ON THE OSCILLATION OF SUMS OF RANDOM VARIABLES(1)

BY

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1. **Introduction.** Let X be a random variable with distribution function $F(x)$ and characteristic function $\phi(z) = \int_{-\infty}^{\infty} e^{izx} dF(x)$. The sequence of partial sums $\{S_n\}$ will be said to be generated by X if $S_n = \sum_{j=1}^{n} X_j$ where $X_1, \cdots, X_n, \cdots$ are independent, identically distributed random variables with distribution function $F(x)$.

Let the abbreviations i.o. and f.o. denote the phrases "infinitely often" and "finitely often," respectively. The sequence $\{S_n\}$ generated by the random variable X is said to *oscillate* if

$$P\{S_n > 0 \text{ i.o.}\} = P\{S_n \leqq 0 \text{ i.o.}\} = 1. \tag{1.1}$$

A sufficient condition for oscillation of the sequence S_n, convenient for the application of results concerning partial limit laws of normed sums of independent and identically distributed random variables, will be obtained. When $E(|X|) < \infty$, the problem considered will be shown to be equivalent to a problem dealt with by K. L. Chung and W. H. J. Fuchs [1](2).

The necessary and sufficient condition for the oscillation of the sequence $\{S_n\}$ generated by X will be obtained. The necessary and sufficient condition is used to obtain a class of random variables each of which generates sums S_n which satisfy

$$\begin{aligned} P\left\{\lim_{n\to\infty} |S_n| = \infty\right\} &= P\left\{\liminf_{n\to\infty} S_n = -\infty\right\} \\ &= P\left\{\limsup_{n\to\infty} S_n = \infty\right\} = 1 \end{aligned} \tag{1.2}$$

and

$$\lim_{n\to\infty} P\{S_n > 0\} = 0 \tag{1.3}$$

simultaneously.

2. **Preliminary results.**

LEMMA 2.1. *If* $\limsup_{n\to\infty} P\{S_n>0\}>0$ $(\limsup_{n\to\infty} P\{S_n<0\}>0)$, *then* $P\{S_n>0 \textit{ i.o.}\}=1$ $(P\{S_n<0 \textit{ i.o.}\}=1)$.

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Now $\{S_n<0 \text{ i.o.}\}=\prod_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \{S_n>0\}$. But $P\{\bigcup_{n=m}^{\infty}\{S_n>0\}\} \geqq \limsup_{n\to\infty} P\{S_n>0\}>0$. Since $\bigcup_{n=m}^{\infty}\{S_n>0\}$ is a monotone decreasing sequence of events,

$$P\{S_n > 0 \text{ i.o.}\} = \lim_{m\to\infty} P\left\{\bigcup_{n=m}^{\infty} \{S_n > 0\}\right\} > 0.$$

P. Lévy has shown (see [4, p. 147]) that if $A_1, A_2, \cdots$ are given constants,

$$P\{S_n > A_n \text{ i.o.}\}$$

can be only either zero or one. Hence

$$P\{S_n > 0 \text{ i.o.}\} = 1.$$

One shows that $\limsup_{n\to\infty} P\{S_n<0\}>0$ implies $P\{S_n<0 \text{ i.o.}\}=1$ in like manner.

K. L. Chung and W. H. J. Fuchs [1] have studied recurrent values of partial sums S_n generated by a random variable X. The value b is said to be recurrent if

$$P\{|S_n - b| < \epsilon \text{ i.o.}\} = 1$$

for every $\epsilon>0$. They have shown that the set of recurrent values is a closed additive group. In particular, 0 is the one and only recurrent value if and only if $X=0$ with probability 1.

LEMMA 2.2. *If the partial sums S_n generated by X have recurrent values and if $X\neq 0$ with positive probability, the sequence $\{S_n\}$ oscillates.*

Zero cannot be the only recurrent value since $X\neq 0$ with positive probability. There must then be two values $a>0$, $b<0$ which are recurrent values, that is, given any $\epsilon>0$

$$P\{|S_n - b| < \epsilon \text{ i.o.}\} = P\{|S_n - a| < \epsilon \text{ i.o.}\} = 1.$$

Let $\epsilon<\min(a, |b|)$. Then

$$P\{S_n > 0 \text{ i.o.}\} = P\{|S_n - a| < \epsilon \text{ i.o.}\} = 1,$$
$$P\{S_n < 0 \text{ i.o.}\} = P\{|S_n - b| < \epsilon \text{ i.o.}\} = 1.$$

THEOREM 2.1. *If $E(|X|)<\infty$ and $X\neq 0$ with positive probability, the problem of oscillation and the Chung-Fuchs problem are equivalent. Under these conditions, oscillation takes place if and only if $E(X)=0$.*

If $X\neq 0$ with positive probability, Lemma 2.2 implies that the Chung-Fuchs problem is included in the problem of oscillation. Chung and Fuchs [1] have shown that if $E(|X|)<\infty$ and $E(X)=0$, there are recurrent values and hence there is oscillation. But if $E(X)=m\neq 0$, the strong law of large numbers tells us that

$$P\left\{\lim_{n\to\infty} \frac{S_n}{n} = m\right\} = 1$$

and hence (1.1) cannot be true. The equivalence of the two problems when $E(|X|) < \infty$ is demonstrated.

