THE AUTOMORPHISM GROUP OF A LIE GROUP

BY

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Introduction. The group $A(G)$ of all continuous and open automorphisms of a locally compact topological group G may be regarded as a topological group, the topology being defined in the usual fashion from the compact and the open subsets of G (see §1). In general, this topological structure of $A(G)$ is somewhat pathological. For instance, if G is the discretely topologized additive group of an infinite-dimensional vector space over an arbitrary field, then $A(G)$ already fails to be locally compact.

On the other hand, if G is a connected Lie group, we shall show without any difficulty that the compact-open topology of $A(G)$ coincides with the topology obtained by identifying $A(G)$ with a closed subgroup of the linear group of automorphisms of the Lie algebra of G, as was done by Chevalley (in [1]) in order to make $A(G)$ into a Lie group. We shall then deduce that $A(G)$ is a Lie group whenever the group of its components, G/G_0, is finitely generated(1), where G_0 denotes the component of the identity element in G.

The other questions with which we shall be concerned are the following: Let $I(G_0)$ denote the group of the inner automorphisms of G_0, and let $E(G_0, G)$ denote the natural image in $A(G_0)$ of $A(G)$. Regard $I(G_0)$ and $E(G_0, G)$ as subgroups of $A(G_0)$. Are these subgroups closed in $A(G_0)$? Is $E(G_0, G)$ *topologically*, as well as group-theoretically, isomorphic with the corresponding factor group of $A(G)$?

We shall show, under the assumption that G/G_0 is finitely generated, that these questions are related as follows: The natural continuous homomorphism of $A(G)$ onto $E(G_0, G)$ is open if and only if $E(G_0, G)$ is closed in $A(G_0)$. Under the stronger assumption that G/G_0 is finite, a sufficient condition for $E(G_0, G)$ to be closed in $A(G_0)$ is that $I(G_0)$ be closed in $A(G_0)$. Finally, in order to throw some light on the difficulties which are involved here, we shall give a simple example in which $I(G_0)$ and $E(G_0, G)$ are not closed in $A(G_0)$. In this example, G has only two components and G_0 is homeomorphic with Euclidean 5-space.

1. **Topological preparation.** We shall describe the topology of a group G in terms of a fundamental system $\mathfrak{V}$ of neighborhoods V of the identity element. A system $\mathfrak{V}$ of subsets of G will define a Hausdorff topology consistent with the group operations if and only if it satisfies the following conditions(2):

Received by the editors May 1, 1951.

(1) I am indebted to the referee for the remark that my original requirement, "G/G_0 finite", can be relaxed to the present one.

(2) We are taking these from §2 of [4].

I. The intersection of all $V \in \mathfrak{V}$ is the set consisting of the identity element of G only.

II. If V_1 and V_2 are sets belonging to $\mathfrak{V}$, there is a $V \in \mathfrak{V}$ such that $V \subseteq V_1 \cap V_2$.

III. For every $V \in \mathfrak{V}$ there is a $W \in \mathfrak{V}$ such that $W^{-1}W \subseteq V$.

IV. For every $g \in G$ and $V \in \mathfrak{V}$ there is a $W \in \mathfrak{V}$ such that $W \subseteq gVg^{-1}$.

The neighborhoods of the identity element are then all the sets containing a set belonging to $\mathfrak{V}$.

If C is any compact subset of G and $V \in \mathfrak{V}$, we denote by $N(C, V)$ the set of all $\alpha \in A(G)$ for which $\alpha(x)x^{-1} \in V$ and $\alpha^{-1}(x)x^{-1} \in V$, whenever $x \in C$. We claim that if G is locally compact, the system of these $N(C, V)$ satisfies conditions I–IV above. In fact, I holds quite evidently. If $V \subseteq V_1 \cap V_2$, we clearly have $N(C_1 \cup C_2, V) \subseteq N(C_1, V_1) \cap N(C_2, V_2)$, so that II is satisfied. In order to verify III we proceed as follows: Since G is locally compact, given $V \in \mathfrak{V}$, there is a compact set C_0 and a $V_0 \in \mathfrak{V}$ such that $V_0 \subseteq C_0$ and $V_0V_0V_0 \subseteq V$. From the identity $(\beta^{-1}\alpha)(c)c^{-1} = [\beta^{-1}(\alpha(c)c^{-1})(\alpha(c)c^{-1})^{-1}] \cdot [\alpha(c)c^{-1}][\beta^{-1}(c)c^{-1}]$ we can see immediately that we have then $N(C \cup C_0, V_0)^{-1} N(C \cup C_0, V_0) \subseteq N(C, V)$, which shows that III is satisfied. Finally, we have, with $\alpha \in A(G)$, $\alpha^{-1}N(\alpha(C), \alpha(V))\alpha \subseteq N(C, V)$, whence IV holds.

