DEGREE OF APPROXIMATION TO FUNCTIONS ON A JORDAN CURVE

BY

J. L. WALSH

If C is an arbitrary Jordan curve of the z-plane containing the origin in its interior, and if $f(z)$ is an arbitrary function continuous on C, it is known [1, p. 39] that $f(z)$ can be represented on C by a uniformly convergent sequence of polynomials in z and $1/z$ of the form

$$(1) \qquad p_n(z) \equiv \sum_{k=-n}^{n} a_{nk} z^k.$$

Components of these functions are defined for z interior to C (assumed rectifiable) by the equations

$$(2) \qquad p_{1n}(z) \equiv \sum_{k=0}^{n} a_{nk} z^k \equiv \frac{1}{2\pi i}\int_C \frac{p_n(t)dt}{t-z}, \qquad f_1(z) \equiv \frac{1}{2\pi i}\int_C \frac{f(t)dt}{t-z},$$

where the integrals are taken counterclockwise, and for z exterior to C by the equations

$$(3) \qquad p_{2n}(z) \equiv \sum_{k=-n}^{-1} a_{nk} z^k \equiv \frac{1}{2\pi i}\int_C \frac{p_n(t)dt}{t-z}, \qquad f_2(z) \equiv \frac{1}{2\pi i}\int_C \frac{f(t)dt}{t-z},$$

where the integrals are taken clockwise; it follows that we have for z interior to C

$$(4) \qquad \lim_{n\to\infty} p_{1n}(z) = f_1(z),$$

and for z exterior to C

$$(5) \qquad \lim_{n\to\infty} p_{2n}(z) = f_2(z).$$

It is the object of the present note to indicate that under suitable conditions (Theorem 1) various degrees of approximation of $p_n(z)$ to $f(z)$ on C imply certain continuity properties of $f(z)$, and (Theorem 2) under suitable conditions equations (4) and (5) in more precise form are valid even on C itself. We consider also (Theorems 3, 4, 5) approximation on C by analytic functions more general than polynomials $p_n(z)$, and prove the analogues of (4) and (5) in more precise form. We mention briefly (Theorems 6 and 7) approximation in a multiply-connected region.

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All numbers A and B with or without subscripts are constants independent of n and z.

THEOREM 1. *Let C be an analytic Jordan curve containing the origin in its interior, and let the function $f(z)$ be defined on C. A necessary and sufficient condition that $f(z)$ be of class $L(p, \alpha)$ for $0<\alpha<1$ or of class Z_p for $\alpha=1$ on C is that there exist polynomials $p_n(z)$ given by* (1) *such that we have* $(n>0)$

$$|f(z) - p_n(z)| \leqq A/n^{p+\alpha}, \qquad z \text{ on } C. \tag{6}$$

Here p is a non-negative integer and $L(p, \alpha)$ and Z_p are respectively the classes of functions continuous on C whose pth derivatives with respect to arc length on C are continuous and satisfy on C a Lipschitz condition of order α or a condition of form

$$|F(s+h) + F(s-h) - 2F(s)| \leqq A_1 |h| \tag{7}$$

uniformly with respect to arc length s and h; this latter condition has been investigated especially by Zygmund.

The existence of the $p_n(z)$ satisfying (6) has already been established [2], by separate approximation in the two closed regions bounded by C of the respective components of $f(z)$.

Conversely, if inequalities (6) are valid there exists some constant A_2 such that we have $|p_n(z)| \leqq A_2$, z on C. In a suitably chosen annulus containing C in its interior we then have [1, p. 259]

$$|p_n(z)| \leqq A_2 R^n. \tag{8}$$

Inequalities (6) and (8) imply [3, Theorem 4] that $f(z)$ is of class $L(p, \alpha)$ if $0<\alpha<1$ and is of class Z_p if $\alpha=1$ on each of two open Jordan subarcs of C which overlap at both ends and cover C; this implies the conclusion of Theorem 1(1).