If $f(z)$ is a complex-valued function, let Re $f(z)$ and Im $f(z)$ denote the real and imaginary parts of $f(z)$, respectively.

Let

$$\log_2 z = \log \{-\log z\} \text{ and } \log_n z = \log \{\log_{n-1} z\}, \qquad n > 2.$$

LEMMA 2.3. *The sequence $\{S_n\}$ generated by X has no recurrent values if the characteristic function $\phi(z)$ of X is such that*

$$(2.1) \qquad -\liminf_{z\to 0+} \frac{1 - \operatorname{Re} \phi(z)}{z \log z \cdots \log_{n-1} z \log_n^{1+\epsilon} z} > k > 0$$

for some $\epsilon > 0$ and an integer $n \geqq 1$.

A necessary and sufficient condition that there be recurrent values [1] is that

$$\lim_{\rho\to 1-} \int_{-\delta}^{\delta} \frac{1}{1 - \rho\phi(z)} dz = \infty, \qquad \delta > 0.$$

But

$$\lim_{\rho\to 1-} \int_{-\delta}^{\delta} \frac{1}{1 - \rho\phi(z)} dz = \lim_{\rho\to 1-} \int_{-\delta}^{\delta} \frac{1 - \rho \operatorname{Re} \phi(z)}{(1 - \rho \operatorname{Re} \phi(z))^2 + (\rho \operatorname{Im} \phi(z))^2} dz$$
$$\leqq \lim_{\rho\to 1-} \int_{-\delta}^{\delta} \frac{1}{1 - \rho \operatorname{Re} \phi(z)} dz$$
$$\leqq \int_{-\delta}^{\delta} \frac{1}{1 - \operatorname{Re} \phi(z)} dz.$$

But this last integral converges if (2.1) is satisfied.

LEMMA 2.4. *A necessary and sufficient condition that S_n oscillate is that*

$$(2.2) \qquad \sum_{j=1}^{\infty} P\{S_j > 0, S_{j+1} \leqq 0\} = \infty$$

and

$$(2.3) \qquad \sum_{j=1}^{\infty} P\{S_j \leqq 0, S_{j+1} > 0\} = \infty.$$

The necessity is trivial since the theorem of Borel-Cantelli tells us that

$$\sum_{j=1}^{\infty} P\{S_j > 0, S_{j+1} \leqq 0\} < \infty$$

or

$$\sum_{j=1}^{\infty} P\{S_j \leqq 0, S_{j+1} > 0\} < \infty$$

implies that there is no oscillation with probability 1.

Now

$$\begin{aligned} P\{S_n > 0 \text{ f.o.}\} &= P\{S_n \leqq 0; n = 1, 2, \cdots\} \\ &\quad + \sum_{j=1}^{\infty} P\{S_j > 0, S_{j+n} \leqq 0; n = 1, 2, \cdots\} \\ &\geqq P\{S_n \leqq 0; n = 1, 2, \cdots\} \\ &\quad + \sum_{j=1}^{\infty} P\{S_j > 0, S_{j+1} \leqq 0, S_{j+n} - S_{j+1} \leqq 0; n = 1, 2, \cdots\} \\ &= P\{S_n \leqq 0; n = 1, 2, \cdots\} \\ &\quad + \sum_{j=1}^{\infty} P\{S_j > 0, S_{j+1} \leqq 0\} P\{S_n \leqq 0; n = 1, 2, \cdots\} \\ &= P\{S_n \leqq 0; n = 1, 2, \cdots\} \left[1 + \sum_{j=1}^{\infty} P\{S_j > 0, S_{j+1} \leqq 0\}\right]. \end{aligned}$$

The divergence of $\sum_{j=1}^{\infty} P\{S_j>0, S_{j+1}\leqq 0\}$ implies that $P\{S_n \leqq 0; n=1, 2, \cdots\}=0$. But

$$P\{S_n > 0 \text{ f.o.}\} \leqq \lim_{m\to\infty} P\{S_n \leqq 0; n = 1, 2, \cdots\} \cdot \left[1 + \sum_{j=1}^{m} P\{S_j > 0\}\right].$$

Hence the divergence of $\sum_{j=1}^{\infty} P\{S_j>0, S_{j+1}\leqq 0\}$ implies that

$$P\{S_n > 0 \text{ f.o.}\} = 0.$$

An analogous argument indicates that

$$\sum_{j=1}^{\infty} P\{S_j \leqq 0, S_{j+1} > 0\} = \infty \quad \text{implies that} \quad P\{S_n \leqq 0 \text{ f.o.}\} = 0.$$

3. **Application of the sufficient condition.** One makes use of Lemma 2.1 and the study of the infinitely divisible laws as partial limit laws of normed sums of independent random variables to characterize a class of random variables each of which generates partial sums S_n that oscillate. Random variables with no finite first moment are of interest since Chung and Fuchs have completely solved the problem for random variables having a first moment.