From now on, if G is any locally compact group, $A(G)$ will denote the group of all continuous and open automorphisms of G, with the topology defined by the $N(C, V)$, where C ranges over the compact subsets of G and V over the set of neighborhoods of the identity element in G.

Next we shall prove two elementary results which we shall need later on.

LEMMA 1. *Let G be a topological group, U a neighborhood of the identity element, C a compact subset of G. Then the intersection of all the sets $c^{-1}Uc$, with $c \in C$, is a neighborhood of the identity element.*

Proof. We can find $V \in \mathfrak{V}$ such that $VVV^{-1} \subseteq U$. Then we have $y^{-1}Uy \supseteq V$, for every $y \in V$. Since C is compact, there are elements $c_1, \cdots, c_n$ in C such that C is contained in the union of the n sets Vc_i. Now if c is any element in C, we write $c = yc_i$, with $y \in V$. Then $c^{-1}Uc = c_i^{-1}y^{-1}Uyc_i \supseteq c_i^{-1}Vc_i$. Hence the intersection of all the sets $c^{-1}Uc$ contains the finite intersection of the $c_i^{-1}Vc_i$ and is therefore a neighborhood of the identity element.

LEMMA 2. *Suppose that G is connected and locally compact. Let C be a compact subset of G, V a neighborhood of the identity, and S a compact neighborhood of the identity. Then there exists a neighborhood W of the identity such that $N(S, W) \subseteq N(C, V)$.*

Proof. Since G is connected and S a neighborhood of the identity, we have $G = \bigcup_{n=1}^{\infty} S^n$. Since C is compact and since $S^n \subseteq S^{n+1}$, it follows that $C \subseteq S^m$, for some m. Now choose a neighborhood T of the identity such that $T^m \subseteq V$, and put $W = \bigcap_{x \in S^m} x^{-1}Tx$. By Lemma 1, W is a neighborhood of the identity, for S^m is compact(3). Now let $\alpha \in N(S, W)$ and $c \in C$. We have $c = x_1, \cdots, x_m$, with $x_i \in S$. Put $c_k = x_1 \cdots x_k$, and suppose we have already shown that $\alpha(c_k)c_k^{-1} \in T^k$. Then we have $\alpha(c_{k+1})c_{k+1}^{-1} = (\alpha(c_k)c_k^{-1})c_k(\alpha(x_{k+1})x_{k+1}^{-1})c_k^{-1} \in T^k c_k W c_k^{-1} \subseteq T^{k+1}$. Hence we get $\alpha(c)c^{-1} \in T^m \subseteq V$, which clearly suffices to establish our lemma.

2. **Automorphism groups.** Let G be a connected Lie group. An automorphism $\alpha \in A(G)$ induces an automorphism $\dot{\alpha}$ of the Lie algebra $\mathfrak{G}$ of G. We denote by $A(\mathfrak{G})$ the group of automorphisms of $\mathfrak{G}$, with the topology induced by that of the full linear group of which $A(\mathfrak{G})$ is clearly a closed subgroup. It is shown in [1] that the mapping $\alpha \rightarrow \dot{\alpha}$ is a group isomorphism of $A(G)$ onto a closed subgroup of $A(\mathfrak{G})$. (If G is simply-connected the image of of $A(G)$ coincides with $A(\mathfrak{G})$.) We shall prove the following result:

THEOREM 1. *Let G be a connected Lie group. Then the group isomorphism $\alpha \rightarrow \dot{\alpha}$ of $A(G)$ onto the corresponding closed subgroup of $A(\mathfrak{G})$ is also a homeomorphism.*

Proof. We denote by e the "exponential mapping" of $\mathfrak{G}$ into G(4). We have then, for $\alpha \in A(G)$, $e\dot{\alpha} = \alpha e$, and e gives an analytic isomorphism between a neighborhood of 0 in $\mathfrak{G}$ and a neighborhood of the identity in G. Let $z_1, \cdots, z_n$ be a linear basis for $\mathfrak{G}$ such that the corresponding solid sphere $\mathfrak{Z}_2$ of radius 2, in the Euclidean metric defined by our basis, around 0 in $\mathfrak{G}$, is mapped by e 1-1 and analytically onto the canonical sphere $Z_2 = e(\mathfrak{Z}_2)$ around the identity element in G. For any positive real number p, $\mathfrak{Z}_p$ will denote the closed solid sphere of radius p around 0 in $\mathfrak{G}$, and we set $Z_p = e(\mathfrak{Z}_p)$.

Now if N is any neighborhood of the identity in $A(\mathfrak{G})$, there is a real number s such that $0 < s < 1$ and such that every $\tau \in A(\mathfrak{G})$ satisfying $\tau(z_i) - z_i \in \mathfrak{Z}_s$ belongs to N. It follows from the elementary properties of the exponential mapping e that there is a real number $q > 1$ and a real number r, $s > r > 0$, such that, for all $a, b \in \mathfrak{Z}_r$, we have $e(a)e(b) = e(a+b+c)$, with $|c| < q \cdot |a| \cdot |b|$, where $|u|$ denotes the distance of u from 0 in $\mathfrak{G}$. Let $\alpha \in N(Z_r, Z_{r^2/2q})$. Then, for $0 \leqq t \leqq r$, $\alpha(e(tz_i))e(tz_i)^{-1} = e(u_i(t))$, where $u_i(t) \in \mathfrak{Z}_{r^2/2q}$. Hence $e(t\dot{\alpha}(z_i)) = e(u_i(t))e(tz_i) = e(u_i(t) + tz_i + v_i(t))$, with $|v_i(t)| < q|u_i(t)|t \leqq r^3/2$. In particular, this shows that $t\dot{\alpha}(z_i)$ remains in $\mathfrak{Z}_{2r}$ as t varies from 0 to r, and that we must have $r(\dot{\alpha}(z_i) - z_i) = u_i(r) + v_i(r)$. Therefore, $|\dot{\alpha}(z_i) - z_i| \leqq r/2q + r^2/2 < r < s$, whence $\dot{\alpha} \in N$. Thus we have shown that $N(Z_r, Z_{r^2/2q})^{\cdot} \subseteq N$, and this implies that the mapping $\alpha \rightarrow \dot{\alpha}$ is continuous.

On the other hand, given any neighborhood V of the identity in G, we can find a real number $r > 0$ such that $e(u+z) \in Ve(z)$ whenever $|z| \leqq 1$ and $|u| \leqq r$, because e is uniformly continuous on $\mathfrak{Z}_2$. Hence if $\dot{\alpha}$ is such that $\dot{\alpha}(z) - z \in \mathfrak{Z}_r$ for all $z \in \mathfrak{Z}_1$, then $\alpha(x)x^{-1} \in V$ for all $x \in Z_1$. Since the $N(Z_1, V)$

(3) If A and B are compact, so is AB; see §3 of [4].

(4) See §VIII, Chap. IV, of [1].

constitute a fundamental system of neighborhoods of the identity in $A(G)$, by Lemma 2, and since the $\dot{\alpha}$ satisfying the above conditions make up a neighborhood of the identity in the image of $A(G)$ in $A(\mathfrak{G})$, we conclude that the mapping $\dot{\alpha}\rightarrow\alpha$ is also continuous. This completes the proof of Theorem 1.

