin{align*}
M &= - (b + \lambda_3) x + x_{uu}, \\
M_u &= [a \alpha_2 - b u + (2b - \beta_2) \lambda_2 + \lambda_2^2 - \delta_{n-2} \lambda_{n+1}] x \\
&\quad - (b - \beta_2 + \lambda_2)x_u + \delta_{n-2} y_{n-2}, \\
M_v &= (-ab + \lambda_2 \lambda_3) x + ax_u - \lambda_2 x_v, \\
M_{uv} &= [a \alpha_2 - a u b - ab u + (ab - a u) \lambda_2 + (\beta_2 - 2b) \lambda_2 \lambda_3 \\
&\quad + \delta_{n-2} \lambda_3 \lambda_{n+1} - 2\lambda_2 \lambda_3] x + (ab_2 - ab + a u - \lambda_2 + \lambda_2 \lambda_3)x_u \\
&\quad + [(b - \beta_3) \lambda_2 + \lambda_2^2 - \delta_{n-2} \lambda_{n+1}] x_v + a \delta_{n-2} y_{n-2},
\end{align*}
$$

from which it is easily seen that the coordinates of the point $M$ satisfy the equation of Laplace

$$
(4.2) \quad M_{uu} = CM + AM_u + BM_v,
$$
where
\begin{align*}
\mathcal{A} &= a, \\
\mathcal{B} &= \beta_2 - b - \lambda_3 + \rho_{n-2}^2 \lambda_{n+1}/\lambda_2, \\
\mathcal{C} &= a(b - \beta_2) + a_u + a\lambda_2 + \lambda_2\lambda_3 - a\rho_{n-2}^2 \lambda_{n+1}/\lambda_2. 
\end{align*}

Thus the point \( M \) describes a conjugate net \( N_M \) in the fixed hyperplane \( \pi \). The Laplace transformed points \( M_x, M_{-x} \) and the Laplace-Darboux invariants \( \mathcal{K}, \mathcal{K}_x \) at the point \( M \) of this net \( N_M \) are given by the equations
\begin{align}
M_x &= -\lambda_2 M, \\
M_{-x} &= b\rho_{n-2} \frac{\lambda_{n+1}}{\lambda_2} x - \rho_{n-2}^2 \frac{\lambda_{n+1}}{\lambda_2} x_u + \rho_{n-2}^2 y_{n-2}, \\
\mathcal{K} &= \lambda_2 \lambda_3, \\
\mathcal{K}_x &= \rho_{n-2}^2 \lambda_{n+1} \frac{\lambda_{n+1}}{\lambda_2} - \rho_{n-2}^2 (D_{n-2} + bF_{n-2}).
\end{align}

By means of the substitution (1.5) we can write out immediately the similar equations for the point \( \overline{M} \). Combining the above results and the similar one for a conjugate net in an ordinary space we arrive at Theorem 1.

From equations (3.4), (4.4), it is easily seen that if every point \( x_{-1} \) lies in the fixed hyperplane \( \pi \), then the net \( N_M \) coincides with the net \( N_{-1} \), described by the point \( x_{-1} \), which reduces to a \( u \)-curve. Similarly, if every point \( x_1 \) lies in the fixed hyperplane \( \pi \), then \( H = 0, \mathcal{K} = 0 \), and therefore the first Laplace transformed net of the net \( N_M \) coincides with the net \( N_1 \), described by the point \( x_1 \), which reduces to a \( v \)-curve. In each of these two special cases, the fixed hyperplane \( \pi \) is uniquely determined for the net \( N_x \).

Finally, it should be noted that the net \( N_M \) has equal and nonzero Laplace-Darboux invariants \( \mathcal{K}, \mathcal{K}_x \) if and only if
\begin{equation}
\lambda_2^2 \lambda_3 = \rho_{n-2}^2 K\lambda_{n+1} - \rho_{n-2}^2 (D_{n-2} + bF_{n-2})\lambda_2.
\end{equation}

5. A conjugate net associated with the net \( N_x \) in a fixed linear space \( S_{n-2} \) of \( n - 2 \) dimensions. In this section we consider in the space \( S_n \) a fixed linear subspace \( S_{n-2} \) of \( n - 2 \) dimensions determined by two fixed hyperplanes given respectively by equations (3.4) and
\begin{equation}
\xi_1 + \sum_{i=2}^{n+1} \mu_i \xi_i = 0,
\end{equation}
where \( \mu_2, \ldots, \mu_{n+1} \) are functions of \( u, v \). In order that the second hyperplane (5.1) be fixed in the space \( S_n \) it is necessary and sufficient that \( \mu_2, \ldots, \mu_{n+1} \)

satisfy a system of equations similar to (3.6) and the analogous ones obtainable by the substitution (1.5). From these two systems of equations it is easily seen that for a general net \( N \lambda_2 \neq \mu_2 \), as otherwise the two hyperplanes (3.4), (5.1) would be coincident. Similarly, \( \lambda_3 \neq \mu_3 \).

