REPRESENTATIONS OF PRIME RINGS

BY

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This paper is a continuation of the study of prime rings started in [2]. We recall that a prime ring is a ring having its zero ideal as a prime ideal.

A right (left) ideal I of a prime ring R is called *prime* if $\mathfrak{a}\mathfrak{b}\subseteq I$ implies that $\mathfrak{a}\subseteq I$ ($\mathfrak{b}\subseteq I$), $\mathfrak{a}$ and $\mathfrak{b}$ right (left) ideals of R with $\mathfrak{b}\neq 0$ ($\mathfrak{a}\neq 0$). We denote by $\mathfrak{P}_r$ ($\mathfrak{P}_l$) the set of all prime right (left) ideals of R. For any subset A of R, A^r (A^l) denotes the right (left) annihilator of A; A^r (A^l) is a right (left) *annihilator ideal* of R. The set of all right (left) annihilator ideals of R is denoted by $\mathfrak{A}_r$ ($\mathfrak{A}_l$).

For the prime rings R studied in [2], it was assumed that there existed a mapping $I\rightarrow I^*$ of the set of all right (left) ideals of R onto a subset $\mathfrak{R}$ ($\mathfrak{L}$) of $\mathfrak{P}_r$ ($\mathfrak{P}_l$) having the following seven properties:

(P1) $\quad I^* \supseteq I.$ $\qquad$ (P2) $\quad I^{**} = I^*.$

(P3) $\quad$ If $I \supseteq I'$, then $I^* \supseteq I'^*.$ $\qquad$ (P4) $\quad 0^* = 0.$

(P5) $\quad$ If $I \cap I' = 0$, then $I^* \cap I'^* = 0.$

(P6) $\quad aI^* \subseteq (aI)^* \quad (I^*a \subseteq (Ia)^*), \quad a \in R.$

(P7) $\quad \mathfrak{R}$ ($\mathfrak{L}$) is atomic.

That the above properties arise naturally may be seen by letting $I^*=p(I)$, the least prime right (left) ideal of R containing I. Then properties (P1)–(P6) are known to hold [2]. Thus (P1)–(P7) hold for any ring having minimal prime right (left) ideals. In particular, these properties hold for a primitive ring with minimal right ideals.

A subset $\mathfrak{R}$ ($\mathfrak{L}$) of $\mathfrak{P}_r$ ($\mathfrak{P}_l$) satisfying (P1)–(P7) will be called a *right structure* (*left structure*) of R. A right (left) structure $\mathfrak{R}$ ($\mathfrak{L}$) of R may be made into a lattice in the usual way. Thus for any I, I' in $\mathfrak{R}$ ($\mathfrak{L}$), define $I\cap I'$ as the intersection of these ideals and $I\cup I'$ as $(I+I')^*$. It follows from [2] that $\mathfrak{R}$ ($\mathfrak{L}$) is a modular lattice under these operations. A consequence of [2, p. 803] is that $\mathfrak{A}_r\subseteq\mathfrak{R}$ ($\mathfrak{A}_l\subseteq\mathfrak{L}$). Since $(I+I')^l=(I\cup I')^l$ by (P6), it is evident that $(I\cup I')^l=I^l\cap I'^l$ for any I, I' in $\mathfrak{R}$, and similarly for $\mathfrak{L}$.

It is assumed in this paper that the prime ring R has both a right and a left structure. Some properties of structures, in addition to those given in [2], are developed in the first section. Next, atoms of these structures are used for dual representation spaces of R. It is shown that these structures in R have isomorphic structures in their dual representation spaces. Finally, the

Presented to the Society, September 5, 1952; received by the editors May 24, 1952.

given ring is shown to be an n-fold transitive ring of transformations on these spaces in a certain restricted sense.

1. **Right-left structure relations.** We assume that the prime ring R has both a right structure $\mathfrak{R}$ and a left structure $\mathfrak{L}$. Each of the results of this section has a dual obtained by interchanging the roles of $\mathfrak{R}$ and $\mathfrak{L}$.

