THE FUNDAMENTAL SOLUTION OF A DEGENERATE PARTIAL DIFFERENTIAL EQUATION OF PARABOLIC TYPE[1]

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1. **Introduction.** The equations studied in this paper arise in the probability treatment of diffusion problems and were first introduced by Kolmogoroff [1][2]. Kolmogoroff showed that under certain conditions the probability density of a system with $2n$ degrees of freedom satisfies a parabolic differential equation of Fokker-Planck type. The ordinary Fokker-Planck equation in $2n$ dimensions is

$$(1.1)\qquad \sum_{i,j}^{n} a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i}^{n}\left(x_i\frac{\partial u}{\partial y_i} + a_i\frac{\partial u}{\partial x_i}\right) + au + \frac{\partial u}{\partial t} = 0 \quad (a_{ij} = a_{ji})$$

a_{ij}, a_i, a are functions of x, y, t. It is degenerate in the sense that the second derivatives in y do not appear in the equation. The $2n$-dimensional space is the phase space of a system, where y is the position and x the velocity vector. For a more recent discussion of stochastic processes giving rise to equations of that type, see S. Chandrasekhar [2]. The more general equation

$$(1.2)\qquad \sum_{i,j}^{n} a_{ij}\frac{\partial^2 u}{\partial \xi_i \partial \xi_i} + \sum_{i}^{n}\left(b_i\frac{\partial u}{\partial \eta_i} + a_i\frac{\partial u}{\partial \xi_i}\right) + au + \frac{\partial u}{\partial \tau} = 0$$

can be reduced to (1.1) by the substitution $x_k = b_k(\xi, \eta, \tau)$ provided

$$\frac{\partial b_i}{\partial \xi_k}, \quad \frac{\partial b_i}{\partial \eta_k}, \quad \frac{\partial b_i}{\partial \tau}$$

exist for all i and k and the transformation

$$(1.3)\qquad x_k = b_k(\xi, \eta, \tau), \qquad y_k = \eta_k, \qquad t = \tau$$

represents a continuous one-to-one mapping of the ξ, η, τ-plane on the x, y, t-plane. Here the relation between the position η and the velocity ξ is given by $\dot{\eta}_i = b_i$.

The construction of a solution of (1.1) depends on the determination of the fundamental solution. It is the purpose of this paper to obtain the fundamental solution of (1.1) for any given open region R of phase space under

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[2] Numbers in brackets refer to the bibliography at the end of the paper.

certain conditions on the coefficients; these conditions will be given in §3. (The region R is not necessarily bounded, it may cover the whole phase-space.) Our development follows closely the methods of Feller [3] and Dressel [4; 5], going in some essentials back to Gevrey [6; 7].

Notations. Throughout the text the following notations are used: When there is no misunderstanding multiple integrals will be indicated by a single integral sign. We shall write dx and dy for $dx_1 \cdots dx_n$ and $dy_1 \cdots dy_n$ respectively. Integration with respect to t will always be indicated separately. The notation x will be used for $x_1 \cdots x_n$ and similarily y, ξ, η, μ, ν represent points in n-space. All summations, unless otherwise indicated, extend from 1 to n.

2. **Definition of the fundamental solution and the problem of uniqueness.** Let (1.1) be defined over some open region R in phase-space (x, y) and for $t_0 \leqq t \leqq t_1$, with uniformly bounded and continuous coefficients a_{ij}, a_i, a. Let $\partial a_{ij}/\partial x_i$ be uniformly bounded and continuous. We define the fundamental solution u of (1.1) by the following three properties:

(I) For $t_0 \leqq t < \tau \leqq t_1$ and each pair of points (x, y) and (ξ, η) in R, $u(x, y, t; \xi, \eta, \tau)$ is a regular solution of (1.1), that is, it possesses, as a function of (x, y, t), the continuous derivatives occurring in equation (1.1).

(II) For $x = \xi$, $y = \eta$, $t = \tau$ the function $u(x, y, t; \xi, \eta, \tau)$ possesses a singularity such that for every subregion D of R and every continuous bounded function $f(x, y)$

$$\lim_{t \to \tau-} \int_D u(x, y, t; \xi, \eta, \tau) f(x, y) dx dy = \begin{cases} f(\xi, \eta) & \text{if } (\xi, \eta) \text{ is interior to } D, \\ 0 & \text{if } (\xi, \eta) \text{ is exterior to } D. \end{cases} \tag{2.1}$$

(III) For fixed ξ, η, τ, t with $t_0 \leqq t < \tau \leqq t_1$ the functions $u(x, y, t; \xi, \eta, \tau)$ and $x_i u(x, y, t; \xi, \eta, \tau)$ are absolutely integrable over R and $\partial u/\partial x_i$ are bounded.

The equation

$$L^*(u) = \sum_{i,j} \frac{\partial^2 a_{ij} u}{\partial x_i \partial x_j} - \sum_i \left(\frac{\partial a_i u}{\partial x_i} + x_i \frac{\partial u}{\partial y_i} \right) + au - \frac{\partial u}{\partial \tau} = 0 \tag{2.2}$$

defines the adjoint to (1.1). In §3 we shall give sufficient conditions on the coefficients of (1.1) to ensure the existence of a fundamental solution. These conditions will automatically entail the existence of a fundamental solution of (2.2).