THEOREM 2. *Let G be a Lie group, and let G_0 denote the component of the identity in G. Suppose that the group of components, G/G_0, is finitely generated. Then $A(G)$ is a Lie group and has at most countably many components.*

Proof. Our assumption on G/G_0 means that there is a finite set $g_1, \cdots, g_m$ of elements of G such that every component P of G is of the form $P=pG_0$, where p is a product of g_i's (with repetitions allowed). Now let B denote the subgroup of $A(G)$ which consists of all those automorphisms that map each component of G onto itself. It is clear from the form pG_0 of a component that we have $N((g_1, \cdots, g_m), G_0)\subseteq B$, whence we see that B is open in $A(G)$. (Note that G_0 is open in G.) Furthermore, since $A(G)/B$ is isomorphic with a subgroup of the automorphism group $A(G/G_0)$ of the finitely generated group G/G_0, it follows that $A(G)/B$ has at most countably many elements. Hence it will suffice to prove that B is a Lie group with at most countably many components.

Let H be the semi-direct product $(G_0^m\times A(G_0))_A$ whose elements are the $m+1$-tuples $(c_1, \cdots, c_m, \alpha)$,with $c_i\in G_0$ and $\alpha\in A(G_0)$, and where products are defined by the formula $(c_1, \cdots, c_m, \alpha)$ $(d_1, \cdots, d_m, \beta)=(c_1\alpha(d_1), \cdots, c_m\alpha(d_m), \alpha\beta)$. If we topologize H by the natural product topology it is evident, since $A(G_0)$ is a Lie group, that H is a Lie group. Furthermore, since $A(G_0)$ may, according to Theorem 1, be identified with a subgroup of the full linear group, its topology satisfies the second axiom of countability, and the same holds therefore for H.

Now we define a mapping ϕ of B into H by setting $\phi(b)=(g_1^{-1}b(g_1), \cdots, g_m^{-1}b(g_m), \beta)$, where β is the restriction of b to G_0. It is immediately seen that ϕ is a continuous isomorphism of B into H. We show next that ϕ^{-1} maps $\phi(B)$ continuously onto B. For this it suffices to show that, if V is any neighborhood of the identity in G and C any compact subset of G, there is a neighborhood M of the identity in H for which $\phi^{-1}(M\cap\phi(B))\subseteq N(C, V)$. Now, since C is compact, there is a finite set $p_1, \cdots, p_k$ of products of the g_i's such that $C\subseteq\bigcup_{j=1}^k p_jG_0$. Let $S=\bigcup_{j=1}^k(p_j^{-1}C)\cap G_0$, and let W be a neighborhood of the identity in G_0. Note that S is compact, so that the corresponding subset $N_0(S, W)$ of $A(G_0)$ is a neighborhood of the identity in $A(G_0)$. Put $U=W\times\cdots\times W\times N_0(S, W)$. Then U is a neighborhood of the identity in H, and so is $M=U\cap U^{-1}$. Since every element of C can be written in the form $c=p_js$, with $s\in S$, it is not hard to see that, with a suitable choice of W, the neighborhood M will satisfy our above requirement. Hence ϕ is a homeomorphism of B onto $\phi(B)$.

Now for each component P of G select an explicit product p of g_i's such that $P = pG$. Let $\eta = (c_1, \cdots, c_m, \alpha)$ be an arbitrary element H. Let p' denote the element of G which is obtained from p by replacing each g_i by $g_i c_i$. For $x \in G_0$, define $\tilde{\eta}(px) = p'\alpha(x)$. Then $\tilde{\eta}$ is evidently a homeomorphism of G onto itself. It follows that η will belong to $\phi(B)$ if and only if $\tilde{\eta}(uv) = \tilde{\eta}(u)\tilde{\eta}(v)$, for all $u, v \in G$. This shows that $\phi(B)$ is closed in H. Hence $\phi(B)$ is a Lie group. Since H satisfies the second axiom of countability, so does $\phi(B)$. Since the components of a Lie group are open sets, it follows that the number of components of B is at most countable. Hence also B is a Lie group with at most countably many components, and our proof is complete.

