ON LOCAL STRUCTURE OF FINITE-DIMENSIONAL GROUPS(1)

BY

SZE-TSEN HU

1. **Introduction.** Let G be a connected locally compact separable metrizable topological group of finite dimension n. When G is compact or solvable, it is known that G is locally the direct product of an invariant compact zero-dimensional subgroup Z of G and an invariant local Lie subgroup(2) S of G [4; 6; 8; 10; 11](3). This same fact is probably true in general but is unproved. However, in a recent paper of D. Montgomery [9], a long and successful stride has been taken in this direction. Namely, he proved that G is locally the topological product of a compact zero-dimensional subset Z of G and a connected locally connected invariant local subgroup S of dimension n of G (see §2). His result does not imply that the subset Z can be selected to be a subgroup of G.

In the present paper, Montgomery's result just mentioned will be strengthened for the particular case that the center of G is locally connected. In fact, the following Local Structure Theorem is proved in the sequel.

THEOREM I. *If the center of G is locally connected, then G is locally the semi-direct product of a compact zero-dimensional subgroup and a connected locally connected invariant local subgroup of dimension n. In greater detail: G contains a compact zero-dimensional subgroup Z and a connected locally connected invariant local subgroup S of dimension n such that $U=ZS$ is an open neighborhood of the neutral element e in G and the map $F:Z\times S\to U$ of the topological product space $Z\times S$ onto U defined by*

$$F(z, s) = zs \qquad (z \in Z,\ s \in S)$$

is a homeomorphism of $Z\times S$ onto U. Hence every element $u\in U$ is decomposed uniquely and continuously into the product $u=zs$ where $z\in Z$ and $s\in S$.

It should be emphasized that the local decomposition of G given in the theorem is only a semi-direct product instead of a direct product. In fact, the author is not able to prove that the subgroup Z can be selected to be invariant.

Presented to the Society, February 23, 1952; received by the editors February 6, 1952.

(1) A major part of this work was done under Project M801 of Engineering Research Institute, University of Michigan, during the summer session of 1951.

(2) Following L. Pontrjagin [10], closed subgroups and closed local subgroups will be simply called subgroups and local subgroups.

(3) Numbers in brackets refer to the bibliography at the end of the paper.

The proof of Theorem I given in the following sections is based on the results of D. Montgomery and some assertions given in a previous paper [7] of the author together with a reduction theorem (see §6) on local group extensions which was suggested by an analogous theorem of Eilenberg and MacLane [5] on abstract group extensions.

If G does not have arbitrarily small subgroups, then it follows immediately that the center of G is locally connected and that the compact zero-dimensional subgroup Z of G in Theorem I must be discrete. This implies that the local subgroup S is an open neighborhood of the neutral element e in G. Since S is locally connected, we obtain the following theorem(4) as a corollary of Theorem I.

THEOREM II. *If G does not have arbitrarily small subgroups, then G must be locally connected.*

2. Montgomery's results. Throughout the present paper, G will denote a locally compact separable metric topological group of finite dimension n. The assertions given in this section are proved by D. Montgomery in one of his recent works [9].

(2.1) *There exists an open neighborhood V of the neutral element e in G with compact closure such that the component R of V which contains e is locally connected and n-dimensional and an invariant local subgroup of V* [9, p. 598].

Choose an open neighborhood W of e in G satisfying the following conditions:

(i) $$W \text{ is symmetric;}$$

(ii) $$\overline{W}^4 \text{ is in } V;$$

(iii) $$(\overline{W}^2 \cap R)^2 \subset R, \qquad (\overline{W}^2 \cap R)^{-1} \subset R;$$

(iv) $$x \in \overline{W} \quad \text{and} \quad y \in \overline{W}^2 \cap R \quad \text{imply} \quad xyx^{-1} \in R.$$

(2.2) *The quotient local group $Q = W/R$* [9, p. 596] *is zero-dimensional* [9, p. 599].

The canonical map $\pi: W \rightarrow Q$ which maps each $w \in W$ onto the coset $wR \cap W$ is continuous and open and a local homomorphism of W onto Q. We shall denote by $q_0 = \pi(e)$ the neutral element of the local group Q.

(2.3) *There exists a local cross-section of Q into W* [9, p. 600]. *In more detail, there exists a continuous map $\chi: M \rightarrow W$ defined on a compact open neighborhood M of q_0 in Q such that $\pi\chi(x) = x$ for every element x in M. Moreover, we may assume that $\chi(q_0) = e$.*

Choose S as a symmetric open connected subset of R which includes e and is such that

$$S^3 \subset R \cap W.$$

(4) The author was informed by K. Iwasawa that Theorem II is also known to M. Kuranishi.

Then S *is an invariant connected, locally connected, locally compact n-dimensional local subgroup of* G. Choose a compact open neighborhood N of q_0 in M such that

$$\chi(N)S \subset W.$$

Call $Z=\chi(N)$. Then Z *is a compact zero-dimensional set in* G.

(2.4) *The subset* $U=ZS$ *of* G *is an open neighborhood of* e *in* G *and the map* $F: Z\times S\rightarrow U$ *of the topological product space* $Z\times S$ *onto* U *defined by*

$$F(z, s) = zs \qquad (z \in Z,\ s \in S)$$

is a homeomorphism of $Z\times S$ *onto* U [9, p. 600].

Let G_0 denote the smallest subgroup of G in the algebraic sense generated by the elements of R. The abstract group G_0 may be topologized so as to form a topological group G^* by taking as a basis of the open neighborhoods of the neutral element e in G^* those open subsets of R which contain e. Let $h: G^*\rightarrow G$ denote the identity map.

(2.5) G^* *is a connected locally compact locally connected n-dimensional group and* $h: G^*\rightarrow G$ *is a one-to-one continuous homomorphism of* G^* *onto a subgroup* $H=h(G^*)$ *of* G, *not necessarily closed, which is dense in the neutral component of* G [9, p. 602].

3. Locally compact zero-dimensional local groups.

THEOREM 3.1. *Let* Q *be a locally compact zero-dimensional local group*(5). *If* M *is a given neighborhood of the neutral element* q_0 *in* Q, *then there exists a compact open neighborhood* N *of* q_0 *in* Q *such that* $N\subset M$ *and* N *is a (global) group under the multiplication defined in the local group* Q.

Proof(6). Let M be a given neighborhood of the neutral element q_0 in the local group Q. Since Q is locally compact and zero-dimensional, there exists a compact open neighborhood K of q_0 in Q such that $K\subset M$ and that, for every pair of elements x and y in K, xy and x^{-1} are defined and depend continuously on x and y simultaneously.

Let x be an arbitrary element in K. Since K is open and $q_0x=x\in K$, there exist an open neighborhood U_x of q_0 and an open neighborhood V_x of x such that

$$U_xV_x \subset K.$$

Since K is compact, there exist a finite number of elements of K, say

(5) A local group Q is said to be zero-dimensional if, for any given neighborhood M of the neutral element q_0 in Q, there is an open and closed subset N of Q such that $q_0\in N\subset M$. We do not assume Q to be separable metric.