THEOREM 2. *Let C be an analytic Jordan curve containing the origin in its interior, and suppose for a sequence of polynomials $p_n(z)$ given by* (1) *and for some function $f(z)$ we have* $(n>0)$

$$|f(z) - p_n(z)| \leqq A_2/n^{p+\alpha}, \quad z \text{ on } C, \tag{9}$$

where p is a non-negative integer, $0<\alpha\leqq 1$. Then in the notation of (2) *and* (3) *we have, for a suitably chosen A_3,* $(n>1)$

$$\begin{aligned} |f_1(z) - p_{1n}(z)| &\leqq A_3 \log n/n^{p+\alpha}, \quad z \text{ on } C, \\ |f_2(z) - p_{2n}(z)| &\leqq A_3 \log n/n^{p+\alpha}, \quad z \text{ on } C. \end{aligned} \tag{10}$$

Several lemmas are convenient in the proof of Theorem 2.

(1) We remark incidentally that Theorem 1 and numerous other results [3] assuming various degrees of approximation can be extended to apply under the hypothesis of more general degrees of approximation, by application of classical theorems due to de la Vallée Poussin.

LEMMA 1. *Let the function $F(z)$ be measurable and in modulus not greater than M on C: $|z|=1$, with the formal development on C*

$$F(z) \sim \sum_{k=-\infty}^{\infty} a_k z^k, \qquad a_k = \frac{1}{2\pi i}\int_C \frac{F(z)dz}{z^{k+1}}. \tag{11}$$

Then there exists an absolute constant B such that $(n>1)$

$$\left| \sum_{k=-n}^{n} a_k z^k \right| \leqq B\cdot M\cdot \log n, \quad z \text{ on } C. \tag{12}$$

On C we write $z=e^{i\theta}$,

$$\cos k\theta = \frac{z^k + z^{-k}}{2}, \qquad \sin k\theta = \frac{z^k - z^{-k}}{2i},$$

so the summation in (12) is precisely the sum of the first $n+1$ terms of the formal Fourier expansion of $F(z)$; inequality (12) is merely another formulation of a well known inequality due to Lebesgue.

LEMMA 2. *Under the hypothesis of Lemma* 1, *there exists an absolute constant B' such that*

$$\left| \sum_{k=0}^{n} a_k z^k \right| \leqq B'M \log n, \quad z \text{ on } C,$$

$$\left| \sum_{k=-n}^{-1} a_k z^k \right| \leqq B'M \log n, \quad z \text{ on } C.$$

We note that the formal expansion on C of $z^\nu F(z)$ is

$$z^\nu F(z) \sim \sum_{k=-\infty}^{\infty} a_k z^{k+\nu},$$

for we have by (11)

$$\frac{1}{2\pi i}\int_C \frac{z^\nu F(z)dz}{z^{k+1}} = \frac{1}{2\pi i}\int_C \frac{F(z)dz}{z^{k-\nu+1}} = a_{k-\nu};$$

of course on C we have $|z^\nu F(z)| \leqq M$, and we have $|a_k| \leqq M$. If $n(>2)$ is even there follows by applying Lemma 1 to the function $z^{-n/2}F(z)$

$$\left| \sum_{k=0}^{n} a_k z^k \right| \leqq BM \log (n/2), \quad z \text{ on } C; \tag{13}$$

if $n(>3)$ is odd we have by use of (13)

$$\left| \sum_{k=0}^{n-1} a_k z^k + a_n z^n \right| \leqq BM \log [(n-1)/2] + M, \quad z \text{ on } C. \tag{14}$$

The conclusion of Lemma 2 now follows from (13), (14), similar inequalities for the case $n=2$ and $n=3$, and from (12).

LEMMA 3. *Let C be an analytic Jordan curve containing the origin $z=0$ in its interior; there exists a constant $B'(C)$ depending only on C such that if $F(z)\equiv\sum_{k=-n}^{n} a_k z^k$ and if we have $|F(z)|\leqq M$ on C, then we have on C $(n>1)$*

$$\left|\sum_{k=0}^{n} a_k z^k\right| \leqq B'(C)M\log n.$$

Here it is convenient to use Faber's polynomials $\phi_n(z)$, $n=0, 1, 2, \cdots$, of respective degrees n belonging to C. Let $w=\phi(z)$, $z=\psi(w)$ map the exterior of C onto the exterior of γ: $|w|=1$ with $\phi(\infty)=\infty$. Then we have for z interior to C (the details are discussed in more detail in [2])

