ON A CLASS OF MARKOV PROCESSES(1)

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1. **Introduction.** *Let $x_k(t)$, $x_k(0)=0$, $0 \leqq t < \infty$ $(k=1, \cdots, n)$ be elements of n independent Wiener spaces. Let $\bar{x}(t)$ be an element of the product space of the n independent Wiener spaces. Throughout the paper we shall assume $V(t, \bar{x})$ to be a Borel measurable function of t and the n-dimensional vector $\bar{x}=(x_1, \cdots, x_n)$ that is bounded in every finite Euclidean sphere of the $(t, \bar{x})$ space,* $0 \leqq t < \infty$. These assumptions will not be restated in the rest of the paper. We shall often assume that $V(t, \bar{x})$ satisfies additional regularity conditions specified later on in the paper.

Let

$$y(t) = \int_0^t V(\tau, \bar{x}(\tau))d\tau. \tag{1.1}$$

We wish to relate the study of the Markov process

$$(\bar{x}(t), y(t)) \tag{1.2}$$

to the study of certain differential and integral equations. Such a study is of interest for the following reasons. We obtain information about Markov processes of type (1.2). For certain choices of the function $V(t, \bar{x})$, the study yields interesting information about the problem of the absorbing barrier in diffusion theory. Moreover, if we look at the problem from a different point of view, we obtain a general method of getting limit theorems of a certain type [2](2).

Consider the distribution function

$$\sigma(\alpha; t) = \Pr \left\{ y(t) \leqq \alpha \right\}. \tag{1.3}$$

Let $\overline{G}_m = (G_m^{(1)}, \cdots, G_m^{(n)})$, where $G_1^{(k)}, G_2^{(k)}, \cdots (k=1, \cdots, n)$ are independent, normally distributed random variables with mean 0 and variance 1. The distribution function of

$$\frac{1}{m} \sum_{k/m \leqq t} V\left(\frac{k}{m}, \frac{\overline{G}_1 + \cdots + \overline{G}_k}{m^{1/2}}\right)$$

is the same as that of

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$$\frac{1}{m}\sum_{k/m\leqq t} V\left(\frac{k}{m},\ \bar{x}\left(\frac{k}{m}\right)\right) \tag{1.4}$$

by definition of Wiener measure. The distribution function of (1.4) approaches the distribution function $\sigma(\alpha;\ t)$ at every continuity point of the latter as $m\to\infty$, given that $V(t,\ \bar{x})$ is a limit of continuous functions.

In many instances, it is possible to show that $\sigma(\alpha;\ t)$ is not only the limiting distribution of (1.4) but also that of

$$\frac{1}{m}\sum_{k/m\leqq t} V\left(\frac{k}{m},\ \frac{\overline{X}_1+\cdots+\overline{X}_k}{m^{1/2}}\right) \tag{1.5}$$

where $X_1^{(k)},\ X_2^{(k)},\ \cdots\ (k=1,\ \cdots,\ n)$ are general, independent, identically distributed random variables with mean 0 and variance 1 and $\overline{X}_m = (X_m^{(1)},\ \cdots,\ X_m^{(n)})$. In the last section we shall consider the case of a function $V(t,\ \bar{x}) = V(\bar{x})$ for which such an invariance is easily proved.

Let $V(t,\ \bar{x})$ be bounded below by the constant M in $(t,\ \bar{x})$-space. Then

$$Q(t,\ \bar{x}) = E\{\exp\left(-uy(t)\right) \mid \bar{x}(t) = \bar{x}\}\ \frac{\exp\left(-\bar{x}^2/2t\right)}{(2\pi t)^{n/2}}, \qquad u \geqq 0, \tag{1.6}$$

exists since

$$\begin{aligned} Q(t,\ \bar{x}) &\leqq E\{\exp\left(u\,|\,M\,|\,t\right) \mid \bar{x}(t) = \bar{x}\}\ \frac{\exp\left(-\bar{x}^2/2t\right)}{(2\pi t)^{n/2}} \\ &\leqq \exp\left(u\,|\,M\,|\,t\right)\ \frac{\exp\left(-\bar{x}^2/2t\right)}{(2\pi t)^{n/2}}. \end{aligned} \tag{1.7}$$

Set

$$C(\bar{\xi},\ \tau;\ \bar{x},\ t) = \frac{\exp\left(-\,|\,\bar{x}-\bar{\xi}\,|^2/2(t-\tau)\right)}{(2\pi(t-\tau))^{n/2}}. \tag{1.8}$$

If $V(t,\ \bar{x})$ is bounded in $(t,\ \bar{x})$-space, $Q(t,\ \bar{x})$ satisfies the following integral equation

$$Q(t,\ \bar{x}) + u\int_0^t\int_{-\infty}^{\infty} C(\bar{\xi},\ \tau;\ \bar{x},\ t)V(\tau,\ \bar{\xi})Q(\tau,\ \bar{\xi})\,d\bar{\xi}\,d\tau = C(\bar{0},\ 0;\ \bar{x},\ t) \tag{1.9}$$

where the integration from $-\infty$ to ∞ denotes integration over all of $\bar{x}$ space and $\bar{0}$ denotes the origin of the $\bar{x}$-space. M. Kac proved this result in the 1-dimensional case and the proof for the n-dimensional case follows in a completely analogous manner [1].

DEFINITION 1. *The function $g(t,\ \bar{x})$ is said to satisfy a Hölder condition at $(t',\ \bar{x}')$ if there are numbers $K,\ \alpha>0$ such that*

$$(1.10) \qquad | g(t, \bar{x}) - g(t', \bar{x}') | \leqq K\{ | t - t' |^{\alpha} + | \bar{x} - \bar{x}' |^{\alpha}\}$$

in a neighborhood of $(t', \bar{x}')$.

DEFINITION 2. *The function* $g(t, \bar{x})$ *is said to satisfy a uniform Hölder condition at* $(t', \bar{x}')$ *if there are numbers* $K, \alpha, \delta > 0$ *such that*

$$| g(t + \Delta t, \bar{x} + \bar{\Delta}\bar{x}) - g(t, \bar{x}) | \leqq K\{ | \Delta t |^{\alpha} + | \bar{\Delta}\bar{x} |^{\alpha}\}$$

when

$$| \Delta t | + | \bar{\Delta}\bar{x} | < \delta$$

for all points $(t, \bar{x})$ *in a neighborhood of* $(t', \bar{x}')$.