(6) The author is grateful to the referee for this quick proof of Theorem 3.1. The original proof was much more complicated and was essentially the same as that of L. Pontrjagin [10, p. 77] given to the analogous theorem for locally compact zero-dimensional global groups.

$x_1, \cdots, x_n$, such that K is the union of the open sets $V_{x_1}, \cdots, V_{x_n}$. Let us denote by U the intersection of the open neighborhoods $U_{x_1}, \cdots, U_{x_n}$. Then U is an open neighborhood of q_0 in Q such that $UK \subset K$.

Choose an open neighborhood W of q_0 in Q such that

$$W^{-1} = W \subset U.$$

Then we have $WK \subset K$. This implies that W^n is defined and is contained in K for every natural number $n = 1, 2, \cdots$. Let

$$N = \bigcup_{n=1}^{\infty} W^n.$$

Since every W^n is an open neighborhood of q_0 contained in K, so is their union N. Since K is compact (and hence closed), the closure $\mathrm{Cl}\,(N)$ of N is contained in K. Now let $x \in \mathrm{Cl}\,(N)$. Since xW is an open neighborhood of x, it must contain some point y of N. Then there is a natural number m such that $y \in W^m$. $y \in xW$ implies that $y = xw$ for some element w of W. It follows that

$$x = yw^{-1} \in W^{m+1} \subset N.$$

This proves that N is a closed subset of K and hence it is compact.

It remains to show that N forms an abstract global group under the multiplication defined in Q. For this purpose, let x and y be any pair of elements in N. Then there exist natural numbers m and n such that $x \in W^m$ and $y \in W^n$. On account of the relation $W^{-1} = W$, we have

$$xy^{-1} \in W^{m+n} \subset N.$$

This completes the proof.

NOTE. As a consequence of Theorem 3.1, we may assume that the compact open neighborhood N of q_0 chosen before the assertion (2.4) is actually a global subgroup of the local group Q.

4. Local group extensions. For the remainder of the paper, we shall assume that G is a *connected* locally compact separable metric topological group and shall use all the notations given in §2. Since G is connected, it follows from (2.5) that the image H of the one-to-one continuous homomorphism $h: G^* \to G$ is dense in G.

As in (2.1), let R denote the invariant locally connected n-dimensional local subgroup of G. An element $c \in R$ is said to be *central* if there is an open neighborhood T of e in R such that tct^{-1} is defined and is equal to c for every element $t \in T$. The totality of the central elements of the local group R is called *the center* C of R. Let B denote the center of the global group G. Since H is dense in G, it follows easily that

$$C = R \cap B$$

and hence C is an abelian local subgroup of R.

Choose an open neighborhood W of e in G satisfying the conditions (i)–(iv) of §2. Then, by (2.2), the quotient local group $Q=W/R$ is zero-dimensional. Let $\pi: W\to Q$ denote the canonical map. According to (2.3), there exists a local cross-section of Q into W, that is to say, there is a continuous map $\chi: M\to W$ defined on an open neighborhood M of the neutral element q_0 in Q such that $\pi\chi(x)=x$ for each $x\in M$ and that $\chi(q_0)=e$.

According to the condition (iv) of §2, for each $x\in M$ and each $y\in \overline{W}^2\cap R$, $\chi(x)y\chi(x)^{-1}$ is in R. Hence the correspondence

$$y \to \mathfrak{a}_x(y) = \chi(x)y\chi(x)^{-1}$$

defines a local automorphism $\mathfrak{a}_x$ of the local group R. The correspondence

$$\mathfrak{a}: M \to A(R)$$

of M into the group $A(R)$ of local automorphisms of R defined by $\mathfrak{a}(x)=\mathfrak{a}_x$ for each $x\in M$ will be called a *collective character*. Obviously, $\mathfrak{a}$ depends on the local cross-section $\chi: M\to W$. Let $\chi': M'\to W$ be another local cross-section of Q into W and let $x\in M\cap M'$. Since $\pi\chi(x)=x=\pi\chi'(x)$, it follows that the element $\rho(x)=\chi'(x)\chi(x)^{-1}$ is in R. Hence

$$\mathfrak{a}_x'(y) = \chi'(x)y\chi'(x)^{-1} = \rho(x)\mathfrak{a}_x(y)\rho(x)^{-1},$$

that is to say, $\mathfrak{a}_x$ and $\mathfrak{a}_x'$ differ only by an inner local automorphism of R. Hereafter, we shall choose a local cross-section $\chi: M\to W$ of Q into W and call it *our basic local cross-section*. The collective character $\mathfrak{a}: M\to A(R)$ determined by χ will be called *our basic collective character*. Since the center C of R is contained in the center B of G, $\mathfrak{a}_x$ operates trivially on C.

A *local group extension* of the local group R by the local group Q is a triple (E, ϕ, ψ) where E is a local group, ϕ is an open continuous local homomorphism of E onto Q, and ψ is an open continuous local isomorphism of R onto the kernel of ϕ [7, §14].

A *local cross-section* of a local group extension (E, ϕ, ψ) of R by Q is a continuous map $u: M\to E$ defined on an open neighborhood M of q_0 in Q in such a way that $\phi u(x)=x$ for each $x\in M$ and that $u(q_0)$ is the neutral element e_0 of E. A local group extension (E, ϕ, ψ) is said to be *fibered* [2] if it has a local cross-section; it is said to be *inessential* [3] if it has a local cross-section u which is a local homomorphism of Q into E.

Let (E, ϕ, ψ) be a fibered local group extension of R by Q. Choose a local cross-section $u: M\to E$ of (E, ϕ, ψ). Take an open neighborhood M_0 of q_0 in M and an open neighborhood R_0 of e such that, for each $x\in M_0$ and each $y\in R_0$,

$$\psi^{-1}(u(x)\psi(y)u(x)^{-1})$$

is a well-defined element of R. (E, ϕ, ψ) is said to be *corresponding to our basic* *collective character* $\mathfrak{a}$ if the local cross-section $u: M \to E$ and the open neighborhood M_0 and R_0 can be so chosen that

$$\psi^{-1}(u(x)\psi(y)u(x)^{-1}) = \mathfrak{a}_x(y)$$

for each $x \in M_0$ and each $y \in R_0$. Such a local cross-section $u: M \to E$ will be called *admissible.*

Let ι denote the identity local isomorphism of R onto the kernel of $\pi: W \to Q$. Then, as shown above, (W, π, ι) is a fibered local group extension of R by Q corresponding to our basic collective character $\mathfrak{a}$ with our basic local cross-section $\chi: M \to W$ as an admissible local cross-section of Q into W. Hereafter, we shall call (W, π, ι) *our basic local group extension* of R by Q. The purpose of these elaborations is to see whether or not (W, π, ι) is inessential.