$$\sum_{k=0}^{n} a_k z^k \equiv \frac{1}{2\pi i}\int_C \frac{F(t)dt}{t-z} \equiv \sum_{k=0}^{n} c_k\phi_k(z),$$

$$c_k = \frac{1}{2\pi i}\int_\gamma \frac{F[\psi(w)]dw}{w^{k+1}};$$

this expansion is valid also on C. It is to be noted that the coefficients c_k of order greater than n all vanish, for we have $(k>n)$

$$\int_\gamma \frac{F[\psi(w)]dw}{w^{k+1}} = 0,$$

since the integrand is analytic for $|w|\geqq 1$ and vanishes to at least the second order at infinity.

On C we have $\phi_k(z)\equiv w^k+r_k(z)$, where $|r_k(z)|\leqq A'r^k$, $0<r<1$, A' and r depend only on C, so we can write on C

$$\sum_{k=0}^{n} c_k\phi_k(z) \equiv \sum_{k=0}^{n} c_k w^k + \sum_{k=0}^{n} c_k r_k(z).$$

The first sum in the second member is on γ precisely the first sum of Lemma 2; moreover we have $|c_k|\leqq M$, whence

$$\left|\sum_{k=0}^{n} c_k r_k(z)\right| \leqq MA'/(1-r);$$

thus we have $(n>1)$

$$\left|\sum_{k=0}^{n} c_k\phi_k(z)\right| \leqq [B'\log n + A'/(1-r)]M,$$

from which Lemma 3 follows.

Of course the proof and conclusion of Lemma 3 apply under broader conditions than those mentioned; the conclusion essentially refers to the sum of the first $n+1$ terms of the formal development in Faber polynomials of an arbitrary measurable function on C with bound M.

We are now in a position to complete the proof of Theorem 2. From (9) it follows by Theorem 1 that $f(z)$ is of class $L(p, \alpha)$ or Z_p on C according as we have $0<\alpha<1$ or $\alpha=1$. Consequently, by the theory of the boundary values of the functions represented by the Cauchy integral as first considered by Plemelj [2; 7], it follows that $f_1(z)$ and $f_2(z)$ as defined by (2) and (3) are also of class $L(p, \alpha)$ or Z_p on C. Then there exist [for $0<\alpha<1$ this result is due to Curtiss; see 2, 4] polynomials $P_{1n}(z)$ in z and $P_{2n}(z)$ in $1/z$ of degrees n such that we have on C $(n>0)$

$$(15) \qquad |f_1(z) - P_{1n}(z)| \leqq A_4/n^{p+\alpha}, \quad |f_2(z) - P_{2n}(z)| \leqq A_4/n^{p+\alpha}.$$

However, $f_2(z)$ vanishes at infinity and it is important to choose the polynomials $P_{2n}(z)$ so as to vanish at infinity; this is done by choosing arbitrary polynomials of the requisite degree of convergence on C, a degree of convergence valid throughout the closed exterior of C and hence at infinity, and subtracting from each approximating polynomial its value at infinity; such subtraction does not alter the degree of convergence on C nor the function approximated, namely $f_2(z)$. In the notation of (2) and (3) we write (9) in the form $|f_1(z)+f_2(z)-p_{1n}(z)-p_{2n}(z)| \leqq A_2/n^{p+\alpha}$, z on C, which by (15) gives

$$(16) \qquad |P_{1n}(z) + P_{2n}(z) - p_{1n}(z) - p_{2n}(z)| \leqq A_5/n^{p+\alpha}, \quad z \text{ on } C.$$

Lemma 3, together with a corresponding conclusion concerning $P_{2n}(z)-p_{2n}(z)$, —a direct consequence of hypothesis and conclusion of Lemma 3—now yields $(n>1)$ for z on C

$$|P_{1n}(z) - p_{1n}(z)| \leqq A_6 \log n/n^{p+\alpha},$$
$$|P_{2n}(z) - p_{2n}(z)| \leqq A_6 \log n/n^{p+\alpha},$$

and (10) follows from (15).