We shall now give a uniqueness theorem for the fundamental solution, provided R is the entire phase space

$$S \begin{cases} -\infty < x_i < +\infty, \\ -\infty < y_i < +\infty, \end{cases} \qquad i = 1, 2, \cdots, n. \tag{2.3}$$

THEOREM 1. *Under the assumption of existence of a fundamental solution of*

(2.2), $u(x, y, t; \xi, \eta, \tau)$, *as defined by conditions* I→III *with* $R\equiv S$, *satisfies equation* (2.2) *in the variables* ξ, η, *and* τ *and as a consequence is uniquely determined.*

Proof. The proof is omitted because it follows the same lines as for the ordinary parabolic equation (cf. Dressel [5]).

COROLLARY.

$$\int_S u(x, y, t; \xi, \eta, \tau)u(\mu, \nu, \lambda; x, y, t)dxdy = u(\mu, \nu, \lambda; \xi, \eta, \tau). \tag{2.4}$$

3. **The fundamental solution of equation** (1.1). We determine the fundamental solution as the solution of an integral equation. We assume that in R and for $t_0 \leqq t \leqq t_1$ the coefficients of (1.1) satisfy the following conditions:

(a) The functions $\partial a_{ij}/\partial t$, $\partial^2 a_{ij}/\partial x_k \partial x_e$, $\partial a_i/\partial x_k$, a_i, a, $\partial a_{ij}/\partial y_k$ satisfy a local Lipschitz condition of order γ, $0<\gamma$, and are uniformly bounded.

(b) The characteristic roots of the symmetric matrix $\|a_{ij}\|$ are positive and uniformly bounded both above and away from zero.

Let A_{ik} denote the cofactor of a_{ik} divided by the determinant A. Because of condition (b), A is bounded above and below and so are the characteristic roots of $\|A_{ik}\|$. Then as an immediate consequence of (b) we have:

LEMMA 1. *There exist positive constants* d_1 *and* d_2 *such that for all* u_i *and all* (x, y, t) *in* R

$$d_1 \sum_i u_i^2 \leqq \sum_{i,k} a_{ik}u_iu_k \leqq d_2 \sum_i u_i^2, \tag{3.1}$$

$$d_1 \sum_i u_i^2 \leqq \sum_{i,k} A_{ik}u_iu_k \leqq d_2 \sum_i u_i^2. \tag{3.2}$$

d_1 *is the greatest lower bound of the characteristic roots of both* $\|a_{ik}\|$ *and* $\|A_{ik}\|$ *and* d_2 *the least upper bound.*

In the case of equation (1.2) additional assumptions are to be made on derivatives of the b_i's up to the third order and a_{ij} is to be replaced by

$$\bar{a}_{ij} = \frac{1}{2} \sum_{k,e} a_{ke} \left[\frac{\partial b_e}{\partial x_i} \frac{\partial b_k}{\partial x_j} + \frac{\partial b_e}{\partial x_j} \frac{\partial b_k}{\partial x_i} \right]. \tag{3.3}$$

We now proceed to prove the following theorem.

THEOREM 2. *Under assumptions* (a) *and* (b) *there exists a fundamental solution of* (1.1). *In case* R *is the entire phase space this fundamental solution is unique and satisfies equation* (2.2).

We need some preliminary results:

The equation

$$L(u) = \frac{\partial^2 u}{\partial x^2} + x\frac{\partial u}{\partial y} + \frac{\partial u}{\partial t} = 0 \tag{3.4}$$

has the fundamental solution

$$\begin{aligned} F(x, y, t; \xi, \eta, \tau) = 3^{1/2}2^{-1}\pi^{-1}(\tau - t)^{-2}\exp\Big[-\frac{(\xi - x)^2}{4(\tau - t)} \\ - 3\frac{\{\eta - y - 2^{-1}(\tau - t)(\xi + x)\}^2}{(\tau - t)^3}\Big] \end{aligned} \tag{3.5}$$

given by Kolmogoroff [1], which satisfies all the conditions in §2. We use this function in the construction of the first approximation of the fundamental solution of (1.1).

Let

$$\begin{aligned} R_{ik}(x, y, t; \xi, \eta, \tau) = \frac{(\xi_i - x_i)(\xi_k - x_k)}{4(\tau - t)} \\ + 3\frac{\{\eta_i - y_i - 2^{-1}(\tau - t)(\xi_i + x_i)\}\{\eta_k - y_k - 2^{-1}(\tau - t)(x_k + \xi_k)\}}{(\tau - t)^3}. \end{aligned} \tag{3.6}$$

We choose as first approximation for our fundamental solution

$$\begin{aligned} u_0(x, y, t; \xi, \eta, \tau) \\ = [\phi(\xi, \eta, \tau)]^{-1}(\tau - t)^{-2n}\exp\Big[-\sum_{i,k} A_{ik}(x, y, t)R_{ik}(x, y, t; \xi, \eta, \tau)\Big]. \end{aligned} \tag{3.7}$$

$\phi(x, y, t)$ is defined by

$$\begin{aligned} \phi(x, y, t) = \lim_{\lambda \to t+} \int_Q (\lambda - t)^{-2n} \\ \cdot \exp\Big[-\sum_{i,k} A_{ik}(x, y, t)R_{ik}(x, y, t; \mu, \nu, \lambda)\Big] d\mu d\nu. \end{aligned} \tag{3.8}$$

Q is a small square centered in the point x. By a simple change of variable this limit can be shown to exist. By Lemma 1 and assumption (a) it follows that ϕ is a continuous function bounded away from zero and differentiable with respect to all its variables.

