THE INFLUENCE OF THE MAXIMUM TERM IN THE ADDITION OF INDEPENDENT RANDOM VARIABLES

BY

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1. Introduction. Let $X_1, X_2, \cdots$ be independent random variables having the common distribution function $F(x) = \Pr\{X_i < x\}$, and let $S_n = X_1 + X_2 + \cdots + X_n$. If there exist constants a_n and b_n such that $a_nS_n + b_n$ has a nondegenerate limiting distribution function $G(x)$, then $G(x)$ is necessarily a stable (or quasi-stable) distribution (Lévy [4]). Necessary and sufficient conditions on $F(x)$ for this result to obtain were given by Doeblin [2, Theorem 3.1], and as far as linear normalizations are concerned the subject has been completely exhausted.

It is still meaningful however to ask about the nature of S_n when the conditions of Doeblin's theorem are not met. In particular there may exist a suite of nonlinear functions $\phi_n(x)$ such that $\phi_n(S_n)$ has a nondegenerate limiting distribution. Of course, to avoid trivial results some restriction will have to be put on the ϕ_n—for instance if $F(x)$ is continuous and $\phi_n(x) = \Pr\{S_n < x\}$, then $\phi_n(S_n)$ is uniformly distributed over the interval (0, 1) for every n. It seems natural to require, among other things, that $\phi_n(x)$ does not involve the distribution of S_n, if we are to get any intelligible asymptotic information.

One case when this procedure might be helpful is when the variables X_i have infinite moments of all orders: $E(|X_i|^\rho) = \infty$ for $\rho > 0$. In this case, as noted by Lévy [3], every linear normalization leads to a degenerate limiting distribution. Lévy stated that here the largest term in S_n is the primary factor, outshadowing the contribution of all the rest of the terms. The present paper treats this matter in some detail, and in general attempts to give precise information on the role played by the largest term in the sum S_n.

In §2 the main analytical tool is developed. Letting $X_n^* = \max\{X_1, X_2, \cdots, X_n\}$ we obtain in Lemma 2.1 the characteristic function of the random variable $Z_n = S_n/X_n^*$ under certain conditions on $F(x)$, the distribution function of the X_i. This enables us to find conditions under which $S_n \sim X_n^*$, which is our main theorem. In §4 we obtain two theorems on the limiting distribution of S_n (in the sense given above) when the X_i have moments of no finite order, analogous to the theorem of Doeblin for the stable distributions.

In §5 there are three theorems devoted to assessing the influence of X_n^*

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on S_n when S_n has a limiting stable distribution. As is well known, unless S_n has a limiting Gaussian distribution, the maximum term has a non-negligible contribution to S_n. It turns out that there is a rather intimate asymptotic connection between S_n and X_n^* which reveals some new aspects of the approach to the stable distribution(1).

2. **The principal lemma.** Let $X_1, X_2, \cdots, X_n$ and X_n^* be the random variables described in §1. For convenience we put the further restriction that $X_i \geqq 0$ and that $F(x)$, the distribution of the X_i, is absolutely continuous. These two requirements are not essential for some of the asymptotic theorems to follow, as will be noted in the sequel.

For notational simplicity we shall often use the complementary distribution function $f(x) = 1 - F(x) = \Pr\{X \geqq x\}$ and we denote by $\phi(x)$ the corresponding density, so that $f(x) = \int_x^\infty \phi(z)dz$.

LEMMA 2.1. *Letting $Z_n = S_n/X_n^*$ we have*

$$\xi_n(t) = E(e^{itZ_n}) = ne^{it}\int_0^\infty \left(\beta \int_0^1 e^{it\alpha}\phi(\alpha\beta)d\alpha\right)^{n-1} dF(\beta). \tag{2.1}$$

Proof. There is no loss in generality in assuming $X_1 = X_n^*$ since each X_i has a probability of $1/n$ of being the largest term, and $\Pr\{X_i = X_j\} = 0$ for $i \neq j$ since $F(x)$ is presumed continuous.