Theorem 2 assumes the validity of (9) for *every* $n(>0)$. It is readily shown by standard methods that the validity of (9) for an infinite sequence of indices not possessing arbitrarily large gaps implies the existence of $p_n(z)$ defined for every $n(>0)$ such that (9) is valid, after possible modification of the constant A_2. If the hypothesis of Theorem 2 is modified so as to assume that $f(z)$ is of class $L(p, \alpha)$ or Z_p on C, and if (9) is assumed not for every n but for an infinite sequence of indices n, then (10) is valid for that same sequence.

Of course Theorem 2 cannot be extended to the general case where we assume $p_n(z)$ merely to converge to $f(z)$ uniformly on C and attempt to prove (4) and (5) uniformly on C, for if $f(z)$ is an arbitrary function continuous on C there exists a sequence $p_n(z)$ converging to $f(z)$ uniformly on C, but the functions $f_1(z)$ and $f_2(z)$ defined by (2) and (3) need not be continuous on C. A sufficient condition that (4) and (5) hold uniformly on C is that $p_n(z)$ converge to $f(z)$ on C with a degree of convergence $1/n^\beta$ for some $\beta > 0$.

On the other hand [1, §9.9] it is true that if we replace (9) by the hypothesis

$$\limsup_{n\to\infty} [\max |f(z) - p_n(z)|,\ z \text{ on } C]^{1/n} \leqq 1/R < 1,$$

the inequalities (10) may be replaced by the conclusion

$$\limsup_{n\to\infty} [\max |f_1(z) - p_{1n}(z)|,\ z \text{ on } C]^{1/n} \leqq 1/R,$$

$$\limsup_{n\to\infty} [\max |f_2(z) - p_{2n}(z)|,\ z \text{ on } C]^{1/n} \leqq 1/R.$$

Theorem 2 is related to, and can be extended to include, approximation by functions more general than polynomials:

THEOREM 3. *Let C be an analytic Jordan curve contained in an annular region D. Let the sequence of functions $F_n(z)$ analytic in D satisfy the inequality*

$$|F_n(z)| \leqq A_7 R^n, \quad R > 1, \quad z \text{ in } D, \tag{17}$$

and suppose for some function $f(z)$ defined on C we have $(n > 0)$

$$|f(z) - F_n(z)| \leqq A_8/n^{p+\alpha}, \quad z \text{ on } C. \tag{18}$$

Then on C the function $f(z)$ is of class $L(p, \alpha)$ if $0 < \alpha < 1$, and of class Z_p if $\alpha = 1$. Moreover, if we set as in (2) *and* (3)

$$F_{1n}(z) \equiv \frac{1}{2\pi i}\int_C \frac{F_n(t)dt}{t - z}, \qquad F_{2n}(z) \equiv \frac{1}{2\pi i}\int_C \frac{F_n(t)dt}{t - z}, \tag{19}$$

respectively for z interior and exterior to C, then we have $(n > 1)$

$$\begin{aligned} |f_1(z) - F_{1n}(z)| &\leqq A_9 \log n/n^{p+\alpha}, \quad z \text{ on } C, \\ |f_2(z) - F_{2n}(z)| &\leqq A_9 \log n/n^{p+\alpha}, \quad z \text{ on } C. \end{aligned} \tag{20}$$

Let $w = \phi(z)$ map the exterior of C onto $|w| > 1$ with $\phi(\infty) = \infty$; let $z = 0$ lie interior to C, and let $w = \Phi(z)$ map the interior of C onto $|w| < 1$ with $\Phi(0) = 0$. We assume the functions $F_n(z)$ to be analytic and to satisfy (17) in the closure of D, where D is now bounded by the curve C_ρ: $|\phi(z)| = \rho(>1)$ exterior to C and the curve $C_{1/\rho}$: $|\Phi(z)| = 1/\rho$ interior to C.

We expand $F_{1n}(z)$ in the Faber polynomials belonging to C, where the function is represented by the formula

$$F_{1n}(z) \equiv \frac{1}{2\pi i}\int_{C_\rho} \frac{F_n(t)dt}{t - z}, \quad z \text{ interior to } C_\rho,$$

and the coefficients are found by integration over C_ρ or over $|w| = \rho$. Then there exist [compare 3] polynomials $p_{1,\lambda n}(z)$ in z of respective degrees λn such that we have

$$(21) \qquad |F_{1n}(z) - p_{1,\lambda n}(z)| \leqq B_1(R/\rho^\lambda)^n, \quad z \text{ on } C;$$

here λ is any positive integer such that $R/\rho^\lambda < 1$.