The boundedness of $V(t, \bar{x})$ in $(t, \bar{x})$-space is assumed throughout §2. The boundedness of $V(t, \bar{x})$ and a set of basic estimates imply that $Q(t, \bar{x})$ satisfies a uniform Hölder condition at all $(t, \bar{x}) \neq (0, \bar{0})$. $Q(t, \bar{x})$ is then shown to satisfy the differential equation

$$(1.11) \qquad \frac{1}{2}\Delta Q - \frac{\partial Q}{\partial t} - uV(t, \bar{x})Q = 0$$

at every point $(t, \bar{x}) \neq (0, \bar{0})$ at which $V(t, \bar{x})$ satisfies a Hölder condition. ΔQ denotes the Laplacian of Q in $\bar{x}$-space. Continuity properties of the derivatives of Q are then examined.

Stronger conditions are assumed in §3 to insure the applicability of Green's theorem in obtaining further results. We assume $V(t, \bar{x})$ satisfies a uniform Hölder condition everywhere except in a regular set S. The definition of a regular set is given in §3. We require that $V(t, \bar{x})$ be bounded below in $(t, \bar{x})$ space. $Q(t, \bar{x})$ is then a solution of equation (1.11) at all points $(t, \bar{x})$ not in the set S and satisfies

$$\lim_{t \to 0} \int_{|\bar{x}| < \epsilon} Q(t, \bar{x})d\bar{x} = 1 \qquad \text{for all } \epsilon > 0$$

and a few other auxiliary conditions. $Q(t, \bar{x})$ is the unique solution of equation (1.11), satisfying these conditions if $V(t, \bar{x}) \geqq 0$. Additional remarks are made indicating that an analogous theorem holds for

$$(1.12) \qquad F(t, \bar{x}) = E\{\exp (iuy(t)) \mid \bar{x}(t) = \bar{x}\} \frac{\exp (-\bar{x}^2/2t)}{(2\pi t)^{n/2}} \cdot$$

The transform

$$(1.13) \qquad q(\bar{x}, s) = \int_0^{\infty} e^{-st}Q(t, \bar{x})dt, \qquad s \geqq 0,$$

is considered in §4 when $V(t, \bar{x}) = V(\bar{x}) \geqq 0$. We again assume that $V(\bar{x})$ satisfies a uniform Hölder condition everywhere except in a regular set S. $q(\bar{x}, s)$ then is the unique solution of

$$(1.14) \qquad 2^{-1}\Delta q - (s + uV(\bar{x}))q = 0$$

at all points $\bar{x} \neq \bar{0}$ not in S, satisfying

$$\lim_{\epsilon \to 0} \int_{|\bar{x}|=\epsilon} \frac{\partial q}{\partial n} ds = -2 \qquad \text{for all } \epsilon > 0$$

and a few other auxiliary conditions. $\partial q/\partial n$ is the derivative of q normal to the sphere $|\bar{x}| = \epsilon$. M. Kac proved this theorem in the 1-dimensional case [2].

Let

$$(1.15) \qquad \sigma(\bar{x}, \alpha, t) = \Pr\left\{ y(t) \leqq \alpha \,\middle|\, \bar{x}(t) = \bar{x} \right\}.$$

The integral equation satisfied by $\sigma(\bar{x}, \alpha, t)$ when $V(t, \bar{x})$ is bounded is derived in §5.

Note that

$$(1.16) \qquad \int_0^\infty e^{-u\alpha} d_\alpha \sigma(\alpha; t) = \int_{-\infty}^\infty Q(t, \bar{x}) d\bar{x}$$

when $V(t, \bar{x}) \geqq 0$. *Moreover, if* $V(t, \bar{x}) = V(\bar{x}) \geqq 0$,

$$(1.17) \qquad \int_0^\infty \int_0^\infty e^{-u\alpha - st} d_\alpha \sigma(\alpha; t) dt = \int_{-\infty}^\infty q(\bar{x}, s) d\bar{x}.$$

2. The parabolic differential equation.

LEMMA 1. *Let* $|V(t, \bar{x})| \leqq M$. *Then*

$$(2.1) \qquad \lim_{t \to 0} \int_{|\bar{x}| \leqq \epsilon} Q(t, \bar{x}) d\bar{x} = 1$$

for all $\epsilon > 0$. *Moreover,* $Q(t, \bar{x})$ *satisfies the uniform Hölder condition*

$$(2.2) \qquad |Q(t + \Delta t, \bar{x} + \bar{\Delta}\bar{x}) - Q(t, \bar{x})| \leqq M(t, \bar{x}) \{ |\bar{\Delta}\bar{x}| + |\Delta t| |\lg \Delta t| \}$$

at all $(t, \bar{x}) \neq (0, \bar{0})$.

Equation (1.9) is basic for the required estimates. The function $C(\bar{0}, 0; \bar{x}, t)$ is infinitely differentiable in t and the components of $\bar{x}$ at all points $(t, \bar{x}) \neq (0, \bar{0})$. Moreover $\lim_{t \to 0} \int_{|\bar{x}| \leqq \epsilon} C(\bar{0}, 0; \bar{x}, t) d\bar{x} = 1$ for all $\epsilon > 0$. Hence we need only consider

$$(2.3) \qquad G(t, \bar{x}) = \int_0^t \int_{-\infty}^\infty C(\bar{\xi}, \tau; \bar{x}, t) V(\tau, \bar{\xi}) Q(\tau, \bar{\xi}) d\bar{\xi} d\tau.$$

Inequality (1.7) implies that

$$\int_{|\bar{x}| \leqq \epsilon} G(t, \bar{x}) d\bar{x} \leqq M e^{u|M|t} t \int_{|\bar{x}| \leqq \epsilon} C(\bar{0}, 0; \bar{x}, t) d\bar{x}$$

so that equation (2.1) is easily verified.