1.1 LEMMA. *If I is an atom of $\mathfrak{R}$ and x is any nonzero element of I, then $(Rx)^*$ is an atom of $\mathfrak{L}$.*

To prove this, let L be any atom of $\mathfrak{L}$. The primeness of R implies that $L\cap xR\neq 0$. Select $xa\in L\cap xR$, $xa\neq 0$; since L is an atom, $(xa)^l$ is a maximal element of $\mathfrak{L}$ by [2, 4.11]. Now $I\cap(xa)^{lr}\neq 0$, and therefore $I\subseteq(xa)^{lr}$. Thus $(x)^l=(xa)^l$ and $(Rx)^*$ is an atom of $\mathfrak{L}$ by [2, 4.11].

The ring union of all atoms of $\mathfrak{R}$ is shown in [2, 4.2] to be an ideal of R. The above lemma shows that this ideal is also the ring union of all atoms of $\mathfrak{L}$.

1.2 THEOREM. *If I is an atom of $\mathfrak{R}$, then I^l is a maximal element of $\mathfrak{L}$, while if I is a maximal element of $\mathfrak{R}$ for which $I^l\neq 0$, then I^l is an atom of $\mathfrak{L}$.*

If I is an atom of $\mathfrak{R}$, then $I^l=(x)^l$ for any nonzero x in I, and hence I^l is maximal in $\mathfrak{L}$ by the proof of the above lemma.

On the other hand, if I is maximal in $\mathfrak{R}$ and $I^l\neq 0$, then $(x)^r=I$ for any nonzero x in I^l. Thus $(xR)^*$ is an atom of $\mathfrak{R}$ by [2, 4.11]. Since x is in $(xR)^*$ [2, 1.2], we have by 1.1 that $(Rx)^*$ is an atom of $\mathfrak{L}$ for every nonzero x in I^l. If I^l is not an atom of $\mathfrak{L}$, it must contain atoms L_1 and L_2 such that $L_1\cap L_2=0$ [2, 4.3]. Let x_1 be any nonzero element of L_1. Since $(x_1)^lL\neq 0$ due to the primeness of R, there must exist a nonzero element x_2 in L_2 such that $(x_1)^l\neq(x_2)^l$. Then $R(x_1+x_2)\cap Rx_i\neq 0$, $i=1, 2$, and therefore $(R(x_1+x_2))^*=(Rx_1)^*=(Rx_2)^*$. This contradicts the assumption that $L_1\cap L_2=0$, and proves 1.2.

It is a corollary of 1.2 that the atoms of $\mathfrak{R}$ ($\mathfrak{L}$) are contained in $\mathfrak{A}_r$ ($\mathfrak{A}_l$).

1.3 THEOREM. *If I is an atom of $\mathfrak{R}$ and I' is any element of $\mathfrak{A}_r$, then $I\cup I'$ also is in $\mathfrak{A}_r$.*

Since $(I\cup I')^{lr}=(I^l\cap I'^l)^r$, what we wish to prove is that $(I^l\cap I'^l)^r=I\cup I'$. Clearly $I\cup I'\subseteq(I^l\cap I'^l)^r$, so that we need only prove that $(I^l\cap I'^l)^r\subseteq I\cup I'$. In view of [2, 4.3], this can be accomplished by showing that every atom I_1 of $\mathfrak{R}$ contained in $(I^l\cap I'^l)^r$ is also contained in $I\cup I'$.

So let us assume that $I_1\subseteq(I^l\cap I'^l)^r$, I_1 an atom of $\mathfrak{R}$. If either $I_1=I$ or $I_1\subseteq I'$, nothing remains to be proved; henceforth we shall assume that $I_1\neq I$ and $I_1\not\subseteq I'$. Then necessarily $I\not\subseteq I'$ and $I'^l\not\subseteq I^l$. Hence there exists an atom L of $\mathfrak{L}$ [2, 4.3] such that $L\subseteq I'^l$, $L\cap I^l=0$. Since I^l is maximal in $\mathfrak{L}$ by 1.2, evidently $L\cup I^l=R$. From the modularity of $\mathfrak{L}$ we see that $L\cup(I^l\cap I'^l)=I'^l$, and therefore that $L^r\cap(I^l\cap I'^l)^r=I'$. Since L^r is a maximal element of $\mathfrak{R}$ and $I\not\subseteq L^r$, clearly $I\cup L^r=R$. Hence it follows from (P5) that $I_1\cap(I+L^r)\neq 0$, and therefore that $(I_1+I)\cap L^r\neq 0$. Since also $I_1+I\subseteq(I^l\cap I'^l)^r$, it follows that $(I_1+I)\cap I'\neq 0$. Thus $I_1\cap(I+I')\neq 0$ and $I_1\subseteq I\cup I'$. This proves the theorem.