The joint density of $X_1, X_2, \cdots, X_n$, given $X_1 = X_n^*$, is

$$g(\beta_1, \beta_2, \cdots, \beta_n) = \begin{cases} n\phi(\beta_1)\phi(\beta_2)\cdots\phi(\beta_n) & \text{if } \beta_1 = \max_i \{\beta_i\}, \\ 0 & \text{otherwise.} \end{cases}$$

Thus

$$\begin{aligned} E(e^{itZ_n}) &= \int\int_0^\infty \cdots \int e^{it(x_1+x_2+\cdots+x_n)/x_1} g(x_1, x_2, \cdots, x_n)dx_1dx_2\cdots dx_n \\ &= ne^{it}\int_0^\infty\int_0^\beta \cdots \int_0^\beta e^{it(\alpha_2+\alpha_3+\cdots+\alpha_n)/\beta}\phi(\alpha_2)\phi(\alpha_3)\cdots \\ &\qquad\qquad \cdot\phi(\alpha_n)\phi(\beta)d\alpha_2\cdots d\alpha_n d\beta \\ &= ne^{it}\int_0^\infty \left\{\int_0^\beta e^{it\alpha/\beta}\phi(\alpha)d\alpha\right\}^{n-1}\phi(\beta)d\beta \end{aligned}$$

which yields the lemma.

3. **The main theorem.** Let us denote by γ the "exponent" of the random variable, i.e., γ is the supremum of all values of $\rho \geqq 0$ for which $E(|X|^\rho) < \infty$.

(1) The author is indebted to the referee for a simplification of the proof of Lemma 2.1 as well as other valuable emendations.

If $\gamma > 2$, then S_n has a linear normalization to a limiting Gaussian distribution. For $0 < \gamma < 2$ we have the following basic theorem of Doeblin [2].

THEOREM 3.1 (DOEBLIN). *If* $0 < \gamma < 2$ *and if* $a_n S_n + b_n$ *has a nondegenerate limiting distribution, it is a stable or quasi-stable distribution. The necessary and sufficient conditions for this are that*

$$F(-x) = \frac{h_1(x)}{x^\gamma}, \qquad 1 - F(x) = \frac{h_2(x)}{x^\gamma}$$

where $h_i(\alpha x)/h_i(x) \to 1$ *for all* $\alpha > 0$, $i = 1, 2$, *and* $h_1(x)/(h_1(x) + h_2(x))$ *tends to a limit when* $x \to \infty$.

Thus the tails of $F(x)$ are of decisive importance, and must asymptotically possess the character of the corresponding stable law. This is because, as pointed out by Lévy [3], the largest term in S_n is not asymptotically negligible, and in point of fact S_n is determined essentially by a finite number of its larger terms, the rest being asymptotically negligible in comparison. The term X_n^* is said to be "asymptotically negligible" if $X_n^*/S_n \to 0$ in probability when $n \to \infty$. Or instead we might say that the terms $X_1, X_2, \cdots, X_n$ are uniformly negligible compared with S_n.

When $\gamma = 0$, the remaining case not covered by Doeblin's theorem, there are moments of no order, and as mentioned before no linear normalization can be obtained (of course in this case, as in the case of $\gamma > 0$, there may exist subsequences S_{n_ν} which are linearly normalizable to limiting infinitely divisible distributions). The following theorem shows that for distributions of this nature which have a certain regularity at infinity, S_n is completely dominated by its largest term, all the rest being asymptotically negligible.

Let us suppose that $X_i \geqq 0$ and as before put $f(x) = 1 - F(x)$. Then we have the following theorem.

THEOREM 3.2. *Suppose that* $X_i \geqq 0$. *Then* $S_n/X_n^* \to 1$ *in the mean of order one if for every* $\alpha > 0$ *we have*

$$\lim_{x \to \infty} \frac{f(\alpha x)}{f(x)} = 1. \tag{3.1}$$

Proof. If we put $Z_n = S_n/X_n^*$, it is sufficient to prove that $E(Z_n) \to 1$ since $Z_n \geqq 1$. Assume initially that $F(x)$ is absolutely continuous, so that Lemma 2.1 is applicable. Since, in the lemma, $\xi_n(t)$ is the characteristic function of a bounded random variable, it is analytic, and it is simple to see that we can differentiate under the integration sign. Differentiating with respect to it, and putting $t = 0$, we obtain after some simplification

$$\begin{aligned} \mu_n &= E(Z_n) \\ &= 1 + n(n-1) \int_0^\infty [F(\beta)]^{n-2} f(\beta) \left\{ \int_0^1 \left(\frac{f(\beta u)}{f(\beta)} - 1 \right) du \right\} dF(\beta). \end{aligned} \tag{3.2}$$

We first show that if we put

$$A(\beta) = \int_0^1 \left(\frac{f(\beta u)}{f(\beta)} - 1\right) du, \tag{3.3}$$

then $\lim_{\beta\to\infty} A(\beta) = 0$. From the hypothesis of the theorem it is plain that for every $\epsilon > 0$

$$\lim_{\beta\to\infty} \int_\epsilon^1 \left(\frac{f(\beta u)}{f(\beta)} - 1\right) du = 0,$$

and it will be sufficient to dominate the integrand by a function, integrable and independent of β.