In the following we shall determine a function $f(\mu, \nu, \lambda; \xi, \eta, \tau)$ such that the fundamental solution of (1.1) can be written as

$$\begin{aligned} u(x, y, t; \xi, \eta, \tau) = u_0(x, y, t; \xi, \eta, \tau) \\ + \int_t^\tau d\lambda \int_R u_0(x, y, t; u, \nu, \lambda) f(\mu, \nu, \lambda; \xi, \eta, \tau) d\mu d\nu. \end{aligned} \tag{3.9}$$

For this purpose we need a set of three formulas collected in the following lemma.

LEMMA 2. *Let $f(\mu, \nu, \lambda)$ satisfy a Lipschitz condition of order γ, $0<\gamma$, for any point (μ, ν) in R and $t \leqq \lambda < \tau$. For any $\epsilon > 0$ let $f(\mu, \nu, \lambda)$ be bounded over any set for $\lambda \leqq \tau - \epsilon$, and absolutely integrable over R for $t \leqq \lambda \leqq \tau$.*

Let

$$U(x, y, t) = \int_t^\tau d\lambda \int_R f(\mu, \nu, \lambda) v_0(x, y, t; \mu, \nu, \lambda) d\mu d\nu \tag{3.10}$$

with $t_0 \leqq t < \tau \leqq t_1$ and $v_0(x, y, t; \mu, \nu, \lambda) = u_0(x, y, t; \mu, \nu, \lambda)\phi(\mu, \nu, \lambda)$. Then we have

$$\begin{aligned} \frac{\partial U}{\partial t} &= - f(x, y, t)\phi(x, y, t) \\ &\quad + \int_t^\tau d\lambda \int_R [f(\mu, \nu, \lambda) - f(x, y, t)] \frac{\partial v_0}{\partial t} d\mu d\nu \\ &\quad + f(x, y, t) \int_{\to t+}^\tau d\lambda \int_R \frac{\partial v_0}{\partial t} d\mu d\nu, \end{aligned} \tag{3.11}$$

$$\begin{aligned} \frac{\partial U}{\partial y_k} &= \int_t^\tau d\lambda \int_R [f(\mu, \nu, \lambda) - f(x, y, t)] \frac{\partial v_0}{\partial y_k} \\ &\quad + f(x, y, t) \int_{\to t+}^\tau d\lambda \int_R \frac{\partial v_0}{\partial y_k} d\mu d\nu, \end{aligned} \tag{3.12}$$

$$\begin{aligned} \frac{\partial^2 U}{\partial x_i \partial x_k} &= \int_t^\tau d\lambda \int_R [f(\mu, \nu, \lambda) - f(x, y, t)] \frac{\partial^2 v_0}{\partial x_i \partial x_k} d\mu d\nu \\ &\quad + f(x, y, t) \int_{\to t+}^\tau d\lambda \int_R \frac{\partial^2 v_0}{\partial x_i \partial x_k} d\mu d\nu. \end{aligned} \tag{3.13}$$

Each of the last integrals means $\lim_{\epsilon \to 0} \int_{t+\epsilon}^\tau$.

Equation (3.11) is an extension of Theorem 1 of Dressel [4] and (3.13) of Theorem 2. Our fundamental solution differs from his by the normalization factor and the second exponential term.

Proof of (3.11). Let Q be a $2n$-dimensional square of side length 2η and $R-Q$ the remainder of the region considered. We write for $\Delta t > 0$

$$\frac{\Delta_t U}{\Delta t} = - \frac{1}{\Delta t} \int_t^{t+\Delta t} d\lambda \int_R f v_0 d\mu d\nu + \int_{t+\Delta t}^\tau d\lambda \int_R f \frac{\Delta_t v_0}{\Delta t} d\mu d\nu = I_1 + I_2.$$

To shorten the formulas we omit the variables on which depend U, f, v_0; they are to be found explicitly in (3.10). Δ_t means an increment where t alone is varied.

The part of I_1 for which the space integral extends over $R-Q$ tends to zero with Δt. The remaining integral, with the space integration over Q, tends to $-f(x, y, t)\phi(x, y, t)$, because of the definition of ϕ and the continuity of f in the point (x, y, t). I_2 can be split up into three parts, the first an integration over $R-Q$ and from $t+\Delta t$ to $\tau-\epsilon$, the second integrated over R and from $\tau-\epsilon$ to τ, and the third over Q and from $t+\Delta t$ to $\tau-\epsilon$, where $\tau-\epsilon>t+\Delta t$ and $\epsilon>0$. In the first two integrals we can pass to the limit $\Delta t\to 0$ under the integral sign. For the third integral we get

$$J = \int_{t+\Delta t}^{\tau-\epsilon} d\lambda \int_Q [f(\mu, \nu, \lambda) - f(x, y, t)] \frac{\Delta_t v_0}{\Delta t} d\mu d\nu$$

$$+ f(x, y, t) \int_{t+\Delta t}^{\tau-\epsilon} d\lambda \int_Q \frac{\Delta_t v_0}{\Delta t} d\mu d\nu = J_1 + J_2.$$