3. **The restriction homomorphism.** Let G be a Lie group, G_0 the component of the identity element in G. We assume that G/G_0 is finitely generated, or—which is equivalent—that G is generated by a compact subset. The restriction of automorphisms to G_0 evidently gives a continuous homomorphism, ρ say, of $A(G)$ onto a subgroup $E(G_0, G)$ of $A(G_0)$. Let R denote the kernel of ρ. It is natural to inquire under what conditions $A(G)/R$ is isomorphic, as a *topological* group, with $E(G_0, G)$, or, equivalently, under what conditions ρ is open. A superficial answer is given by the following theorem:

THEOREM 3. *The restriction homomorphism ρ of $A(G)$ onto $E(G_0, G)$ is open if and only if $E(G_0, G)$ is closed in $A(G_0)$.*

Proof. Suppose first that ρ is open. Then $E(G_0, G)$ is homeomorphic with $A(G)/R$ and hence is locally compact. It follows from this, as is well known, that $E(G_0, G)$ is closed in $A(G_0)$(5). Conversely, if $E(G_0, G)$ is closed in $A(G_0)$, then it is locally compact. Furthermore, as a subspace of $A(G_0)$, it satisfies the second axiom of countability. By Theorem 2, $A(G)$ is locally compact and satisfies the second axiom of countability. By a well known result(6), the continuous homomorphism ρ must therefore be open. This completes the proof.

We shall now give an example in which $E(G_0, G)$ is not closed in $A(G_0)$: Let C denote the additive group of the complex numbers, R the additive group of the real numbers, both with the ordinary topology. We form the semi-direct product $G_0 = (C \times C \times R)_h$ with the multiplication

$$(c_1, c_2, r)(c_1', c_2', r') = (c_1 + e^{2\pi i r}c_1', c_2 + e^{2\pi i h r}c_2', r + r'),$$

where h is a fixed irrational real number. Evidently, G_0 is a Lie group, and its underlying space is Euclidean 5-space.

Next we construct a semi-direct product G of G_0 by a group of order 2, with a generator g such that g^2 is the identity element $(0, 0, 0)$ of G_0 and G,

(5) See §3 of [4].

(6) Theorem 13, chap. III, in [3].

and $g(c_1, c_2, r)g^{-1}=\gamma(c_1, c_2, r)=(\bar{c}_1, \bar{c}_2, -r)$, where $\bar{c}$ denotes the complex conjugate of c.

With s, t arbitrary real numbers, let $\alpha=\alpha_{s,t}$ be the automorphism of G_0 which is defined by setting $\alpha(c_1, c_2, r)=(e^{2\pi is}c_1, e^{2\pi it}c_2, r)$. We shall determine the extensions of α to G. Let $\tilde{\alpha}\in A(G)$ be such that $\rho(\tilde{\alpha})=\alpha$. We must have $\tilde{\alpha}(g)=gz$, with $z\in G_0$. If we write down the equations which express the fact that $\tilde{\alpha}$ is a homomorphism, we find that we must have: (1) $\gamma(z)z=(0, 0, 0)$, and (2) $z\alpha\gamma(x)=\gamma\alpha(x)z$, for all $x\in G_0$.

Conversely, every element $z\in G_0$ which satisfies these conditions defines an extension $\tilde{\alpha}\in A(G)$ of α.

Write $z=(c_1, c_2, r)$. If we write down conditions (2) with $x=(0, 0, u)$, we find that we must have $c_1=0=c_2$. Then condition (1) holds with arbitrary $r\in R$. If we rewrite conditions (2) with $z=(0, 0, r)$ and all $x\in G_0$, we find that they are equivalent to the condition that $r+2s$ and $hr+2t$ be integers. Now, since h is irrational, we can find a sequence of integers k_n such that the congruence class mod 1 of $hk_n/2$ approaches the congruence class of $1/3$ as n becomes large. Put $s_n=1/2hn$ and $t_n=1/2n+hk_n/2$. Let $\alpha_n=\alpha_{s_n,t_n}$. If $r_n=-k_n-1/hn$, then it satisfies our above conditions on r, whence we conclude that each α_n belongs to $E(G_0, G)$. On the other hand, α_n evidently approaches the automorphism $\alpha_{0,1/3}$ in $A(G_0)$, and we claim that $\alpha_{0,1/3}\notin E(G_0, G)$. In fact, the conditions for the number r needed for extending $\alpha_{0,1/3}$ become $r=m$, and $hr=n-2/3$, where m and n are integers. But these conditions are incompatible, because h is irrational. Hence $E(G_0, G)$ is not closed in $A(G_0)$.