If $f(x)$ satisfies (3.1), the function $q(x) = x^\epsilon f(x)$, $\epsilon > 0$, is said to be of "regular growth," and if we let $Q(x) = \max_{0 \leqq z \leqq x} q(z)$, we have from a theorem of Karamata (cf. Knopp [5, No. 20]) the result that $Q(x)/q(x) \to 1$, $x \to \infty$. This implies that

$$\frac{\max_{0\leqq u\leqq 1} \{u^\epsilon f(u\beta)\}}{f(\beta)} \to 1, \qquad \beta \to \infty.$$

Since the expression on the left is bounded for $\beta < \infty$, we obtain

$$\frac{\max_{0\leqq u\leqq 1} \{u^\epsilon f(u\beta)\}}{f(\beta)} < C, \qquad \frac{f(u\beta)}{f(\beta)} < Cu^{-\epsilon}, \qquad 0 \leqq u \leqq 1,$$

for all β, the constant C depending only on ϵ. Choosing $\epsilon = 1/2$, say, we have dominated the integrand of (3.3) by an integrable function independent of β. Thus $A(\beta) \to 0$.

Now

$$\mu_n = E(Z_n) = 1 + n(n-1)\int_0^\infty (F(\beta))^{n-2} f(\beta) A(\beta) dF(\beta) = 1 + \int_0^\infty I(\beta) dF(\beta).$$

Choose $\epsilon > 0$ arbitrary, and choose β_0 such that $A(\beta_0) < \epsilon$ for $\beta > \beta_0$, then choose n_0 such that $n(n-1)F(\beta)^{n-2}A(\beta) < \epsilon$ for $n > n_0$ and all $\beta < \beta_0$. This can clearly be accomplished for $F(\beta) < 1$ for all $\beta < \infty$. Then if $n > n_0$,

$$\begin{aligned}
\mu_n &= 1 + \int_0^\infty I(\beta) dF(\beta) = 1 + \int_0^{\beta_0} + \int_{\beta_0}^\infty I dF(\beta) \\
&\leqq 1 + \epsilon \int_0^{\beta_0} f(\beta) dF(\beta) + \epsilon \int_{\beta_0}^\infty n(n-1)(F(\beta))^{n-2} f(\beta) dF(\beta) \\
&\leqq 1 + \frac{\epsilon}{2} + \epsilon \int_0^1 n(n-1) u^{n-2}(1-u) du = 1 + \frac{3}{2}\epsilon,
\end{aligned}$$

and since ϵ was arbitrary, $\mu_n \to 1$.

The condition that $F(x)$ be absolutely continuous may be waived, for continuity is needed only to ensure that the maximum term is unique. But $X_n^* \to \infty$ with probability 1, and thus the limiting probability is 1 that the maximum term is assumed by only one summand X_i. Also the condition $X_i \geqq 0$ can be relaxed to $\Pr\{X_i < -x\} = o(f(x))$, $x \to \infty$.

4. **The distribution of S_n.** From Theorem 3.2 we should expect the limiting distribution of S_n and X_n^* to be related, and this turns out to be the case if condition 3.1 is met.

THEOREM 4.1. *Let $X_i \geqq 0$ and*

$$\lim_{x \to \infty} \frac{f(\alpha x)}{f(x)} = 1, \qquad \alpha > 0.$$

Then

$$\lim_{n \to \infty} \Pr \{nf(S_n) < y\} = 1 - e^{-y}. \tag{4.1}$$

Proof. Choose $\epsilon > 0$, $\delta > 0$ arbitrary, and then n_0 such that $n > n_0$ implies $\Pr\{S_n/X_n^* > 1 + \epsilon\} < \delta$. This is possible by virtue of Theorem 3.2. Abbreviating, as before, $Z_n = S_n/X_n^*$ we have $S_n \geqq X_n^*$ and $f(x) \searrow 0$, so that

$$\begin{aligned}
\Pr \{nf(X_n^*) < z\} &\leqq \Pr \{nf(S_n) < z\} \\
&= \Pr \{nf(S_n) < z, Z_n > 1 + \epsilon\} \\
&\quad + \Pr \{nf(S_n) < z, Z_n \leqq 1 + \epsilon\} \\
&\leqq \delta + \Pr \{nf(S_n) < z, Z_n \leqq 1 + \epsilon\} \\
&\leqq \delta + \Pr \{nf(X_n^*(1 + \epsilon)) < z\}.
\end{aligned}$$

