

ON THE SYMMETRY AND BOUNDED CLOSURE OF LOCALLY CONVEX SPACES

BY

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Preface. For some time it has been recognized that the properties of symmetry and bounded closure are of cardinal importance in the theory of locally convex spaces. The purpose of this paper is to discuss symmetry and bounded closure with the aid of a new topology, and to discuss the new topology itself.

The introduction includes the preliminaries to the rest of the paper. The account of the basic theory of locally convex spaces is fairly complete, but brief. We have given proofs only when they represent some contribution. As the references indicate, most of the theorems of the introduction can be found in the literature.

In §1 we introduce the new topology, called the *GF* topology, and develop the theory of the *GF* space. A space is a *GF* space if and only if it is symmetric and boundedly closed. Study of the *GF* topology on duals leads to information about interior maps and quotient spaces, information which is not obvious from direct consideration of symmetry and bounded closure. We conclude the section with several examples.

In view of the general results of the first section it is important to know what spaces are *GF* spaces. In §2 we take up this question for direct products and their duals, and also the questions of symmetry and bounded closure. §3 serves the same purpose for continuous function spaces. Finally in §4 we discuss metric and *LF* spaces and their duals.

The influence of the works of G. W. Mackey, J. Dieudonné, and L. Schwartz is apparent throughout the paper, and it is to these authors that we owe our interest in locally convex spaces. The terminology used in the paper is almost exclusively that employed in the works of Dieudonné, Schwartz, and Bourbaki. For the convenience of the reader we list the exceptions. We distinguish between weak and weak-star topologies; our total bounded is the French *precompact* and corresponds to the definition given in [14]; we owe the term relatively strong to Mackey [13]; where the French authors use filters we make use of nets, as in Kelley [10]; and for a uniform structure topology the natural definition of a Cauchy net leads to a concept of completeness which is equivalent to the French term and stronger than that given in [14]. There are no French equivalents for symmetry and bounded closure.

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Introduction. Closely associated with any linear space E over the complex numbers are the linear forms on E , the functions $f(x)$ from E to the complex numbers satisfying, for x and y in E and λ a complex number, $f(x+y) = f(x) + f(y)$ and $f(\lambda x) = \lambda f(x)$. When addition and scalar multiplication are defined in the obvious fashion, the class of linear forms becomes a linear space, which we call the algebraic dual of E and denote by \bar{E} . If E is a topological space as well as a linear space, those linear forms which are continuous are of particular importance. The set of continuous linear forms is a subspace of \bar{E} , which we call the dual of E and denote by E' .

Let E be a linear space and let F be a subspace of its algebraic dual which contains for each $x \neq 0$ in E an f such that $f(x) \neq 0$. Then E is a subspace of the algebraic dual of F with the same property. To exhibit the duality between E and F we write (f, x) rather than $f(x)$. The polar in F of a set A in E , written A° , is the set of all f in F such that if x is in A , then $|(f, x)| \leq 1$. The polar in E of a set A in F is defined similarly. F defines on E a natural topology called the weak topology and represented by the symbol $W(E, F)$. Its neighborhoods of the point 0 are the polars in E of finite sets in F . Its neighborhoods of the point x are the sets $x + V$, where V is a neighborhood of 0. In the same way E defines on F a natural topology called the weak-star topology and represented by the symbol $W(F, E)$.

The weak topology on E is defined by neighborhoods of the origin; each neighborhood of the origin is convex ($V + V \subseteq 2V$) and circled ($e^{i\alpha}V \subseteq V$), and the operations of addition and scalar multiplication are jointly continuous. Any topology on E having these properties is called a locally convex topology. A linear space with a locally convex topology is a locally convex space.

Let E with topology T be a locally convex space. We indicate that V is a neighborhood of the origin in the topology T by saying that V is in T or that V is a T neighborhood or just that V is a neighborhood. The semi-norm associated with the neighborhood V is the function $v(x) = \inf \{|\lambda| : (1/\lambda)x \text{ in } V\}$. The closure of V is the set where $v(x) \leq 1$, so that v is continuous, and the topology T can be described by specifying its semi-norms as well as by specifying its neighborhoods. Every semi-norm has the properties $v(x+y) \leq v(x) + v(y)$ and $v(\lambda x) = |\lambda|v(x)$. A linear form f on E is continuous if and only if it is bounded on a neighborhood, and hence if and only if it satisfies $|(f, x)| \leq v(x)$ for some semi-norm v . With the notion of semi-norm it is easy to prove the following theorem.

THEOREM 1. *If A is a closed convex circled subset of E and y is in $E - A$, then there is an f in E' such that $(f, y) > 1$ and $|(f, x)| \leq 1$ for every x in A [7].*

Any theorem of this type which describes the separation of a point and a convex set by a hyperplane is called a Hahn-Banach theorem. The Hahn-Banach theorem implies, in particular, that E' has the property previously required of F : for each $x \neq 0$ in E it contains an f such that $(f, x) \neq 0$. A Hahn-Banach theorem is also valid for E' in the weak-star topology.

THEOREM 2. *If A is a weak-star closed convex circled subset of E' and g is in $E' - A$, then there is an x in E such that $(g, x) > 1$ and $|(f, x)| \leq 1$ for every f in A [5; 7].*

A set A in E is T bounded, or simply bounded, if to each T neighborhood V corresponds a number λ such that $\lambda A \subseteq V$. A linear form on E is bounded if it is bounded on each bounded set. The bounded sets in E are used to define another topology on E' . It is called the ordinary topology, is labeled T' , and its neighborhoods are the polars of the T bounded sets. Unqualified topological terms such as bounded, closed, continuous refer either to the given topology T of E or to the ordinary topology T' of E' . Since $W(E, E')$, $W(E', E)$, T , and T' are all locally convex topologies, it is proper to speak of boundedness in any of them.

THEOREM 3. *Weak bounded and T bounded are the same. If E is sequentially complete, weak-star bounded and T' bounded are the same [12; 13].*

Every locally convex topology is a uniform structure and the usual definition of total bounded is employed.

THEOREM 4. *In weak or weak-star topologies bounded and total bounded are the same [7].*

Using Theorem 4 it is easy to prove the following theorem.

THEOREM 5. *If V is a neighborhood in E , then V° is weak-star compact [1; 5].*

It is a consequence of the Hahn-Banach theorem that, for any set A in E (E'), $A^{\circ\circ}$ is the smallest closed (weak-star closed) convex circled set containing A [7]. It follows that the topology T , as well as all of the others which have been mentioned, can be defined by taking as neighborhoods the polars of certain convex circled and weak or weak-star compact sets. The polars of all convex circled weak-star compact sets form the neighborhoods for a locally convex topology on E which we call the relatively strong topology and write $T(E, E')$. $T(E, E')$ is clearly stronger than T , and its precise relation to the several topologies on E is given by the fundamental Mackey-Arens theorem [2; 12; 13].

THEOREM 6. *$T(E, E')$ is the strongest locally convex topology on E such that E' is the dual of E .*

As the known proofs of the Mackey-Arens theorem are complicated, we offer a simple proof of our own. In view of the preceding remarks, if $T(E, E')$ does yield E' as the dual of E , it is the strongest topology doing so.

A convex circled weak-star compact subset of E' is a convex circled weak-star closed subset of \bar{E} . Thus if C is such a set, C is the polar in \bar{E} of C° , and any linear form which is bounded on C° is a scalar multiple of an element in C , and hence lies in E' .

It is a simple matter to deduce from the Mackey-Arens theorem the following theorem.

THEOREM 7. *Every convex circled weak-star compact subset of E' is bounded.*

The foregoing discussion reveals that any linear space E can be converted into a locally convex space having as dual a prescribed weak-star dense subspace F of \bar{E} . Among the topologies which serve the weakest is $W(E, F)$ and the strongest is $T(E, F)$. In comparing the different topologies it is convenient to use the words "stronger than" to mean "at least as strong" and to give this meaning to the symbol \geq .