The derivatives

$$\frac{\partial G(t,\bar{x})}{\partial x_k}=\int_0^t\int_{-\infty}^{\infty}\frac{\partial C(\bar{\xi},\tau;\bar{x},t)}{\partial x_k}V(\tau,\bar{\xi})Q(\tau,\bar{\xi})d\bar{\xi}d\tau \tag{2.4}$$

($k=1,\cdots,n$) exist and are continuous everywhere. Hence, the derivatives $\partial Q(t,\bar{x})/\partial x_k$ ($k=1,\cdots,n$) exist and are continuous at all $(t,\bar{x})\neq(0,\bar{0})$. This in turn implies that $Q(t,\bar{x})$ satisfies the uniform Hölder condition

$$\left|Q(t,\bar{x}+\bar{\Delta}\bar{x})-Q(t,\bar{x})\right|\leqq M(t,\bar{x})\left|\bar{\Delta}\bar{x}\right| \tag{2.5}$$

at all $(t,\bar{x})\neq(0,\bar{0})$. We derive the Hölder condition in t again making use of inequality (1.7).

$$\begin{aligned}G(t+\Delta t,\bar{x})&-G(t,\bar{x})\\&=\int_t^{t+\Delta t}\int_{-\infty}^{\infty}C(\bar{\xi},\tau;\bar{x},t+\Delta t)V(\tau,\bar{\xi})Q(\tau,\bar{\xi})d\bar{\xi}d\tau\\&\quad+\int_0^t\int_{-\infty}^{\infty}\left\{C(\bar{\xi},\tau;\bar{x},t+\Delta t)-C(\bar{\xi},\tau;\bar{x},t)\right\}V(\tau,\bar{\xi})Q(\tau,\bar{\xi})d\bar{\xi}d\tau\\&=I_1+I_2, \qquad \Delta t>0.\end{aligned}$$

Now

$$\begin{aligned}\left|I_1\right|&\leqq C(\bar{0},0;\bar{x},t+\Delta t)Me^{uMt}\Delta t,\\ I_2&=\int_{t-\Delta t}^t\int_{-\infty}^{\infty}+\int_0^{\Delta t}\int_{-\infty}^{\infty}+\int_{\Delta t}^{t-\Delta t}\int_{-\infty}^{\infty}(C(\bar{\xi},\tau;\bar{x},t+\Delta t)\\&\qquad -C(\bar{\xi},\tau;\bar{x},t))V(\tau,\bar{\xi})Q(\tau,\bar{\xi})d\bar{\xi}d\tau\\&=I_3+I_4+I_5.\end{aligned}$$

Clearly

$$\left|I_3+I_4\right|\leqq 2\left\{C(\bar{0},0;\bar{x},t+\Delta t)+C(\bar{0},0;\bar{x},t)\right\}Me^{uMt}\Delta t$$

while

$$\begin{aligned}\left|I_5\right|&\leqq \Delta tMe^{uMt}\int_{\Delta t}^{t-\Delta t}\int_{-\infty}^{\infty}\left|\frac{\partial C(\bar{\xi},\tau;\bar{x},t+\theta\Delta t)}{\partial t}\right|C(\bar{0},0;\bar{\xi},\tau)d\bar{\xi}d\tau\\&\leqq \Delta tMe^{uMt}\int_{\Delta t}^{t-\Delta t}\int_{-\infty}^{\infty}\frac{n}{2}\,\frac{(t-\tau+\Delta t)^{n/2}}{(t-\tau)^{(n+2)/2}}C(\bar{\xi},\tau;\bar{x},t+\Delta t)C(\bar{0},0;\bar{\xi},\tau)d\bar{\xi}d\tau\\&\quad+\Delta tMe^{uMt}\int_{\Delta t}^{t-\Delta t}\int_{-\infty}^{\infty}\frac{1}{2}\,\frac{\left|\bar{x}-\bar{\xi}\right|^2(t-\tau+\Delta t)^{n/2}}{(t-\tau)^{(n+4)/2}}C(\bar{\xi},\tau;\bar{x},t+\Delta t)\\&\quad\cdot C(\bar{0},0;\bar{\xi},\tau)d\bar{\xi}d\tau=I_6+I_7, \qquad 0\leqq\theta\leqq 1.\end{aligned}$$

Now

$$
\begin{aligned}
|I_6| &\leqq \Delta t M e^{uMt} C(\bar{0}, 0; \bar{x}, t+\Delta t) \int_{\Delta t}^{t-\Delta t} \frac{(t+\Delta t-\tau)^{n/2}}{(t-\tau)^{(n+2)/2}} d\tau \\
&\leqq M(t, \bar{x})\Delta t \left| \lg \Delta t \right|, \\
|I_7| &\leqq \frac{4\Delta t M e^{uMt}}{(\pi t)^{(n+4)/2}} \int_{\Delta t}^{t/2} \int_{-\infty}^{\infty} |\bar{x} - \bar{\xi}|^2 C(\bar{0}, 0; \bar{\xi}, \tau) d\bar{\xi} d\tau \\
&\quad + \frac{\Delta t M e^{uMt}}{(\pi t)^{n/2}} \int_{t/2}^{t-\Delta t} \frac{(t+\Delta t-\tau)^{(n+2)/2}}{(t-\tau)^{(n+4)/2}} d\tau \int_{-\infty}^{\infty} \bar{u}^2 e^{-\bar{u}^2/2} d\bar{u} \\
&\leqq M(t, \bar{x})\Delta t \left| \lg \Delta t \right|.
\end{aligned}
$$

Therefore

$$
|G(t+\Delta t, \bar{x}) - G(t, \bar{x})| \leqq M(t, \bar{x})\Delta t \left| \lg \Delta t \right|. \tag{2.6}
$$

Inequalities (2.5), (2.6) imply that $Q(t, \bar{x})$ satisfies the uniform Hölder condition (2.2) at all $(t, \bar{x}) \neq (0, \bar{0})$. Lemma 1 is thereby proved.

LEMMA 2. *Let $V(t, \bar{x})$ be a bounded function satisfying a Hölder condition at $(t, \bar{x}) \neq (0, \bar{0})$. Then the function*

$$
H(t, \bar{x}) = \int_0^t \int_{|\bar{x}-\bar{\xi}| \leqq a} V(\tau, \bar{\xi}) C(\bar{\xi}, \tau; \bar{x}, t) d\bar{\xi} d\tau, \qquad a > 0,
$$

satisfies the differential equation

$$
\frac{1}{2}\Delta H - \frac{\partial H}{\partial t} + V(t, \bar{x}) = 0
$$

at $(t, \bar{x})$. The derivatives $\partial^2 H/\partial x_k^2$ $(k = 1, \cdots, n)$ are given by

$$
\frac{\partial^2 H}{\partial x_k^2} = \int_0^t \int_{|\bar{x}-\bar{\xi}| \leqq a} \frac{\partial^2 C(\bar{\xi}, \tau; \bar{x}, t)}{\partial x_k^2} (V(\tau, \bar{\xi}) - V(t, \bar{x})) d\bar{\xi} d\tau.
$$

This lemma is proved on p. 229 of Levi [4] when $\bar{x}$ is 1-dimensional. The proof in the n-dimensional case completely parallels the proof of Levi.