5. **Equivalence classes of the extensions.** Let us consider the set Ω of all fibered local group extensions of R by Q corresponding to our basic collective character $\mathfrak{a}$. Any two of such extensions (E_1, ϕ_1, ψ_1) and (E_2, ϕ_2, ψ_2) are said to be *equivalent* if there exists an open continuous local isomorphism

$$\sigma: E_1 \approx E_2$$

such that there are an open neighborhood E_1' of the neutral element e_1 in E_1 and an open neighborhood R' of e in R such that

$$\sigma\psi_1(y) = \psi_2(y), \qquad \phi_2\sigma(z) = \phi_1(z)$$

for each $y \in R'$ and each $z \in E_1'$. If $u_1: M_1 \to E_1$ is an admissible local cross-section of (E_1, ϕ_1, ψ_1), then the continuous map $u_2: M_2 \to E_2$ defined by $u_2(x) = \sigma u_1(x)$ for each $x \in M_2 = u_1^{-1}(E_1') \subset M_1$ is clearly an admissible local cross-section of (E_2, ϕ_2, ψ_2).

This equivalence relation divides the local group extensions Ω into disjoint equivalence classes. Our main concern in the sequel is to enumerate these equivalence classes. In the next section, we shall reduce the enumeration of these equivalence classes to that of the equivalence classes of the central fibered local group extensions of the abelian local group C, the center of R, by the local group Q. For the convenience of the reader, we shall recall in the rest of the section the necessary definitions about central local group extensions of C by Q.

As in §4, a *local group extension* of the abelian local group C by the local group Q is a triple (B, ξ, η) where B is a local group, ξ is an open continuous local homomorphism of B onto Q, and η is an open continuous local isomorphism of C onto the kernel of ξ. A *local cross-section* of (B, ξ, η) is a continuous map $v: N \to B$ defined on an open neighborhood N of q_0 in Q in such a way that $\xi v(x) = x$ for each $x \in N$ and that $v(q_0)$ is the neutral element b_0 of B. *Fibered* and *inessential* extensions of C by Q are defined verbally as in §4.

Let (B, ξ, η) be a fibered local group extension of C by Q. Choose a local cross-section $v: N \to B$ of (B, ξ, η). Take an open neighborhood N_0 of q_0 in N and an open neighborhood C_0 of e in C such that, for each $x \in N_0$ and each $z \in C_0$,

$$\mathfrak{c}_x(z) = \eta^{-1}(v(x)\eta(z)v(x)^{-1})$$

is a well-defined element of C. (B, ξ, η) is said to be *central* if N_0 and C_0 can be so chosen that $\mathfrak{c}_x(z) = z$ for each $x \in N_0$ and each $z \in C_0$. It follows from the commutativity of C that the above notion of central extensions does not depend on the choice of the local cross-section $v: N \to B$.

Let Γ denote the set of all central fibered local group extensions of C by Q. An equivalence relation can be defined between the extensions in Γ just as for the extensions in Ω given at the beginning of the present section. This equivalence relation divides the central extensions Γ into disjoint equivalence classes.

6. **The reduction theorem.** We shall denote by $\Omega^{\#}$ the set of all equivalence classes of the fibered local group extensions Ω of R by Q corresponding to our basic collective character $\mathfrak{a}$. Similarly, $\Gamma^{\#}$ will denote the set of all equivalence classes of the central fibered local group extensions of the abelian local group C by Q. The reduction theorem reduces the enumeration of $\Omega^{\#}$ to that of $\Gamma^{\#}$. It can be simply stated as follows(7).

THEOREM 6.1. *$\Omega^{\#}$ and $\Gamma^{\#}$ have the same cardinal number.*

Proof. The proof depends on a construction of a product of local group extensions, analogous to the one used in abstract group extensions by R. Baer [1] and others.

Let $\mathfrak{W} = (W, \pi, \iota)$ be our basic local group extension of R by Q and let $\mathfrak{B} = (B, \xi, \eta)$ be an arbitrarily given central fibered local group extension of C by Q. In the direct product $W \times B$ of the local groups W and B [10, p. 85] consider the local subgroup E^* of all pairs (w, b) with $\pi(w) = \xi(b)$ and define a local homomorphism ϕ^* of E^* onto Q by taking

$$\phi^*(w, b) = \pi(w) = \xi(b), \qquad (w, b) \in E^*.$$

It is easily seen that ϕ^* is open and continuous. Let E_0^* denote the invariant local subgroup of E^* consisting of all pairs $(\iota(c), \eta(c)^{-1})$ for $c \in C$. Let

$$E^{\#} = E^*/E_0^*$$

(7) The proof of the Reduction Theorem 6.1 actually handles a situation more general than implied by the standing hypotheses. More precisely, let R and Q be any given local groups such that the center C of R is abelian. Let $\mathfrak{a}$: $M \to A(R)$ be a given collective character in R and $\mathfrak{b}$ the collective character in C induced by $\mathfrak{a}$ by means of $\mathfrak{b}_x = \mathfrak{a}_x | C$. Let Ω denote the set of all fibered local group extensions of R by Q corresponding to $\mathfrak{a}$ and Γ denote the set of all fibered local group extensions of C by Q corresponding to $\mathfrak{b}$. If $\Omega^{\#}$ and $\Gamma^{\#}$ denote the sets of equivalence classes, then Theorem 6.1 is true for this general case.

denote the quotient local group of E^* over E_0^* [10, p. 85]. Since ϕ^* maps E_0^* into the neutral element q_0 of Q, it induces an open continuous local homomorphism $\phi^\#$ of $E^\#$ onto Q. Let p denote the canonical map of the local group E^* onto the quotient local group $E^\#$; then p is an open continuous local homomorphism of E^* onto $E^\#$. Define a local homomorphism $\psi^\#$ of R into $E^\#$ by taking

$$\psi^\#(y) = p(\iota(y), b_0)$$

for every element y in a sufficiently small open neighborhood of the neutral element e in R, where b_0 denotes the neutral element in B. It is easily verified that $\psi^\#$ is an open continuous local isomorphism of R onto the kernel of $\phi^\#$. Thus we obtain a local group extension

$$\mathfrak{E}^\# = (E^\#, \phi^\#, \psi^\#)$$

of R by Q called the product of $\mathfrak{W}$ and $\mathfrak{B}$, in notation:

$$\mathfrak{E}^\# = \mathfrak{W} \otimes \mathfrak{B}.$$

Let $\chi: M \to W$ be our basic local cross-section of (W, π, ι) and $v: M_0 \to B$ be a local cross-section of (B, ξ, η). Choose a sufficiently small open neighborhood $M^\#$ of q_0 in Q and define a continuous map $u^\#: M^\# \to E^\#$ by taking

$$u^\#(x) = p(\chi(x), v(x))$$

for every element x in $M^\#$. From this definition it follows immediately that

$$\phi^\# u^\#(x) = \phi^\# p(\chi(x), v(x)) = \phi^*(\chi(x), v(x)) = \pi\chi(x) = x$$

for each $x \in M^\#$. Hence $u^\#$ is a local cross-section of $(E^\#, \phi^\#, \psi^\#)$. This proves that $\mathfrak{E}^\#$ is a fibered local group extension of R by Q. Moreover, we have

$$\begin{aligned} \psi^{\#-1}[u^\#(x)\psi^\#(y)u^\#(x)^{-1}] &= \psi^{\#-1}[p(\chi(x), v(x))p(\iota(y), b_0)p(\chi(x), v(x))^{-1}] \\ &= \psi^{\#-1}p(\chi(x)y\chi(x)^{-1}, v(x)b_0v(x)^{-1}) = \psi^{\#-1}p(\mathfrak{a}_x(y), b_0) \\ &= \psi^{\#-1}\psi^\#\mathfrak{a}_x(y) = \mathfrak{a}_x(y) \end{aligned}$$

whenever $x \in M^\#$ and $y \in R$ are sufficiently near the neutral elements. This proves that $\mathfrak{E}^\#$ is corresponding to our basis collective character $\mathfrak{a}$ and that $u^\#: M^\# \to E^\#$ is an admissible local cross-section of $\mathfrak{E}^\#$.