Much of our discussion concerns the continuity of linear forms. We have observed that for a linear form f the assertions that f is bounded on a neighborhood and that f is continuous are equivalent. A third equivalent assertion is that the null space of f is closed.

For a linear transformation from one locally convex space to another the three assertions are not equivalent in general, but the first implies the second, which in turn implies the third. If T is a linear transformation from the locally convex space E into the locally convex space F , its adjoint T^* is the linear transformation from F' into \bar{E} defined by the equation $(T^*f, x) = (f, Tx)$. T^* is a linear transformation from F' into E' if and only if T is continuous when E and F have the weak topologies: i.e., T is weak continuous. If T is weak continuous, then T^* is both continuous and weak-star continuous.

The dual of the locally convex space E' with topology T' is called the second dual of E and is written E'' . As a dual E'' has an ordinary topology T'' . According to the previous definition, the polar in E'' of an arbitrary bounded set in E' is a neighborhood in T'' . If the polar in E of an arbitrary bounded set in E' is a neighborhood in T , then the space E and the topology T are called symmetric.

We have noted that E is a subspace of \bar{E}' ; as the topology T' is clearly stronger than the topology $W(E', E)$, E is a subspace of E'' . The identity mapping from E as a locally convex space to E as a subspace of E'' is termed the canonical mapping. As the neighborhoods in T are the polars of certain convex circled weak-star compact sets in E' , and as each of these is bounded, T is weaker than T'' . Clearly the two are identical if and only if E is symmetric. The considerations which apply to T also apply to $T(E, E')$, and

$T(E, E')$ is weaker than T'' . Thus if E is symmetric, it has the relatively strong topology, and every bounded set in E' is contained in a convex circled weak-star compact set. Evidently the canonical mapping is continuous if and only if E is symmetric.

If the canonical mapping of E into E'' is onto, i.e., if the elements of E are the only continuous linear forms on E' , then clearly $T' = T(E', E)$. When this is the case we say that E is semi-reflexive, and it is clear that E is semi-reflexive if and only if any bounded set in E is contained in a convex circled weak compact set. The space E is called reflexive when E and E'' coincide both algebraically and topologically, that is, when E is both symmetric and semi-reflexive. If E is semi-reflexive, then E' is symmetric, for a bounded set in E'' is also a bounded set in E , hence its polar in E' contains a neighborhood.

A locally convex space is called a metric space if there exists a fundamental set of neighborhoods which is countable. The topology of such a space is indeed a metric topology. The metric spaces are the only locally convex spaces which have a fundamental set of neighborhoods linearly ordered under set inclusion. A metric space is called an F space if it is complete.

The locally convex space E is said to be boundedly closed if every bounded linear form on E is continuous. We use the term bounded closure to describe the property of being boundedly closed, and not in the way it was used originally by Mackey [13].

THEOREM 8. *Every metric space is symmetric and boundedly closed.*

The bounded closure is established in [12]. A trivial modification of the argument establishes the symmetry. Since every LF space is also symmetric [7], we have the following corollary.

COROLLARY. *Every semi-reflexive metric or LF space is reflexive.*

We have no example of a symmetric space which is not boundedly closed. Boundedly closed spaces which are not symmetric are common, but we have:

THEOREM 9. *Every boundedly closed space is symmetric in the relatively strong topology.*

Proof. If B is bounded in E' , then C , the smallest weak-star closed convex circled set containing B , is also bounded, and hence weak-star total bounded. We show that C is a weak-star complete, therefore weak-star compact, and this fact implies that $B^\circ = C^\circ$ is a relatively strong neighborhood. If f_α is a Cauchy net in C , the linear form $(f, x) = \lim_\alpha (f_\alpha, x)$ is bounded on bounded sets, hence is continuous.

THEOREM 10. *There is at most one symmetric topology on E yielding a prescribed dual.*

Proof. We have already observed that if E admits a symmetric topology with dual E' , then $T = T'' = T(E, E')$.

COROLLARY. *There is at most one metric topology on E yielding a prescribed dual.*

We conclude this introduction with some representation theorems which are responsible for our interest in direct products and continuous function spaces.

THEOREM 11. *Every locally convex space is isomorphic to a subspace of a direct product of Banach spaces [8].*

Proof. If $v(x)$ is a semi-norm on E and H_v is its null space, then E_v , the completion of E/H_v with norm v , is a Banach space. The transformation which maps x in E into the function $x(v)$, where $x(v)$ is the canonical image of x in E_v , is an algebraic isomorphism between E and a subspace of the direct product of the E_v . Convergence, both in the direct product and in E , is convergence with respect to each v , so that the algebraic isomorphism is also a topological isomorphism.

THEOREM 12. *Every locally convex space is isomorphic to a subspace of a $C(S)$: the continuous functions on a locally compact Hausdorff space, S , in the compact-open topology.*

Proof. Because of the previous theorem, it is enough to show this result for a direct product of Banach spaces. By a result of Alaoglu [1] every Banach space E_v is a closed subspace of the space of continuous functions on some compact Hausdorff space S_v . Let S be the set union of the S_v , and topologize S by assigning to each point the neighborhood system which it has in the S_v to which it belongs. It follows that S is a locally compact Hausdorff space, and that the direct product of the E_v is a subspace of $C(S)$. Convergence in $C(S)$ implies convergence in each $C(S_v)$, hence convergence in the product. On the other hand, every compact set in S is contained in the union of finitely many S_v , whence convergence in the product implies convergence in $C(S)$.

COROLLARY. *Every separable metric space is isomorphic to a subspace of $C(R)$, where R is the real axis in the usual topology.*

Proof. We argue as before, using the theorem that a separable Banach space is isomorphic to a subspace of $C([0, 1])$ [3]. The remainder of the argument is clear.

1. GF spaces. In 1949 in a paper entitled *La dualité dans les espaces F et LF* [7], Dieudonné and Schwartz introduced a new type of locally convex space called the *LF* space. In this section we generalize the concept of *LF* space. We develop a theory for the new spaces and examine the concepts of symmetry and bounded closure. We show how the new *GF* spaces arise nat-

urally in the study of locally convex spaces and, indeed, that many of the common spaces are *GF* spaces.

A Dieudonné-Schwartz *LF* space is a locally convex space E whose topology is defined by an ascending sequence of *F* spaces E_n . It is supposed that the *F* spaces have compatible topologies and that their set union is E . A neighborhood of the origin in E is then any convex circled set whose intersection with each E_n is a neighborhood of the origin in E_n .

To generalize the *LF* space we proceed as follows. Let E be any vector space, and let E' be a subspace of its algebraic dual which contains all linear forms bounded in the topology $W(E, E')$. Such an E' can be obtained by choosing a weak-star dense subspace F of the algebraic dual and letting E' consist of all linear forms bounded in the topology $W(E, F)$. If V is a neighborhood in the ordinary topology on E' , then the space E_v of all x in E such that $v(x) = \sup \{ |(f, x)| : f \in V \}$ is finite is a normed linear space with v as norm. A set is a neighborhood of the origin in the *GF* topology defined by the E_v if it is convex and circled and its intersection with each E_v is a norm neighborhood in E_v . The next theorem shows that with the *GF* topology defined by the E_v , E is a symmetric and boundedly closed locally convex space.

THEOREM 1. *The GF topology defined by the E_v is the topology $T(E, E')$.*

Proof. Let W be a neighborhood in $T(E, E')$. For each V , the unit sphere in E_v , being the polar of V , is bounded in E . For each V , therefore, a contraction of the unit sphere in E_v is contained in W ; and W is a *GF* neighborhood.

Let W be a *GF* neighborhood. Each bounded set in E is contained in the unit sphere of some E_v and therefore in a multiple of W . It follows that a linear form which is bounded on W is bounded, which implies that W° is weak-star complete. W° is clearly weak-star bounded. From the equivalence of weak-star bounded and weak-star total bounded we deduce that W° is weak-star compact. In this way we see that $W^{\circ\circ}$ is a $T(E, E')$ neighborhood and we must show that W itself is.