THEOREM 3.1. *Let the sequence S_n be generated by a random variable X with distribution function $F(x)$. If there are monotone sequences $n_k \to \infty$ and $a_k \to \infty$ of positive integers and positive numbers respectively such that*

$$\text{(i)} \qquad \lim_{\epsilon\to 0}\lim_{k\to\infty}\frac{n_k}{a_k}\left[\int_{|x|<\epsilon a_k} x\,dF(x) + \int_{|x|<\epsilon a_k}\frac{x}{1+(x/a_k)^2}\,dF(x)\right] = m,$$

$$\text{(ii)} \qquad \lim_{\epsilon\to 0}\lim_{k\to\infty}\frac{n_k}{a_k^2}\left[\int_{|x|<\epsilon a_k} x^2\,dF(x) - \left(\int_{|x|<\epsilon a_k} x\,dF(x)\right)^2\right] = \sigma^2,$$

$$\text{(iii)} \qquad \lim_{k\to\infty} n_k(1 - F(xa_k)) = -\Omega(x), \qquad x > 0,$$

$$\lim_{k\to\infty} n_k F(-xa_k) = \Omega(-x), \qquad x > 0,$$

where $\Omega(x)$ is finite for $|x| > 0$ and $\Omega(-\infty) = \Omega(\infty) = 0$,

(iv) . $\Omega(x)$ *increases somewhere in both the ranges $x > 0$ and $x < 0$,*

then the sequence $\{S_n\}$ oscillates.

W. Doeblin [2] has shown that if conditions (i), (ii), (iii) are satisfied, the normed subsequence S_{n_k}/a_k has a limiting distribution function as $k \to \infty$ whose characteristic function is

$$\exp\left\{imz - \frac{\sigma^2}{2}z + \int_{-\infty}^{\infty}\left(e^{izx} - 1 - \frac{izx}{1+x^2}\right)d\Omega(x)\right\}.$$

Note that $\Omega(x)$ is nondecreasing in the ranges $x > 0$ and $x < 0$. If $\Omega(x)$ increases somewhere in both the ranges $x > 0$ and $x < 0$, the limiting distribution function attributes positive probability to both positive and negative values, that is,

$$\lim_{k\to\infty} P\left\{\frac{S_{n_k}}{a_k} > 0\right\} > 0, \qquad \lim_{k\to\infty} P\left\{\frac{S_{n_k}}{a_k} < 0\right\} > 0.$$

This follows from the fact that the characteristic function

$$\exp\left\{\int_{-\infty}^{\infty}\left(e^{izx} - 1 - \frac{izx}{1+x^2}\right)d\Omega(x)\right\}$$

is the limit of a sequence each element of which is the characteristic function of a weighted sum of properly centered independent and non-identically distributed Poisson random variables (see [4, pp. 173–180]).

But then Lemma 2.1 implies that the sequence $\{S_n\}$ oscillates.

4. Application of the necessary and sufficient condition. The object of this section is to obtain Theorems 4.2 and 4.3 which give a characterization of a class of random variables satisfying (1.2) and (1.3) simultaneously.

LEMMA 4.1. *Let $F(x)$ be a distribution function with $\int |x|^\alpha dF(x) < \infty$ for some α, $0<\alpha<1$. Given $Y>0$, let*

$$F(x;Y) = \begin{cases} 0 & \text{if } x \leqq -Y, \\ F(x) - F(-Y) & \text{if } |x| < Y, \\ F(Y) - F(-Y) & \text{if } x \geqq Y, \end{cases}$$

and $\phi(z;Y)$ be the Fourier-Stieltjes transform of $F(x;Y)$. Then $\operatorname{Im} \phi(z;Y) = o(|z|^\alpha)$ *at $z=0$ uniformly for all $Y>0$.*

Now $|\operatorname{Im} \phi(z,Y)| = \int_{|x|<Y} \sin zx dF(x) \leqq \int_{|x|<Y} |xz|^\alpha |\sin xz|^{1-\alpha} dF(x) = |z|^\alpha o(1)$ uniformly for all $Y>0$ by the Lebesgue theorem on dominated convergence.