Now consider the difference

$$\begin{aligned}
\Pr \{nf(X_n^*(1 + \epsilon)) < z\} - \Pr \{nf(X_n^*) < z\} & \\
&= \Pr \{nf(X_n^*)(1 + \epsilon)) < z \leqq nf(X_n^*)\} \\
&\leqq \Pr \left\{\frac{f(X_n^*(1 + \epsilon))}{f(X_n^*)} < 1\right\}
\end{aligned}$$

and this latter expression approaches zero as $n \to \infty$ since $X_n^* \to \infty$ with probability one and condition 3.1 is satisfied. Since δ was arbitrary, we have shown

$$\lim_{n \to \infty} \Pr \{nf(S_n) < z\} = \lim_{n \to \infty} \Pr \{nf(X_n^*) < z\}$$

provided the second of these limits exists. But the distribution of the extreme is well known, and we have (cf. Cramér [1])

$$\lim_{n\to\infty} \Pr\left\{nf(X_n^*) < z\right\} = 1 - e^{-z}$$

as can also readily be deduced from the simple formula

$$\Pr\left\{X_n^* < z\right\} = (1 - f(z))^n.$$

This concludes the proof of Theorem 4.1.

It seems difficult to give "natural" restrictions on the admissible class of normalizing functions $\phi_n(x)$, as we have noted in §1, and for this reason it appears meaningless to attempt to say under what circumstances (3.1) would be a necessary condition for a limiting theorem to obtain.

For the distribution given by Lévy [4], $f(x) = (\log x)^{-1}$, $x > e$, as an example for which S_n could not be linearly normalized we have however by Theorem 4.1 the following limiting expression:

$$\lim_{n\to\infty} \Pr\left\{S_n^{1/n} < y\right\} = \exp\left\{-\frac{1}{\log x}\right\}, \qquad x > 1.$$

It is possible to remove the restriction $X_i \geqq 0$ (which in Theorem 4.1 could be relaxed to $\Pr\left\{X_i < -x\right\} = o(f(x))$, $x \to \infty$). As a direct extension of Doeblin's theorem (Theorem 3.1) we shall prove the following theorem.

THEOREM 4.2. *Let $F(x)$, the distribution of the X_i, be such that*

(i) $$F(x) = 1 - f_1(x), \qquad x \geqq 0,$$

(ii) $$F(x) = f_2(-x), \qquad x < 0,$$

(iii) $$\lim_{x\to\infty} \frac{f_i(\alpha x)}{f_i(x)} = 1, \qquad i = 1, 2, \ \alpha > 0,$$

(iv) $$\lim_{x\to\infty} \frac{f_1(x)}{f_1(x) + f_2(x)} = p = 1 - q.$$

Then, letting

$$f(x) = \begin{cases} f_1(x) & x \geqq 0, \\ f_2(-x) & x < 0, \end{cases}$$

we obtain

$$\lim_{n\to\infty} \Pr\left\{nf(S_n) < y\right\} = 1 - pe^{-y/p} - qe^{-y/q}. \tag{4.2}$$

Proof. As before we may presume without loss in generality that $f_1(x)$ and $f_2(x)$ are continuous. Assume initially that $0 < p < 1$ and let X_n^* be the term of maximum modulus among $X_1, X_2, \cdots, X_n$. It is clear that the result of Theorem 4.1 applies to each tail of $F(x)$ separately and $S_n \sim X_n^*$ in probability if the conditions of the theorem are met.

To see this we note that $\Pr\{|X_i|>x\}=f_1(x)+f_2(x)$ and that this function satisfies condition (3.1) by virtue of (iii) and (iv). Hence by Theorem 3.2, $|X_1|+|X_2|+\cdots+|X_n|\sim|X_n^*|$ in probability. If we denote by $\sum^*$ a sum omitting the term X_n^*, we have $|S_n-X_n^*|=|\sum^* X_i|\leqq\sum^*|X_i| = o(|X_n^*|)$, and hence $|S_n/X_n^*-1|\to 0$ in probability and $S_n\sim X_n^*$ in probability as asserted.