For any x in E we obtain from the Hahn-Banach theorem a linear form f on E satisfying $(f, x) = p(x)$ and $|(f, y)| \leq p(y)$ for all y in E , where p is the semi-norm associated with W . The latter inequality implies that f is bounded, and that it in fact lies in W° . Thus if x is in $W^{\circ\circ}$, $p(x) = (f, x) \leq 1$, so that p is continuous $T(E, E')$, and W is a $T(E, E')$ neighborhood.

We abstract the essential elements of the above *GF* topology to obtain the general definition.

Let E be a linear space, and let E_v be a family of symmetric and boundedly closed locally convex spaces. For every v let T_v be a linear transformation of E_v into E , and let the union of the sets $T_v(E_v)$ span E . The *GF* topology on E defined by $\{E_v, T_v\}$ is the strongest locally convex topology on E making the mappings T_v continuous, provided such a topology exists.

THEOREM 2. *If E admits any locally convex topology making the mappings*

T_v continuous, then the GF topology on E defined by $\{E_v, T_v\}$ exists.

We omit the obvious proof.

Suppose that a GF topology exists on the space E , and denote it by T^* . If W is a set in T^* , the continuity of the transformation T_v implies that $T_v^{-1}(W)$ contains a neighborhood of the origin in E_v . It is also clear that any set W having this property belongs to T^* . The attempt to define the GF topology directly in terms of the neighborhoods identified above leads to a topology on E which satisfies all of the axioms characterizing a locally convex space except the separation axiom. Accordingly it seems preferable to define the GF space in the manner adopted. We give an example of a system $\{E, E_v, T_v\}$ where the GF topology does not exist. Let E be a two-dimensional vector space spanned by the vectors i and j . Let E_i and E_j be infinite-dimensional Hilbert space. Let f be a continuous linear form on E_i , and let g be a discontinuous linear form on E_j . Define $T_i(x) = (f, x)i$ and $T_j(x) = (g, x)j$. If S is any sphere in E_j , $T_j(S)$ contains the whole subspace spanned by the vector j . Hence there is no separated topology making T_j continuous.

A locally convex space is called a GF space if its topology is a GF topology.

THEOREM 3. Every GF space is symmetric and boundedly closed. Every symmetric and boundedly closed space is a GF space determined by a family of normed linear spaces.

Proof. Suppose f is a bounded linear form on E . The function $(f, T_v(x))$ is a bounded linear form on E_v , since the continuous mapping T_v carries bounded sets in E_v into bounded sets in E . The hypothesis that E_v is boundedly closed thus implies that $(f, T_v(x))$ is continuous on E_v . If V is the set of all z in E for which $|(f, z)| \leq 1$, then $T_v^{-1}(V)$ is the set of all x in E_v for which $|(f, T_v(x))| \leq 1$, which, as we have seen, contains a neighborhood in E_v . Hence V is a GF neighborhood and f is continuous.

The argument to show that E is symmetric is similar. Let B be a bounded set in E' . The linear forms $(f, T_v(x))$ for f in B form a bounded set in E'_v , which by the symmetry of E_v is contained in the polar of a neighborhood U_v in E_v . Since $T_v^{-1}(B^\circ)$ contains U_v , B° is a neighborhood in the GF topology of E .

That every symmetric and boundedly closed space is a GF space determined by a family of normed linear spaces follows directly from Theorem 1, and the discussion preceding it. These considerations, together with Theorem 3, give us:

THEOREM 4. There is at most one GF topology on E yielding a prescribed dual. In particular every GF topology is of the type described at the beginning of this section.

The concept of a GF topology is most fruitful when applied to the dual of a locally convex space. There are in fact two natural GF topologies on the

dual. One is obtained by taking as E'_v the normed linear space of all f in E' such that $v(f) = \sup \{ |(f, x)| : x \text{ in } V \}$ is finite, where V is a neighborhood in E . The other is obtained in the same way, but where V is a neighborhood in E'' . In each case it is clear that the identity transformation T_v from E'_v to E' is continuous with respect to the ordinary topology of E' so that the GF topology exists. It follows from Theorem 1 that the latter, which is the weaker of the two, is the topology $T(E', E'^*)$, E'^* being the space of bounded linear forms on E' . If E is symmetric, the two GF topologies coincide, and the interesting case occurs when they do coincide and coincide also with the ordinary topology of E' . When this situation obtains, we say that E has a GF dual. If E is not symmetric, the two GF topologies may not coincide. An example is Banach space in its weak topology.

We obtain an interior mapping theorem for spaces with GF duals. The first step is the following lemma, which also plays a critical role in several later theorems.

LEMMA. *Let A and B be closed convex circled subsets of a locally convex space E . Then $(A \cap B)^\circ$ is the weak-star closure of the convex envelope of A° and B° ; if both A and B are compact, their convex envelope is compact; if either is compact, their sum $A + B$ is closed.*

Proof. The first statement is proved by Dieudonné and Schwartz [7], the second assertion is elementary, while the last is a standard theorem.

THEOREM 5. *Let T be an isomorphism of E into any locally convex space F such that $T(E)$ is closed in F . If E has a GF dual, then the adjoint T^* is a homomorphism of F' onto E' .*

Proof. By definition a linear transformation T of E into F is a homomorphism of E into F if it is continuous and interior between E and $T(E)$; that is, continuous and such that to each neighborhood V in E corresponds a neighborhood U in F with $U \cap T(E) \subseteq T(V)$. T is an isomorphism if it is a homomorphism and one to one. It is well known that the adjoint of a continuous linear transformation is a continuous linear transformation itself [7]. It remains to show that T^* is onto and interior. That T^* is onto follows from a theorem of Dieudonné [5]. On the other hand, it is as difficult to show that the theorem of Dieudonné is applicable as to give a direct proof. Let f be arbitrary in E' . The equation $(g, Tx) = (f, x)$ defines g as a linear form on $T(E)$. If f is bounded on the neighborhood V in E , then g is bounded on $T(V)$ which is a neighborhood in $T(E)$, so that g is continuous on $T(E)$. Let h be a continuous extension of g to all of F . Then for each x in E , $(T^*h, x) = (g, Tx) = (f, x)$, and $T^*h = f$.

Finally, T^* is interior. Let V be an arbitrary neighborhood in E and choose the neighborhood U in F so that $U \cap T(E) \subseteq T(V)$. As U and $T(E)$ are both closed and U° is weak-star compact, it follows from the lemma

that $T(V)^\circ \subseteq (U \cap T(E))^\circ = U^\circ + T(E)^\circ$. Recalling that T^* is onto we see that $V^\circ \subseteq T^*((TV)^\circ)$. The collected inequalities and the fact that $T(E)^\circ$ is the null space of T^* give $V^\circ \subseteq T^*(U^\circ)$. If W is an arbitrary neighborhood in F' , there is a number λ such that $U^\circ \subseteq \lambda W$, so that $V^\circ \subseteq \lambda T^*(W)$. In other words, the image of an arbitrary neighborhood in F' is a *GF* neighborhood in E' .

COROLLARY. *If E is a closed subspace of the locally convex space F , and E has a *GF* dual, then $E' = F'/E^\circ$.*

Proof. The identity transformation from E to F satisfies the conditions of the theorem. Since the null space of T^* is $T(E)^\circ = E^\circ$, T^* can be regarded as a transformation from F'/E° to E' . It is a direct consequence of the definition of the quotient topology that T^* is an isomorphism.

REMARK. If E is a closed subspace of the locally convex space F , if F' is boundedly closed, and if $E' = F'/E^\circ$, then E' is boundedly closed.

Proof. Every bounded linear form on E' has an obvious extension to a bounded linear form on F' . The form on E' is bounded on the canonical image of the neighborhood on which the extension is bounded.