Therefore, our product construction gives a map $\kappa: \Gamma \to \Omega$ defined by the correspondence $\mathfrak{B} \to \mathfrak{W} \otimes \mathfrak{B}$. We are going to show that κ maps equivalent extensions in Γ into equivalent extensions in Ω. Indeed, let

$$\mathfrak{B}_1 = (B_1, \xi_1, \eta_1), \qquad \mathfrak{B}_2 = (B_2, \xi_2, \eta_2)$$

denote any two equivalent extensions in Γ. Then, by definition, there is an open continuous local isomorphism

$$\sigma: B_1 \approx B_2$$

such that there are an open neighborhood B_1' of the neutral element b_1 in B_1 and an open neighborhood C' of the neutral element e in C such that

$$\sigma\eta_1(y) = \eta_2(y), \quad \xi_2\sigma(z) = \xi_1(z)$$

for each $y \in C'$ and each $z \in B_1'$. Let us denote by

$$\mathfrak{E}_i^{\#} = (E_i^{\#}, \phi_i^{\#}, \psi_i^{\#}) = \mathfrak{W} \otimes \mathfrak{V}_i \qquad (i = 1, 2)$$

the product extensions in Ω constructed as above. Then an open continuous local isomorphism

$$\tau : E_1^{\#} \approx E_2^{\#}$$

can be defined by taking

$$\tau p_1(w, z) = p_2(w, \sigma(z))$$

for each $p_1(w, z)$ in a sufficiently small open neighborhood E_1' of the neutral element e_1 in $E_1^{\#}$, where $p_i : E_i^* \to E_i^{\#}$ $(i = 1, 2)$ denote the canonical maps. It is easily verified that

$$\tau\psi_1^{\#}(y) = \psi_2^{\#}(y), \qquad \phi_2^{\#}\tau(z) = \phi_1^{\#}(z)$$

for each $z \in E_1'$ and each y in a sufficiently small open neighborhood R' of e in R. This proves that $\mathfrak{E}_1^{\#}$ and $\mathfrak{E}_2^{\#}$ are equivalent. Hence the map κ maps equivalent extensions in Γ into equivalent extensions in Ω.

Thus the map $\kappa : \Gamma \to \Omega$ induces a map $\kappa^{\#} : \Gamma^{\#} \to \Omega^{\#}$ of the equivalence classes. The theorem is proved if we have shown that $\kappa^{\#}$ maps $\Gamma^{\#}$ onto $\Omega^{\#}$ in a one-to-one fashion. This follows from the two auxiliary lemmas in the following sections.

7. The first auxiliary lemma.

Lemma 7.1. *Every extension in Ω is equivalent with an extension of the form $\mathfrak{W} \otimes \mathfrak{V}$ with some $\mathfrak{V}$ in Γ.*

Proof. As in §6, let $\mathfrak{W} = (W, \pi, \iota)$ denote our basic extension and $\chi : M \to W$ denote our basic local cross-section of $\mathfrak{W}$. Choose an open neighborhood M_0 of q_0 in Q and an open neighborhood R_0 of the neutral element e in R so small that the following relations (i)–(iv) are true. For every $x \in M_0$ and $y \in R_0$, we have

$$\text{(i)} \qquad \chi(x) y \chi(x)^{-1} = \mathfrak{a}_x(y),$$

where $\mathfrak{a}$ denotes our basic collective character. Since $\pi\chi(x) = x$ for each $x \in M$ and M_0 is sufficiently small, we may define a continuous map

$$f : M_0 \times M_0 \to R$$

of the topological product space $M_0 \times M_0$ into R by taking

(ii) $$f(x_1, x_2) = \chi(x_1)\chi(x_2)(\chi(x_1x_2))^{-1}$$

for any pair of elements x_1 and x_2 in M_0. The associativity on the multiplication in W as well as the relations (i) and (ii) imply that

(iii) $$f(x_1, x_2)f(x_1x_2, x_3) = \mathfrak{a}_{x_1}(f(x_2, x_3))f(x_1, x_2x_3)$$

for x_1, x_2, x_3 in M_0 and

(iv) $$\mathfrak{a}_{x_1}\mathfrak{a}_{x_2}(y) = f(x_1, x_2)\mathfrak{a}_{x_1x_2}(y)f(x_1, x_2)^{-1}$$

for x_1, x_2 in M_0 and y in R_0.

Now let $\mathfrak{E}=(E, \phi, \psi)$ be an arbitrary local group extension of R by Q in Ω and $u: M_* \rightarrow E$ be an admissible cross-section of $\mathfrak{E}$. Choose an open neighborhood M_1 of q_0 in Q and an open neighborhood R_1 of e in R so small that the following relations (v)–(viii) are defined and true. For every $x \in M_1$ and $y \in R_1$, we have

(v) $$\psi^{-1}(u(x)\psi(y)u(x)^{-1}) = \mathfrak{a}_x(y).$$

Since $\phi u(x)=x$ for each $x \in M_*$ and M_1 is sufficiently small, we may define a continuous map

$$g: M_1 \times M_1 \rightarrow R$$

of the topological product space $M_1 \times M_1$ into R by taking

(vi) $$g(x_1, x_2) = \psi^{-1}(u(x_1)u(x_2)(u(x_1x_2))^{-1})$$

for every pair of elements x_1 and x_2 in M_1. The relations of (v) and (vi) and the associativity of the multiplication in E imply that

(vii) $$g(x_1, x_2)g(x_1x_2, x_3) = \mathfrak{a}_{x_1}(g(x_2, x_3))g(x_1, x_2x_3)$$

for x_1, x_2, x_3 in M_1 and

(viii) $$\mathfrak{a}_{x_1}\mathfrak{a}_{x_2}(y) = g(x_1, x_2)\mathfrak{a}_{x_1x_2}(y)g(x_1, x_2)^{-1}$$

for x_1, x_2 in M_1 and y in R_1.