1.4 COROLLARY. *If $I_1, \cdots, I_n$ are atoms of $\mathfrak{R}$, then $I_1\cup\cdots\cup I_n$ is in $\mathfrak{A}_r$.*

The corollary follows by mathematical induction.

2. R-**modules.** If M is a right (left) R-module and A is a subset of M, we shall again use the notation A^r (A^l) to denote the annihilator of A in R. A right (left) R-module M is called *prime* if $A^r=0$ ($A^l=0$) for every nonzero submodule A of M. A submodule M' of M is called a *prime submodule* of M if $M-M'$ is a prime module. If the ring R has a right (left) structure $\mathfrak{R}$ ($\mathfrak{L}$), then a right (left) R-module M is called *admissible* relative to $\mathfrak{R}$ ($\mathfrak{L}$) if M is prime and $(x)^r\in\mathfrak{R}$ ($(x)^l\in\mathfrak{L}$) for every $x\in M$. For any I in $\mathfrak{R}$ ($\mathfrak{L}$), both I and $R-I$ are examples of admissible right (left) R-modules.

It is shown in [2, p. 804] that an admissible right R-module M has a structure much the same as R does. For any submodule N of M, define

$$N^* = \{x;\, x\in M,\ [(N:x)]^* = R\}.$$

Here $(N:x)$ denotes the annihilator in R of the element $x+N$ in $M-N$. Then the set $\mathfrak{M}$ of all submodules N^* of M is a structure of M in that it possesses the properties analogous to (P1)–(P7). Naturally, similar remarks hold for admissible left R-modules.

Let us assume now that R is a ring with a right structure $\mathfrak{R}$ and a left structure $\mathfrak{L}$, and that N is a fixed atom of $\mathfrak{R}$. Select an atom M of $\mathfrak{L}$ so that

$$M\cdot N\neq 0.$$

Such an M must exist, since the ring union S of all atoms of $\mathfrak{L}$ is an ideal of R [2, 4.2], and $S\cdot N\neq 0$ due to the primeness of R. Let

$$K = M\cap N,$$

a nonzero subring of R. If we consider the rings K, M, and N as modules, it is evident that K is an (N, M)-module, that M is an (R, K)-module, and that N is an (K, R)-module. Clearly $N\cdot M\subseteq K$.

2.1 LEMMA. *For x in M and y in N, $xy=0$ if and only if $x=0$ or $y=0$.*

If $x\neq 0$, then $(x)^r$ is an element of $\mathfrak{R}$ and therefore either $(x)^r\cap N=0$, in which case the desired conclusion follows immediately, or $N\subseteq(x)^r$. In this latter case evidently $N^l\cap M\neq 0$ and $M\subseteq N^l$, which is contrary to the choice of M. This proves 2.1.

An obvious corollary of this lemma is that K is an integral domain.

2.2 LEMMA. *The integral domain K possesses a quotient division ring D.*

If x and y are nonzero elements of K, then $(xN)^*=(yN)^*=N$ in view of 2.1 and the fact that N is an atom. Thus $xN\cap yN\neq 0$ by (P5), and hence $xN\cap yN\cap M\neq 0$. However, $xN\cap yN\cap M\subseteq K$ so that evidently $xK\cap yK\neq 0$. This proves that K has a right quotient division ring D. That D also is the left quotient of K follows by duality.