THEOREM 6. *Every closed reflexive subspace of a reflexive space with a *GF* dual has itself a *GF* dual.*

Proof. Let F be the space and E the subspace. Since E is reflexive, the neighborhoods in E' are the polars of weak-star compact sets in $E'' = E$. Every reflexive space is obviously symmetric, so that, in view of Theorem 1, it is sufficient to show that E' is boundedly closed. Dieudonné and Schwartz show that whenever F is reflexive and E is a closed subspace of F , $E' = F'/E^\circ$ [7]. The remark above completes the proof.

THEOREM 7. *If H is a closed subspace of the symmetric space E , then E/H is symmetric.*

Proof. Let T be the canonical mapping of E on E/H . If B is a bounded subset of $(E/H)'$, then $T^*(B)$ is a bounded subset of E' . When E is symmetric, $T^*(B)$ is bounded on a neighborhood V , and B is bounded on the neighborhood $T(V)$ in E/H .

If it is also true that E/H has a *GF* dual whenever E has a *GF* dual, then the interior mapping theorem is valid for homomorphisms T as well as isomorphisms. We do not know if this is the case.

One might suspect that an analogous theorem holds for H : that is, if E is symmetric, then the closed subspace H is symmetric. We have shown that every locally convex space is a subspace of a direct product of Banach spaces, and it is clear that a complete space is a closed subspace of the direct product. We show later that a product of Banach spaces is always symmetric; we

know, however, that there exist complete nonsymmetric spaces. An example of one will be given later.

The succeeding sections are devoted to discussion of the *GF* topology and the related concepts of symmetry and bounded closure in certain general types of locally convex space: direct products and their duals, continuous function spaces and their duals, and the duals of *F* and *LF* spaces. This section is concluded with a few particular examples.

1. Every metric or *LF* space is a *GF* space, for such spaces are symmetric and boundedly closed.

2. The space L^ω is a *GF* space and it has a *GF* dual. L^ω itself is a reflexive *F* space, and therefore a *GF* space. We show later that every reflexive *F* space has a *GF* dual. The dual of L^ω , however, is especially interesting as an example of the general theory, so we discuss it in some detail here.

Let E be the space L^ω , consisting of those measurable functions on the closed interval $(0, 1)$ which are p -summable for all $p > 1$. E can be regarded as the intersection of all the spaces L^p for $p > 1$. E is topologized by taking the L^p norms as a family of semi-norms. It is easy to see that E is an *F* space in this topology.

Every f in E' is bounded on a neighborhood in E , and hence can be regarded as an element of the dual of some L^p , which is an L^q . On the other hand it is clear that every element of an L^q is in E' ; and E' is the union of the spaces L^q for $q > 1$. Let T_q be the identity transformation from L^q into E' . The unit sphere in L^q , being the polar of a neighborhood in E , is bounded in E' , so that the mapping T_q is continuous, and the *GF* topology defined by $\{L^q, T_q\}$ exists. This *GF* topology is clearly the stronger of the two natural *GF* topologies. At the same time E is metric, hence symmetric, so that the two natural *GF* topologies coincide, and we can speak of the *GF* topology on E' .

The dual of E' with respect to the *GF* topology is the space of bounded linear forms on E' . Every *GF* continuous linear form is continuous on each L^q , hence is in each L^p , hence belongs to E . We conclude that E' with its ordinary topology is boundedly closed and that E is semi-reflexive. It follows that E' is symmetric (introduction). Thus the ordinary topology is a *GF* topology whose dual is the same as that of the *GF* topology above. Because of the uniqueness of the *GF* topologies, the two coincide.

3. The ring of entire functions is a *GF* space and it has a *GF* dual. The space E of entire functions is again a reflexive *F* space, and one which profitably can be examined directly. Its topology is defined by a family of semi-norms p_n , $p_n(x)$ being the upper bound over the circle of radius n of $|x(z)|$. The fact that the neighborhood $p_n(x) \leq 1$ is compact with respect to the metric $p_{n-1}(x)$ is used to show that the *GF* and ordinary topologies of the dual coincide on bounded sets. The well known fact that a subspace in the dual of an *F* space is weak-star closed if its intersection with each weak-star compact set is weak-star closed and the reflexivity of E then imply that the *GF* dual of

E' is E itself. The remainder of the argument parallels the last part of the argument for L^ω .

4. The space D' of distributions is a GF dual. D' is the dual of the reflexive LF space of all functions defined on Euclidean n -space which are infinitely differentiable and vanish outside compact sets [15]. We rely on our general theory, which asserts that every LF space which is reflexive has a GF dual, rather than discuss D' directly; but we mention the example because of its current interest.

5. A purely algebraic situation, discussed in part by Mackey [13], is of some interest. Let E be an arbitrary linear space. If H is a Hamel basis for E , then E can be regarded as the space of all complex-valued functions on H which vanish outside finite sets. Let E' be the algebraic dual of E . It is easy to see that E' is the space of all complex-valued functions on H , and that the weak-star topology on E' is that of pointwise convergence on H . In other words, E' is the direct product, with H as index set, of complex planes. In the next section we show that a bounded set in the dual of a direct product contains only a finite number of nonzero components. Thus every ordinary neighborhood in E' , as the polar of a bounded set in E , is a weak-star neighborhood, and the ordinary and weak-star topologies on E' coincide.

With respect to the topology $T(E, E')$ the mappings from the finite-dimensional subspaces of E to E are continuous so that the GF topology defined by the finite-dimensional subspaces, which are of course Euclidean, exists and is stronger than $T(E, E')$. The GF topology must actually be the same as $T(E, E')$ because its dual can be no larger than E' , which is the algebraic dual.

2. **Direct products.** In this section we examine the direct product of an arbitrary family of locally convex spaces. Let A be an index set, t the generic point of A , E_t a locally convex space, and $E(A)$ the direct product of the E_t . $E(A)$ is composed of functions $x(t) = x_t$ defined on A for which $x_t \in E_t$. A neighborhood of the origin in $E(A)$ is determined by a finite set F of indices and neighborhoods V_t in E_t for t in F . It consists of all $x(t)$ for which t in F implies x_t in V_t .

We use consistently the following notation: $E = E(A)$, N is the subspace of E consisting of functions with only a finite number of nonzero components, and N' is the corresponding subspace of the direct product of the spaces E'_t . By the identity mapping of E_t into E we mean the transformation I_t which carries x_t in E_t into that x in E which is x_t at the index t and 0 elsewhere. We call its adjoint, which carries E' onto E'_t , the projection of E' on E'_t . Since the identity is continuous, projection is both strong and weak-star continuous. Using the identity mapping, we can regard E_t as a subspace of E .

We proceed to identify the dual E' .

THEOREM 1. *E' and N' are algebraically isomorphic.*

Proof. If $f(t)$ is in N' , we write $(f(t), x) = \sum (f_t, x_t)$ to obtain a one to one linear mapping of N' into E' . To show that the mapping is onto, we note that every f in E' is bounded on a neighborhood, so that only a finite number of its projections are different from 0, and $f(t) = I_t^* f$ belongs to N' . Clearly the image of $f(t)$ and f coincide on N , which is dense in E .

COROLLARY. *The topology $W(E, E')$ is the product of the topologies $W(E_t, E'_t)$.*

As in the case of E , we may now speak of the identity transformation of E'_t into E' . Since the image of E'_t consists of linear forms which vanish on every E_t save one, it is clear that the identity mapping is continuous: a net in E'_t converging uniformly on bounded sets in E_t certainly has for an image a net converging uniformly on bounded sets in E . Hence T' , the ordinary topology of E' , is stronger than the product of the topologies T'_t . The identity is also weak-star continuous, so that its adjoint projects E onto E_t . We note that a set in E is bounded if and only if each of its projections is bounded.