If $M_2 \subset M_0 \cap M_1$ and $R_2 = R_0 \cap R_1$ are sufficiently small open neighborhoods of the neutral elements, then one can easily derive the following relation by means of the formulas (iv) and (viii):

$$\begin{aligned}
&g(x_1, x_2)f(x_1, x_2)^{-1}y(g(x_1, x_2)f(x_1, x_2)^{-1})^{-1} \\
&\qquad = g(x_1, x_2)\left[f(x_1, x_2)^{-1}yf(x_1, x_2)\right]g(x_1, x_2)^{-1} \\
&\qquad = g(x_1, x_2)\left[\mathfrak{a}_{x_1x_2}\mathfrak{a}_{x_2}^{-1}\mathfrak{a}_{x_1}^{-1}(y)\right]g(x_1, x_2)^{-1} \\
&\qquad = \mathfrak{a}_{x_1}\mathfrak{a}_{x_2}\mathfrak{a}_{x_1x_2}^{-1}\mathfrak{a}_{x_1x_2}\mathfrak{a}_{x_2}^{-1}\mathfrak{a}_{x_1}^{-1}(y) = y
\end{aligned}$$

for every pair of elements x_1, x_2 in M_2 and every y in R_2. This implies that $g(x_1, x_2)f(x_1, x_2)^{-1}$ is an element in the center C of the local group R and hence we may define a continuous map

$$d:M_2 \times M_2 \to C$$

of the topological product space $M_2 \times M_2$ into C by taking

$$d(x_1, x_2) = g(x_1, x_2)f(x_1, x_2)^{-1}$$

for each pair of elements in M_2. It follows directly from (ii) and (vi) that

$$d(x, q_0) = d(q_0, x) = d(q_0, q_0) = e$$

for every x in M_2. Since $\mathfrak{a}_x$ operates trivially on the center C of R, one can easily verify by means of (iii) and (vii) that

$$d(x_1, x_2)d(x_1x_2, x_3) = d(x_2, x_3)d(x_1, x_2x_3) \tag{ix}$$

for x_1, x_2, x_3 in M_2.

By means of the continuous map d, we are going to construct a local group extension $\mathfrak{B} = \{B, \xi, \eta\}$ of C by Q as follows. The space of B is the topological product space $Q \times C$. The multiplication in B is defined by

$$(x_1, y_1)(x_2, y_2) = (x_1x_2, y_1y_2d(x_1, x_2))$$

whenever x_1, $x_2 \in Q$ and y_1, $y_2 \in C$ are sufficiently near the neutral elements. One observes that (ix) implies that this multiplication is associative and hence it is easily verified that B is a local group. The local homomorphism $\xi: B \to Q$ is defined by

$$\xi(x, y) = x, \qquad (x, y) \in B.$$

Since ξ is the projection of $Q \times C$ onto Q, it is open and continuous. The kernel B_0 of ξ is the set of all pairs (q_0, y) for all $y \in C$. The open continuous local isomorphism $\eta: C \to B_0$ of C onto B_0 is defined by

$$\eta(y) = (q_0, y), \qquad y \in C.$$

The local group extension $\mathfrak{B}$ constructed above has an obvious local cross-section $v: Q \to B$ defined by

$$v(x) = (x, e), \qquad x \in Q.$$

When $x \in Q$ and $y \in C$ are sufficiently near the neutral elements, we always have

$$\begin{aligned}\eta^{-1}[v(x)\eta(y)v(x)^{-1}] &= \eta^{-1}[(x, e)(q_0, y)(x, e)^{-1}] = \eta^{-1}[(x, y)(x, e)^{-1}] \\ &= \eta^{-1}[(q_0, y)(x, e)(x, e)^{-1}] = \eta^{-1}(q_0, y) = y.\end{aligned}$$

Hence $\mathfrak{B} = (B, \xi, \eta)$ is a central fibered local group extension of C by Q; in other words, $\mathfrak{B}$ is in Γ. Now let

$$\mathfrak{E}^\# = (E^\#, \phi^\#, \psi^\#) = W \otimes \mathfrak{B}$$

denote the product extension of $\mathfrak{W}$ and $\mathfrak{B}$ constructed in the proof of Theorem

6.1. We are going to show that the given extension $\mathfrak{E}$ is equivalent with $\mathfrak{E}^\#$.

As shown in §6, $\mathfrak{E}^\#$ has an admissible local cross-section $u^\#: M^\# \to E^\#$ defined on an open neighborhood $M^\# \subset M_2$ of q_0 in Q by

$$u^\#(x) = p(\chi(x), v(x))$$

for each $x \in M^\#$. Choose an open neighborhood E_0 of the neutral element z_0 in E so small that all the expressions involved in the following construction are well-defined. First, define a continuous map $\beta: E_0 \to R$ by taking, for each $z \in E_0$,

$$\beta(z) = \psi^{-1}[z(u\phi(z))^{-1}].$$

Next, define a continuous map $\sigma: E_0 \to E^\#$ by taking, for each $z \in E_0$,

$$\sigma(z) = \psi^\# \beta(z) u^\# \phi(z).$$

Obviously, σ maps an open neighborhood of z_0 in E topologically onto an open neighborhood of the neutral element $z_0^\#$ in $E^\#$. To prove that σ is an open continuous local isomorphism of E onto $E^\#$, it needs only to be shown that σ is locally homomorphic

Let z_1, z_2 be any two elements of E near the neutral element z_0. Call

$$x_1 = \phi(z_1), \qquad x_2 = \phi(z_2), \qquad y_1 = \beta(z_1), \qquad y_2 = \beta(z_2).$$

Then, for each $i = 1, 2$, we have

$$\sigma(z_i) = \psi^\#(y_i)u^\#(x_i) = p(\iota(y_i), b_0)p(\chi(x_i), v(x_i)) = p(y_i\chi(x_i), v(x_i)).$$

Hence we obtain

$$\sigma(z_1)\sigma(z_2) = p(y_1\chi(x_1)y_2\chi(x_2), v(x_1)v(x_2)) = p(y_1\mathfrak{a}_{x_1}(y_2)\chi(x_1)\chi(x_2), v(x_1)v(x_2)).$$

On the other hand, we have

$$\begin{aligned}
\phi(z_1z_2) &= \phi(z_1)\phi(z_2) = x_1z_2,\\
\beta(z_1z_2) &= \psi^{-1}[z_1z_2(u(x_1x_2))^{-1}] = \psi^{-1}[\psi(y_1)u(x_1)\psi(y_2)u(x_2)(u(x_1x_2))^{-1}]\\
&= \psi^{-1}[\psi(y_1)\psi(\mathfrak{a}_{x_1}(y_2))\psi(g(x_1, x_2))] = y_1\mathfrak{a}_{x_1}(y_2)g(x_1, x_2).
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
\sigma(z_1z_2) &= \psi^\#\beta(z_1z_2)u^\#\phi(z_1z_2) = p[y_1\mathfrak{a}_{x_1}(y_2)g(x_1, x_2)\chi(x_1x_2), v(x_1x_2)]\\
&= p[y_1\mathfrak{a}_{x_1}(y_2)d(x_1, x_2)f(x_1, x_2)\chi(x_1x_2), v(x_1x_2)]\\
&= p[y_1\mathfrak{a}_{x_1}(y_2)f(x_1, x_2)\chi(x_1x_2), d(x_1, x_2)v(x_1x_2)]\\
&= p[y_1\mathfrak{a}_{x_1}(y_2)\chi(x_1)\chi(x_2), v(x_1)v(x_2)] = \sigma(z_1)\sigma(z_2).
\end{aligned}$$

This proves that σ is locally homomorphic and hence an open continuous local isomorphism of E onto $E^\#$.

