BANACH SPACES WITH THE EXTENSION PROPERTY

BY

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It is the object of this note to complete a characterization of those Banach spaces $B$ with the Hahn-Banach extension property: each bounded linear function $F$ on a subspace of any Banach space $C$ with values in $B$ has a linear extension $F'$ carrying all of $C$ into $B$ such that $\|F'\| = \|F\|$. It is shown here that:

**Theorem.** Each such space $B$ is equivalent to the space $C_X$ of continuous real-valued functions on an extremally disconnected compact Hausdorff space $X$, $C_X$ having the usual supremum norm.

Recently, in these Transactions, Nachbin [N] and, independently, Goodner [G] have shown that if $B$ has the extension property and if its unit sphere has an extreme point, then $B$ is equivalent to a function space of this sort; both authors have also proved that such a function space has the extension property. The above theorem simply omits the extreme point hypothesis, and so establishes the equivalence.

My original proof, of which the proof given here is a distillate, depends on an idea of Jerison [J]. Briefly, letting $X$ be the weak* closure of the set of extreme points of the unit sphere of the adjoint $B^*$, $B$ can be shown equivalent to the space of all weak* continuous real functions $f$ on $X$ such that $f(x) = -f(-x)$, and then properties of $X$ are deduced which imply the theorem. The same idea occurs implicitly in the proof below.

**Note.** Goodner asks [G, p. 107] if every Banach space having the extension property is equivalent to the conjugate of an abstract $(L)$-space. It is known (this is not my contribution) that the Birkhoff-Ulam example ([B, p. 186] or [HT, p. 490]) answers this question in the negative, the pertinent Banach space being the bounded Borel functions on $[0, 1]$ modulo those functions vanishing except on a set of the first category, with $\|f\| = \inf \{K : |f(x)| \leq K \text{ save on a set of first category}\}$.

1. Preliminary definitions and remarks. A point $x$ is an extreme point of a convex subset $K$ of a real linear space if $x$ is not an interior point of any line segment contained in $K$ (i.e., if $x = ty + (1-t)z$, $0 < t < 1$, $y \in K$, and $z \in K$, then $x = y = z$). A set $L$ is a support of $K$ if $L$ is a convex, nonvoid subset of $K$ such that each line segment contained in $K$ which has an interior point in $L$ is contained in $L$. If $x$ is an extreme point of $L$ and $L$ is a sup-

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If $F$ is a linear function carrying a convex set $K$ into a convex set $M$ and $L$ is a support of $M$, then $F^{-1}(L) \cap K$ is either void or a support of $K$.

For each Banach space $B$ the adjoint space is denoted by $B^*$ and the weak* topology for $B^*$ is the topology of pointwise convergence of functionals. Each convex, norm-bounded, weak* closed subset $K$ (convex, weak* compact subset) is, according to the classic theorem of Krein and Milman, the smallest convex weak* closed set which contains all extreme points of $K$. (See, for example, [K].) If $F$ is a bounded linear function on $B$ to a Banach space $C$, then $F^*$, the adjoint function, carries $C^*$ into $B^*$ in a weak* continuous fashion, and in particular, the image of the unit sphere of $C^*$ is weak* compact.

A compact Hausdorff space is extremally disconnected if the closure of each open set is open. If $X$ is a compact Hausdorff space, then $C_X$ is the Banach space of all real-valued continuous functions on $X$, with the usual supremum norm. For each $x \in X$ there is assigned a functional $e_x$, by setting $e_x(f) = f(x)$ for $f \in C_X$. This functional $e_x$ is the evaluation at $x$. It is known (see [AK]) that the set of extreme points of the unit sphere of $C_X^*$ is precisely $E \cup (-E)$, where $E$ is the set of all evaluations. Moreover, if $E$ has the relativized weak* topology, then the function $e$ carrying $x$ into $e_x$ maps $X$ homeomorphically onto $E$.

2. Proof of the theorem. Let $B$ be a Banach space with the property: if $H$ is a linear isometry of $B$ into a Banach space $C$, then there is a linear map $G$ of norm one carrying $C$ onto $B$ such that $GH$ is the identity map of $B$ onto itself. Let $X$ be the weak* closure of the set of all extreme points of the unit sphere of $B^*$. Then $X$ is weak* compact. In what follows, a subset of $X$ is “open” if it is “open in the relativized weak* topology for $X$,” and the closure $\overline{U}$ of a subset $U$ of $X$ is the weak* closure of $U$.

Suppose, now, that $U$ and $V$ are open subsets of $X$ such that both $U \cap V$ and $[-(U \cup V)] \cap (U \cup V)$ are void, and $[-(U \cup V)] \cup (U \cup V)$ is dense in $X$. We construct a space $Y$, by setting $Y = (\{0\} \times U^c) \cup (\{1\} \times V^c)$, so that $Y$ consists of disjoint copies of $U^c$ and $V^c$. The set $Y$ is topologized by agreeing that if $U_1$ is open in $U^c$ and $V_1$ is open in $V^c$, then $\{0\} \times U_1$ and $\{1\} \times V_1$ are each open in $Y$. Let $H$ be the map of $B$ into $C_Y$ defined, for $b \in B$, $u \in U^c$, $v \in V^c$ by: $H(b)((0, u)) = u(b)$, $H(b)((1, v)) = v(b)$. The basic result about this construction is:

**Lemma.** The map $H$ is a linear isometry of $B$ onto $C_Y$. Moreover, $U^c \cap V^c$ and $[-(U^c \cup V^c)] \cup (U^c \cup V^c)$ are void, and $H^*$ maps the set of evaluations in $C_Y^*$ weak* homeomorphically onto $U^c \cap V^c$.