Now

$$\Pr\{0 < X_n^* < x\} = P_n(x) = -n\int_0^x (1-f_1(t)-f_2(t))^{n-1}df_1(t)$$

and from (iv) we have $f_1(x)+f_2(x)=(f_1(x)/p)\,(1+o(1))$, $x\to\infty$. Let us define $a(x)$ so that $f_1(x)+f_2(x)=(f_1(x)/p)(1+a(x))$ so that $a(x)\to 0$, $x\to\infty$. Choose $\epsilon>0$ arbitrary and let x_0 be such that $|a(x)|<\epsilon$, $x>x_0$. Then for $x>x_0$

$$P_n(x) = P_n(x_0) - n\int_{x_0}^x \left(1-\frac{f_1(t)}{p}(1+a(t))\right)^{n-1} df_1(t)$$

and

$$\begin{aligned} P_n(x_0) - n\int_{x_0}^x \left(1-\frac{f_1(t)}{p}(1+\epsilon)\right)^{n-1} df_1(t) &\leqq P_n(x) \\ &\leqq P_n(x_0) - n\int_{x_0}^x \left(1-\frac{f_1(t)}{p}(1-\epsilon)\right)^{n-1} df_1(t), \end{aligned}$$

$$\begin{aligned} P_n(x_0) - \frac{p}{1+\epsilon}\left(\frac{1-f_1(x_0)}{p}(1+\epsilon)\right)^n + \frac{p}{1+\epsilon}\left(1-\frac{f_1(x)}{p}(1+\epsilon)\right)^n \\ \leqq P_n(x) \leqq P_n(x_0) - \frac{p}{1-\epsilon}\left(1-\frac{f_1(x_0)}{p}(1-\epsilon)\right)^n \\ + \frac{p}{1-\epsilon}\left(1-\frac{f_1(x)}{p}(1-\epsilon)\right)^n. \end{aligned} \tag{4.3}$$

Now

$$P_n(x_0) \leqq -n\int_0^{x_0}(1-f_1(t))^{n-1}df_1(t) = (1-f_1(x_0))^n - (1-f_1(0))^n,$$

so that $P_n(x_0)\to 0$ as $n\to\infty$. Let y be fixed and greater than 0 and determine x such that $f_1(x)=y/n$, choosing n large enough so that $x>x_0$. If $f_1(x)$ is not strictly monotone, we choose the largest such x so that we can write $x=f_1^{-1}(y/n)$ unambiguously. Then (4.3) becomes

$$P_n(x_0) - \frac{p}{1+\epsilon}\left(1 - \frac{f_1(x_0)}{p}(1+\epsilon)\right)^n + \frac{p}{1+\epsilon}\left(1 - \frac{y}{np}(1+\epsilon)\right)^n$$
$$\leqq P_n(\overset{-1}{f_1}(y/n)) \leqq P_n(x_0) - \frac{p}{1-\epsilon}\left(1 - \frac{f_1(x_0)}{p}(1-\epsilon)\right)^n$$
$$+ \frac{p}{1-\epsilon}\left(1 - \frac{y}{np}(1-\epsilon)\right)^n,$$

and letting $n\to\infty$ we obtain

$$\frac{p}{1+\epsilon} e^{-y/p(1+\epsilon)} \leqq \liminf_{n\to\infty} P_n(f^{-1}(y/n))$$
$$\leqq \limsup_{n\to\infty} P_n(f^{-1}(y/n)) \leqq \frac{p}{1-\epsilon} e^{-y/p(1-\epsilon)},$$

and since ϵ is arbitrary, $\lim_{n\to\infty} P_n(f^{-1}(y/n)) = pe^{-y/p}$.

Recalling the definition of $P_n(x)$ above we have

$$P_n(\overset{-1}{f_1}(y/n)) = \Pr\left\{0 < X_n^* < \overset{-1}{f_1}(y/n)\right\} = \Pr\left\{nf_1(X_n^*) > y,\ X_n^* > 0\right\}$$

and using the fact that $S_n \sim X_n^*$ we obtain as in Theorem 4.1

$$\lim_{n\to\infty} \Pr\left\{nf_1(S_n) > y,\ S_n > 0\right\} = pe^{-y/p}.$$

In a similar manner we prove

$$\lim_{n\to\infty} \Pr\left\{nf_2(-S_n) > y,\ S_n < 0\right\} = qe^{-y/q}$$

and adding these last two equalities we obtain the conclusions to the theorem when $pq>0$.