Let X be a lattice random variable, i.e., there is a largest $h>0$ such that

$$P\{X = kh\} = p_k \geqq 0, \qquad k = 0, \pm 1, \pm 2, \cdots,$$

and

$$\sum_{k=-\infty}^{\infty} p_k = 1.$$

Let

$$\phi_1(z) = \sum_{j=-\infty}^{-1} p_j e^{ijhz}, \qquad \phi_2(z) = \sum_{j=1}^{\infty} p_j e^{ijhz}.$$

THEOREM 4.1. *Let X be a lattice random variable with $E(|X|^\alpha) < \infty$ for some α, $0<\alpha<1$, and let the partial sums generated by X have no recurrent values. A necessary and sufficient condition that the partial sums generated by X oscillate is that*

$$\begin{aligned} \lim_{\rho\to 1-} \Bigg[& \frac{h(1-\rho)}{4\pi} \int_{-\pi/h}^{\pi/h} \frac{\sin zh}{1-\cos zh} \frac{\operatorname{Im} \phi_2(z)}{|1-\rho\phi(z)|^2} dz \\ & + \frac{\rho h}{4\pi} \int_{-\pi/h}^{\pi/h} \frac{\sin zh}{1-\cos zh} \Bigg(\frac{\{\phi_1(0) - \operatorname{Re} \phi_1(z)\} \operatorname{Im} \phi_2(z)}{|1-\rho\phi(z)|^2} \\ & \qquad - \frac{\{\phi_2(0) - \operatorname{Re} \phi_2(z)\} \operatorname{Im} \phi_1(z)}{|1-\rho\phi(z)|^2} \Bigg) dz \Bigg] \end{aligned} \tag{4.1}$$

$$\begin{aligned} = \lim_{\rho\to 1-} \Bigg[& -\frac{h(1-\rho)}{4\pi} \int_{-\pi/h}^{\pi/h} \frac{\sin zh}{1-\cos zh} \frac{\operatorname{Im} \phi_1(z)}{|1-\rho\phi(z)|^2} dz \\ & + \frac{\rho h}{4\pi} \int_{-\pi/h}^{\pi/h} \frac{\sin zh}{1-\cos zh} \Bigg(\frac{\{\phi_1(0) - \operatorname{Re} \phi_1(z)\} \operatorname{Im} \phi_2(z)}{|1-\rho\phi(z)|^2} \\ & \qquad - \frac{\{\phi_2(0) - \operatorname{Re} \phi_2(z)\} \operatorname{Im} \phi_1(z)}{|1-\rho\phi(z)|^2} \Bigg) dz \Bigg] = \infty. \end{aligned} \tag{4.2}$$

Lemma 2.4 implies that

$$\lim_{\rho\to 1-} \sum_{j=1}^{\infty} \rho^j P\{S_j > 0, S_{j+1} \leqq 0\} = \infty, \tag{4.3}$$

$$\lim_{\rho\to 1-} \sum_{j=1}^{\infty} \rho^j P\{S_j \leqq 0, S_{j+1} > 0\} = \infty \tag{4.4}$$

is a necessary and sufficient condition for oscillation.

First consider (4.3).

$$\sum_{j=1}^{\infty} \rho^j P\{S_j > 0, S_{j+1} \leqq 0\} = \sum_{j=1}^{\infty} \rho^j \sum_{k=1}^{\infty} P\{S_j = kh\} \sum_{n=-\infty}^{-k} P\{X = nh\}. \tag{4.5}$$

But $P\{S_j = kh\} = (h/2\pi)\int_{-\pi/h}^{\pi/h} \phi^j(z)e^{-ikhz}dz$ where $\phi(z)$ is the characteristic function of X. Expression (4.5) can then be rewritten as

$$\sum_{j=1}^{\infty} \rho^j \sum_{k=1}^{\infty} \frac{h}{2\pi}\int_{-\pi/h}^{\pi/h} \phi^j(z)e^{-ikhz}dz \sum_{n=-\infty}^{-k} p_n$$

$$= \sum_{k=1}^{\infty} \frac{h}{2\pi}\int_{-\pi/h}^{\pi/h} e^{-ikhz}\frac{\rho\phi(z)}{1-\rho\phi(z)}\,dz \sum_{j=-\infty}^{-k} p_j$$

$$= \sum_{j=-1}^{-\infty} \sum_{k=1}^{-j} \frac{h}{2\pi}\int_{-\pi/h}^{\pi/h} e^{-ikhz}\frac{\rho\phi(z)}{1-\rho\phi(z)}\,dz p_j$$

$$= \sum_{j=-1}^{-\infty} \frac{h}{2\pi}\int_{-\pi/h}^{\pi/h} \frac{e^{-ihz} - e^{i(j-1)hz}}{1-e^{-ihz}}\, p_j \frac{\rho\phi(z)}{1-\rho\phi(z)}\,dz$$

$$= \frac{h}{2\pi}\int_{-\pi/h}^{\pi/h} \frac{e^{-ihz}\{\phi_1(0) - \phi_1(z)\}}{1-e^{-ihz}}\, \frac{\rho\phi(z)}{1-\rho\phi(z)}\,dz \tag{4.6}$$

where the last interchange of summation and integration follows by applying Lemma 4.1. Now expression (4.6) can be rewritten as

$$-\frac{h}{2\pi}\int_{-\pi/h}^{\pi/h} \{\phi_1(0) - \phi_1(z)\}\, \frac{\rho\phi(z)}{1-\rho\phi(z)}\,dz \tag{4.7}$$

$$+\frac{h}{2\pi}\int_{-\pi/h}^{\pi/h} \frac{\phi_1(0) - \phi_1(z)}{1-e^{-ihz}}\, \frac{\rho\phi(z)}{1-\rho\phi(z)}\,dz. \tag{4.8}$$