THEOREM 2. *Every bounded set in E' is contained in the direct product of finitely many E'_t .*

Proof. Let B be a bounded set in E' , and let $p(x) = \sup \{ |(f, x)| : f \text{ in } B \}$; by hypothesis $p(x)$ is bounded on bounded sets in E . If the theorem is not true, there exist sequences t_n in A and x_n in E_{t_n} for which $p(Ix_n) = n$. However, such a sequence Ix_n is bounded.

COROLLARY. *The topology $T(E, E')$ is the product of the topologies $T(E_t, E'_t)$.*

Proof. Theorem 1 implies that if each factor is provided with the relatively strong topology, E still has the dual $E' = N'$, whence $T(E, E')$ is stronger than the product of the $T(E_t, E'_t)$. Now if C is a convex circled weak-star compact set in E' , the projection C_t of C on E'_t is $W(E'_t, E_t)$ compact by virtue of the weak-star continuity of projection. Moreover, C_t is 0 for all but finitely many t in A , since C is bounded and Theorem 2 applicable. It follows that C is uniformly bounded on the product neighborhood determined by the non-trivial C_t , and the product of $T(E_t, E'_t)$ is stronger than $T(E, E')$.

THEOREM 3. *E is symmetric if and only if each E_t is symmetric.*

Proof. If each E_t is symmetric, and B is bounded in E' , the projection B_t of B on E'_t is bounded, since projection is continuous. B_t is 0 except for a finite number of t because of Theorem 2. The direct product of $(B_t)^\circ$ is a neighborhood in E upon which B is uniformly bounded, i.e., E is symmetric.

If the factor E_{t_0} is not symmetric, there exists a bounded set B_0 in E'_{t_0} which is not uniformly bounded on a neighborhood in E_{t_0} . The identity

mapping of E'_t into E' provides a bounded set in E' which is bounded on no neighborhood in E , whence E is not symmetric.

THEOREM 4. *If each E_t has a GF dual, then E has a GF dual.*

Proof. Let W be a neighborhood in the stronger of the two natural GF topologies. Then $W \cap E'_t$ is a GF neighborhood in E'_t , and by hypothesis there is a bounded set B_t in E_t such that $W \cap E'_t$ contains $B_t^o \cap E'_t$. Let B be the direct product of the sets B_t . B is bounded and $B^o \cap E' = B_t^o \cap E'_t \subseteq W$.

Let F be any set of n points in A , and let E'_F be the direct product of the spaces E'_t for t in F . Let f be an arbitrary element of $B^o \cap E'_F$. The projection f_t of f on E'_t is in B_t^o . Set $\alpha_t = \sup \{(f_t, x_t) : x_t \text{ in } B_t\}$. Since we assume B circled, for $\epsilon > 0$ there exists x_t in B_t with $(f_t, x_t) > \alpha_t - \epsilon/n$. The x in E which is x_t if t is in F and is 0 otherwise is in B so that $1 \geq (f, x) = \sum (f_t, x_t) \geq \sum \alpha_t - \epsilon$; it follows that $\sum \alpha_t \leq 1$. Writing $f = \sum \alpha_t (f_t / \alpha_t)$ and noting that f_t / α_t is in W , we conclude that f is in W . Since every f in B^o is in some $B^o \cap E'_F$, we have $B^o \subseteq W$.

This implies that the ordinary topology, which is weaker than the weak GF topology, is also stronger than the strong GF topology.

THEOREM 5. *Every bounded linear form on E is continuous on N .*

Proof. If this were not so, there would exist sequences t_n in A and x_n in E_{t_n} such that $(f, Ix_n) = n$. The sequence Ix_n , however, is bounded.

COROLLARY. *E is boundedly closed if and only if every bounded linear form on E is uniquely determined by its behavior on N .*

Proof. The restriction of f to N is continuous there; since N is dense in E , f has a continuous extension over E .

COROLLARY. *E is boundedly closed if and only if N is dense in E in the weak topology determined on E by the space of bounded linear forms.*

COROLLARY. *E is not boundedly closed if and only if there exists a nonzero bounded linear form on E which vanishes on N .*

We suppose now that the factor spaces E_t are metric spaces and consider the question of the bounded closure of E . Some of the results which follow have been obtained in a less general context by Mackey [11] and Ulam [17]; these authors, however, employ arguments essentially different from ours.

We suppose that f is some fixed nontrivial bounded linear form on $E = E(A)$, and that L is the class of subsets K of A such that $x(t) = 0$ on K implies $(f, x) = 0$. It is convenient to present our argument in a sequence of lemmas.

LEMMA 1. *If A is the union of an ascending sequence of sets A_n , there exists N such that $n \geq N$ implies A_n is in L .*

Proof. If no A_n is in L , there exists, for every n , an x_n in E which vanishes on A_n and for which $(f, x_n) = n$. This cannot happen, however, since any such sequence is bounded. It follows that some A_n is in L , and obviously any set containing a set in L is also in L .

LEMMA 2. *There exist neighborhoods V_t in E_t such that f is bounded on V , the direct product of the V_t .*

Proof. Write the neighborhood system in E_t as V_{tn} , and for each n let V_n be the direct product of the sets V_{tn} . If f is unbounded on V_n , there exists x_n in V_n with $(f, x_n) = n$. The sequence x_n converges to 0, hence is a bounded set, and $(f, x_n) = n$ is impossible.

LEMMA 3. *f is sequentially continuous.*

Proof. We suppose that there is a sequence x_n converging to 0 in E for which $|(f, x_n)| > 1$ and deduce a contradiction. Choose the neighborhoods V_t in E_t so that $|(f, x)| \leq 1$ for all x in V , their direct product. Let A_n be the set of all t in A for which $m \geq n$ implies $x_m(t) \in V_t$. A is the union of the ascending sequence A_n , and by Lemma 1 there exists N such that A_N is in L . Let y_n be the element of E which coincides with x_n on A_N and vanishes elsewhere; clearly $(f, y_n) = (f, x_n)$. For every $n \geq N$, y_n is in V , whence $|(f, x_n)| = |(f, y_n)| \leq 1$.

LEMMA 4. *The family L is closed under finite or countable set intersection.*

Proof. If B and D are sets in L , and x in E vanishes on $B \cap D$, let y be that element of E which vanishes on B and coincides with x elsewhere. Writing $x = y + x - y$, we see that $(f, x) = (f, y) + (f, x - y) = 0$, since y vanishes on B and $x - y$ on D . It follows that L is closed under finite intersection.

If the sets D_n form a descending sequence of sets in L and x in E vanishes on $\bigcap D_n$, let y_n be that element of E which vanishes on D_n and coincides with x elsewhere. Then y_n converges to x and $(f, y_n) = 0$ for all n ; the sequential continuity of f implies that $(f, x) = 0$.

LEMMA 5. *Let Q be the class of all subsets of A which intersect every set in L . Q contains at most finitely many disjoint sets.*

Proof. If B_n is a sequence of disjoint sets in Q , let A_0 denote the complement of their union, and let $A_n = \bigcup_{i=1}^n (A_0 \cup B_i)$. The A_n form an ascending sequence of sets whose union is A . By Lemma 1 there exists N such that A_N is in L . If $n > N$, B_n fails to intersect A_N , hence is not in Q .

LEMMA 6. *If M is a set in Q which is the union of finitely many disjoint sets M_i , then at least one M_i is in Q .*

Proof. If not, then for every M_i there is a set A_i in L such that $M_i \cap A_i$ is empty. $A_0 = \bigcap A_i$ is in L and $M \cap A_0$ is empty, so that M is not in Q .

LEMMA 7. *Let M_i be a maximal collection of disjoint sets in Q and suppose that N_k is another such collection. There exists a maximal collection M_i^* which is a refinement of both M_i and N_k .*

Proof. Consider the family of sets $M_i \cap N_k$. It is easy to see that one and only one of these sets is in Q . Call it M_1^* . Form similarly the other sets M_i^* . Evidently each M_i^* is contained in the intersection of an M_i and an N_k , and the collection M_i^* is a maximal collection of disjoint sets in Q .