If p or $q=0$, the above derivation breaks down, but in that case one of the tails is negligible and the result of Theorem 4.1 is applicable. We also obtain (4.1) as a limiting case ($p\to 1$) of (4.2).

5. **The approach to the stable distribution.** When the sum S_n has a linear normalization to a stable distribution (i.e., when the conditions of Theorem 3.1 are fullfilled), the largest term X_n^* forms a sensible proportion of the entire sum S_n, and in reality S_n is essentially dominated by the contribution of a finite number of its larger terms, the remainder being asymptotically negligible. These results were announced by Lévy [3] in 1935, and in point of fact he demonstrated that the necessary and sufficient condition that the central limit obtain is that the largest term be asymptotically negligible (even in the case of nonidentically distributed components). Thus only in the case of the limiting Gaussian distribution can we truly say that the distribution of S_n is composed of infinitely many infinitesimal components.

These remarks are capable of quantitative analysis, and we give in this section three theorems which show in what manner X_n^* influences the sum S_n.

THEOREM 5.1. *If $X_i \geqq 0$ and S_n has a limiting stable distribution with exponent $0<\gamma<1$, then*

$$\lim_{n\to\infty} \Pr \{S_n < yX_n^*\} = G(y)$$

where $G(y)$ has the characteristic function

$$\int_0^\infty e^{ity} dG(y) = \frac{e^{it}}{1 - \gamma \int_0^1 (e^{it\alpha} - 1) \frac{d\alpha}{\alpha^{\gamma+1}}}.$$

Proof. Here again we have supposed $X_i \geqq 0$, but as before if $\Pr \{X_i < -x\} = o(x^{-\gamma})$, $x \to \infty$, the negative tail is negligible. We may also presume, as before, that $F(x)$ is continuous. Let us put

$$(5.1) \qquad \psi(\beta) = \beta \int_0^1 e^{it\alpha} \phi(\alpha\beta) d\alpha$$

so that by (2.1) the characteristic function of $Z_n = S_n/X_n^*$ becomes

$$\xi_n(t) = -ne^{it} \int_0^\infty (\psi(\beta))^{n-1} df(\beta).$$

By a slight transformation we find for (5.1)

$$\psi(\beta) = 1 - f(\beta) + it \int_0^1 e^{it\alpha}(f(\alpha\beta) - f(\beta)) d\alpha.$$

It follows from Doeblin's theorem that $f(x)$ is $h(x)/x^\gamma$ where $h(x)$ satisfies $h(ax)/h(x) = 1 + o(1)$ for $x \to \infty$ and $a > 0$; from this it follows that if $\alpha > 0$,

$$(5.2) \qquad f(\alpha\beta) = (1/\alpha^\gamma) f(\beta)(1 + o(1))$$

for large β. Hence

$$\begin{aligned} it \int_0^1 e^{it\alpha}(f(\alpha\beta) - f(\beta)) d\alpha &= itf(\beta)(1 + o(1)) \int_0^1 e^{it\alpha}\left(\frac{1}{\alpha^\gamma} - 1\right) d\alpha \\ &= f(\beta)\gamma \int_0^1 (e^{it\alpha} - 1) \frac{d\alpha}{\alpha^{\gamma+1}} + o(f(\beta)) \end{aligned}$$

when $\beta \to \infty$ and $|t|$ is bounded. Putting

$$\phi_1(t) = \phi_1 = \gamma \int_0^1 (e^{it\alpha} - 1) \frac{d\alpha}{\alpha^{\gamma+1}}$$

we obtain

$$\psi(\beta) = 1 - (1 - \phi_1)f(\beta) + o(f(\beta)), \qquad \beta \to \infty.$$

Since $|\psi(\beta)| < 1$ for bounded β, it is seen that the early portion of the integral for $\xi_n(t)$ is negligible for sufficiently large n. Operating formally for the moment we use the preceding asymptotic development for $\psi(\beta)$ in the integral for $\xi_n(t)$, and make the change of variable $nf(\beta) = v$ to give

$$\xi_n(t) \sim e^{it} \int_0^n \left(1 - \frac{v}{n}(1 - \phi_1) + vo\left(\frac{1}{n}\right)\right)^{n-1} dv$$

$$\to e^{it} \int_0^\infty e^{-v(1-\phi_1)} dv = \frac{e^{it}}{1 - \phi_1}$$

which yields the conclusions to the theorem.