LEMMA 3. *Let $V(t, \bar{x})$ be a bounded function satisfying a Hölder condition at a point $(t, \bar{x}) \neq (0, \bar{0})$. $Q(t, \bar{x})$ then satisfies the differential equation* (1.11) *at $(t, \bar{x})$.*

$V(t, \bar{x})Q(t, \bar{x})$ satisfies a Hölder condition at $(t, \bar{x})$ by Lemma 1. The function $C(\bar{0}, 0; \bar{x}, t)$ satisfies the differential equation

$$
\frac{1}{2}\Delta C - \frac{\partial C}{\partial t} = 0.
$$

Equation (1.9) and Lemma 2 then imply that $Q(t, \bar{x})$ satisfies equation (1.11) at $(t, \bar{x})$.

LEMMA 4. *Let $V(t, \bar{x})$ be a bounded function satisfying a uniform Hölder condition at a point $(t, \bar{x}) \neq (0, \bar{0})$. The derivatives $\partial^2 Q/\partial x_k^2$ $(k=1, \cdots, n)$, $\partial Q/\partial t$ exist in a neighborhood of $(t, \bar{x})$ and are continuous at $(t, \bar{x})$.*

The derivatives $\partial^2 Q/\partial x_k^2$ $(k=1, \cdots, n)$, $\partial Q/\partial t$ exist in a neighborhood of $(t, \bar{x})$ by Lemma 3. Now

$$\frac{\partial Q}{\partial t} = \frac{1}{2}\Delta Q - uVQ$$

so that we need only verify the continuity of $\partial^2 Q/\partial x_k^2$ $(k=1, \cdots, n)$ at $(t, \bar{x})$.

$$\begin{aligned}
&\frac{\partial^2 G(t, \bar{x})}{\partial x_k^2} - \frac{\partial^2 G(t+\Delta t, \bar{x}+\bar{\Delta}\bar{x})}{\partial x_k^2} \\
&\quad = \left\{ \iint_{|\bar{x}-\bar{\xi}|+|t-\tau|<\delta} + \iint_{|\bar{x}-\bar{\xi}|+|t-\tau|\geqq\delta} \right\} \frac{(\partial^2 C(\bar{\xi}, \tau; \bar{x}, t)}{\partial x_k^2} (V(\tau, \bar{\xi})Q(\tau, \bar{\xi}) \\
&\quad - V(t, \bar{x})Q(t, \bar{x})) - \frac{\partial^2 C(\bar{\xi}, \tau; \bar{x} \mp \Delta\bar{x}, t+\Delta t)}{\partial x_k^2} (V(\tau, \bar{\xi})Q(\tau, \bar{\xi}) \\
&\quad - V(t+\Delta t, \bar{x}+\bar{\Delta}\bar{x})Q(t+\Delta t, \bar{x}+\bar{\Delta}\bar{x}))d\bar{\xi}d\tau \\
&\quad = I_8 + I_9.
\end{aligned}$$

I_9 vanishes as $|\Delta t|$, $|\bar{\Delta}\bar{x}| \to 0$. The uniform Hölder condition allows us to obtain the same estimate for each of the two terms of I_8. We find that

$$\begin{aligned}
|I_8| &\leqq 2\iint_{|\bar{x}-\bar{\xi}|+|t-\tau|<2\delta} \frac{\partial^2 C(\bar{\xi}, \tau; \bar{x}, t)}{\partial x_k^2} (V(\tau, \bar{\xi})Q(\tau, \bar{\xi}) - V(t, \bar{x})Q(t, \bar{x}))d\bar{\xi}d\tau \\
&\leqq 2K \int_{|t-\tau|<2\delta} \int_{-\infty}^{\infty} e^{-\bar{u}^2}(1+u_k^2)\left\{ |t-\tau|^{\alpha-1} + |u_k|^\alpha |t-\tau|^{\alpha/2-1} \right\} d\bar{u}d\tau \\
&\leqq \frac{M\delta^{\alpha/2}}{\alpha}.
\end{aligned}$$

The continuity is verified on letting $\delta \to 0$.

3. Unbounded $V(t, \bar{x})$.

DEFINITION 3. *We shall say that the set S is regular if the following conditions are satisfied for every pair $(T, \bar{y})$, $T \geqq 0$, and all $b > b(T, \bar{y}) > 0$:*

(1) *For every $\epsilon > 0$ there is a set $R(\epsilon)$, the sum of a finite number of nonoverlapping closed parallelepipeds each of diameter less than ϵ with sides parallel to the coordinate axes, in $\{0 \leqq t \leqq T, |\bar{x}-\bar{y}| \leqq b\}$ and covering*

$$\{0 \leqq t \leqq T, |\bar{x}-\bar{y}| \leqq b\} \cap S.$$

(2) *The volume of* $R(\epsilon)$ *is less than* ϵ.

(3) *The distance between the complement of* $R(\epsilon)$ *in* $\{0 \leqq t \leqq T, |\bar{x}-\bar{y}| \leqq b\}$ *and S is positive.*

(4) *The total surface area of the parallelepipeds of* $R(\epsilon)$ *is bounded for all* ϵ.

THEOREM 1. *Let* $V(t, \bar{x})$ *be a function bounded below and satisfying a uniform Hölder condition everywhere except in a regular set S. Then* $Q(t, \bar{x})$ *is a solution of equation* (1.11) *at all points* $(t, \bar{x}) \neq (0, \bar{0})$ *not in S and satisfies the following conditions:*

(1) $Q(t, \bar{x}) \to 0$ *as* $|\bar{x}| \to \infty$.

(2) $\lim_{t \to 0} \int_{|\bar{x}| \leqq \epsilon} Q(t, \bar{x}) d\bar{x} = 1$ *for all* $\epsilon > 0$.

(3) $\partial Q/\partial x_k$ $(k=1, \cdots, n)$ *are continuous at all* $(t, \bar{x}) \neq (0, \bar{0})$. $\partial^2 Q/\partial x_k^2$ $(k=1, \cdots, n)$, $\partial Q/\partial t$ *are continuous at all* $(t, \bar{x}) \neq (0, \bar{0})$ *not in S.*

If $V(t, \bar{x}) \geqq 0$, $Q(t, \bar{x})$ *is the unique solution of equation* (1.11), *satisfying conditions* (1)–(3).