We introduce a set of measures associated with a maximal family of disjoint sets in Q . If S is a subset of A , write $m_i(S) = 1$ if $M_i \cap S$ is in Q , $m_i(S) = 0$ otherwise.

LEMMA 8. *Each m_i is a countably additive measure on A for which every subset of A is measurable.*

Proof. It is clear that we have only to show the countable additivity. Let S be the union of a sequence of disjoint sets S_n , and let m be one of the m_i . At most one of the S_n has positive measure, since $m(S_n) = m(S_k) = 1$ implies that $M_i \cap S_n$ and $M_i \cap S_k$ are disjoint sets in Q , contrary to the maximality of the family M_i . If one S_n , say S_0 , does have positive measure, clearly $m(S) = m(S_0) = 1$. If no S_n has positive measure, then there exists for each n a set A_n in L such that $M_i \cap S_n \cap A_n$ is empty. Because of Lemma 4, $A_0 = \bigcap A_n$ is in L , and as $M_i \cap S_n \cap A_0$ is empty for all n , $M_i \cap S \cap A_0$ is empty. Thus $M_i \cap S$ fails to intersect a set in L and cannot lie in Q so that $m(S) = \sum m(S_n) = 0$.

LEMMA 9. *The measures m_i depend only on the linear form f .*

Proof. The set families L and Q depend only on f ; we have to show that the m_i do not depend on the particular choice of the maximal family M_i . Let N_k be another family of disjoint sets in Q which is also maximal, and let M_i^* be the refinement of M_i and N_k provided by Lemma 7. Now $m_i(S) = 1$ if and only if $M_i \cap S$ is in Q , which happens if and only if $M_i^* \cap S$ is in Q , i.e., $m_i(S) = 1$ if and only if $m_i^*(S) = 1$. Similarly, after the subscripts have been suitably arranged, $n_k(S) = 1$ if and only if $m_k^*(S) = 1$. The measures m_i and n_k then coincide, except possibly for order.

LEMMA 10. *If x is in E and D is a subset of A outside of which $x(t) = 0$, then $m_i(D) = 0$ for all i implies $(f, x) = 0$.*

Proof. For each i there exists A_i in L such that $M_i \cap D \cap A_i$ is empty. If A_0 is the intersection of the A_i , then A_0 is in L and $M_i \cap D \cap A_0$ is empty for all i . Since the M_i are maximal, $D \cap A_0$ cannot be in Q and there exists A^* in L such that $D \cap A_0 \cap A^*$ is empty. $A_0 \cap A^*$ is a set in L lying in the complement of D ; hence $x(t) = 0$ on a set in L , and $(f, x) = 0$.

We sum up our results in the following theorem.

THEOREM 6. *If f is a bounded linear form on E , the direct product of metric spaces, f is sequentially continuous; there is associated with f a finite set of atomic measures m_i on A such that $m_i(A) = 1$, every subset of A is m_i -measurable, m_i is countably additive, and if $x(t)$ in E vanishes on a set of positive m_i measure for all i , then $(f, x) = 0$.*

It is easy to see that if the bounded linear form f is continuous, then to each i there corresponds a point t_i of A , and this collection of points forms the unique maximal collection of disjoint sets in Q . If, however, f is not continuous, then at least one of the associated measures vanishes on finite sets. The problem of the existence of such a measure on an arbitrary set A has been considered by Ulam [17]. Let us agree to call a measure m on A a U -measure if m is countably additive, $m(A) = 1$, m is atomic and vanishes on finite sets, and every subset of A is m -measurable. It is clear that if the set A admits no U -measure, then $E(A)$ is boundedly closed. It is easy to establish the converse. If m is a U -measure on A , then every complex-valued function on A is measurable and indeed integrable. If for every t in A we choose a nonzero f_t in E'_t , and define $(g, x) = \int_A (f_t, x_t) dm$, it is clear that g is a bounded discontinuous linear form on $E(A)$. Thus we have the following:

COROLLARY. *$E(A)$, the direct product of metric spaces, is boundedly closed if and only if A admits no U -measure.*

We are now in a position to drop the hypothesis that the factor spaces E_t are metric spaces and to obtain the following theorem.

THEOREM 7. *$E(A)$, the direct product of boundedly closed locally convex spaces, is boundedly closed if and only if A admits no U -measure.*

Proof. That the existence of a U -measure on A implies that $E(A)$ is not boundedly closed has been established in the proof of the corollary above, where no use was made of the hypothesis that the spaces E_t were metric.

If $E(A)$ is not boundedly closed, by the corollary of Theorem 5 there exists a bounded linear form f vanishing on N . Choose x in $E(A)$ such that $(f, x) \neq 0$, and for every complex-valued function $y(t)$ on A write $(g, y) = (f, \sum y(t)x_t)$ where x_t is the projection of x on E_t . g is a bounded discontinuous linear form on the direct product of complex planes; hence by the corollary above, there exists a U -measure on A .

From Theorem 7 it follows that the bounded closure of $E(A)$ depends only on the cardinality of the set A , and in no way on the character of the locally convex spaces E_t .

Suppose that \aleph' is the smallest cardinal number for which there exists a set A of that cardinality which admits a U -measure. Ulam [17] has shown: (1) \aleph' is a strongly inaccessible cardinal, and (2) $\aleph < \aleph'$ implies $2^\aleph < \aleph'$, whence \aleph' is at least as great as the cardinal $\sum_{n=0}^{\infty} C(n)$, where $C(0) = \aleph_0$ and $C(n) = 2^{C(n-1)}$.

3. Continuous function spaces. In this section, we investigate the status of bounded closure, symmetry, and *GF* topologies in continuous function spaces and their duals. We use the following notation.

S is any locally compact Hausdorff space. $E = C(S)$ is the vector space of all complex-valued continuous functions on S . The topology on E is called the compact-open topology, and is that obtained by taking as neighborhoods of the origin all sets of the form: $V(C, \epsilon) = \{x(t) \in E \mid |x(t)| \leq \epsilon \text{ for all } t \in C\}$, where C is a compact subset of S and ϵ is positive. Convergence in E is uniform convergence on compact subsets of S . $H(C)$ is the subspace of E consisting of all functions which vanish on C ; or, equivalently, the null space of the semi-norm associated with the neighborhood $V(C, \epsilon)$.

Let $V = V(C, \epsilon)$ be a neighborhood in E and let $v(x)$ be the associated semi-norm. $E_v = E/H(C)$ with $v(x)$ as norm is the usual Banach space of continuous functions on the compact Hausdorff space C . Its dual $(E_v)'$ is the space $(E')_v$ appearing in the definition of the stronger of the two natural *GF* topologies on E' . Each f in E' lies in some $(E')_v = (E_v)'$ and is therefore a measure on the set C ; or, if we wish, a measure on S which vanishes outside C . Thus E' can be described as the space of all Borel measures on S which vanish outside a compact set.

It is easy to see that the bounded sets in E' which are bounded on a neighborhood consist of measures all of which vanish outside a common compact set. For some Hausdorff spaces S there exist bounded subsets of E' without this property, as we shall show by an example, so there exist spaces $E = C(S)$ which are not symmetric. On the other hand, if S is a countable union of compact sets, E is metric and surely symmetric. In addition, we can state the following theorem.

THEOREM 1. *If S is paracompact, E is symmetric.*

Proof. By a result of Dieudonné, the paracompactness of S implies that E is a direct product of metric spaces, hence symmetric. We prefer, however, to give our own proof.

The paracompactness and local compactness insure the existence of a covering (O_α) of S with the properties: (a) O_α is open and has compact closure and (b) no compact set intersects more than a finite number of the O_α .