To justify the limiting process it suffices, first of all, to consider only $-ne^{it}\int_A^\infty \psi^{n-1} df(\beta)$ which for every A will differ arbitrarily little from $\xi_n(t)$ for n sufficiently large. Hence

$$\xi_n(t) \sim e^{it} \int_0^{nf(A)} \left(1 - \frac{v}{n}(1 - \phi_1) + vo\left(\frac{1}{n}\right)\right)^{n-1} dv$$

for each A, and if we write the last integral as $\int_0^{nf(A)} = \int_0^M + \int_M^{nf(A)}$, the second term can be made arbitrarily small by choosing M, n sufficiently large.

For the case when the exponent of the distribution is larger than 1 the mean value $E(X_i) = \mu$ exists, so that by the strong law of large numbers $S_n \sim n\mu$ with probability one, and this term will dominate X_n^*. In this case there are two theorems—one which compares S_n directly with X_n^* and another which compares the deviation $S_n - n\mu$ with X_n^*.

THEOREM 5.2. *Let $X_i \geqq 0$ have an exponent $1 < \gamma < 2$ and such that S_n has a limiting stable distribution, and let a sequence $\{c_n\}$ be determined by the relation $f(c_n) = 1/n$. Then $\mu = E(X_i)$ exists, and if $a_n = n/c_n$, we have*

$$\lim_{n\to\infty} \Pr\left\{\frac{S_n}{X_n^*} < a_n x\right\} = 1 - e^{-(x/\mu)^\gamma}.$$

Proof. We use the same formula for $\xi_n(t)$ as before, but we need a different estimate for $\psi(\beta)$. We have

$$\psi(\beta) = 1 - f(\beta) + it\int_0^1 e^{it\alpha}(f(\alpha\beta) - f(\beta))d\alpha$$

$$= 1 - f(\beta) + it\int_0^1 (e^{it\alpha} - 1)(f(\alpha\beta) - f(\beta))d\alpha - itf(\beta) + it\int_0^1 f(\alpha\beta)d\alpha.$$

Now

$$\int_0^1 f(\alpha\beta)d\alpha = \frac{1}{\beta}\int_0^\beta f(z)dz = \frac{1}{\beta}\int_0^\infty f(z)dz - \frac{1}{\beta}\int_\beta^\infty f(z)dz$$
$$= \frac{\mu}{\beta} - \int_1^\infty f(\alpha\beta)d\alpha$$
$$= \frac{\mu}{\beta} - f(\beta)(1 + o(1))\int_1^\infty \frac{d\alpha}{\alpha^\gamma}$$
$$= \frac{\mu}{\beta} - \frac{f(\beta)}{\gamma - 1} + o(f(\beta))$$

where $\mu = E(X_i)$. We next estimate $it\int_0^1(e^{it\alpha} - 1)(f(\alpha\beta) - f(\beta))d\alpha$ as in Theorem 5.1, obtaining for it

$$itf(\beta)(1 + o(1))\int_0^1 (e^{it\alpha} - 1)\left(\frac{1}{\alpha^\gamma} - 1\right)d\alpha$$
$$= f(\beta)\int_0^1 (e^{it\alpha} - 1 - it\alpha)\frac{d\alpha}{\alpha^{\gamma+1}} + ito(f(\beta)).$$

As a consequence we obtain for large β and bounded $|t|$

$$\psi(\beta) = 1 + \frac{it\mu}{\beta} - f(\beta)(1 - \phi_2(t)) - itf(\beta)\frac{\gamma}{\gamma - 1} + ito(f(\beta)) \tag{5.3}$$

where

$$\phi_2(t) = \gamma\int_0^1 (e^{it\alpha} - 1 - it\alpha)\frac{d\alpha}{\alpha^{\gamma+1}}\cdot \tag{5.4}$$