Now let $y \in R$ and $z \in E$ be given elements sufficiently near the neutral elements. Then we have

$$\begin{aligned}\sigma\psi(y) &= \psi^{\#}\beta\psi(y)u^{\#}\phi\psi(y) = \psi^{\#}(y)u^{\#}(q_0) = \psi^{\#}(y),\\ \phi^{\#}\sigma(z) &= \phi^{\#}[\psi^{\#}\beta(z)u^{\#}\phi(z)] = \phi^{\#}p[\beta(z)\chi\phi(z), v\phi(z)]\\ &= p^*[\beta(z)\chi\phi(z), v\phi(z)] = \xi v\phi(z) = \phi(z).\end{aligned}$$

This proves that $\mathfrak{E}$ and $\mathfrak{E}^{\#}=\mathfrak{W}\otimes\mathfrak{B}$ are equivalent and hence the proof of the lemma is complete.

8. The second auxiliary lemma.

LEMMA 8.1. *Two extensions $\mathfrak{B}_1$ and $\mathfrak{B}_2$ in Γ are equivalent if and only if $\mathfrak{W}\otimes\mathfrak{B}_1$ and $\mathfrak{W}\otimes\mathfrak{B}_2$ are equivalent.*

Proof. Let $\mathfrak{B}_i=(B_i, \xi_i, \eta_i)$ and choose a local cross-section $v_i: M_i\rightarrow B_i$ for each $i=1, 2$. Take a sufficiently small open neighborhood $M_0\subset M_1\cap M_2$ of q_0 in Q and define continuous maps

$$d_i: M_0 \times M_0 \rightarrow C \qquad (i = 1, 2)$$

of the topological product space $M_0\times M_0$ into the center C of R by taking

$$\text{(i)} \qquad d_i(x_1, x_2) = \overset{-1}{\eta_i}(v_i(x_1)v_i(x_2)(v_i(x_1x_2))^{-1})$$

for every pair of elements x_1 and x_2 in M_0. It follows from the associativity of the multiplication in B_i that

$$\text{(ii)} \qquad d_i(x_1, x_2)d_i(x_1x_2, x_3) = d_i(x_2, x_3)d_i(x_1, x_2x_3)$$

for x_1, x_2, x_3 in M_0.

As in §6, $\mathfrak{W}=(W, \pi, \iota)$ will denote our basic extension of R by Q with our basic local cross-section $\chi: M\rightarrow W$. We may assume $M_0\subset M$. Let

$$\mathfrak{E}_i^{\#} = W \otimes \mathfrak{B}_i = (E_i^{\#}, \phi_i^{\#}, \psi_i^{\#}) \qquad (i = 1, 2)$$

denote the product extensions as constructed in §6 and

$$u_i^{\#}: M_0 \rightarrow E_i^{\#} \qquad (i = 1, 2)$$

the local cross-sections defined by

$$u_i^{\#}(x) = p_i(\chi(x), v_i(x)) \qquad (i = 1, 2)$$

for each $x\in M_0$, where $p_i: E_i^*\rightarrow E_i^{\#}$ denotes the canonical map of E_i^* onto $E_i^{\#}$ as in §6. Then, when M_0 is sufficiently small, we have

$$\begin{aligned}&\psi_i^{\#-1}[u_i^{\#}(x_1)u_i^{\#}(x_2)(u_i^{\#}(x_1x_2))^{-1}]\\ \text{(iii)} \qquad &= \psi_i^{\#-1}p_i[\chi(x_1)\chi(x_2)(\chi(x_1x_2))^{-1}, v_i(x_1)v_i(x_2)(v_i(x_1x_2))^{-1}]\\ &= \psi_i^{\#-1}p_i[f(x_1, x_2), \eta_i d_i(x_1, x_2)] = d_i(x_1, x_2)f(x_1, x_2)\end{aligned}$$

for every pair of elements x_1 and x_2 in M_0, where $f: M_0 \times M_0 \to R$ is the continuous map defined by (ii) of §7.

The necessity of the lemma is already proved in §6. It remains to show the sufficiency of the lemma. Hence we assume that $\mathfrak{E}_1^\#$ and $\mathfrak{E}_2^\#$ are equivalent; that is to say, there exists an open continuous local isomorphism

$$\tau: \overset{\#}{E}_1 \approx \overset{\#}{E}_2$$

such that there are an open neighborhood E_1' of the neutral element $e_1^\#$ in $E_1^\#$ and an open neighborhood R' of e in R such that

$$\tau\overset{\#}{\psi}_1(y) = \overset{\#}{\psi}_2(y), \qquad \overset{\#}{\phi}_2\sigma(z) = \overset{\#}{\phi}_1(z)$$

for each $y \in R'$ and each $z \in E_1'$. When M_0 is small enough, we may define a continuous map

$$\mathfrak{b}: M_0 \to R$$

of M_0 into R by taking

$$\text{(iv)} \qquad \mathfrak{b}(x) = \overset{\#}{\psi}{}_2^{-1}\left[\tau\overset{\#}{u}_1(x)\left(\overset{\#}{u}_2(x)\right)^{-1}\right]$$

for each $x \in M_0$. Since the local cross-sections $u_1^\#$ and $u_2^\#$ are admissible, we have

$$\text{(v)} \qquad \overset{\#}{u}_1(x)\overset{\#}{\psi}_1(y) = \overset{\#}{\psi}_1\mathfrak{a}_x(y)\overset{\#}{u}_1(x),$$

$$\text{(vi)} \qquad \overset{\#}{u}_2(x)\overset{\#}{\psi}_2(y) = \overset{\#}{\psi}_2\mathfrak{a}_x(y)\overset{\#}{u}_2(x),$$

for each $x \in M_0$ and $y \in R'$ provided that M_0 and R' are chosen sufficiently small. On applying τ to the equation (v) we find

$$\text{(vii)} \qquad \overset{\#}{\psi}_2\mathfrak{b}(x)\overset{\#}{u}_2(x)\overset{\#}{\psi}_2(y) = \overset{\#}{\psi}_2\mathfrak{a}_x(y)\overset{\#}{\psi}_2\mathfrak{b}(x)\overset{\#}{u}_2(x)$$

for each $x \in M_0$ and $y \in R'$. It follows easily from (vi) and (vii) that

$$\text{(viii)} \qquad \mathfrak{b}(x)\mathfrak{a}_x(y) = \mathfrak{a}_x(y)\mathfrak{b}(x)$$

for each $x \in M_0$ and $y \in R'$ provided that M_0 and R' are chosen sufficiently small. Since $\mathfrak{a}_x(R')$ is a neighborhood of the neutral element e in R, (viii) implies that $\mathfrak{b}(x)$ is a central element of R and hence $\mathfrak{b}(M_0) \subset C$.