Let

$$V_N(t, \bar{x}) = \begin{cases} V(t, \bar{x}) & \text{if } |V(t, \bar{x})| \leqq N, \\ N \text{ otherwise} \end{cases}$$

and

$$Q_N(t, \bar{x}) = E\left\{\exp\left(-u \int_0^t V_N(\tau, \bar{x}(\tau)) d\tau\right) \middle| \bar{x}(t) = \bar{x}\right\} \frac{\exp(-\bar{x}^2/2t)}{(2\pi t)^{n/2}} \cdot$$

Lemma 3 implies that $Q_N(t, \bar{x})$ satisfies

$$\frac{1}{2}\Delta Q_N - \frac{\partial Q_N}{\partial t} - uV_N(t, \bar{x})Q_N = 0$$

at all points $(t, \bar{x}) \neq (0, \bar{0})$ not in S. Inequality (1.7) and Lemmas 1 and 4 imply that $Q_N(t, \bar{x})$ satisfies conditions (1) to (3). Let

$$(3.1) \quad f(\bar{\xi}, \tau; \bar{x}, t) = C(\bar{\xi}, \tau; \bar{x}, t) - \int_0^{t-\tau} \frac{e^{-b^2/2(t-\tau-z)}}{(2\pi(t-\tau-z))^{n/2}} v_n(|\bar{x} - \bar{\xi}|, z) dz$$

where

$$(3.2) \qquad v_n(r, t) = \sum_{j=1}^{\infty} 2 \exp\left(-\alpha_j^{(n)^2} t/2b^2\right) J_{(n-2)/2}(\alpha_j^{(n)} r/b)/J'_{(n-2)/2}(\alpha_j^{(n)})$$

(see §6). The numbers $\alpha_j^{(n)}$ $(j=1, \cdots)$ are the positive zeros of the Bessel function $J_{(n-2)/2}(x)$. $f(\bar{\xi}, \tau; \bar{x}, t)$ is the fundamental solution of

$$\frac{1}{2}\Delta f(\cdot, \cdot; \bar{x}, t) - \frac{\partial f}{\partial t} = 0, \qquad \frac{1}{2}\Delta f(\bar{\xi}, \tau; \cdot, \cdot) + \frac{\partial f}{\partial \tau} = 0$$

with boundary value zero at the spheres of radius b about $\bar{\xi}$, $\bar{x}$ respectively.

Let $b>0$ be such that we can find sets $R(\epsilon)$ satisfying conditions (1) to (4) of definition 3 covering

$$\{0 \leqq \tau \leqq t, \ |\bar{x} - \bar{\xi}| \leqq b\} \cap S.$$

Let

$$R(\epsilon, \delta) = R(\epsilon) \cap \{\delta \leqq \tau \leqq t - \delta, \ |\bar{x} - \bar{\xi}| \leqq b\}, \qquad \delta > 0.$$

Call the surfaces consisting of the upper and lower faces respectively of the parallelepipeds of $R(\epsilon, \delta)$ perpendicular to the t axis, $U(\epsilon)$ and $L(\epsilon)$. Call the surface consisting of the faces of the parallelepipeds of $R(\epsilon, \delta)$ parallel to the t axis $P(\epsilon)$. Let

$$B(W) = \frac{1}{2}\Delta W - \frac{\partial W}{\partial \tau} - uV_N W,$$

$$C(W) = \frac{1}{2}\Delta W + \frac{\partial W}{\partial \tau}\cdot$$

We apply Green's theorem making use of the continuity of $\partial^2 Q_N/\partial x_k^2$ $(k=1, \cdots, n)$, $\partial Q/\partial t$ away from the set S and obtain

$$\begin{aligned}
0 = &\left\{\int_\delta^{t-\delta}\int_{|\bar{x}-\bar{\xi}|\leqq b} - \iint_{R(\epsilon,\delta)}\right\}(f(\bar{\xi}, \tau; \bar{x}, t)B(Q_N(\tau, \bar{\xi})) \\
&- Q_N(\tau, \bar{\xi})C(f(\bar{\xi}, \tau; \cdot, \cdot)))d\bar{\xi}d\tau \\
= &-\frac{1}{2}\int_\delta^{t-\delta}\int_{|\bar{x}-\bar{\xi}|=b}\frac{\partial f(\bar{\xi}, \cdot; \cdot, \cdot)}{\partial n}Q_N ds d\tau \\
&- \left\{\int_\delta^{t-\delta}\int_{|\bar{x}-\bar{\xi}|\leqq b} - \iint_{R(\epsilon,\delta)}\right\} ufV_N Q_N d\bar{\xi}d\tau \\
&+ \frac{1}{2}\int_{P(\epsilon)}\left(\frac{\partial f(\bar{\xi}, \tau; \cdot, \cdot)}{\partial n}Q_N - \frac{\partial Q_N(\tau, \bar{\xi})}{\partial n}f\right)ds \\
&- \left\{\int_{U(\epsilon)} - \int_{L(\epsilon)}\right\} f(\bar{\xi}, \tau; \cdot, \cdot)Q_N ds \\
&+ \int_{|\bar{x}-\bar{\xi}|\leqq b}(f(\bar{\xi}, \delta; \bar{x}, t)Q_N(\delta, \bar{\xi}) - f(\bar{\xi}, t-\delta; \bar{x}, t)Q_N(t-\delta, \bar{\xi}))d\bar{\xi}.
\end{aligned}$$

Let $\epsilon \to 0$. Condition (2) of Definition 3 and the boundedness of $fV_N Q_N$ over the $R(\epsilon, \delta)$ imply that the volume integral over $R(\epsilon, \delta)$ vanishes in the limit. The bounded surface area of $P(\epsilon)$, $U(\epsilon)$, and $L(\epsilon)$, the symmetry of the parallelepipeds, and the continuity of $\partial f/\partial n$, $\partial Q_N/\partial n$, Q_N, f imply that the surface integrals over $P(\epsilon)$, $U(\epsilon)$, $L(\epsilon)$ vanish in the limit. We then let $\delta \to 0$ and obtain

$$Q_N(t, \bar{x}) + u \int_0^t \int_{|\bar{x}-\bar{\xi}| \leqq a} f(\bar{\xi}, \tau; \bar{x}, t) Q_N(\tau, \bar{\xi}) V_N(\tau, \bar{\xi}) d\bar{\xi} d\tau$$
$$= f(\bar{0}, 0; \bar{x}, t) - \frac{1}{2} \int_0^t \int_{|\bar{x}-\bar{\xi}|=a} \frac{\partial f(\bar{\xi}, \tau; \bar{x}, t)}{\partial n} Q_N(\tau, \bar{\xi}) ds d\tau. \tag{3.3}$$