In the integral for $\xi_n(t)$ we propose to make the substitution $nf(\beta) = v$ as before; however we need an asymptotic solution to this equation for β when $n\to\infty$. We determine c_n such that $f(c_n) = 1/n$; then $vf(c_n) = v/n = f(\beta)$ and from (5.2) it follows that $f(\beta) = f(v^{-1/\gamma}c_n)(1 + o(1))$ and we may easily infer $1/\beta = v^{1/\gamma}/c_n + o(1/c_n)$ which is the required asymptotic solution when v is bounded and $n\to\infty$. Now let $a_n = n/c_n$ and replace t by t/a_n in $\psi(\beta)$, and notice that $f(\beta)/a_n = o(1/n)$ so that

$$\psi\left(\beta, \frac{t}{a_n}\right) = 1 + \frac{it\mu v^{1/\gamma}}{n} - \frac{v}{n}(1 - \phi_2(t/a_n)) + o\left(\frac{1}{n}\right)$$
$$= 1 + \frac{it\mu v^{1/\gamma}}{n} - \frac{v}{n} + o\left(\frac{1}{n}\right)$$

since $a_n \nearrow \infty$ and $\phi_2(t) = o(t)$ for $t\to 0$.

The argument now proceeds as in Theorem 5.1, and the formal justification of the limiting process is identical. For any A we obtain

$$\xi_n(t/a_n) \sim e^{it/a_n} \int_0^{nf(A)} \left(1 + \frac{it\mu v^{1/\gamma}}{n} - \frac{v}{n} + o\left(\frac{1}{n}\right)\right)^{n-1} dv$$

$$\to \int_0^\infty e^{it\mu v^{1/\gamma} - v} dv .$$

and the latter expression is the characteristic function for the random variable $\mu W^{1/\gamma}$ where W has the exponential distribution $1-e^{-x}$. An application of the continuity theorem for characteristic functions concludes the proof of the theorem.

An idea of the size of the normalizing constants a_n can be obtained from the fact that for any $\epsilon > 0$ there is an n_0 such that

$$n^{1-1/\gamma-\epsilon} < a_n < n^{1-1/\gamma+\epsilon}, \qquad n > n_0.$$

For the case when the mean exists there is, in addition, a theorem analogous to Theorem 5.1, and it turns out that the normalized sum $S_n - n\mu$ is of the order of magnitude of X_n^*.

THEOREM 5.3. *Let $X_i \geqq 0$ have an exponent $1 < \gamma < 2$, and let $\mu = E(X_i)$. Then if S_n has a limiting stable distribution.*

$$\lim_{n\to\infty} \Pr\left\{S_n < n\mu + yX_n^*\right\} = G(y)$$

where the characteristic function of $G(y)$ is

$$\int_0^\infty e^{ity} dG(y) = \frac{e^{it}}{1 + \dfrac{it\gamma}{\gamma - 1} - \gamma \displaystyle\int_0^1 (e^{it\alpha} - 1 - it\alpha) \frac{d\alpha}{\alpha^{\gamma+1}}} .$$

Proof. We first need a result analogous to Lemma 2.1, and a line of reasoning similar to that employed before shows that

$$E\left(\exp\left(it\frac{S_n - (n-1)}{X_n^*}\mu\right)\right) = -ne^{it}\int_0^\infty \left\{\int_0^\beta e^{it(x-\mu)/\beta}\phi(x)\right\}^{n-1} df(x)$$

and putting

$$\zeta(\beta) = \int_0^\beta e^{it(x-\mu)/\beta}\phi(x)dx = e^{-it\mu/\beta}\psi(\beta)$$

where $\psi(\beta)$ is given by (5.1).

Noting that

$$1/\beta^2 = o(f(\beta))$$

we obtain by using (5.3)

$$\begin{aligned}\zeta(\beta) &= \left\{1 - \frac{it\mu}{\beta} + o(f(\beta))\right\} \\ &\quad\cdot \left\{1 + \frac{it\mu}{\beta} - f(\beta)\left(1 + \frac{it\gamma}{\gamma - 1} - \phi_2(t)\right) + o(f(\beta))\right\} \\ &= 1 - f(\beta)\left(1 + \frac{it\gamma}{\gamma - 1} - \phi_2(t)\right) + o(f(\beta))\end{aligned}$$

where $\phi_2(t)$ is given by (5.4). Reasoning identical to that of Theorem 5.1 shows that

$$\begin{aligned}\lim_{n\to\infty} - ne^{it}\int_0^\infty \zeta^{n-1}(\beta)df(\beta) &= e^{it}\int_0^\infty e^{-v(1+it\gamma/(\gamma-1)-\phi_2(t))}dv \\ &= \frac{e^{it}}{1 + it\gamma/(\gamma - 1) - \phi_2(t)}\end{aligned}$$

and Theorem 5.3 is proved.

An intermediate case corresponding to $\gamma = 1$ could also be studied by these methods.

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