But expression (4.7) equals

$$-P\{X<0\}\sum_{n=1}^{\infty} \rho^n P\{S_n = 0\} + \sum_{j=-1}^{-\infty} p_j \sum_{n=1}^{\infty} \rho^n P\{S_n = -jh\}$$

which is bounded by

$$\sum_{j=1}^{\infty} P\{S_j = 0\} < \infty$$

in absolute value. The assumption that the partial sums have no recurrent values implies that $\sum_{j=1}^{\infty} P\{S_j=0\} < \infty$ [1]. Hence we need consider only (4.8) which can be written as

$$(4.9) \qquad -\frac{h}{2\pi}\int_{-\pi/h}^{\pi/h} \frac{\phi_1(0) - \phi_1(z)}{1 - e^{-ihz}} dz$$

$$(4.10) \qquad +\frac{h}{2\pi}\int_{-\pi/h}^{\pi/h} \frac{\phi_1(0) - \phi_1(z)}{1 - e^{-ihz}} \frac{1}{1 - \rho\phi(z)} dz.$$

But (4.9) equals

$$- P\{X < 0\}$$

so that only (4.10) need be considered. Expression (4.10) equals

$$(4.11) \qquad \frac{h}{4\pi}\int_{-\pi/h}^{\pi/h} \frac{\phi_1(0) - \phi_1(z)}{1 - \rho\phi(z)} dz$$

$$(4.12) \qquad -\frac{hi}{4\pi}\int_{-\pi/h}^{\pi/h} \frac{\sin zh}{1 - \cos zh} \frac{\phi_1(0) - \phi_1(z)}{1 - \rho\phi(z)} dz.$$

But (4.11) is bounded in absolute value by

$$1 + \sum_{j=1}^{\infty} P\{S_j = 0\} < \infty$$

so that only (4.12) need be considered. Expression (4.12) can be rewritten as

$$\begin{aligned}
& -\frac{hi}{4\pi}\int_{-\pi/h}^{\pi/h} \frac{\sin zh}{1 - \cos zh} \frac{\phi_1(0) - \phi_1(z)}{|1 - \rho\phi(z)|^2} (1 - \rho\bar{\phi}(z))dz \\
&= -\frac{hi}{4\pi}\int_{-\pi/h}^{\pi/h} \frac{\sin zh}{1 - \cos zh} \frac{\phi_1(0) - \phi_1(z)}{|1 - \rho\phi(z)|^2} (1 - \rho)dz \\
&\quad -\frac{hi}{4\pi}\int_{-\pi/h}^{\pi/h} \frac{\sin zh}{1 - \cos zh} \frac{\phi_1(0) - \phi_1(z)}{|1 - \rho\phi(z)|^2} (1 - \bar{\phi}(z))\rho dz
\end{aligned}$$

$$\begin{aligned}
(4.13) \qquad &= -\frac{h(1 - \rho)}{4\pi}\int_{-\pi/h}^{\pi/h} \frac{\sin zh}{1 - \cos zh} \frac{\operatorname{Im} \phi_1(z)}{|1 - \rho\phi(z)|^2} dz \\
&\quad + \frac{\rho h}{4\pi}\int_{-\pi/h}^{\pi/h} \frac{\sin zh}{1 - \cos zh} \left(\frac{\{\phi_1(0) - \operatorname{Re} \phi_1(z)\} \operatorname{Im} \phi_2(z)}{|1 - \rho\phi(z)|^2} \right. \\
&\qquad \left. - \frac{\{\phi_2(0) - \operatorname{Re} \phi_2(z)\} \operatorname{Im} \phi_1(z)}{|1 - \rho\phi(z)|^2} \right) dz.
\end{aligned}$$

Hence equation (4.3) is true if and only if the limit of (4.13) as $\rho \to 1-$ is infinite. One derives condition (4.1) in a completely analogous manner from equation (4.4).

Given the distribution function $F(x)$, consider the two auxiliary distribution functions

$$F_2(x) = \begin{cases} 0 & \text{if} \quad x \leqq 0, \\ \dfrac{F(x)}{1 - F(0+)} & \text{if} \quad x > 0 \end{cases}$$

and

$$F_1(x) = \begin{cases} \dfrac{F(x)}{F(0-)} & \text{if} \quad x < 0, \\ 1 & \text{if} \quad x \geqq 0. \end{cases}$$

Let $\phi(z)$, $\Phi_1(z)$, $\Phi_2(z)$ be the characteristic functions of $F(x)$, $F_1(x)$, and $F_2(z)$, respectively.