The Lipschitz condition on f ensures the existence and convergence to zero with ϵ of

$$\int_t^{t+\epsilon} d\lambda \int_Q [f(\mu, \nu, \lambda) - f(x, y, t)] \frac{\partial v_0}{\partial t} d\mu d\nu,$$

which implies that one can pass to the limit under the integral sign in J_1. We now show that $\lim_{\Delta t\to 0} J_2$ exists. Consider

$$J_3 = f(x, y, t) \int_{t+\Delta t}^{t+\Delta t+\epsilon} d\lambda \int_Q \frac{\Delta_t v_0}{\Delta t} d\mu d\nu.$$

In the integral obtained by subtracting J_3 from J_2 one can pass to the limit under the integral sign. In J_3 we change λ into $\lambda+\Delta t$ in $v_0(x, y, t+\Delta t; \mu, \nu, \lambda)$ and obtain

$$J_3 = f(x, y, t) \Bigg\{ \int_t^{t+\Delta t} d\lambda/\Delta t \int_Q (\lambda - t)^{-2n}$$

$$\cdot \exp\left[- \sum_{i,k} A_{ik}(x, y, t + \Delta t) R_{ik}(x, y, t; \mu, \nu, \lambda)\right] d\mu d\nu + \int_{t+\Delta t}^{t+\Delta t+\epsilon} d\lambda/\Delta t$$

$$\cdot \int_Q \frac{\exp\left[- \sum_{i,k} A_{ik}(x, y, t + \Delta t) R_{ik}\right] - \exp\left[- \sum_{i,k} A_{ik}(x, y, t) R_{ik}\right]}{(\lambda - t)^{2n}} d\mu d\nu$$

$$- \int_{t+\epsilon}^{t+\Delta t+\epsilon} d\lambda/\Delta t \int_Q (\lambda - t)^{-2n} \exp\left[- \sum_{i,k} A_{ik}(x, y, t + \Delta t) R_{ik}\right] d\mu d\nu \Bigg\}.$$

Passing to the limit Δt tending to zero, we get

$$\lim_{\Delta t\to 0} J_3 = f(x, y, t)\phi(x, y, t) + f(x, y, t)\int_t^{t+\epsilon} d\lambda \int_Q - \sum_{i,k} \frac{\partial A_{ik}}{\partial t} R_{ik} v_0 d\mu d\nu$$
$$- f(x, y, t)\int_Q v_0(x, y, t; \mu, \nu, t+\epsilon) d\mu d\nu.$$

The passage to the limit $\epsilon \to 0$ completes the proof of (3.11) for $\Delta t > 0$. For $\Delta t < 0$ we write

$$\frac{\Delta_t U}{\Delta t} = \int_{t+\Delta t}^{t} d\lambda/\Delta t \int_R f v_0(x, y, t+\Delta t; \mu, \nu, \lambda) d\mu d\nu - \int_t^{\tau} d\lambda \int_R f \frac{\Delta_t v_0}{\Delta t} d\mu d\nu$$

and follow a proof completely analogous to the preceding one.

Proof of (3.12). Let $\epsilon > 0$ and $t+\epsilon < \tau - \epsilon$. We form the ratio $\Delta_y U/\Delta y_k$ and split the integration from t to τ into three parts, from $\tau-\epsilon$ to τ, $t+\epsilon$ to $\tau-\epsilon$, and t to $t+\epsilon$, thus obtaining three integrals of which the first two are regular. In these the limit can be taken under the integral sign. The third can be transformed into

$$\int_t^{t+\epsilon} d\lambda \int_R [f(\mu, \nu, \lambda) - f(x, y, t)] \frac{\Delta_{y_k} v_0}{\Delta y_k} d\mu d\nu + f(x, y, t)\int_t^{t+\epsilon} d\lambda \int_R \frac{\Delta_{y_k} v_0}{\Delta y_k} d\mu d\nu.$$

Because of the Lipschitz condition on $f(x, y, t)$, the passage to the limit can be effected under the first integral. Its contribution tends to zero with ϵ. Introducing in the second the new variable $\bar{\nu}_k = \nu_k - \Delta y_k$ and calling $\overline{R}$ the transformed region, we obtain

$$f(x, y, t)\int_t^{t+\epsilon} d\lambda$$
$$\cdot \int_{\overline{R}} \frac{\exp\left[-\sum_{i,e} A_{ie}(x, y_k + \Delta y_k, t) R_{ie}\right] - \exp\left[-\sum_{i,e} A_{ie}(x, y, t) R_{i,e}\right]}{\Delta y_k (\lambda - t)^{2n}} d\mu d\nu.$$

We let $\Delta y_k \to 0$ and obtain

$$f(x, y, t)\int_t^{t+\epsilon} d\lambda \int_R - \sum_{i,e} \frac{\partial A_{ie}}{\partial y_k} R_{ie} v_0 d\mu d\nu.$$

Now we let $\epsilon \to 0$ and, combining the results of this paragraph, we obtain (3.12).