It will be apparent from the next theorem that in the above example the group $I(G_0)$ of the inner automorphisms of G_0 is not closed in $A(G_0)$; a fact which could also be shown quite directly by considering the above automorphisms $\alpha_{s,t}$.

THEOREM 4. *If G/G_0 is finite and $I(G_0)$ is closed in $A(G_0)$, then the restriction homomorphism ρ of $A(G)$ onto $E(G_0, G)$ is open.*

Proof. Let B denote the subgroup of $A(G)$ whose elements map each component g_iG_0, $i=1, \cdots, m$, of G onto itself. We claim that it suffices to show that the restriction of ρ to B is an open homomorphism of B onto $\rho(B)$. In fact, if this has been proved, we may conclude that $\rho(B)$ is homeomorphic with $B/R\cap B$, where R is the kernel of ρ, and hence that $\rho(B)$ is locally compact. This implies that $\rho(B)$ is closed in $E(G_0, G)$. Since $A(G)/B$ is finite, so is $E(G_0, G)/\rho(B)$. Hence the complement of $\rho(B)$ in $E(G_0, G)$ is the union of a finite number of cosets of $\rho(B)$ and hence is closed. Hence $\rho(B)$ is open in $E(G_0, G)$. It follows that ρ is an open homomorphism of $A(G)$ onto $E(G_0, G)$, which proves our claim.

Now let us observe that our assumption on $I(G_0)$ implies that the natural homomorphism of G_0 onto $I(G_0)$ is open, as well as continuous. Indeed, the continuity is independent of our assumption; for, given a compact subset

C of G_0 and a neighborhood V of the identity in G_0, it is clear that for each $c \in C$ we can find a neighborhood V_c of the identity such that $uxu^{-1}x^{-1} \in V$ for all $u \in V_c$ and $x \in cV_c$. Since C is compact, there is a finite subset $c_1, \cdots, c_q$ of C such that $C \subseteq \bigcup_{i=1}^{q} c_i V_{c_i}$. Then, if $W = \bigcap_{i=1}^{q} V_{c_i}$, we have $uxu^{-1}x^{-1} \in V$, for all $u \in W$ and $x \in C$, which proves that the natural homomorphism of G_0 onto $I(G_0)$ is always continuous. Now if $I(G_0)$ is closed in $A(G_0)$, then it is a connected Lie group. Since G_0 is also a connected Lie group, the continuous natural homomorphism of G_0 onto $I(G_0)$ must automatically be open.

It will be convenient to identify B with the closed subgroup $\phi(B)$ of the group H which we introduced in the proof of Theorem 2. It is easy to check that $\phi(R \cap B)$ is precisely the set of elements $(z_1, \cdots, z_m, 1) \in H$, where 1 stands for the identity automorphism of G_0, and where the elements z_i belong to the center, Z_0, say, of G_0 and satisfy the relations $g_j^{-1} z_i g_j z_j = z_{k(i,j)}$, the index $k(i, j)$ being determined by the relation $g_i g_j \in g_{k(i,j)} G_0$.

What we still have to prove therefore amounts to the following: Given a neighborhood U of the identity in G_0, there is a neighborhood M of the identity in $A(G_0)$, such that, for every element $(c_1, \cdots, c_m, \alpha) \in \phi(B)$ with $\alpha \in M$, we can find elements z_i as above and such that $z_i c_i \in U$(7).

Since Z_0 is a Lie group, there is a neighborhood D of the identity in G_0 which has the following properties:

(1) For every $z \in D \cap Z_0$, there is an element $t \in D \cap Z_0$ such that $t^m = z$.

(2) If s, t belong to $(g_j^{-1} D g_j D D^{-1}) \cap Z_0$ and $s^m = t^m$, then $s = t$.

(3) $DD \subseteq U$.