LEMMA 4.2. *Let $F(x)$ be such that*

$$1 - F_2(x) = x^{-\alpha} h_2(x), \qquad F_1(x) = x^{-\beta} h_1(x),$$

$0 < \alpha, \beta < 1$ *where*

$$\lim_{x \to \infty} \frac{h_2(cx)}{h_2(x)} = \lim_{x \to -\infty} \frac{h_1(cx)}{h_1(x)} = 1$$

for every constant $c > 0$. Then

$$\frac{1 - \operatorname{Re} \Phi_2(z)}{\Gamma(1-\alpha) \cos(\pi\alpha/2)}, \quad \frac{\operatorname{Im} \Phi_2(z)}{\sin(\pi\alpha/2)\Gamma(1-\alpha)} \sim 1 - F_2\left(\frac{1}{z}\right) \quad as \quad z \to 0+$$

and

$$\frac{1 - \operatorname{Re} \Phi_1(z)}{\Gamma(1-\alpha) \cos(\pi\alpha/2)}, \quad \frac{-\operatorname{Im} \Phi_1(z)}{\sin(\pi\alpha/2)\Gamma(1-\alpha)} \sim F_1\left(-\frac{1}{z}\right) \quad as \quad z \to 0+.$$

Let $\{a_k\}$, $\{b_k\}$ be positive sequences such that

$$1 - F_2(a_k) \sim 1/k, \qquad F_1(-b_k) \sim 1/k$$

as $k \to \infty$. Let $Y_1, Y_2, \cdots$ be independent random variables with common distribution function $F_2(x)$, while $Z_1, Z_2, \cdots$, are independent random variables with common distribution function $F_1(x)$. Let S_n and T_n be their corresponding partial sums. Then [2, 3]

$$P\{S_n \leqq x a_n\} \to G_\alpha(x), \qquad P\{T_n \leqq x b_n\} \to H_\beta(x)$$

where $G_\alpha(x)$, $H_\beta(x)$ have the characteristic functions $\gamma_\alpha(z)$, $\delta_\beta(z)$ and

$$\delta_\beta(z) = \bar{\gamma}_\beta(z),\ \gamma_\alpha(z) = \exp - \left\{ |z|^\alpha \left(\cos \frac{\pi\alpha}{2} - i \sin \frac{\pi\alpha}{2} \operatorname{sgn} z \right) \Gamma(1-\alpha) \right\}.$$

Then

$$n(1 - \operatorname{Re} \Phi_2(z/a_n)) \to z^\alpha \cos \frac{\pi\alpha}{2} \Gamma(1-\alpha),$$

$$n \operatorname{Im} \Phi_2(z/a_n) \to z^\alpha \sin \frac{\pi\alpha}{2} \Gamma(1-\alpha)$$

as $n \to \infty$, $z > 0$. But the choice of $\{a_k\}$ and the properties of $h_2(x)$ imply that

$$n(1 - F_2(a_n x)) \to \frac{1}{x^\alpha}$$

as $n \to \infty$. This implies that

$$\frac{1 - \operatorname{Re} \Phi_2(z)}{\cos(\pi\alpha/2)\Gamma(1-\alpha)}, \quad \frac{\operatorname{Im} \Phi_2(z)}{\sin(\pi\alpha/2)\Gamma(1-\alpha)} \sim 1 - F_2\left(\frac{1}{z}\right)$$

as $z \to 0+$. The analogous result for $1 - \operatorname{Re} \Phi_1(z)$, $\operatorname{Im} \Phi_1(z)$ follows in like manner.

THEOREM 4.2. *Let $F(x)$ be such that*

$$1 - F_2(x) = x^{-\alpha} h_2(x), \qquad F_1(x) = x^{-\beta} h_1(x),$$

$0 < \alpha, \beta < 1$, *where*

$$\lim_{x\to\infty} \frac{h_2(cx)}{h_2(x)} = \lim_{x\to-\infty} \frac{h_1(cx)}{h_1(x)} = 1$$

for every constant $c > 0$. There is no oscillation of the partial sums generated by a random variable X with distribution function $F(x)$ if $\alpha \neq \beta$. Assume $\alpha = \beta$. Then a sufficient condition for the partial sums generated to oscillate is that there be a positive integer n and constants $k_1, k_2 > 0$ such that

$$\text{(A)} \qquad \frac{-k_2}{\log z \cdots \log_n z} < \frac{h_1(-1/z)}{h_2(1/z)} < -k_1 \log z \cdots \log_n z$$

as $z \to 0+$. On the other hand, if there is a positive integer n and there are constants $\epsilon, k > 0$ such that