Proof of (3.13). It is easy to see that $\partial U/\partial x_k$ can be obtained by a differentiation under the integral sign. It is sufficient therefore to examine the derivative with respect to x_m of

$$T = \int_t^{t+\epsilon} d\lambda \int_Q f(\mu, \nu, \lambda) R_e(x, y, t; \mu, \nu, \lambda) v_0(x, y, t; \mu, \nu, \lambda) d\mu d\nu,$$

where we let

$$R_e = -\sum_{i,k} \frac{\partial A_{ik}(x, y, t)}{\partial x_e} R_{ik}(x, y, t; \mu, \nu, \lambda) + \sum_i A_{ie}\left[2\,\frac{\mu_i - x_i}{4(\lambda - t)} - 3\,\frac{\nu_i - y_i - 2^{-1}(\mu_i + x_i)(\lambda - t)}{(\lambda - t)^2}\right].$$

We increase x_m by Δx and keep all the other variables fixed; then we have

$$\frac{\Delta_x T}{\Delta x} = \frac{1}{\Delta x}\int_t^{t+\epsilon} d\lambda \int_Q f R_e(x_m + \Delta x, y, t; \mu, \nu, \lambda)\left[v_0(x_m + \Delta x, y, t; \mu, \nu, \lambda) - \bar{v}_0\right] d\mu d\nu + \frac{1}{\Delta x}\int_t^{t+\epsilon} d\lambda \int_Q f\left[R_e(x_m + \Delta x, y, t; \mu, \nu, \lambda)\bar{v}_0 - R_e(x, y, t; \mu, \nu, \lambda) v_0(x, y, t; \mu, \nu, \lambda)\right] d\mu d\nu,$$

where $\bar{v}_0$ denotes the function obtained from $v_0(x, y, t; \mu, \nu, \lambda)$ by replacing the x_m occurring in $A_{ik}(x, y, t)$ by x_m and the x_m elsewhere by $x_m + \Delta x$. To the first integral apply the mean value theorem and pass to the limit under the integral sign. It is easy to see that this integral tends to zero with ϵ. In the other integral write for $f(\mu, \nu, \lambda)$ the sum $[f(\mu, \nu, \lambda) - f(x, y, t)] + f(x, y, t)$ and in this way obtain two integrals. Because of the Lipschitz condition on f, we can pass to the limit under the first integral. This integral then tends to zero with ϵ. In the second integral I_2 we introduce the variables $\bar{\mu}_m = \mu_m - \Delta x$, $\bar{\nu}_m = \nu_m - 2(\lambda - t)\Delta x$. Writing Q' for those integration limits in the integral which remain unchanged, we obtain.

$$I_2 = f(x, y, t)/\Delta x \int_t^{t+\epsilon} d\lambda \cdot \int_{Q'} \left\{ \int_{x_m-\eta-\Delta x}^{x_m+\eta-\Delta x} \int_{y_m-\eta-2\Delta x(\lambda-t)}^{y_m+\eta-2\Delta x(\lambda-t)} - \int_{x_m-\eta}^{x_m+\eta} \int_{y_m-\eta}^{y_m+\eta} \right\} R_e v_0 d\mu d\nu$$

$$= f(x, y, t)/\Delta x \int_t^{t+\epsilon} d\lambda \int_{Q'} \left\{ \int_{x_m-\eta-\Delta x}^{x_m+\eta-\Delta x} \left[\int_{y_m-\eta-2\Delta x(\lambda-t)}^{y_m-\eta} - \int_{y_m+\eta-2\Delta x(\lambda-t)}^{y_m+\eta} \right] + \int_{y_m-\eta}^{y_m+\eta} \left[\int_{x_m-\eta-\Delta x}^{x_m-\eta} - \int_{x_m+\eta-\Delta x}^{x_m+\eta} \right] \right\} R_e v_0 d\mu d\nu.$$

If Δx is small enough, that is, satisfying $\Delta x < \min[n/2, \eta/4(\lambda - t)]$, the integrand is continuous and we can pass to the limit $\Delta x \to 0$ which gives us

$$\lim_{\Delta x \to 0} I_2 = f(x, y, t) \int_t^{t+\epsilon} d\lambda \int_Q \left[\frac{\partial R_e}{\partial \nu_m} + \frac{\partial R_e}{\partial \mu_m}\right] v_0 d\mu d\nu.$$

This gives the essential points in the proof of (3.13).

LEMMA 3. *The function U defined by* (3.10) *satisfies the equation*

$$L(U) = -f(x, y, t)\phi(x, y, t) + \int_t^\tau d\lambda \int_R f(\mu, \nu, \lambda)L(v_0)d\mu d\nu, \tag{3.14}$$

where L is the operator defined in (1.1)