According to (iii) for $i=1$, we have

$$\overset{\#}{u}_1(x_1)\overset{\#}{u}_1(x_2) = \overset{\#}{\psi}_1[d_1(x_1, x_2)f(x_1, x_2)]\overset{\#}{u}_1(x_1x_2)$$

for each pair of elements x_1 and x_2 in M_0. Applying τ to this equation and using the fact that $\mathfrak{b}(x_2)$ is in C and that $u_2^\#$ is an admissible local cross-section, we obtain

$$\overset{\#}{\psi}_2\mathfrak{b}(x_1)\overset{\#}{\psi}_2\mathfrak{b}(x_2)\overset{\#}{u}_2(x_1)\overset{\#}{u}_2(x_2) = \overset{\#}{\psi}_2[d_1(x_1, x_2)f(x_1, x_2)]\overset{\#}{\psi}_2\mathfrak{b}(x_1x_2)\overset{\#}{u}_2(x_1x_2).$$

But, according to (iii) for $i=2$,

$$\overset{\#}{u}_2(x_1)\overset{\#}{u}_2(x_2) = \overset{\#}{\psi}_2[d_2(x_1, x_2)f(x_1, x_2)]\overset{\#}{u}_2(x_1x_2),$$

hence, we obtain

$$\text{(ix)} \qquad d_1(x_1, x_2)\mathfrak{b}(x_1x_2) = \mathfrak{b}(x_1)\mathfrak{b}(x_2)d_2(x_1, x_2)$$

for each pair of elements x_1 and x_2 in M_0.

Choose an open neighborhood B_1' of the neutral element b_1 in B_1 so small that all the expressions involved in the following construction are well-defined. First, define a continuous map $\beta: B_1' \to C$ by taking

$$\beta(z) = \eta_1^{-1}[z(v_1\xi_1(z))^{-1}]$$

for each $z \in B_1'$. Next, let us define a continuous mapping $\sigma: B_1' \to B_2$ by taking

$$\sigma(z) = \eta_2\beta(z)\eta_2\mathfrak{b}\xi_1(z)v_2\xi_1(z)$$

for each $z \in B_1'$. Obviously, σ maps an open neighborhood of b_1 in B_1 topologically onto an open neighborhood of the neutral element b_2 in B_2. To prove that σ is an open continuous local isomorphism of B_1 onto B_2, it needs only to be shown that σ is locally homomorphic.

Let z_1, z_2 be any two elements of B_1 near the neutral element b_1. Call

$$x_1 = \xi_1(z_1), \qquad x_2 = \xi_1(z_2), \qquad y_1 = \beta(z_1), \qquad y_2 = \beta(z_2).$$

Then, for each $i=1, 2$, we have

$$\sigma(z_i) = \eta_2\beta(z_i)\eta_2\mathfrak{b}\xi_1(z_i)v_2\xi_1(z_i) = \eta_2[y_i\mathfrak{b}(x_i)]v_2(x_i).$$

Hence we obtain

$$\sigma(z_1)\sigma(z_2) = \eta_2[y_1\mathfrak{b}(x_1)]v_2(x_1)\eta_2[y_2\mathfrak{b}(x_2)]v_2(x_2) = \eta_2[y_1y_2\mathfrak{b}(x_1)\mathfrak{b}(x_2)]v_2(x_1)v(x_2).$$

On the other hand, we have

$$\begin{aligned}
\xi_1(z_1z_2) &= \xi_1(z_1)\xi_1(z_2) = x_1x_2,\\
\beta(z_1z_2) &= \eta_1^{-1}[z_1z_2(v_1(x_1x_2))^{-1}] = \eta_1^{-1}[\eta_1(y_1)v_1(x_1)\eta_1(y_2)v_1(x_2)(v_1(x_1x_2))^{-1}]\\
&= \eta_1^{-1}[\eta_1(y_1)\eta_1(y_2)v_1(x_1)v_1(x_2)(v_1(x_1x_2))^{-1}] = y_1y_2d_1(x_1, x_2).
\end{aligned}$$

Hence, using (ix) and (i), we obtain

$$\begin{aligned}
\sigma(z_1z_2) &= \eta_2\beta(z_1z_2)\eta_2\mathfrak{b}\xi_1(z_1z_2)v_2\xi_1(z_1z_2) = \eta_2[y_1y_2d_1(x_1, x_2)\mathfrak{b}(x_1x_2)]v_2(x_1x_2)\\
&= \eta_2[y_1y_2\mathfrak{b}(x_1)\mathfrak{b}(x_2)]\eta_2d_2(x_1, x_2)v_2(x_1x_2) = \eta_2[y_1y_2\mathfrak{b}(x_1)\mathfrak{b}(x_2)]v_2(x_1)v_2(x_2)\\
&= \sigma(z_1)\sigma(z_2).
\end{aligned}$$

This proves that σ is locally homomorphic and hence an open continuous local isomorphism of B_1 onto B_2.

Now let $y\in C$ and $z\in B_1$ be given elements sufficiently near the neutral elements. Then we have

$$\sigma\eta_1(y) = \eta_2\beta\eta_1(y)\eta_2\mathfrak{b}\xi_1\eta_1(y)v_2\xi_1\eta_1(y) = \eta_2(y),$$
$$\xi_2\sigma(z) = \xi_2\left[\eta_2\beta(z)\eta_2\mathfrak{b}\xi_1(z)v_2\xi_1(z)\right] = \xi_1(z).$$

This proves that B_1 and B_2 are equivalent and hence the proof of the lemma is complete.

Thus, we have completely proved our reduction theorem 6.1.

9. The trivial local group extension in Ω. There is a *trivial* local group extension $\mathfrak{E}_0=(E_0, \phi_0, \psi_0)$ in Ω which will be constructed as follows. The space E_0 is the topological product space $Q\times R$ of Q and R. The multiplication in E_0 is defined

$$(x_1, y_1)(x_2, y_2) = (x_1x_2, y_1\mathfrak{a}_{x_1}(y_2))$$

whenever $x_1, x_2\in Q$ and $y_1, y_2\in R$ are sufficiently near the neutral elements. One verifies easily that this multiplication is associative and makes E_0 a local group. The local homomorphism $\phi_0:E_0\to Q$ is defined by

$$\phi_0(x, y) = x, \qquad (x, y) \in E_0.$$

Since ϕ_0 is the projection of $Q\times R$ onto Q, it is open and continuous. The kernel R_0 of ϕ_0 is the set of all pairs (q_0, y) for all $y\in R$. The open continuous local isomorphism $\psi_0:R\to R_0$ of R onto R_0 is defined by

$$\psi_0(y) = (q_0, y), \qquad y \in R.$$

The local group extension $\mathfrak{E}_0$ constructed above has an obvious local cross-section $u_0:Q\to E_0$ defined by

$$u_0(x) = (x, e), \qquad x \in Q.$$

When $x\in Q$ and $y\in R$ are sufficiently near the neutral elements, we always have

$$\begin{aligned}\psi_0^{-1}\left[u_0(x)\psi_0(y)u_0(x)^{-1}\right] &= \psi_0^{-1}\left[(x, e)(q_0, y)(x, e)^{-1}\right] = \psi_0^{-1}\left[(x, \mathfrak{a}_x(y))(x, e)^{-1}\right]\\ &= \psi_0^{-1}\left[(q_0, \mathfrak{a}_x(y))(x, e)(x, e)^{-1}\right] = \psi_0^{-1}\left[\psi_0\mathfrak{a}_x(y)\right] = \mathfrak{a}_x(y).\end{aligned}$$