$$\text{(B)} \qquad \frac{h_1(-1/z)}{h_2(1/z)} > -k \log z \cdots \log_{n-1} z \log_n^{1+\epsilon} z$$

or

$$\text{(C)}\qquad \frac{h_2(1/z)}{h_1(-1/z)} > -k \log z \cdots \log_{n-1} z \log_n^{1+\epsilon} z$$

as $z \to 0+$, *there is no oscillation.*

Let X first be a lattice random variable. If X has a distribution function of the form specified above, Lemmas 4.2 and 2.3 imply that there are no recurrent values. In investigating expressions (4.1) and (4.2) as $\rho \to 1-$, it is clear that one need only consider the indicated integrations over a neighborhood $(-\epsilon, \epsilon)$, $\epsilon > 0$, of zero. Lemma 4.2 implies that

$$(4.14)\qquad \frac{\phi_2(0) - \operatorname{Re}\phi_2(z)}{\cos(\pi\alpha/2)}, \quad \frac{\operatorname{Im}\phi_2(z)}{\sin(\pi\alpha/2)} \sim z^\alpha h_2\left(\frac{1}{z}\right)\Gamma(1-\alpha)(1 - F(0+))$$

and

$$(4.15)\qquad \frac{\phi_1(0) - \operatorname{Re}\phi_1(z)}{\cos(\pi\alpha/2)}, \quad \frac{-\operatorname{Im}\phi_1(z)}{\sin(\pi\alpha/2)} \sim z^\beta h_1\left(-\frac{1}{z}\right)\Gamma(1-\alpha)F(0-)$$

as $z \to 0+$. Now

$$0 \leqq -\frac{h(1-\rho)}{4\pi}\int_{-\epsilon}^{\epsilon} \frac{\sin zh}{1-\cos zh}\,\frac{\operatorname{Im}\phi_1(z)}{|1-\rho\phi(z)|^2}\,dz$$

$$(4.16)\qquad \leqq -\frac{h}{4\pi}\int_{-\epsilon}^{\epsilon}\frac{\sin zh}{1-\cos zh}\,\frac{\operatorname{Im}\phi_1(z)}{1-\operatorname{Re}\phi(z)}\,dz.$$

The non-negativity of the integrand and Fatou's lemma imply that

$$\begin{aligned}\liminf_{\rho\to 1-}\int_{-\epsilon}^{\epsilon}\frac{\sin zh}{1-\cos zh}\Bigg(&\frac{\{\phi_1(0)-\operatorname{Re}\phi_1(z)\}\operatorname{Im}\phi_2(z)}{|1-\rho\phi(z)|^2}\\ &-\frac{\{\phi_2(0)-\operatorname{Re}\phi_2(z)\}\operatorname{Im}\phi_1(z)}{|1-\rho\phi(z)|^2}\Bigg)dz\\ \geqq \int_{-\epsilon}^{\epsilon}\frac{\sin zh}{1-\cos zh}\Bigg(&\frac{\{\phi_1(0)-\operatorname{Re}\phi_1(z)\}\operatorname{Im}\phi_2(z)}{|1-\phi(z)|^2}\\ &-\frac{\{\phi_2(0)-\operatorname{Re}\phi_2(z)\}\operatorname{Im}\phi_1(z)}{|1-\phi(z)|^2}\Bigg)dz.\end{aligned}$$

However

$$\frac{1}{|1-\rho\phi(z)|^2} \leqq \frac{1}{(1-\operatorname{Re}\phi(z))^2}, \qquad 0 < \rho \leqq 1.$$

But $|1-\phi(z)|$ and $1-\operatorname{Re}\phi(z)$ have the same behavior at zero as can be seen by making use of (4.14) and (4.15). Hence

$$\int_{-\epsilon}^{\epsilon} \frac{\sin zh}{1-\cos zh} \frac{\{\phi_1(0) - \operatorname{Re} \phi_1(z)\} \operatorname{Im} \phi_2(z) - \{\phi_2(0) - \operatorname{Re} \phi_2\} \operatorname{Im} \phi_1(z)}{|1-\rho\phi(z)|^2} dz$$

diverges as $\rho \to 1-$ if and only if

$$\int_0^{\epsilon} \frac{1}{z} \frac{z^{\alpha+\beta} h_2(1/z) h_1(-1/z)}{z^{2\alpha} h_1^2(-1/z) + z^{2\beta} h_2(1/z)} dz \tag{4.17}$$

diverges.

Let $\beta > \alpha$. By making use of (4.14) and (4.15), we see that (4.16) is bounded by a multiple of

$$\int_0^{\epsilon} z^{\beta-\alpha-1} \frac{h_1(-1/z)}{h_2(1/z)} dz$$

which converges since

$$\frac{h_1(-1/z)}{h_2(1/z)} = o(z^{\gamma})$$

as $z \to 0+$ for every $\gamma < 0$. Expression (4.17) converges for the same reason. Hence, (4.2) converges and there is no oscillation. If $\alpha > \beta$, a similar argument shows that (4.1) converges.