In a precisely similar way, we expand $F_{2n}(z)$ defined now by the formula

$$F_{2n}(z) \equiv \frac{1}{2\pi i}\int_{C_{1/\rho}} \frac{F_n(t)dt}{t - z}, \quad z \text{ exterior to } C_{1/\rho},$$

in polynomials in $1/z$, the expansion being valid throughout the closed exterior of $C_{1/\rho}$. The function $F_{2n}(z)$ vanishes at infinity and thus, as in the proof of Theorem 2, for suitably chosen polynomials $p_{2,\lambda n}(z)$ in $1/z$ of respective degrees λn which vanish at infinity we have

$$(22) \qquad |F_{2n}(z) - p_{2,\lambda n}(z)| \leqq B_2(R/\rho^\lambda)^n, \quad z \text{ on } C.$$

The functions $f_1(z)$ and $f_2(z)$ represented by (1) and (2) are [3, Theorem 4] on C of classes $L(p, \alpha)$ if $0<\alpha<1$ or Z_p if $\alpha=1$. Consequently there exist polynomials $P_{1,\lambda n}(z)$ in z of degrees λn and polynomials $P_{2,\lambda n}(z)$ in $1/z$ of degrees λn vanishing at infinity such that we have $(n>0)$

$$(23) \qquad \begin{aligned} |f_1(z) - P_{1,\lambda n}(z)| &\leqq B_3/(\lambda n)^{p+\alpha}, \quad z \text{ on } C, \\ |f_2(z) - P_{2,\lambda n}(z)| &\leqq B_3/(\lambda n)^{p+\alpha}, \quad z \text{ on } C. \end{aligned}$$

On C the equations $f(z) \equiv f_1(z) + f_2(z)$, $F_n(z) \equiv F_{1n}(z) + F_{2n}(z)$ and inequalities (18), (21), (22), (23) yield $(n>0)$

$$|P_{1,\lambda n}(z) + P_{2,\lambda n}(z) - p_{1,\lambda n}(z) - p_{2,\lambda n}(z)| \leqq B_4/(\lambda n)^{p+\alpha}, \quad z \text{ on } C.$$

From Lemma 3 we have $(n>1)$

$$\begin{aligned} |P_{1,\lambda n}(z) - p_{1,\lambda n}(z)| &\leqq B_5 \log n/n^{p+\alpha}, \quad z \text{ on } C, \\ |P_{2,\lambda n}(z) - p_{2,\lambda n}(z)| &\leqq B_5 \log n/n^{p+\alpha}, \quad z \text{ on } C. \end{aligned}$$

Inequalities (20) now follow by (21), (22), and (23).

We add the remark that Theorem 3 is related to previous work [5], concerning approximation on an arbitrary rectifiable Jordan curve C by certain functions $F_n(z)$ not required to be analytic on C itself, and where we assume

$$\limsup_{n\to\infty} [\max |f(z) - F_n(z)|, z \text{ on } C]^{1/n} < 1.$$

Theorem 3 includes Theorem 2, for in the proof of Theorem 1 we have indicated that (6) implies (8), and (8) is precisely (17) for the functions $p_n(z)$. Still more can be concluded.

LEMMA 4. *Let C be a Jordan curve, let $w=\phi(z)$ with $\phi(\infty)=\infty$ map the exterior of C onto $|w|>1$, and let C_R denote generically the Jordan curve*

$|\phi(z)| = R(>1)$ *exterior to* C. *Let the function* $R_n(z)$ *be analytic exterior to* C *and continuous in the corresponding closed region except possibly for poles not more than* n *in number all of which lie in the closed exterior of* C_ρ. *Then the inequality*

$$|R_n(z)| \leqq L, \quad z \text{ on } C,$$

implies $(R_1 < \rho)$

$$|R_n(z)| \leqq L\left(\frac{\rho R_1 - 1}{\rho - R_1}\right)^n, \quad z \text{ on } C_{R_1}.$$

Lemma 4 can be proved by applying in the w-plane a known method of proof [1, p. 231, Lemma I].