Let us choose a neighborhhod S of the identity in G_0 such that $S^{2m} \subseteq D$. Let γ_i denote the inner automorphism $u \to g_i^{-1} u g_i$, and choose a neighborhood V of the identity in G_0 such that $V = V^{-1}$ and $V \gamma_i \gamma_j \gamma_k^{-1}(VV) \subseteq S$, for all i, j, k.

Now observe that if $\phi(b) = (c_1, \cdots, c_m, \alpha)$, then $\gamma_i \alpha \gamma_i^{-1} \alpha^{-1}(x) = c_i x c_i^{-1}$, for every $x \in G_0$. Since the homomorphism of G_0 onto $I(G_0)$ is open, we can find a neighborhood M' of the identity in $A(G_0)$ such that every automorphism belonging to $I(G_0) \cap M'$ is effected by an element belonging to V. Choose a neighborhood M of the identity in $A(G_0)$ such that, for every $\alpha \in M$, we have $\gamma_i \alpha \gamma_i^{-1} \alpha^{-1} \in M'$, for each i, and also $\alpha(a(i, j)) a(i, j)^{-1} \in V$, for each pair (i, j), where $a(i, j) = g_{k(i,j)}^{-1} g_i g_j$.

Now let $\alpha \in M$ and $(c_1, \cdots, c_m, \alpha) = \phi(b)$. Then the inner automorphism of G_0 which is effected by c_i is also effected by an element $v_i \in V$, whence $c_i = t_i v_i$, with $t_i \in Z_0$. We have $\gamma_j(c_i) c_j = g_j^{-1} g_i^{-1} b(g_i g_j) = g_j^{-1} g_i^{-1} g_{k(i,j)} c_{k(i,j)} \alpha(a(i, j))$

(7) For the argument which now follows I am indebted to T. Nakayama. The idea of this proof is that if α is "small" enough the c_i can be replaced by "small" elements. This possibility is due to the fact that a factor set, defined on G/G_0, and with sufficiently small values in Z_0, must be a transformation set, because unique divisibility holds near the identity in Z_0. Such a device has been used by Iwasawa on p. 510 of [2] in dealing with the automorphisms of a compact group.

$=a(i, j)^{-1}c_{k(i,j)}\alpha(a(i, j))$, whence $\gamma_j(t_i)t_jt_{k(i,j)}^{-1}=a(i, j)^{-1}v_{k(i,j)}\alpha(a(i, j))v_j^{-1}\gamma_j(v_i^{-1})$ $=\gamma_j\gamma_i\gamma_k^{-1}{}_{(i,j)}(v_{k(i,j)}\alpha(a(i, j))a(i, j)^{-1})v_j^{-1}\gamma_j(v_i^{-1})\in S^2$. Hence $\prod_{i=1}^{m}\gamma_j(t_i)t_jt_k^{-1}{}_{(i,j)}$ $\in S^{2m}\subseteq D$, i.e., $\gamma_j(t)t^{-1}t_j^m\in D\cap Z_0$, where $t=t_1\cdots t_m$. By property (1) of D, there are elements $u_j\in D\cap Z_0$ such that $u_j^m=\gamma_j(t)t^{-1}t_j^m$, and then we have $(\gamma_j(u_i)u_ju_{k(i,j)}^{-1})^m=(\gamma_j(t_i)t_jt_{k(i,j)}^{-1})^m$. By property (2) of D, it follows that $\gamma_j(u_i)u_ju_{k(i,j)}^{-1}=\gamma_j(t_i)t_jt_{k(i,j)}^{-1}$. Set $z_i=u_it_i^{-1}$. Then $g_j^{-1}z_ig_jz_j=z_{k(i,j)}$, and $z_ic_i=u_iv_i$ $\in DD\subseteq U$. This is what we had to prove in order to establish Theorem 4.

COROLLARY. *If G_0/Z_0 is compact, or if G_0 is semisimple, then the restriction homomorphism of $A(G)$ onto $E(G_0, G)$ is open.*

Proof. If G_0/Z_0 is compact, so is $I(G_0)$, and hence $I(G_0)$ is closed in $A(G_0)$. If G_0 is semisimple, then $I(G_0)$ coincides with the component of the identity in $A(G_0)$, and hence is closed in $A(G_0)$.

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