Let $\alpha = \beta$. Assume condition (C). Then both (4.16) and (4.17) are bounded by a multiple of

$$-\int_0^{\epsilon} \frac{dz}{z \log z \cdots \log_{n-1} z \log_n^{1+\epsilon} z}$$

which converges. Expression (4.2) converges and there is no oscillation. If condition (B) were valid, a similar argument would show that (4.1) converges.

Let $\alpha = \beta$ and assume condition (A). Expression (4.16) is bounded below by zero and (4.17) is greater than

$$\int_0^{\epsilon} \frac{dz}{z \log z \cdots \log_n z}$$

which diverges. Hence (4.2) diverges. A similar argument shows that (4.1) diverges. It then follows that the partial sums generated oscillate.

Now let X be a random variable whose distribution function satisfies the assumptions of Theorem 4.2. Let $C(x)$ denote the greatest integer less than or equal to x. Consider the two auxiliary lattice random variables

$$X_- = hC(X/h), \qquad X_+ = X_- + h,$$

where h is such that $P\{X_-=h\}$, $P\{X_+=h\}>0$. Now

$$X_- \leqq X \leqq X_+. \tag{4.18}$$

X_-, X, X_+ have distribution functions with the same behavior at $-\infty$ and $+\infty$. Therefore X_-, X, X_+ have characteristic functions with the same behavior as $z\to 0$. In view of what has been proved, X_- and X_+ generate partial sums which either oscillate together or are both positive finitely often with probability one or are both negative finitely often with probability one. Hence, the theorem applies to X in view of (4.18).

THEOREM 4.3. *Let $F(x)$ be such that*

$$1 - F_2(x) = x^{-\alpha}h_2(x), \qquad F_1(x) = x^{-\alpha}h_1(x), \qquad 0 < \alpha < 1,$$

where

$$\lim_{x\to\infty} \frac{h_2(cx)}{h_2(x)} = \lim_{x\to-\infty} \frac{h_1(cx)}{h_1(x)} = 1$$

for every constant $c>0$. Assume condition (A) *and*

$$\lim_{x\to\infty} \frac{h_2(x)}{h_1(-x)} = 0, \qquad \left(\lim_{x\to\infty} \frac{h_1(-x)}{h_2(x)} = 0\right).$$

Then

$$\begin{aligned} P\left\{\lim_{k\to\infty} |S_k| = \infty\right\} &= P\left\{\liminf_{k\to\infty} S_k = -\infty\right\} \\ &= P\left\{\limsup_{k\to\infty} S_k = \infty\right\} = 1, \end{aligned} \tag{4.19}$$

$$\lim_{k\to\infty} P\{S_k > 0\} = 0, \qquad \left(\lim_{k\to\infty} P\{S_k < 0\} = 0\right) \tag{4.20}$$

simultaneously.

If condition (A) is valid, the partial sums generated oscillate but have no recurrent values. Whenever there is oscillation and there are no recurrent values (4.19) is true. For then

$$\sum_{k=1}^{\infty} P\{|S_k| < m\} < \infty$$

for all $m>0$ [1] and hence by the theorem of Borel-Cantelli

$$P\left\{\lim_{k\to\infty} |S_k| = \infty\right\} = 1.$$

The fact that there is oscillation implies that

$$P\left\{\liminf_{k\to\infty} S_k = -\infty\right\} = P\left\{\limsup_{k\to\infty} S_k = \infty\right\} = 1.$$

Now assume

$$\lim_{x\to\infty} \frac{h_2(x)}{h_1(-x)} = 0.$$

Let $\{b_k\}$ be such that

$$F_1(-b_k) \sim 1/k$$

as $k + \infty$. Then

$$1 - F_2(b_k) = o(1/k)$$

as $k \to \infty$ and

$$P\{S_k < xb_k\} \to H_\alpha(x)$$

as $n \to \infty$ by [2]. But $H_\alpha(x)$ is the distribution function of a random variable that is nonpositive. Hence

$$\lim_{k\to\infty} P\{S_k > 0\} = 0.$$

A similar argument shows that

$$\lim_{x\to\infty} \frac{h_1(-x)}{h_2(x)} = 0$$

implies that

$$\lim_{k\to\infty} P\{S_k < 0\} = 0.$$

References

1. K. L. Chung and W. H. J. Fuchs, *On the distribution of sums of random variables*, Memoirs of the American Mathematical Society, No. 6, 1951, pp. 1–12.

2. W. Doeblin, *Sur l'ensemble de puissances d'une loi de probabilité*, Studia Mathematica vol. 9 (1941) pp. 71–96.

3. W. Feller, *Fluctuation theory of recurrent events*, Trans. Amer. Math. Soc. vol. 67 (1949) pp. 98–119.

4. P. Lévy, *Théorie de l'addition de variables aléatoires*, Paris, 1937.

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