Proof. $L(v_0)$ can be written

$$\begin{aligned}(\lambda - t)^{2n}L(v_0) = & \left\{ \sum_{i,k} R_{ik}\left(- \sum_{e,m} \frac{\partial^2 A_{ik}}{\partial x_e \partial x_m} a_{em} + \sum_{e,m,r,s} \frac{\partial A_{ik}}{\partial x_e} \frac{\partial A_{rs}}{\partial x_m} a_{em} R_{rs} \right.\right. \\ & \left.\left. - \sum_e \frac{\partial A_{ik}}{\partial x_e} a_e \right) + a \right\} \exp\left[- \sum_{i,k} A_{ik} R_{ik} \right] \\ & + (\lambda - t)^{-1/2} \left\{ \sum_i \left[\frac{\mu_i - x_i}{2(\lambda - t)^{1/2}} \right.\right. \\ & \left. + 3\, \frac{\nu_i - y_i - 2^{-1}(\mu_i + x_i)(\lambda - t)}{(\lambda - t)^{3/2}} \right] \\ & \left. \cdot \left[2 \sum_{e,m} a_{em} \frac{\partial A_{im}}{\partial x_e} - 4 \sum_{r,k,e,m} a_{em} \frac{\partial A_{rk}}{\partial x_e} A_{im} R_{rk} + \sum_e a_e A_{ie} \right] \right\} \\ & \cdot \exp\left[- \sum_{i,k} A_{ik} R_{ik} \right].\end{aligned} \tag{3.15}$$

The terms of higher order, that is, those containing $(\lambda - t)^{-2n-1}$ and $(\lambda - t)^{-(2n+3/2)}$, disappear because of the choice of u_0. This enables us to operate under the integral sign and derive (3.14) by Lemma 2.

We are now ready to construct $f(\mu, \nu, \lambda; \xi, \eta, \tau)$ of formula (3.9) as the solution of the integral equation

$$\begin{aligned} & f(x, y, t; \xi, \eta, \tau) \\ & \quad = L(u_0) + \int_t^\tau d\lambda \int_R L[u_0(x, y, t; \mu, \nu, \lambda)] f(\mu, \nu, \lambda; \xi, \eta, \tau) d\mu d\nu. \end{aligned} \tag{3.16}$$

In successive approximations we write:

$$f_0(x, y, t; \xi, \eta, \tau) = L(u_0),$$

$$f_m(x, y, t; \xi, \eta, \tau) = \int_t^\tau d\lambda \int_R L[u_0(x, y, t; \xi, \eta, \tau)] f_{m-1}(\mu, \nu, \lambda; \xi, \eta, \tau) d\mu d\nu \qquad (m \geqq 1),$$

and put

$$f(x, y, t; \xi, \eta, \tau) = \sum_{k=0}^{\infty} f_k(x, y, t; \xi, \eta, \tau). \tag{3.17}$$

We prove first uniform and absolute convergence of the series in (3.17).

Let

$$G_i(x, y, t; \xi, \eta, \tau) = \left[\frac{(\xi_i - x_i)^2}{4(\tau - t)} + 3\,\frac{[\eta_i - y_i - 2^{-1}(\xi_i + x_i)(\tau - t)]^2}{(\tau - t)^3}\right].$$

By (3a), Lemma 1, and (3.15) there are constants d and M such that

$$|f_0| \leqq M(\tau - t)^{-2n-1/2} \exp\left[- d^2 \sum_i G_i(x, y, t; \xi, \eta, \tau)\right]$$

at fixed ξ, η, and $t<\tau$. In order to compute bounds on the terms f_k, we need an estimate on the integral

$$I = \int_t^\tau d\lambda \int_R (\lambda - t)^{-2n-1/2}(\tau - \lambda)^{-2n-1/2}$$

$$\cdot \exp\left\{- d^2 \sum_i \left[G_i(x, y, t; \mu, \nu, \lambda) + G_i(\mu, \nu, \lambda; \xi, \eta, \tau)\right]\right\} d\mu d\nu.$$

We change $d\mu_i = \mathrm{M}_i$, $d\nu_i = \mathrm{N}_i$, and in the same way x_i, y_i, ξ_i, η_i into $d^{-1}X_i$, $d^{-1}Y_i$, $d^{-1}\Xi_i$, $d^{-1}\mathrm{H}_i$. The integral becomes

$$I = d^{-2n} \int_t^\tau (\lambda - t)^{-1/2}(\tau - \lambda)^{-1/2} d\lambda \int_R (\lambda - t)^{-2n}(\tau - \lambda)^{-2n}$$

$$\cdot \exp\left\{- \sum_i \left[G_i(X, Y, t; \mathrm{M}, \mathrm{N}, \lambda) + G_i(\mathrm{M}, \mathrm{N}, \lambda; \Xi, \mathrm{H}, \tau)\right]\right\} d\mathrm{M} d\mathrm{N}.$$

I is less than or equal to the integral obtained by replacing R by the whole phase-space S. We also know that the function defined in (3.5) satisfies the corollary of Theorem 1. Now the integral over S is nothing but a product of integrals as in (2.4) and therefore we get

$$I \leqq (3^{1/2}d)^{-2n}(2\pi)^{2n} \int_t^\tau (\lambda - t)^{-1/2}(\tau - \lambda)^{-1/2}(\tau - t)^{-2n}$$

$$\cdot \exp\left[- \sum_i G_i(X, Y, t; \Xi, \mathrm{H}, \tau)\right] d\lambda,$$

$$I \leqq (d3^{1/2})^{-2n}(2\pi)^{2n}\pi(\tau - t)^{-2n} \exp\left[- d^2 \sum_i G_i(x, y, t; \xi, \eta, \tau)\right].$$