The tangent plane of the net \( N \) at the point \( x \) intersects the fixed subspace \( S_{n-2} \) in a point \( T \) whose coordinates are given by

\[
T = [a(\lambda_2 - \mu_2) - b(\lambda_3 - \mu_3) + \lambda_2 \mu_3 - \lambda_3 \mu_2]x
+ (\lambda_3 - \mu_3)x_u - (\lambda_2 - \mu_2)x_v.
\]

Differentiating the expression (5.2) and making use of certain equations obtained in §3, we may show that the coordinates of the point \( T \) satisfy the equation of Laplace

\[
T_{uv} = C^*T + A^*T_u + B^*T_v,
\]

where we have placed

\[
\begin{align*}
\mathcal{A}^* &= \gamma_2 - a - (\lambda_3 + \mu_3) + \frac{q_1^2(\lambda_4 - \mu_4)}{\lambda_3 - \mu_3}, \\
\mathcal{B}^* &= \beta_2 - b - (\lambda_2 + \mu_2) + \frac{p_{n-2}(\lambda_{n+1} - \mu_{n+1})}{\lambda_2 - \mu_2}, \\
\mathcal{C}^* &= a_u + b_v + c - 2ab + a\beta_2 + b\gamma_2 - \beta_2\gamma_2 + (\gamma_2 - a)(\lambda_2 + \mu_2) \\
&+ (\beta_2 - b)(\lambda_3 + \mu_3) - \lambda_2 \mu_3 - \lambda_3 \mu_2 \\
&+ \frac{q_1^2(\lambda_4 - \mu_4)}{\lambda_3 - \mu_3} (b - \beta_2 + \lambda_2 + \mu_2) \\
&+ \frac{p_{n-2}(\lambda_{n+1} - \mu_{n+1})}{\lambda_2 - \mu_2} (a - \gamma_2 + \lambda_3 + \mu_3) \\
&- \frac{p_{n-2}^2}{\lambda_2 - \mu_2} (\lambda_4 - \mu_4)(\lambda_{n+1} - \mu_{n+1}) \\
&- \frac{p_{n-2}^2 q_1}{(\lambda_2 - \mu_2)(\lambda_3 - \mu_3)}.
\end{align*}
\]

Thus we obtain Theorem 2.

The Laplace-Darboux invariants \( \mathcal{C}^* \), \( \mathcal{K}^* \) of the net \( N \) at the point \( T \) are given by the equations

\[
\begin{align*}
\mathcal{C}^* &= \frac{q_1^2}{(\lambda_3 - \mu_3)^2} (\lambda_2 - \mu_2)(\lambda_3 \mu_4 - \lambda_4 \mu_3), \\
\mathcal{K}^* &= \frac{p_{n-2}^2}{(\lambda_2 - \mu_2)^2} (\lambda_3 - \mu_3)(\lambda_2 \mu_{n+1} - \lambda_{n+1} \mu_2).
\end{align*}
\]

It is obvious that \( \mathcal{C}^* = 0 \) if, and only if, the line \( x_1y_1 \) corresponding to each
point $x$ of the net $N_x$ intersects the fixed subspace $S_{n-2}$. We can easily show that in this case the termination of the Laplace sequence determined by the net $N_T$ in the fixed subspace $S_{n-2}$ is that of Laplace, that is, its first Laplace transformed net reduces to a $v$-curve. Similarly, the minus-first Laplace transformed net of the net $N_T$ reduces to a $u$-curve in case the line $x_iy_{n-2}$ corresponding to each point $x$ of the net $N_x$ intersects the fixed subspace $S_{n-2}$. Moreover, the Laplace sequence determined by the net $N_T$ in the fixed subspace $S_{n-2}$ terminates in both directions after one transformation of Laplace according to the case of Laplace if, and only if, the lines $x_1y_1, x_{i-1}y_{n-2}$ corresponding to each point $x$ of the net $N_x$ both intersect the fixed subspace $S_{n-2}$.