**Proof.** We first verify that $H$ is a linear isometry. The unit sphere $S$ of $B^*$ is weak* compact and, for each $b \in B$, the linear functional $b'$, whose value at $z \in B^*$ is $z(b)$, is weak* continuous, and maps $S$ onto the closed interval
The set of points at which the functional \( b' \) assumes the value \( \|b\| \) is a support of \( S \) and hence contains an extreme point \( x \), which is a member of \( X \). Either \( x \) or \( -x \) belongs to \( U \cap V^c \), and consequently \( \|H(b)\| \geq |x(b)| = \|b\| \). On the other hand, since \( U \cap V^c \) is a subset of the unit sphere of \( B^* \), \( \|H(b)\| \leq \|b\| \), so that \( H \) is an isometry.

Next, a small calculation. Suppose \( e_{(0,u)} \in C^*_x \) is the evaluation at \((0, u)\), and that \( b \in B \). Then \( H^*(e_{(0,u)})(b) \) is, by definition of \( H^* \), \( e_{(0,u)}(H(b)) \), which from the definition of \( e_{(0,u)} \) is \( H(b)((0, u)) \), and using the definition of \( H \) this is \( u(b) \). Consequently, the valuation at \((0, u)\) maps under \( H^* \) onto \( u \), and similarly the evaluation at \((1, v)\) maps onto \( v \).

If \( u \in U \) and \( u \) is an extreme point of the unit sphere \( S \) of \( B^* \), then \( H^{*-1}(u) \) intersects the unit sphere \( T \) of \( C^*_x \) in a set which is a support of \( S \). This support, being weak* compact, consists of a single point or else contains at least two extreme points (the Krein-Milman theorem). Each extreme point of the support is also an extreme point of \( T \). But the extreme points of \( T \) are \( \pm \) evaluations, and since \( u \in V^c \), the only extreme point which can map onto \( u \) under \( H^* \) is \( e_{(0,u)} \), in view of the preceding paragraph. Consequently, \( H^{*-1}(u) \cap T \) consists of the single point \( e_{(0,u)} \) and similarly, if \( v \in V \) and \( v \) is an extreme point of \( S \), then \( H^{*-1}(v) \cap T = \{e_{(1,v)}\} \).

Now let \( G \) be a linear function of norm one carrying \( C_T \) onto \( B \) so that \( GH \) is the identity on \( B \). Then \( G^* \) carries the unit sphere \( S \) of \( B^* \) into the unit sphere \( T \) of \( C^*_x \) and \((GH)^* = H^*G^* \) is the identity on \( B^* \). If \( u \in U \) and \( u \) is an extreme point of \( S \), then necessarily \( G^*(u) = e_{(0,u)} \), in view of the preceding paragraph, and if \( v \in V \) and \( v \) is an extreme point of \( S \), then \( G^*(v) = e_{(1,v)} \). Because such points are dense in \( U \) and in \( V \) the function \( G^* \) carries a dense subset of \( X \) onto a weak* dense subset of \( E \cup (-E) \), where \( E \) is the set of evaluations. Because \( X \) and \( E \cup (-E) \) are weak* compact \( G^* \) carries \( X \) onto \( E \cup (-E) \). Now \( H^*G^* \) is the identity on \( B^* \), and if \( u \in U \) and \( u \) is an extreme point of \( S \), then \( G^*H^*(e_{(0,u)}) = G^*(u) = e_{(0,u)} \), and similarly for \( v \in V \) and \( v \) extreme, so that \( G^*H^* \) is the identity on a dense subset of \( E \cup (-E) \). Consequently \( G^* \) is, on \( X \), a homeomorphism, and \( H^* \) is, on \( E \cup (-E) \), the inverse of this homeomorphism. From the structure of \( E \cup (-E) \) it follows (see preliminary remarks) that \( U \cap V^c \) and \( -\left( U \cap V^c \right) \) are void, and it is also clear that \( H^* \) maps \( E \) homeomorphically onto \( U \cap V^c \).

It remains to show that \( H \) maps \( B \) onto \( C_T \). The image \( G^*(S) \) of the unit sphere \( S \) of \( B^* \) is convex and weak* compact, and each extreme point of the unit sphere \( T \) of \( C^*_x \), as was shown in the preceding paragraph, belongs to \( G^*(S) \). From the Krein-Milman theorem it follows that \( T \subset G^*(S) \), and since \( G^* \) has norm one, \( T = G^*(S) \). Since \( H^*G^* \) is the identity on \( B^* \) and since \( G^* \) maps \( B^* \) onto \( C^*_x \), it follows that \( H^* \) is 1-1. Because \( H^* \) is 1-1 it is true that \( H \) maps \( B \) onto \( C_T \), for otherwise there is a nonzero linear functional on \( C_T \) which vanishes on the range of \( H \) (a closed subspace) and \( H^* \) applied to this functional gives the zero of \( B^* \). The proof of the lemma is then complete.
The theorem is now established as follows. Choose, using Zorn's Lemma, an open subset $W$ of $X$ maximal with respect to the property that $(-W) \cap W$ be void. Then $(-W) \cup W$ is dense in $X$. Applying the lemma to $U=W$, $V=\text{void set}$, it follows that $(-W^c) \cap (W^c)$ is void, and that $W^c$ is open as well as closed in $X$. Moreover $H$ is an isometry of $B$ onto $C_Y$, where $Y$ is homeomorphic to $W^c$. Proceeding, let $U$ be any open subset of $W^c$ and let $V=W^c \setminus U^c$. Applying the lemma again, we see that $U^c \cap V^c$ is void so that $U^c$ is open and it is proven that $W^c$ is extremally disconnected, which establishes the theorem.

References


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