Therefore

$$|f_1| \leqq \pi(d3^{1/2})^{-2n}(2\pi)^{2n}M^2(\tau - t)^{-2n} \exp\left[- d^2 \sum_i G_i(x, y, t; \xi, \eta, \tau)\right].$$

Estimates on the remaining terms of the series are obtained by induction. We get

$$(3.18)\quad |f_k| \leqq \pi^{(k-1)/2}(d3^{1/2})^{-2nk}(2\pi)^{2nk}M^{k+1}(\tau - t)^{1/2(k-1)-2n}\frac{\exp\left[-d^2\sum_i G_i\right]}{\Gamma(k/2+1/2)}$$

and therefore

$$(3.19)\quad |f| \leqq \text{const } (\tau - t)^{-(2n+1/2)} \exp\left[-d^2\sum_i G_i\right].$$

For $t \leqq \tau - \epsilon$ the series (3.17) is uniformly and absolutely convergent. Therefore (3.17) defines a continuous function for $t \leqq \tau - \epsilon$ and by (3.19) this function is absolutely integrable over R for $t < \tau$. We still have to prove that the function f satisfies a Lipschitz condition of order γ, $0 < \gamma$. Because of (3a) and (3.15), $L(u_0)$ satisfies a Lipschitz condition. It is therefore sufficient to prove that

$$f^*(x, y, t) = \int_t^\tau d\lambda \int_R L[u_0(x, y, t; \mu, \nu, \lambda)]f(\mu, \nu, \lambda; \xi, \eta, \tau)d\mu d\nu$$

satisfies a Lipschitz condition.

We keep x_k, $k \neq i$, fixed and write for $x_i^{(1)} < x_i^{(2)}$, both in R,

$$f^*(x_i^{(1)}, y, t) - f^*(x_i^{(2)}, y, t) = \int_t^\tau d\lambda \int_R \Delta L(u_0) f d\mu d\nu,$$

where we introduce the notation

$$\Delta L(u_0) = L[u_0(x_i^{(1)}, y, t; \mu, \nu, \lambda)] - L[u_0(x_i^{(2)}, y, t; \mu, \nu, \lambda)].$$

Outside the region E

$$E\begin{cases} x_i^{(1)} \leqq \mu \leqq x_i^{(2)}, & \\ t \leqq \lambda \leqq t + a & (a > 0,\ t + a < \tau), \\ y - b \leqq \nu \leqq y + b & (b > 0), \\ x_k - b \leqq \mu_k \leqq x_k + b & (k \neq i), \end{cases}$$

$L(u_0)$ satisfies a Lipschitz condition and $f(\mu, \nu, \lambda; \xi, \eta, \tau)$ is absolutely integrable. Therefore it is sufficient to show that

$$N\int_t^{t+a} d\lambda \int_E |\Delta L(u_0)|\, d\mu d\nu$$

satisfies a Lipschitz condition, N is the bound on $f(\mu, \nu, \lambda; \xi, \eta, \tau)$ in R.

According to (3.15)

$$|\Delta L(u_0)| \leqq \text{const. } (\lambda - t)^{-2n-1/2}\left\{\exp\left[-d^2\sum_k G_k(x_i^{(1)}, y, t; \mu, \nu, \lambda)\right] - \exp\left[-d^2\sum_k G_k(x_i^{(2)}, y, t; \mu, \nu, \lambda)\right]\right\}.$$

We split E into two parts according to

$$x_i^{(1)} \leqq \mu \leqq (x_i^{(1)} + x_i^{(2)})/2, \qquad (x_i^{(1)} + x_i^{(2)})/2 \leqq \mu \leqq x_i^{(2)}.$$

We divide the integral inside by $(\mu - x_i^{(1)})^\gamma$ and $(x_i^{(2)} - \mu)^\gamma$, $0<\gamma<1$, respectively and multiply outside both integrals by $|x_i^{(1)} - x_i^{(2)}|^\gamma$. We obtain bounded integrals and have

$$\left|\int_t^{t+a} d\lambda \int_E f\Delta L(u_0) d\mu d\nu\right| \leqq \text{const. } |x_i^{(1)} - x_i^{(2)}|^\gamma.$$

For $y_i^{(1)} - y_i^{(2)}$ the proof is analogous.

For $t<t^{(1)}<t^{(2)}<\tau$ let us consider

$$u(x, y, t^{(1)}) - u(x, y, t^{(2)}) = \int_{t^{(1)}}^{\tau} d\lambda \int_R fL[u_0(x, y, t^{(1)}; \mu, \nu, \lambda)] d\mu d\nu - \int_{t^{(2)}}^{\tau} d\lambda \int_R fL[u_0(x, y, t^{(2)}; \mu, \nu, \lambda)] d\mu d\nu.$$

We write

$$\begin{aligned} u(x, y, t^{(1)}) - u(x, y, t^{(2)}) &= \int_{t^{(1)}}^{m} d\lambda \int_R fL[u_0(x, y, t^{(1)}; \mu, \nu, \lambda)] d\mu d\nu \\ &+ \int_{t^{(2)}}^{m} d\lambda \int_R fL[u_0(x, y, t^{(2)}; \mu, \nu, \lambda)] d\mu d\nu \\ &+ \int_{m}^{\tau} d\lambda \int_R f\{L[u_0(x, y, t^{(1)}; \mu, \nu, \lambda)] \\ &- L[u_0(x, y, t^{(2)}; \mu, \nu, \lambda)]\} d\mu\nu \end{aligned}$$

where $m=$ minimum of $(3t^{(2)} - t^{(1)})/2$ and τ.