Finally, from equations (5.5) it follows immediately that the net $N_T$ has equal and nonzero Laplace-Darboux invariants $\mathcal{K}^*, \mathcal{K}'^*$ in case

$$p_{n-2}^2(\lambda_3 - \mu_3)(\lambda_2\mu_{n+1} - \lambda_{n+1}\mu_2) = q_1(\lambda_2 - \mu_2)^3(\lambda_3 \mu_4 - \lambda_4 \mu_3).$$

6. Conjugate nets with equal and nonzero Laplace-Darboux invariants.

It is known that as $u, v$ vary the Laplace transformed points $x_{-1}, X_1$ given by equations (3.1) at the point $x$ of the conjugate net $N_x$ in the space $S_n$ generate two surfaces $S_{-1}, S_1$, on which the parametric curves also form two conjugate nets $N_{-1}, N_1$. As usual, we call the surfaces $S_{-1}, S_1$ and the nets $N_{-1}, N_1$ respectively, the minus-first and first Laplace transformed surfaces and nets of $N_x$. In this section we shall first find the power series expansions of the surfaces $S_{-1}, S_1$ at the points $x_{-1}, x_1$.

From the system (1.2), equations (1.3), (1.4), (3.1), and the substitution (1.5) by differentiation and substitution, any derivative of $x_{-1}$ can be expressed as a linear combination of $x, x_u, x_v, y_1, \ldots, y_{n-2}$. In particular, one obtains

$$x_{-1u} = (\alpha_2 - b_u)x + (\beta_2 - b)x_u + p_{n-2}^2y_{n-2},$$
$$x_{-1v} = (c - b_v)x + ax_u,$$
$$x_{-1uu} = (\alpha_3 - b_{uu} - b\alpha_2)x + (\beta_3 - 2b_u - b\beta_2)x_u + p_{n-3}^3y_{n-3} + (p_{n-2}^3 - b\beta_{n+2})y_{n-2},$$
$$x_{-1uv} = (c_u - b_{uv} + ac)x + (a_u - b_v + c + a\beta_2)x_u + a\beta_{n+2}^2y_{n-2},$$
$$x_{-1vv} = (c_v - b_{vv} + ac)x + (a_v + a^2)x_u + Kx_v.\quad (6.1)$$

The coordinates $X$, where

$$X = x_{-1}(u + Du, v + Dv),$$

of any point $X$ near the point $x_{-1}$ on the surface $S_{-1}$ can be represented by the Taylor's expansion as power series in the increments $Du, Dv$ corresponding to displacement on the surface $S_{-1}$ from the point $x_{-1}$ to the point $X$:
\[
X = x_{-1} + x_{-1u}\Delta u + x_{-1v}\Delta v
\]

\[
+ \frac{1}{2} (x_{-1uu}\Delta u^2 + 2x_{-1uv}\Delta u\Delta v + x_{-1vv}\Delta v^2) + \cdots.
\]

Substituting the expressions (6.1) in equation (6.2) we obtain the local coordinates \(\xi_1, \ldots, \xi_{n+1}\) of the point \(X\):

\[
\xi_1 = K\Delta v + \cdots,
\]

\[
\xi_2 = 1 + (\beta_2 - b)\Delta u + a\Delta v + \cdots,
\]

\[
\xi_3 = \frac{1}{2} K\Delta v^2 + \cdots,
\]

\[
\cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots
\]

\[
\xi_n = \frac{1}{2} p_n^2 \Delta u + \cdots,
\]

\[
\xi_{n+1} = p_n \Delta u + \frac{1}{2} (p_n^2 - bp_{n-2})\Delta u^2 + a_{2n-2}\Delta u\Delta v + \cdots,
\]

where the unwritten coordinates are of at least third degree in \(\Delta u, \Delta v\). By means of (6.3) it follows that a hyperquadric with the general equation

\[
\sum_{i,k=1}^{n+1} a_{ik}\xi_i\xi_k = 0 \quad (a_{ik} = a_{ki})
\]

has second order contact with the surface \(S_{-1}\) at the point \(x_{-1}\) in case

\[
a_{12} = a_{1,n+1} = a_{22} = a_{2,n+1} = 0, \quad a_{23} = -Ka_{11},
\]

\[
a_{n+1,n+1} = -\frac{p_{n-2}^3}{(p_{n-2})^2} a_{2n}.
\]

Similarly, the hyperquadric (6.4) has second order contact with the surface \(S_1\) at the point \(x_1\) if, and only if,

\[
a_{13} = a_{14} = a_{33} = a_{34} = 0, \quad a_{23} = -Ha_{11},
\]

\[
a_{44} = -\frac{q_1^3}{(q_1^2)^2} a_{33}.
\]

Combining the conditions (6.5), (6.6) we thus reach Theorem 3(3).