2.3 LEMMA. *If x and y are nonzero elements of N, then $Kx\cap Ky\neq 0$ if and only if $(x)^r=(y)^r$.*

If $Kx\cap Ky\neq 0$, then obviously $(x)^r=(y)^r$. Conversely, if $(x)^r=(y)^r$, then x and y are in $(x)^{rl}$, an atom of $\mathfrak{L}$. Now $Mx\neq 0$ and $My\neq 0$, and since both Mx and My are contained in the atom $(x)^{rl}$, necessarily $Mx\cap My\neq 0$ by (P5). Since K is a right M-module, evidently $Kx\cap Ky\neq 0$ as desired.

We shall consider K as having the trivial left and right structures, namely the structures consisting of the set $(0, K)$. In view of 2.2, which guarantees that (P5) holds, it is evident that these structures satisfy (P1)–(P7).

Now N, as an admissible (K, R)-module, has a left structure induced by K and a right structure induced by R. It is clear that for any K-submodule A of N, the closure A^* of A is defined as follows:

$$A^* = \{x;\ x\in N,\ x=0 \text{ or } Kx\cap A\neq 0\}.$$

If $\mathfrak{N}$ denotes the set of all closed K-submodules of N, then $\mathfrak{N}$ is a left structure of N. Since N is an atom of $\mathfrak{N}$, the right structure of N induced by R is the trivial one.

In an analogous way, of course, M has left and right structures induced by R and K respectively. The left structure is trivial; the right structure of M induced by K will be denoted by $\mathfrak{M}$.

The following results, although frequently just stated for $\mathfrak{N}$, have the obvious duals relative to $\mathfrak{M}$.

Any A of $\mathfrak{N}$ is actually a left N-module. For if $x\in N$ and $a\in A$ with $xa\neq 0$, then $(xa)^r=(a)^r$ since both of these right ideals are maximal elements of $\mathfrak{R}$ by [2, 4.11], and $K(xa)\cap Ka\neq 0$ by 2.3. Thus $K(xa)\cap A\neq 0$ and $xa\in A$ since $A^*=A$.

If L is in $\mathfrak{L}$ and kx is a nonzero element of $L\cap N$, $k\in K$ and $x\in N$, then $(Rx)^*\cap L\neq 0$ and, since $(Rx)^*$ is an atom of $\mathfrak{L}$ by 1.1, evidently $(Rx)^*\subseteq L$. Thus x is in $L\cap N$ and we have proved that $L\cap N$ is in $\mathfrak{N}$. Furthermore, if L is an atom of $\mathfrak{L}$, then $L\cap N$ is an atom of $\mathfrak{N}$. This is so since for any nonzero elements x and y of $L\cap N$, $(x)^r=(y)^r=L^r$, and hence $Kx\cap Ky\neq 0$ by 2.3.

On the other hand, if A is an atom of $\mathfrak{N}$, then $Kx\cap Ky\neq 0$ for any nonzero $x, y\in A$. Hence $A^{rl}=(x)^{rl}=L$, an atom of $\mathfrak{L}$, and $A=L\cap N$.

The above remarks constitute part of the proof of the following theorem.

2.4 THEOREM. *The K-submodule A of N is in $\mathfrak{N}$ if and only if $A=L\cap N$ for some L in $\mathfrak{L}$.*

To complete the proof of this theorem, let A be any nonzero element of $\mathfrak{N}$ and let $L=(RA)^*\in\mathfrak{L}$. We shall prove that $A=L\cap N$. If $L_1\subseteq L$, L_1 an atom of

$\mathfrak{L}$, then $L_1\cap RA\neq 0$ so that $N\cdot(L_1\cap RA)\neq 0$ and $L_1\cap NRA\neq 0$. Since $NRA \subseteq NA\subseteq A$, we have proved that $L_1\cap A\neq 0$. Now $L_1\cap N$ is an atom of $\mathfrak{N}$ and therefore $L_1\cap N\subseteq A$. It follows that $L\cap N\subseteq A$, and the proof of the theorem is completed.

2.5 THEOREM. *The lattices* $\{\mathfrak{L};\subseteq,\cup,\cap\}$ *and* $\{\mathfrak{N};\subseteq,\cup,\cap\}$ *are isomorphic under the correspondence* $L\rightarrow L\cap N$. *Dually, the lattices* $\{\mathfrak{N};\subseteq,\cup,\cap\}$ *and* $\{\mathfrak{M};\subseteq,\cup,\cap\}$ *are isomorphic under the correspondence* $I\rightarrow I\cap M$.