Hence $\mathfrak{E}_0=(E_0, \phi_0, \psi_0)$ is a fibered local group extension of R by Q corresponding to our basic collective character $\mathfrak{a}$, that is, $\mathfrak{E}_0$ is in Ω with $u_0:Q\to E_0$ as an admissible local cross-section.

It follows from the definitions of the multiplication in E_0 and the local cross-section $u_0:Q\to E_0$ that u_0 is locally homomorphic. This implies that $\mathfrak{E}_0=(E_0, \phi_0, \psi_0)$ is inessential. Hereafter, $\mathfrak{E}_0$ will be called the *trivial local group extension* in Ω.

10. The local structure theorems. Throughout this last section of the paper, we shall assume one more condition than in the previous sections, namely, that the center C of the locally connected local group R will be assumed to be locally connected and hence an abelian local Lie group. Then C is either discrete or is locally isomorphic with a finite-dimensional vector group. Under this new assumption, we are able to strengthen Montgomery's results as follows.

According to Theorem 3.1, the quotient local group $Q=W/R$ in (2.2) is locally isomorphic with a compact zero-dimensional global topological group Q_0. It follows from the assertions proved in a previous paper [7] of the author, namely, the assertions (7.5), (9.1), (12.2), (12.3), and (14.3) in [7], that equivalence classes $\Gamma^{\#}$ of the central fibered local group extensions of C by Q are in a one-to-one correspondence with the elements of a subgroup of the two-dimensional Čech cohomology group of Q_0 with coefficients in a finite-dimensional vector group. Since Q_0 is zero-dimensional, this cohomology group contains only a single element. This implies $\Gamma^{\#}$ consists of only one equivalence class. Then it follows from our reduction theorem (6.1) that $\Omega^{\#}$ contains only one equivalence class and hence our trivial local group extension $\mathfrak{E}_0=(E_0, \phi_0, \psi_0)$ of R by Q constructed in §9 is equivalent with our basic extension $\mathfrak{W}=(W, \pi, \iota)$.

Therefore, there exists an open continuous local isomorphism $\sigma: E_0 \approx W$ of E_0 onto W such that

$$\sigma\psi_0(y) = \iota(y), \qquad \pi\sigma(z) = \phi_0(z)$$

for every $y \in R$ and $z \in E_0$ sufficiently near the neutral elements. Let $u_0: Q \to E_0$ be the obvious local cross-section of $\mathfrak{E}_0$ constructed in §9. Then we may choose a sufficiently small neighborhood M_0 of q_0 in Q and define a local cross-section

$$\chi_0: M_0 \to W \subset G$$

of $\mathfrak{W}$ by taking $\chi_0(x) = \sigma u_0(x)$ for each $x \in M_0$. Since both u_0 and σ are locally homomorphic, so is χ_0. Thus, we have proved the following assertion which is a strengthened form of (2.3).

(10.1) *There exists a locally homomorphic local cross-section of Q into W. In more detail, there exists a continuous map $\chi_0: M_0 \to W$ defined on a compact open neighborhood M_0 of q_0 in Q such that $\chi_0(q_0)=e$, $\pi\chi_0(x)=x$ for each $x \in M_0$, and $\chi_0(x_1x_2)=\chi_0(x_1)\chi_0(x_2)$ for every pair of elements x_1 and x_2 in M_0 sufficiently near q_0.*

According to Theorem 3.1, we can choose M_0 in such a way that M_0 becomes a compact open global subgroup of the local group Q and that χ_0 becomes an open continuous isomorphism of the global group M_0 into the global group G.

As in §2, choose S as a symmetric open connected subset of R which includes e and is such that

$$S^3 \subset R \cap W.$$

Then S *is an invariant connected, locally connected, locally compact n-dimensional local subgroup of* G. Choose a compact open subgroup N of the compact zero-dimensional group M_0 such that

$$\chi_0(N)S \subset W.$$

Call $Z = \chi_0(N)$. Then Z *is a compact zero-dimensional subgroup of* G. With this new Z, the assertion (2.4) still holds. Hence we have strengthened Montgomery's result (2.4) with additional information that, if C is locally connected, then Z may be assumed to be a subgroup of G.

Now let us assume that the center B of the group G is a Lie group. According to §4, we have

$$C = R \cap B.$$

As a local subgroup of a Lie group B, C is a local Lie group. Hence Theorem I of the introduction is now an obvious consequence of the above information.

Bibliography

1. R. Baer, *Erweiterung von Gruppen und ihren Isomorphismen*, Math. Zeit. vol. 38 (1934) pp. 375–416.

2. L. Calabi and C. Ehresmann, *Sur les extensions de groupes topologiques*, C. R. Acad. Sci. Paris vol. 228 (1949) pp. 1551–1553.

3. L. Calabi, *Sur les extensions de groupes topologiques*, C. R. Acad. Sci. Paris vol. 229 (1949) pp. 413–415.

4. C. Chevalley, *Two theorems on solvable topological groups*, Michigan Lectures in Topology edited by Wilder and Ayres, 1941, pp. 291–292.

5. S. Eilenberg and S. MacLane, *Cohomology theory in abstract groups*. II. *Group extensions with a non-abelian kernel*, Ann. of Math. vol. 48 (1947) pp. 326–341.

6. A. M. Gleason, *The structure of locally compact groups*, Duke Math. J. vol. 18 (1951) pp. 85–104.

7. S. T. Hu, *Cohomology theory in topological groups*, Michigan Journal of Mathematics vol. 1 (1952) pp. 11–60.

8. K. Iwasawa, *On some types of topological groups*, Ann. of Math. vol. 50 (1949) pp. 507–558.

9. D. Montgomery, *Finite dimensional groups*, Ann. of Math. vol. 52 (1950) pp. 591–605.

10. L. Pontrjagin, *Topological groups*, Princeton, 1939.

11. J. von Neumann, *Die Einführung analytischen Parameter in topologischen Gruppen*, Ann. of Math. vol. 34 (1933) pp. 170–190.

University of Michigan,
Ann Arbor, Mich.
Institute for Advanced Study,
Princeton, N. J.