Let B be a bounded subset of E' , and set $p(x) = \sup \{ |(f, x)| : f \text{ in } B \}$. There is a finite set F of indices such that $p(x) = 0$ for every x whose support is contained in an O_α with α not in F . For in the contrary case we can find sequences α_n and x_n , with x_n vanishing outside O_{α_n} and $p(x_n) = n$. Because of (b) above the sequence x_n is bounded uniformly on each compact set, hence bounded in E , and $p(x_n) = n$ is impossible. Let C be the closure of $\bigcup_{\alpha \in F} O_\alpha$.

Let $a_\alpha(t)$ be a partition of unity for the covering O_α [15]. The $a_\alpha(t)$ have the properties: $a_\alpha(t) \geq 0$; $a_\alpha(t) = 0$ outside O_α ; and for all t , $\sum a_\alpha(t) = 1$. Each x in N , the space of functions vanishing outside a compact set, can be written

as a finite sum $x = \sum a_\alpha x$. If x vanishes on C , each term in the sum has support contained in an O_α with α not in F and $p(x) \leq \sum p(a_\alpha x) = 0$. Thus $p(x)$ vanishes on $H(C) \cap N$, $p(x)$, as a sup of continuous functions, is lower semi-continuous, and $H(C) \cap N$ is dense in $H(C)$, so that $p(x)$ vanishes on $H(C)$.

Let D be the bounded set in E of functions at most 1 in absolute value. $p(x)$ is clearly bounded on $H(C) + D$, which, as we shall see, contains a neighborhood.

Let O be an open set with compact closure containing C , and let $e(t)$ be a function in E which vanishes on C , equals 1 on the complement of O , and is between 0 and 1 elsewhere. Any x in $V(\bar{O}, 1)$ can be written as $x(t) = e(t)x(t) + (1 - e(t))x(t)$, and evidently $e(t)x(t)$ is in $H(C)$ while the other term is in D . Thus $V(\bar{O}, 1)$ is contained in $H(C) + D$.

THEOREM 2. *If E is symmetric, then the ordinary topology and the natural GF topology coincide on bounded subsets of E' .*

Proof. Because T' is weaker than the GF topology, it is sufficient to show that every bounded net f_α which converges to 0 for T' also converges to 0 for the GF topology. The f_α , being bounded, lie in the polar of a neighborhood $V(C, \epsilon)$. The f_α then vanish on $H(C)$. Since they converge to 0 uniformly on the set D of functions at most 1 in absolute value, they converge to 0 uniformly on $H(C) + D$, which, as we have seen in the last theorem, contains a neighborhood U . The f_α thus converge to 0 in E'_u , a fortiori in the GF topology.

COROLLARY. *If E is symmetric, then every bounded linear form on E' is continuous on bounded sets. In particular, every bounded linear form is sequentially continuous.*

Proof. By the theorem, every bounded convergent net is GF convergent. At the same time, every bounded linear form is GF continuous.

THEOREM 3. *If S is a countable union of compact sets, then E is symmetric and has a GF dual.*

Proof. As we have observed, the symmetry follows from the fact that E is metric. In view of Theorem 2 it will complete the proof to show that the dual of a metric space does not admit a locally convex topology which is properly stronger than the ordinary topology and which coincides with the latter on bounded sets. A. Grothendieck has announced this result without proof [9]. We therefore provide our own.

THEOREM 4. *Let E be a metric space with a descending sequence of neighborhoods V_n . If W is a convex circled subset of E' such that $W \cap V_n^\circ$ is a neighborhood in V_n° for each n , then W is a neighborhood in E' .*

Proof. By hypothesis we can find for each n a bounded set A_n in E such that $A_n^\circ \cap V_n^\circ \subseteq W$. Let a_{nk} , for $k \geq n$, be a number such that $A_n \subseteq a_{nk} V_k$, and

let c_k be a number such that $(c_k V_k)^\circ \subseteq W$. Setting $a_k = \max(a_{nk}, c_k, 1: n \leq k)$, define $B = \bigcap_{k=1}^{\infty} a_k V_k$, $B_n = \bigcap_{k=n}^{\infty} a_k V_k$, and $C_n = \bigcap_{k=n+1}^{n-1} a_k V_k$. If $k \geq n$, then $a_k \geq a_{nk}$, and $a_k V_k \supseteq a_{nk} V_k \supseteq A_n$, so that $A_n \subseteq B_n$. Therefore $B_n^\circ \subseteq A_n^\circ$, and $B_n^\circ \cap V_n^\circ \subseteq A_n^\circ \cap V_n^\circ \subseteq W$. By the lemma of the first section, $C_n^\circ \subseteq W$. Again by that lemma $B^\circ \cap V_n^\circ = (B_n \cap C_n)^\circ \cap V_n^\circ \subseteq (B_n^\circ + C_n^\circ) \cap V_n^\circ$, and we show that the last set is a subset of $4W$. Every element of the last set is expressible in the form $f+g=h$, where f is in B_n° , g is in C_n° , and h is in V_n° . Since C_n contains V_n , C_n° is contained in V_n° , and $f+h-g$ is in $2V_n^\circ$. Thus f is in $2(V_n^\circ \cap B_n^\circ) \subseteq 2W$, and $f+g$ is in $2W + C_n^\circ \subseteq 4W$. Since, now, $B^\circ \cap V_n^\circ \subseteq 4W$ for every n , $B^\circ \subseteq 4W$, or $(4B)^\circ \subseteq W$.

We give the promised example of a continuous function space which is not symmetric.

THEOREM 5. *If S is sequentially compact but not compact, then E is neither symmetric nor boundedly closed.*

Proof. Each bounded subset B of E consists of functions bounded uniformly on S . If this were not the case, B would fail to be bounded uniformly on some convergent sequence of points in S , which is impossible, since B must be bounded uniformly on each compact set. In particular, each $x(t)$ in E is bounded. We conclude immediately that the set of all point functionals (linear forms f defined by a point t_0 in S and the equation $(f, x) = x(t_0)$) is a bounded set in E' . It is not bounded on a neighborhood, and E is not symmetric.

Suppose for the moment that E is the space of real rather than complex functions. It follows from the first paragraph that every non-negative linear form on E is bounded, and that $\|x\| = \sup \{ |x(t)| : t \in S \}$ is a well defined norm on E . A linear form which is continuous with respect to this norm is a difference of non-negative linear forms [3] and therefore is bounded.

The subspace N of functions which vanish outside a compact set is not norm dense in E , so there exists a nonzero norm continuous linear form f which vanishes on N . f is bounded, but cannot be continuous since N is dense in E .

To extend f to the space of complex functions write the complex function $z(t)$ as $z(t) = x(t) + iy(t)$ where $x(t)$ and $y(t)$ are real. Then define $(f, z) = (f, x) + i(f, y)$.

The representation theorems given in the introduction, Theorem 5, and the results of the preceding section can be used to show that closed subspaces do not inherit the properties of symmetry and bounded closure. A standard example of a Hausdorff space which is sequentially compact but not compact is obtained from the ordinal numbers. Let c be the first uncountable cardinal and let S be the first ordinal with cardinal number c (equivalently, the set of all ordinals with cardinal number less than c). The neighborhoods of a point in S are the intervals containing the point. Each interval $0 \leq t \leq t^*$

is compact and every compact set is contained in such an interval, so that the semi-norms defined by these intervals describe the topology of $C(S)$. Since there are c intervals, E is a subspace of a direct product of c Banach spaces, and the latter, after §2, is symmetric and boundedly closed. Thus $C(S)$ is neither symmetric nor boundedly closed, although it is a closed subspace of a space having both of those properties.

Even the relatively strong topology on $E = C(S)$ is not symmetric. We have shown that the bounded sets in E are the norm bounded sets, which means that E' and E'' are both normed linear spaces. The relatively strong topology on E , however, is not metric, since E is not boundedly closed.

Another example, especially interesting because it is semi-reflexive, of a space which is not symmetric in its relatively strong topology is the space $E = F'$ with the topology $T(F', F)$, where F is a nonreflexive Banach space. E' is F in its norm topology and E'' is F' in its norm topology. As F is not reflexive, the norm topology on F' is properly stronger than $T(F', F)$.