Now

$$\lim_{N \to \infty} Q_N(t, \bar{x}) = Q(t, \bar{x})$$

which is finite since $V(t, \bar{x})$ is bounded below. On taking the limit of equation (3.3) as $N \to \infty$, we have

$$Q(t, \bar{x}) + u \int_0^t \int_{|\bar{x}-\bar{\xi}| \leqq a} Q(\tau, \bar{\xi}) f(\bar{\xi}, \tau; \bar{x}, t) V(\tau, \bar{\xi}) d\bar{\xi} d\tau$$
$$= f(\bar{0}, 0; \bar{x}, t) - \frac{1}{2} \int_0^t \int_{|\bar{x}-\bar{\xi}|=a} \frac{\partial f(\bar{\xi}, \cdot ; \cdot, \cdot)}{\partial n} Q(\tau, \bar{\xi}) ds d\tau. \tag{3.4}$$

This integral equation plays the same role that equation (1.9) did in §2. The surface integral of equation (3.4) is infinitely differentiable in t and the components of $\bar{x}$. It satisfies

$$\frac{1}{2} \Delta C - \frac{\partial C}{\partial t} = 0.$$

The counterparts of Lemmas 1 to 4 with $f(\bar{\xi}, \tau; \bar{x}, t)$ in place of $C(\bar{\xi}, \tau; \bar{x}, t)$ are proved in exactly the same manner as before. Hence $Q(t, \bar{x})$ satisfies equation (1.11) and conditions (1) to (3) of the theorem.

A uniqueness argument for $Q(t, \bar{x})$ as a solution of equation (1.11) can be carried out if $V(t, \bar{x})$ is non-negative. We shall give an example of such a uniqueness argument in the proof of Theorem 2.

THEOREM 2. *Let $V(t, \bar{x})$ satisfy a uniform Hölder condition at all points $(t, \bar{x})$ not in a regular set S. Then $F(t, \bar{x})$ is the unique solution of equation*

$$\frac{1}{2} \Delta F - \frac{\partial F}{\partial t} + iuV(t, \bar{x}) F = 0 \tag{3.5}$$

at all points $(t, \bar{x}) \neq (0, \bar{0})$ not in S with $F(t, \bar{x})$ satisfying the following conditions:

(1) *$F(t, \bar{x}) \to 0$ as $|\bar{x}| \to \infty$.*
(2) *$\lim_{t \to 0} \int_{|\bar{x}| \leqq \epsilon} F(t, \bar{x}) d\bar{x} = 1$ for all $\epsilon > 0$.*
(3) *$\partial F / \partial x_k$ $(k = 1, \cdots, n)$ are continuous at all $(t, \bar{x}) \neq (0, \bar{0})$. $\partial^2 F / \partial x_k^2$*

$(k=1, \cdots, n)$, $\partial F/\partial t$ are continuous at all $(t, \bar{x}) \neq (0, \bar{0})$ not in S.

The proof that $F(t, \bar{x})$ is a solution of (3.5) and satisfies conditions (1) to (3) parallels the proof of Theorem 1. One need not bound $V(t, \bar{x})$ below since

$$\left| F(t, \bar{x}) \right| \leqq \frac{\exp(-\bar{x}^2/2t)}{(2\pi t)^{n/2}}.$$

The analogue of equation (3.4) is

$$\begin{aligned} F(t, \bar{x}) - iu \int_0^t \int_{|\bar{x}-\bar{\xi}| \leqq a} f(\bar{\xi}, \tau; \bar{x}, t) F(\tau, \bar{\xi}) V(\tau, \bar{\xi}) d\bar{\xi} d\tau \\ = f(\bar{0}, 0; \bar{x}, t) - \frac{1}{2} \int_0^t \int_{|\bar{x}-\bar{\xi}|=a} F(\tau, \bar{\xi}) f(\bar{\xi}, \tau; \bar{x}, t) ds d\tau. \end{aligned} \tag{3.6}$$

We now prove that $F(t, \bar{x})$ is the unique solution of equation (3.5) satisfying conditions (1) to (3). Let $F = F_1 - F_2$ be the difference of two such solutions. Let $\tilde{F}$ be the complex conjugate of F. Clearly

$$\frac{1}{2} \Delta \tilde{F} - \frac{\partial \tilde{F}}{\partial t} - iuV(t, \bar{x}) \tilde{F} = 0.$$

Let

$$D(W) = \frac{1}{2} \Delta W - \frac{\partial W}{\partial t} + iuV(t, \bar{x}) W.$$

We apply Green's theorem as in the proof of Theorem 1 and obtain

$$\begin{aligned} \int_0^t \int_{|\bar{x}-\bar{\xi}| \leqq b} \left\{ f(\bar{\xi}, \tau; \bar{x}, t) D(\left| F(\tau, \bar{\xi}) \right|^2) - \left| F(\tau, \bar{\xi}) \right|^2 C(f(\bar{\xi}, \tau; \bar{x}, t)) \right\} d\bar{\xi} d\tau \\ = -\frac{1}{2} \int_0^t \int_{|\bar{x}-\bar{\xi}|=b} \left| F(\tau, \bar{\xi}) \right|^2 \frac{\partial f(\bar{\xi}, \cdot; \cdot, \cdot)}{\partial n} ds d\tau - \left| F(t, \bar{x}) \right|^2 ds d\tau \\ + iu \int_0^t \int_{|\bar{x}-\bar{\xi}| \leqq b} V(\tau, \bar{\xi}) \left| F(\tau, \bar{\xi}) \right|^2 f(\bar{\xi}, \tau; \bar{x}, t) d\bar{\xi} d\tau \\ = \frac{1}{2} \int_0^t \int_{|\bar{x}-\bar{\xi}| \leqq b} f(\bar{\xi}, \tau; \bar{x}, t) \sum_{k=1}^n \left| \frac{\partial F(\tau, \bar{\xi})}{\partial \xi_k} \right|^2 d\bar{\xi} d\tau \\ + iu \int_0^t \int_{|\bar{x}-\bar{\xi}| \leqq b} V(\tau, \bar{\xi}) \left| F(\tau, \bar{\xi}) \right|^2 f(\bar{\xi}, \tau; \bar{x}, t) d\bar{\xi} d\tau. \end{aligned} \tag{3.7}$$

Since $\left| F(t, \bar{x}) \right| \leqq M$, on letting $b \to \infty$ in equation (3.7), we obtain

$$-\left| F(t, \bar{x}) \right|^2 = \frac{1}{2} \int_0^t \int_{-\infty}^{\infty} C(\bar{\xi}, \tau; \bar{x}, t) \sum_{k=1}^n \left| \frac{\partial F(\tau, \bar{\xi})}{\partial \xi_k} \right|^2 d\bar{\xi} d\tau.$$

This cannot be so unless $F(t, \bar{x}) = F_1 - F_2 \equiv 0$. The uniqueness proof is complete. The theorem is thereby proved.