It is sufficient to prove that the mapping $L\rightarrow L\cap N$ of $\mathfrak{L}$ onto $\mathfrak{N}$ is a 1-1 order-preserving mapping in order to prove that these lattices are isomorphic. Clearly the mapping is order-preserving. In order to show that it is a 1-1 mapping, we need only note that if $L_1\not\subseteq L_2$, $L_i\in\mathfrak{L}$, then there exists an atom L of $\mathfrak{L}$ such that $L\subseteq L_1$, $L\cap L_2=0$ by [2, 4.3]. Hence $L\cap N\not\subseteq L_2\cap N$ and therefore $L_1\cap N\not\subseteq L_2\cap N$. This proves 2.5.

In case R is a primitive ring with nonzero socle S, and N and M are simple conjugate right and left R-modules respectively with common centralizer D, this theorem yields the well known isomorphism existing between the lattice of left (right) ideals of S and the lattice of D-submodules of N (M). This application to primitive rings is obtained by letting $\mathfrak{N}$ ($\mathfrak{L}$) be the set of all prime right (left) ideals of R.

If $A\in\mathfrak{N}$, say $A=L\cap N$ for $L\in\mathfrak{L}$, then obviously $A^r\supseteq L^r$. Since, however, $A^{rl}\in\mathfrak{L}$ and $A^{rl}\supseteq A$, evidently $L\subseteq A^{rl}$ and $A^r\subseteq L^r$. Thus $A^r=L^r$. If, in particular, $L\in\mathfrak{A}_l$, then $A=A^{rl}\cap N$. Let us denote by $\mathfrak{N}_l$ ($\mathfrak{M}_r$) the set of all K-submodules of N (M) that are annihilators of right (left) ideals of R. Then $\mathfrak{N}_l=\{A;A\in\mathfrak{N}, A=L\cap N$ for some $L\in\mathfrak{A}_l\}$, and similarly for $\mathfrak{M}_r$.

In view of the isomorphism existing between $\mathfrak{L}$ and $\mathfrak{N}$, Theorem 1.3 has the following counterpart in $\mathfrak{N}$.

2.6 THEOREM. *If A is an atom of $\mathfrak{N}$ and B is any element of $\mathfrak{N}_l$, then $A\cup B$ also is in $\mathfrak{N}_l$.*

A corollary of this theorem is as follows (1.4):

2.7 COROLLARY. *If $A_1,\cdots,A_n$ are atoms of $\mathfrak{N}$, then $A_1\cup\cdots\cup A_n$ is in $\mathfrak{N}_l$.*

For a primitive ring R, analogues of Theorem 2.6 and its corollary can be found in a recent paper by Artin [1, pp. 68, 69]. His results are more general than ours in that his ring R is not assumed to have minimal right ideals. Of course, they are also less general in that they are restricted to apply to primitive rather than prime rings.

3. **Transitivity of R over N.** As usual, the elements $x_1,\cdots,x_n$ of N are called K-linearly independent if and only if $k_1x_1+\cdots+k_nx_n=0$ implies all $k_i=0$, $k_i\in K$. An alternate lattice-theoretic definition is given by the following lemma.

3.1 LEMMA. *The elements $x_1, \cdots, x_n$ of N are K-linearly independent if and only if*

$$(x_j)^r \not\supseteq \bigcap_{i=1, i\neq j}^{n} (x_i)^r, \qquad j = 1, \cdots, n.$$

To prove this lemma, note first that all $A_i = (x_i)^{rl} \cap N = (Kx_i)^*$ are atoms of $\mathfrak{N}$ (assuming, of course, that $x_i \neq 0$). If $k_1x_1 + \cdots + k_nx_n = 0$ with $k_j \neq 0$, then

$$A_j \subseteq \bigcup_{i=1, i\neq j}^{n} A_i,$$

and, since $A_i^r = (x_i)^r$,

$$(x_j)^r \supseteq \bigcap_{i=1, i\neq j}^{n} (x_i)^r.$$

Conversely, if the above inclusion relation holds for some j, then

$$A_j \subseteq \bigcup_{i=1, i\neq j}^{n} A_i,$$

$$Kx_j \cap \sum_{i=1, i\neq j}^{n} Kx_i \neq 0.$$

Thus the elements $x_1, \cdots, x_n$ are K-linearly dependent, and 3.1 follows.