THEOREM 4. *Let* C *be an analytic Jordan curve, let* $F_n(z)$ *be a sequence of rational functions of respective degrees* n *whose poles have no limit point on* C, *and suppose for some function* $f(z)$ *defined on* C *we have* (18). *Then we also have* (17) *for a suitably chosen* D *and* R; *the conclusion of Theorem* 3 *is valid.*

Since the poles of the $F_n(z)$ have no limit point on C, all such poles lie exterior to some annular region D' containing C bounded by the two Jordan curves $|\phi(z)| = \rho$ and $|\Phi(z)| = 1/\rho$, where we assume the origin to lie interior to C and $w = \Phi(z)$ maps the interior of C onto $|w| < 1$ with $\Phi(0) = 0$. It follows from (18) that the functions $F_n(z)$ are uniformly bounded on C, and follows from Lemma 4 as applied first directly and second after map of the interior of C onto $|w| > 1$ that (17) holds with $R = (\rho R_1 - 1)/(\rho - R_1)$ in the annular region D containing C bounded by the two Jordan curves $|\phi(z)| = R_1$ $(1 < R_1 < \rho)$ exterior to C and $|\Phi(z)| = 1/R_1$ interior to C. Thus the conclusion of Theorem 3 is valid. The first part of this conclusion, relative to the properties of $f(z)$ on C, has connections with recent results due to H. M. Elliott [6].

Like Theorem 2, the conclusion of Theorem 4 is valid if we assume (18) not for all $n(>0)$ but merely for an infinite sequence of indices n which does not possess arbitrarily large gaps.

Theorem 3 has application to approximation by bounded analytic functions:

THEOREM 5. *Let* C *be an analytic Jordan curve containing the origin* O *in its interior, and let* D *be a bounded region containing* C *whose closure does not contain* O. *Let the function* $f(z)$ *be continuous on* C. *For each positive* M *let* $F_M(z)$ *be the* (*or a*) *function analytic and of modulus not greater than* M *in* D *such that the measure of approximation*

$$\mu_M = \max\left[\,|f(z) - F_M(z)|,\ z \text{ on } C\right] \tag{24}$$

is least. Then a necessary and sufficient condition that $f(z)$ *be of class* $L(p, \alpha)$

with $0<\alpha<1$ *or of class* Z_p *with* $\alpha=1$ *on* C *is that*

$$\log M\cdot\mu_M^{1/(p+\alpha)} \tag{25}$$

be bounded as $M\to\infty$.

We merely sketch the proof of Theorem 5, for the details are similar to those in the proof of an analogous result [3, Theorem 3]. If (25) is bounded, we choose the sequence of values $M=e^n$, and the conclusion follows from Theorem 3. Conversely, if $f(z)$ is given with the required properties on C, we make use of the polynomials of Theorem 1. Inequality (8) is then valid in D, if R is suitably chosen [1, p. 259], and the boundedness of (25) follows.

It may be noted that if we consider a sequence of extremal functions $F_M(z)$ for values $M=AR^n$, $R>1$, it follows from the boundedness of (25) that the analogue of (18) is valid, and hence the analogue of (20) is also valid.

If $f(z)$ is continuous on C we introduce the definition

$$\liminf_{M\to\infty}\left[\frac{-\log\mu_M}{\log\log M}\right]=\beta;$$

the values $\beta=+\infty$ and $\beta=-\infty$ are not excluded. Then *whenever we have* $p+\alpha<\beta$, *the function* $f(z)$ *is of class* $L(p,\alpha)$ *or* Z_p *on* C *according as* $0<\alpha<1$ *or* $\alpha=1$; *but* $f(z)$ *is of no class* $L(p,\alpha)$ *or* Z_p *on* C *if we have* $\beta\leqq 0$, *and is not of the class* $L(p,\alpha)$ *or* Z_p *on* C *if we have* $p+\alpha>\beta$. If $f(z)$ and β are given, and if we have $p+\alpha<\beta$, then we have for M sufficiently large

$$\frac{-\log\mu_M}{\log\log M}>p+\alpha,$$

from which the boundedness of (25) follows. Conversely, if $f(z)$ is of class $L(p,\alpha)$ or Z_p, the boundedness of (25) implies

$$\log\mu_M+(p+\alpha)\log\log M<B_6,$$

$$p+\alpha\leqq\beta.$$

If $g_M(z)$ is an *arbitrary* function (not necessarily extremal) analytic and of modulus not greater than M in D we set

$$\overset{*}{\mu}_M=\max\left[\,|f(z)-g_M(z)|,\ z\text{ on }C\right],$$

whence $\overset{*}{\mu}_M\geqq\mu_M$,

$$\liminf_{M\to\infty}\left[\frac{-\log\overset{*}{\mu}_M}{\log\log M}\right]\leqq\beta.$$