Consider

$$Q(\tau, \bar{\xi}; t, \bar{x}) = E\{\exp(-u(y(t) - y(\tau))) \mid \bar{x}(t) = \bar{x}, \bar{x}(\tau) = \bar{\xi}\} \cdot \frac{\exp(-|\bar{x} - \bar{\xi}|^2/2(t-\tau))}{(2\pi(t-\tau))^{n/2}},$$

$0 \leqq \tau \leqq t$, $0 \leqq V(t, \bar{x})$. We again assume $V(t, \bar{x})$ satisfies a uniform Hölder condition at all points $(t, \bar{x})$ not in a regular set S. $Q(\tau, \bar{\xi}; t, \bar{x})$ exists and satisfies equation (1.11). Moreover $Q(\tau, \bar{\xi}; t, \bar{x})$ satisfies the following conditions:

(1) $Q(\tau, \bar{\xi}; t, \bar{x}) \to 0$ as $|\bar{x}| \to \infty$.

(2) $\lim_{t \to \tau+} \int_{|\bar{x}-\bar{\xi}| \leqq \epsilon} Q(\tau, \bar{\xi}; t, \bar{x}) d\bar{x} = 1$ for all $\epsilon > 0$.

$Q(\tau, \bar{\xi}; t, \bar{x})$ satisfies the adjoint differential equation and the corresponding conditions in the backward variables $\tau, \bar{\xi}$. The proof is essentially the proof of Theorem 1. Note that $Q(\tau, \bar{\xi}; t, \bar{x})$ satisfies the Chapman-Kolmogorov equation. It does not, however, have the norming of a probability density.

4. **The elliptic differential equation.** Let

$$\psi(s, r) = \frac{1}{((2s)^{1/2}r)^{(n-2)/2}} \left\{ K_{(n-2)/2}((2s)^{1/2}r) - \frac{I_{(n-2)/2}((2s)^{1/2}r) K_{(n-2)/2}((2s)^{1/2}b)}{I_{(n-2)/2}((2s)^{1/2}b)} \right\} \tag{4.1}$$

(see §6). $\psi(s, |\bar{x}|)$ is the fundamental solution of

$$\frac{1}{2} \Delta\psi - s\psi = 0$$

with boundary value zero at $|\bar{x}| = b$. Note that

$$\lim_{\epsilon \to 0} \int_{|\bar{x}| = \epsilon} \frac{\partial \psi}{\partial n} ds = -2. \tag{4.2}$$

Lemma 5. *Let $V(\bar{x})$ be a bounded function satisfying a Hölder condition at the point $\bar{x} \neq 0$. Then the function*

$$H(\bar{x}) = \int_{|\bar{x}-\bar{\xi}| \leqq a} V(\bar{\xi}) \psi(s, |\bar{x} - \bar{\xi}|) d\bar{\xi}$$

satisfies the differential equation

$$\frac{1}{2} \Delta H - sH + V(\bar{x}) = 0$$

at $\bar{x}$.

The proof parallels an argument of Kellogg on p. 153 [3] proving that

$$h(\bar{x}) = \int_{|\bar{x}-\bar{\xi}|\leqq a} \frac{V(\bar{\xi})}{|\bar{x}-\bar{\xi}|} d\bar{\xi}, \qquad \bar{x} = (x_1, x_2, x_3),$$

satisfies

$$\frac{1}{2}\Delta h + V(\bar{x}) = 0$$

at $\bar{x}\neq 0$ if $V(\bar{x})$ satisfies a Hölder condition at $\bar{x}$.

THEOREM 3. *Let $V(\bar{x})\geqq 0$ satisfy a uniform Hölder condition at all $\bar{x}$ not in a regular set S. Then $q(\bar{x}, s)$ is the unique solution of equation* (1.14), *satisfying the following conditions*:

(1) *$q(\bar{x}, s)\rightarrow 0$ as $|\bar{x}|\rightarrow\infty$.*

(2) $\lim_{\epsilon\rightarrow 0}\int_{|\bar{x}|\leqq\epsilon}(\partial q/\partial n)ds = -2$.

(3) *$\partial q/\partial x_k$ $(k=1, \cdots, n)$ are continuous at all $\bar{x}\neq\bar{0}$. $\partial^2 q/\partial x_k^2$ $(k=1, \cdots, n)$ are continuous at all $\bar{x}\neq\bar{0}$ not in S.*

We obtain the integral equation

$$(4.3)\qquad \begin{aligned} q(s, \bar{x}) + u\int_{|\bar{x}-\bar{\xi}|\leqq b} \psi(s, |\bar{x}-\bar{\xi}|)V(\bar{\xi})q(s, \bar{\xi})d\bar{\xi} \\ = \psi(s, |\bar{x}|) - \frac{1}{2}\int_{|\bar{x}-\bar{\xi}|=b} q(s, \bar{\xi})\frac{\partial\psi(s, |\bar{x}-\bar{\xi}|)}{\partial n} ds \end{aligned}$$

by Laplace transforming equation (3.4) with respect to t. $\partial q/\partial x_k$ $(k=1, \cdots, n)$ exist and are continuous at all $\bar{x}\neq\bar{0}$ as can be seen by differentiating equation (4.3). Hence $q(s, \bar{x})\,V(\bar{x})$ satisfies a Hölder condition at $\bar{x}\neq\bar{0}$ when $V(\bar{x})$ does. Lemma 5 implies that $q(s, \bar{x})$ satisfies equation (1.14) at all $\bar{x}\neq\bar{0}$ not in S. Equations (4.2) and (4.3) imply that

$$\lim_{\epsilon\rightarrow 0}\int_{|\bar{x}|=\epsilon}\frac{\partial q}{\partial n}ds = -2.$$

Now

$$0 \leqq q(s, \bar{x}) \leqq \frac{K_{(n-2)/2}((2s)^{1/2}r)}{((2s)^{1/2}r)^{(n-2)/2}}$$

since $V(\bar{x})\geqq 0$. Hence

$$q(s, \bar{x}) \rightarrow 0 \quad \text{as} \quad |\bar{x}| \rightarrow \infty.$$

The proof of the continuity of $\partial^2 q/\partial x_k^2$ $(k=1, \cdots, n)$ proceeds as in Lemma 4. The uniqueness argument for $q(s, \bar{x})$ satisfying conditions (1) to (3) is analogous to the uniqueness argument for $F(t, \bar{x})$ carried out in §3.