4. Metric and LF spaces. We have observed that every metric space is symmetric, so that there is only one natural GF topology on its dual. We show now that a metric space has a GF dual whenever it has a symmetric dual. In particular every reflexive metric space has a GF dual. We show also that every reflexive LF space has a GF dual.

Throughout this section, except in the last theorem, E denotes a locally convex metric space. E'^* denotes the space of bounded linear forms on E' . The sets B° for B a bounded set in E' form a neighborhood system in E'^* , as well as in E'' , and topological terms refer to this topology unless some other is specifically mentioned.

Theorems 1 and 3 are generalizations of theorems announced without proof by A. Grothendieck [9].

THEOREM 1. *Each bounded subset B of E'^* is contained in the weak-star closure (in E'^*) of a bounded subset C of E'' .*

Proof. Let V_n be the descending sequence of neighborhoods in E , and set $a_n = \sup \{ |(z, f)| : f \text{ in } V_n^\circ, z \text{ in } B \}$. The convex envelope W of the sets $1/a_n V_n^\circ$ is a GF neighborhood, so the set W° is bounded in E'^* , and $C = W^\circ \cap E''$ is bounded in E'' .

First we show that each z in B is in the weak-star closure of $\bigcap_{k=1}^n a_k V_k$. If this is not true, there is an f in E' such that $(z, f) > 1$ while $|f| \leq 1$ on $\bigcap_{k=1}^n a_k V_k$. As $a_k V_k$ is closed and $1/a_k V_k^\circ$ is weak-star compact, the set $(\bigcap_{k=1}^n a_k V_k)^\circ$ is the convex envelope of the sets $1/a_k V_k^\circ$ ($k = 1, \dots, n$), which is contained in W . Thus f is in W and, since $|z| \leq 1$ on W , $(z, f) > 1$ is not possible.

Now, given f in E' , we can find x_n in $\bigcap_{k=1}^n a_k V_k$ so that $|(f, x_n) - (z, f)| < 1/n$. The sequence x_n is evidently bounded in E , and has a weak-star cluster point y in E'' . If m is any fixed integer and $n \geq m$, $|x_n| \leq 1$ on the

convex envelope of $1/a_k V_k^\circ$ ($k=1, \dots, m$). Therefore $|y| \leq 1$ on the convex envelope of the same set. Because m is arbitrary, $|y| \leq 1$ on W ; that is, y is in C . At the same time, $(y, f) = (z, f)$. In other words z and C cannot be separated by a hyperplane, and z must lie in the weak-star closure of C .

COROLLARY. *The GF topology has a fundamental system of T' closed neighborhoods.*

Proof. If B is a weak-star compact set in E'^* and C is a bounded set in E'' whose weak-star closure contains B , then the polar of B contains the polar of the weak-star closure of C , which is the same as the polar of C . The polar of C is closed. In this way an arbitrary GF neighborhood contains a closed one.

It is interesting to notice in this connection that the ordinary topology on the dual of any locally convex space is the strongest having a fundamental system of weak-star closed neighborhoods. For a weak-star closed convex and circled set always has the form B° . If B° is to be a neighborhood, B must be bounded, and B° is then an ordinary neighborhood.

As an immediate consequence of Theorem 1 we have:

THEOREM 2. *If the dual of E is symmetric, then it is a GF dual.*

We note, at this point, that A. Grothendieck has discovered an example of an F space with a nonsymmetric dual [8]; hence the dual of a metric space need not be symmetric; nevertheless we do have:

THEOREM 3. *If a bounded subset B of E'' is the union of a sequence of sets B_n , each of which is bounded on a neighborhood in E' , then B is bounded on a neighborhood in E' .*

Proof. Let $c_n = \sup \{ |(z, f)| : f \text{ in } V_n^\circ, z \text{ in } B \}$. By hypothesis there is a bounded set C_n in E such that $B_n \subseteq C_n^\circ \subseteq E''$. There are numbers a_{nk} such that $C_n \subseteq a_{nk} V_k$ for $k \geq n$. Let $b_k = \max(c_k, a_{nk} : n \leq k)$. Now, if $n \leq k$, $a_{nk} \leq b_k$, which implies that $a_{nk} V_k \subseteq b_k V_k$, so that $\bigcap_{k=n+1}^\infty a_{nk} V_k \subseteq \bigcap_{k=n+1}^\infty b_k V_k$, and $(\bigcap_{k=n+1}^\infty b_k V_k)^\circ \subseteq (\bigcap_{k=n+1}^\infty a_{nk} V_k)^\circ \subseteq C_n^\circ$. Hence $B_n \leq 1$ on $(\bigcap_{k=n+1}^\infty b_k V_k)^\circ$. Also $B_n \leq 1$ on $1/b_k V_k^\circ$, and therefore on the convex envelope for $k=1, \dots, n$. Since $b_k V_k$ is closed and $1/b_k V_k^\circ$ is compact, the convex envelope of $1/b_k V_k^\circ$ ($k=1, \dots, n$) is $(\bigcap_{k=1}^n b_k V_k)^\circ$. Finally $B_n \leq 2$ on $(\bigcap_{k=1}^n b_k V_k)^\circ + (\bigcap_{k=n+1}^\infty b_k V_k)^\circ$, which contains $(\bigcap_{k=1}^\infty b_k V_k)^\circ$. The last set is a neighborhood on which all B_n are bounded.

We conclude the discussion of metric spaces with an extension of the interior mapping theorem of §1. A weak-star continuous transformation from F' to E' is one which is continuous when both F' and E' have the weak-star topologies.

THEOREM 4. *If E and F are F spaces and E has a GF dual, then every weak-star continuous linear transformation T of F' onto E' is a homomorphism.*

Proof. Since the duals of E' and F' are, respectively, E and F when E' and F' have the weak-star topologies, the adjoint of T is a continuous linear transformation S from E to F . Furthermore, $S^* = T$. We apply the following result of Dieudonné and Schwartz [7].

Let E and F be F spaces and let u be a continuous linear transformation from E into F . In order that u be an isomorphism of E into F it is necessary and sufficient that $u^*(F) = E'$.

Clearly $S(E)$ is closed in F , for E is complete and S is an isomorphism. The hypotheses of the interior mapping theorem of the first section are satisfied and that theorem asserts that $T = S^*$ is a homomorphism.

It is a consequence of Theorem 2 that every reflexive F space has a GF dual. We establish the same conclusion for reflexive LF spaces.

THEOREM 5. *Every reflexive LF space E has a GF dual.*

Proof. Let E_n be the defining sequence of F spaces and let z be a bounded linear form on E' . We parallel an argument of Dieudonné and Schwartz [7] to show that z vanishes on some E_n° . If it does not, there exists f_n in E_n° with $(z, f_n) = n$. But each x in E is in all but a finite number of E_n so that $(f_n, x) = 0$ for all but a finite number of n , and the sequence f_n is weak-star bounded, hence bounded in E' . Thus $(z, f_n) = n$ is impossible.

If z vanishes on E_n° , then z can be regarded as a linear form on $E'/E_n^\circ = E'_n$. Since E_n is symmetric, the bounded sets in E'_n are the polars of neighborhoods, that is, sets of the type $(V \cap E_n)^\circ = V^\circ + E_n^\circ$, where V is a neighborhood in E . As z is bounded on E'_n , it is bounded on each of these sets and is a bounded linear form on E'_n . E'_n , being the dual of a reflexive F space, is boundedly closed, so that z lies in E_n and hence in E . E' is boundedly closed.

We have observed that the dual of any semi-reflexive space is symmetric. It follows that the ordinary topology of E' is a GF topology. The symmetry of E insures that the two natural GF topologies of E' coincide and have the dual $E'^* = E$. Application of the uniqueness theorem completes the proof.

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