We emphasize the fact that this relation holds for an arbitrary family of functions $g_M(z)$ analytic and of modulus not greater than M in D, and (as is readily shown) holds even if M becomes infinite merely taking the values of a sequence $M=B_0R_1^n$, $R_1>1$.

We turn now from the study of approximation on a Jordan curve to the study of approximation on an annulus or region of higher connectivity, for the methods just developed apply in this new situation with relatively little change.

THEOREM 6. *Let C be a closed point set whose boundary consists of an analytic Jordan curve C_0 and mutually exterior analytic Jordan curves $C_1, C_2, \cdots, C_\nu$ interior to C_0. Let D_k denote that region of the extended plane bounded by C_k containing the interior of C, and let D_k' denote the complementary closed region. Let a point α_k be chosen arbitrarily interior to D_k', $k=0, 1, 2, \cdots, \nu$.*

Let the function $f(z)$ be given continuous on C, analytic in the interior points of C. A necessary and sufficient condition that $f(z)$ be of class $L(p, \alpha)$ for $0<\alpha<1$ or of class Z_p for $\alpha=1$ on all C_k is that there exist rational functions $R_n(z)$ of respective degrees $(\nu+1)n$ having in each point α_k no pole of order greater than n and having no other poles, such that $(n>0)$

$$|f(z)-R_n(z)| \leqq A/n^{p+\alpha}, \quad z \text{ on } C. \tag{26}$$

If a sequence of functions $F_n(z)$ analytic in a region D containing C satisfies the inequality

$$|F_n(z)| \leqq A_1R^n, \quad z \text{ in } D, \tag{27}$$

and if for some function $f(z)$ defined on C we have $(n>0)$

$$|f(z)-F_n(z)| \leqq A_2/n^{p+\alpha}, \quad z \text{ on } C, \tag{28}$$

then on each C_k the function $f(z)$ is of class $L(p, \alpha)$ for $0<\alpha<1$ or of class Z_p for $\alpha=1$. Moreover, each of the $\nu+1$ components $F_{kn}(z)$, $k=0, 1, 2, \cdots, \nu$, of $F_n(z)$ satisfies an inequality $(n>0)$

$$|f_k(z)-F_{kn}(z)| \leqq A_3/n^{p+\alpha}, \quad z \text{ on } C, \tag{29}$$

where $f_k(z)$ is the corresponding component of $f(z)$.

In particular if the functions $F_n(z)$ are rational functions of z of respective degrees n with no limit point of poles on C, then (28) implies (27) in a suitable region D.

A function $f(z)$ analytic interior to C and continuous on C can be split into $\nu+1$ components by the equations (all integrals are to be taken in the positive sense with respect to C)

$$f(z) \equiv \sum_{k=0}^{\nu} f_k(z), \qquad f_k(z) \equiv \frac{1}{2\pi i}\int_{C_k} \frac{f(t)dt}{t-z}; \tag{30}$$

the second of equations (30) is valid for z interior to C, and defines $f_k(z)$ throughout D_k; it will be noted that $f_k(z)$ is analytic at every point of D_k, and $f_k(\infty)=0$ for $k>0$; the first of equations (30), valid for z interior to C, then defines $f_k(z)$ also on C_k, so $f_k(z)$ is continuous on the closure of D_k and if $f(z)$ is of class $L(p, \alpha)$ or Z_p on C_k so also is $f_k(z)$. These comments obviously apply to the $\nu+1$ components $R_{kn}(z)$ and $F_{kn}(z)$ of the functions $R_n(z)$ and $F_n(z)$; each component is analytic at every point of C.