5. The integral equation.

THEOREM 4. *Let $V(t, \bar{x})$ be bounded. Then $\sigma(\bar{x}, \alpha, t)$ satisfies the following integral equation*

$$
\begin{aligned}
&C(\bar{0}, 0; \bar{x}, t) \int_{y}^{y+\lambda} (J(\alpha) - \sigma(\bar{x}, \alpha, t))d\alpha \\
(5.1) \quad &= \int_0^t \int_{-\infty}^{\infty} C(\xi, \tau; \bar{x}, t)C(\bar{0}, 0; \xi, \tau)V(\tau, \xi)\{\sigma(\xi, y+\lambda, \tau) \\
&\qquad\qquad - \sigma(\xi, y, \tau)\} d\xi d\tau
\end{aligned}
$$

where

$$J(x) = \frac{1 + \operatorname{sgn} x}{2}.$$

It is clear that

$$F(t, \bar{x})/C(\bar{0}, 0; \bar{x}, t) = \int_{-\infty}^{\infty} e^{i\alpha u} d_\alpha \sigma(\bar{x}, \alpha, t).$$

Since $V(t, \bar{x})$ is bounded, on letting $a \to \infty$ in equation (3.6) we obtain

$$(5.2) \quad F(t, \bar{x}) - C(\bar{0}, 0; \bar{x}, t) - iu \int_0^t \int_{-\infty}^{\infty} C(\xi, \tau; \bar{x}, t)F(\tau, \xi)V(\tau, \xi)d\xi d\tau = 0.$$

The boundedness of $V(t, \bar{x})$ implies that

$$\int_{-\infty}^{\infty} \alpha d_\alpha \sigma(\bar{x}, \alpha, t)$$

exists. The modified form of the P. Lévy inversion formula [5] used requires the existence of the first moment. Multiply equation (5.1) by $(1 - e^{-i\lambda u}/(iu)^2) \cdot (1/2\pi)e^{-iuy}$ and integrate with respect to y from $-T$ to T. The interchange of order of integration goes through readily. In the limit as $T \to \infty$ we obtain equation(5.1).

Interest in the integral equation (5.1) arises for several reasons. The equation holds without any strong regularity conditions on $V(t, \bar{x})$. Moreover, the density

$$\Pr\{y(t) = y, \bar{x}(t) = \bar{x}\} = \frac{\partial \sigma}{\partial y}(\bar{x}, y, t)$$

need not exist even when very strong regularity conditions are imposed on $V(t, \bar{x})$ so that it makes no sense to speak of the density satisfying the differential equation

$$\frac{\partial P}{\partial t} = \frac{1}{2}\Delta P - V(t, \bar{x})\frac{\partial P}{\partial y}.$$

In particular this is so when $V(t, \bar{x}) \equiv 1$.

6. **An example.** We illustrate the theory in n dimensions by considering the function

$$V(\bar{x}) = \begin{cases} 1, & |\bar{x}| \geqq b, \\ 0, & |\bar{x}| < b. \end{cases}$$

The invariance proof for the distributions of the Wiener functionals $\int_0^t V(\bar{x}(\tau))d\tau$ considered closely parallels that given in [2] for $V(x) = (1+\operatorname{sgn} x)/2$. The computations for the case $n=1$ have been carried out in [2] and elsewhere. We solve the equation

$$\Delta q - 2(s + uV(r))q = 0, \qquad V(r) = \begin{cases} 0, & r < b, \\ 1, & r \geqq b, \end{cases}$$

where $r = |\bar{x}|$. The solution of the differential equation is given in terms of the Bessel functions $I_{(n-2)/2}$, $K_{(n-2)/2}$ by

$$q(s, r) = \begin{cases} \dfrac{1}{((2s)^{1/2}r)^{(n-2)/2}}\left(\alpha K_{(n-2)/2}((2s)^{1/2}r) + \beta I_{(n-2)/2}((2s)^{1/2}r)\right), & r < b, \\ \dfrac{\gamma}{((2(s+u))^{1/2}r)^{(n-2)/2}} K_{(n-2)/2}((2(s+u))^{1/2}r), & r \geqq b, \end{cases}$$

where α, β, γ do not depend on r. We evaluate α, β, γ by making use of the auxiliary conditions. The continuity of q and $\partial q/\partial r$ at b implies that

$$\alpha K_{(n-2)/2}((2s)^{1/2}b) + \beta I_{(n-2)/2}((2s)^{1/2}b) = \gamma\left(\frac{s}{s+u}\right)^{(n-2)/4} K_{(n-2)/2}((2(s+u))^{1/2}b),$$

$$-\alpha K_{n/2}((2s)^{1/2}b) + \beta I_{n/2}((2s)^{1/2}b) = -\gamma\left(\frac{s}{s+u}\right)^{(n-4)/4} K_{n/2}((2(s+u))^{1/2}b).$$

We solve the equations above and find that

$$\beta = \frac{\begin{vmatrix} -K_{(n-2)/2}((2s)^{1/2}b) & -\left(\dfrac{s}{s+u}\right)^{(n-2)/4} K_{(n-2)/2}((2(s+u))^{1/2}b) \\ K_{n/2}((2s)^{1/2}b) & \left(\dfrac{s}{s+u}\right)^{(n-4)/4} K_{n/2}((2(s+u))^{1/2}b) \end{vmatrix}}{D}$$

and

$$\gamma = \frac{\alpha}{(2s)^{1/2}bD}$$

where

$$D = \begin{vmatrix} I_{(n-2)/2}((2s)^{1/2}b) & -\left(\dfrac{s}{s+u}\right)^{(n-2)/4} K_{(n-2)/2}((2(s+u))^{1/2}b) \\ I_{n/2}((2s)^{1/2}b) & \left(\dfrac{s}{s+u}\right)^{(n-4)/4} K_{n/2}((2(s+u))^{1/2}b) \end{vmatrix}.$$

The condition

$$\lim_{\epsilon\to 0} \int_{|\bar{x}|=\epsilon} \frac{\partial q}{\partial n}\, ds = -2$$

implies that $\alpha = 1$. On letting $u \to \infty$, $q(s, r)$ tends to limit $\psi(s, r)$ given by (4.1) for $r < b$. $q(s, r)$ tends to zero for $r \geqq b$. The limiting expression is the Laplace transform of the probability density of diffusion from $\bar{0}$ to a point $\bar{x}$, r units away from $\bar{0}$, when there is an absorbing barrier at the sphere of radius b about $\bar{0}$. We invert

$$I_{(n-2)/2}((2s)^{1/2}r)/I_{(n-2)/2}((2s)^{1/2}b)$$

and obtain $v_n(r, t)$ given by (3.6). Hence the probability density of the diffusion is $f(\bar{0}, 0; \bar{x}, t)$ for $|\bar{x}| \leqq b$ and zero for $|\bar{x}| \geqq b$.

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