The last integral satisfies a Lipschitz condition, because $L(u_0)$ does and f is absolutely integrable.

The first two integrals are bounded and give

$$|u(x, y, t^{(1)}) - u(x, y, t^{(Q)})| \leqq \text{const. } |t^{(2)} - t^{(1)}|^{1/2}.$$

We have therefore proved that f satisfies a Lipschitz condition.

We now write (3.9) and prove that u thereby defined satisfies properties I–III of §2 and this will complete the proof of Theorem 2.

(I) The result of the last paragraph, formula (3.9), and Lemma 3 enable us to write

$$L(u) = L(u_0) - f(x, y, t; \xi, \eta, \tau) + \int_t^\tau d\lambda \int_R L(u_0) f(\mu, \nu, \lambda; \xi, \eta, \tau) d\mu d\nu,$$

which gives, by (3.16), $L(u) = 0$. We notice that for $t \leqq \tau - \epsilon$, u_0 and its derivatives are continuous. So is $\partial u/\partial x_k$, which can be verified by direct differentiation under the integral sign. We need to show only that $\partial^2 u/\partial x_k \partial x_r$ and $\partial u/\partial y_r$ are continuous. This will entail by (1.1) continuity of $\partial u/\partial t$. It is sufficient to examine for continuity formulas (3.12) and (3.13). We can write (3.12)

$$\frac{\partial U}{\partial y_k} = \lim_{\epsilon \to 0} \int_{t+\epsilon}^{\tau-\epsilon} d\lambda \int_R \frac{\partial v_0}{\partial y_k} [f(\mu, \nu, \lambda) - f(x, y, t)] d\mu d\nu$$
$$+ \lim_{\epsilon \to 0} f(x, y, t) \int_{t+\epsilon}^{\tau-\epsilon} d\lambda \int_R \frac{\partial v_0}{\partial y_k} d\mu d\nu.$$

This limit is uniform in (x, y, t) for $t \leqq \tau - \epsilon$. Let (x, y, t) tend to (X, Y, T), (X, Y) in R and $T \leqq \tau$. We can interchange the limits $\epsilon \to 0$ and $\lim (x, y, t) = (X, Y, T)$ and this latter can be taken under the integral sign. This proves continuity of $\partial U/\partial y_k$. The proof is the same for $\partial^2 U/\partial x_i \partial x_k$. Therefore property I is satisfied by u.

(II) We have by (3.9) and (3.19)

$$|u - u_0| \leqq \text{const. } [\phi_{\min}]^{-1} \int_t^\tau (\lambda - t)^{-2n} (\tau - \lambda)^{-2n-1/2} d\lambda$$
$$\cdot \int_R \exp\left[- d^2 \sum_i G_i(x, y, t; \mu, \nu, \lambda) - d^2 \sum_i G_i(\mu, \nu, \lambda; \xi, \eta, \tau) \right] d\mu d\nu;$$

we get

$$(3.20) \quad |u - u_0| \leqq \frac{\text{const.}}{\phi_{\min}} 2^{-1} (\tau - t)^{1/2} \exp\left[- d^2 \sum_i G_i(x, y, t; \xi, \eta, \tau) \right].$$

Therefore for any continuous and bounded function $f(x, y)$ we have

$$(3.21) \quad \lim_{t \to \tau-} \int_D u(x, y, t; \xi, \eta, \tau) f(x, y) dx dy = \lim_{t \to \tau-} \int_D u_0(x, y, t; \xi, \eta, \tau) f(x, y) dx dy$$

where D is finite or infinite. The properties of u_0 immediately yield II.

(III) According to (3.19) we see that $u(x, y, t; \xi, \eta, \tau)$ and $x_i u(x, y, t; \xi, \eta, \tau)$ are absolutely integrable. Differentiation of (3.9) shows that $\partial u/\partial x_k$ are bounded. Hence the results of §2 apply to our fundamental solution.

To illustrate the use of the fundamental solution we consider an initial-value problem. If R is the whole of phase space, the following simple problem can be solved: given a continuous bounded function $\psi(x, y)$ there is for $t>0$ a unique solution of equation (2.2) satisfying

$$\lim_{t\to 0} u(x, y, t) = \psi(x, y),$$

provided that both u and $\partial u/\partial x_k$ are bounded and we have

$$u(x, y, t) = \int_{-\infty}^{+\infty} \psi(\xi, \eta) u(\xi, \eta, \tau; x, y, t) d\xi d\eta. \tag{3.22}$$

(3.22) is easily obtained by use of Green's formula. Uniqueness is consequence of the fact that for $\psi \equiv 0$, $u \equiv 0$. It is obvious that the restrictions on u and $\partial u/\partial x_k$ can be relaxed, because of the fact that $u(x, y, t; \xi, \eta, \tau)$ decreases exponentially as well as $\partial u/\partial x_k$ for large values of the coordinates.

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