Let $f(z)$ of the first part of Theorem 6 be of class $L(p, \alpha)$ or Z_p on all C_k; then $f_k(z)$ is continuous in the closure of D_k and analytic interior to D_k; also $f_k(z)$ is of class $L(p, \alpha)$ or Z_p on C_k and can be approximated in the closure of D_k by a sequence of rational functions of respective degrees n whose poles lie in α_k with degree of approximation $1/n^{p+\alpha}$. If $R_n(z)$ denotes the term-by-term sum of these $\nu+1$ sequences, we have (26). Conversely, if the $R_n(z)$ exist so that (26) is valid, the function $f(z)$ is necessarily continuous on C and analytic interior to C. From Theorem 4 applied to each C_k it follows that $f(z)$ is of class $L(p, \alpha)$ or Z_k on each C_k.

If the functions $F_n(z)$ analytic in D satisfy (27) and (28), then Theorem 3 applies at once, and shows that $f(z)$ is of class $L(p, \alpha)$ or Z_p on each C_k. Moreover we have

$$f_k(z) - F_{kn}(z) \equiv \frac{1}{2\pi i}\int_{C_k} \frac{[f(t) - F_n(t)]dt}{t - z}, \quad z \text{ in } D_k,$$

so on each of the curves C_j with $j \neq k$ we have from (28)

$$|f_k(z) - F_{kn}(z)| \leqq A_4/n^{p+\alpha}. \tag{31}$$

Inequality (28) can be written

$$\left|\sum_{k=0}^{\nu} [f_k(z) - F_{kn}(z)]\right| \leqq A_2/n^{p+\alpha}, \quad z \text{ on } C,$$

so from (31) we have ($n>0$)

$$|f_k(z) - F_{kn}(z)| \leqq A_5/n^{p+\alpha}, \quad z \text{ on } C_k; \tag{32}$$

thus (32) is valid in the closure of D_k and in particular includes (29).

In (32) we have considered that component $F_{kn}(z)$ of $F_n(z)$ which is defined in D_k, not that component $F^*_{kn}(z)$ of $F_n(z)$ defined by

$$F^*_{kn}(z) \equiv \frac{1}{2\pi i}\int_{C_k} \frac{F_n(t)dt}{t - z}, \quad z \text{ interior to } D'_k.$$

However, at a point z of the D'_k, we have

$$0 = \sum_{k=0}^{\nu} \frac{1}{2\pi i}\int_{C_k} \frac{F_n(t)dt}{t - z} = F_{0n}(z) + \cdots + F_{k-1,n}(z)$$
$$+ F^*_{kn}(z) + F_{k+1,n}(z) + \cdots + F_{\nu n}(z),$$

with similar equations for the components $f_k^*(z)$ of $f(z)$. We define $F_{kn}^*(z)$ from the equality of these first and third members so as to be continuous on C_k, hence continuous in D_k', and similarly for $f_k^*(z)$. Thus the behavior of $F_{kn}^*(z)$ in D_k' is readily deduced from (32); we have

$$\left| f_k^*(z) - F_{kn}^*(z) \right| \leqq A_6/n^{p+\alpha}, \quad z \text{ in } D_k'.$$

The last statement in Theorem 6 follows as in the proof of Theorem 4.

By our method of proof of (29) it is clear that any degree of approximation of $F_n(z)$ to $f(z)$ on C assumed in place of (28) implies the *same* degree of approximation of $F_{kn}(z)$ to $f_k(z)$ on C; this is in strong contrast to Theorems 2 and 3.

Theorem 6 admits an application precisely as Theorem 3 admits Theorem 5 as application; we omit the proof:

THEOREM 7. *With the topological notation of Theorem* 6, *let D be a region containing C whose closure contains none of the points α_k. Let the function $f(z)$ be continuous on C, analytic interior to C. For each positive M let $F_M(z)$ be the* (*or a*) *function analytic and of modulus not greater than M in D such that the measure of approximation* (24) *is least. Then a necessary and sufficient condition that $f(z)$ be of class $L(p, \alpha)$ with $0<\alpha<1$ or of class Z_p with $\alpha=1$ on C is that* (25) *be bounded as $M\to\infty$.*

The comments relating to Theorem 5 concerning the boundedness of (25) obviously apply also to Theorem 7.

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HARVARD UNIVERSITY,
CAMBRIDGE, MASS.