3.2 LEMMA. *Let I be any right ideal of R and K' be any left N-submodule of K. Then for any x and y in N such that $xI \neq 0$ and $K'y \neq 0$, also $xI \cap K'y \neq 0$.*

To prove this lemma, let k be any nonzero element of K'. Then, by the primeness of N, $xaky \neq 0$ for some a in I. Now $x(aky) = (xak)y$ where $aky \in I$ and $xak \in K'$, and therefore the lemma is proved.

We now are in a position to prove the main result of our paper, namely that R acts almost as an n-fold transitive ring of K-linear transformations on N for any integer n not exceeding the K-dimension of N. To be more precise, we shall prove the following theorem.

3.3 TRANSITIVITY THEOREM. *If $x_1, \cdots, x_n$ is any set of n K-linearly independent elements of N and if $y_1, \cdots, y_n$ is any set of n elements of N, then there exist an element a of R and a nonzero element k of K such that*

$$x_ia = ky_i, \qquad i = 1, \cdots, n.$$

To aid in the proof of this theorem, let

$$I_j = \bigcap_{i=1, i\neq j}^{n} (x_i)^r.$$

In view of 3.1, evidently $I_j \not\subseteq (x_j)^r$ for any j. Hence, by 3.2, there exist elements $a_j \in I_j$ and $k_j \in K$, $k_j \neq 0$, such that $x_j a_j = k_j y_j \neq 0$ for all j such that $y_j \neq 0$. If $y_j = 0$, select $a_j = 0$. Now for all $y_j \neq 0$, $k_j y_j K$ is a right ideal of K, and $\bigcap_j k_j y_j K \neq 0$ by 2.2. Select $k \in \bigcap_j k_j y_j K$, $k \neq 0$; $k = k_j y_j c_j$ for each j such that $y_j \neq 0$. Then $x_j(a_j c_j y_j) = k y_j$, and if we let $a = a_1 c_1 y_1 + \cdots + a_n c_n y_n$, evidently $x_j a = k y_j$ as desired.

We give now an example of a prime ring of the type considered in this paper. Denote by I the ring of integers and by I_2 the ring of all 2×2 matrices over I. We use the notation E_{ij} for the matrix with 1 in its (i, j) position and zeros elsewhere. Now denote by R the set of all matrices of I_2 having all even or all odd integers for components. It is easily established that R is a prime ring.

The right ideal $N = 2IE_{11} + 2IE_{12}$ is a minimal prime right ideal and the left ideal $M = 2IE_{11} + 2IE_{21}$ is a minimal prime left ideal. Clearly $K = M \cap N = 2IE_{11}$ is an integral domain. The sets of prime right and left ideals of R form right and left structures of R.

As an illustration of the transitivity theorem, let $x_1 = 2E_{11}$, $x_2 = 4E_{12}$, $y_1 = 0$, and $y_2 = 2E_{11}$. Then for $a = 2E_{11}$ and $k = 4E_{11}$ we have $x_1 a = k y_1$ and $x_2 a = k y_2$. We note that there is no a in R such that $x_1 a = y_1$ and $x_2 a = y_2$.

In the case of a primitive ring R, the minimal right ideals are all isomorphic as right R-modules. That such is not the case in general for a prime ring follows from this example. To show this, let $N' = [I(E_{11} + E_{21}) + I(E_{12} + E_{22})] \cap R$. It is not too difficult to show that N' is a minimal prime right ideal of R. If N and N' were isomorphic, then we would have $2aE_{11} \to c(E_{11} + E_{21})$, $2bE_{12} \to d(E_{12} + E_{22})$ for some integers a, b, c, d in order for the annihilators of corresponding elements of N and N' to be the same. But then c and d would have to be even integers, and nothing in N would correspond to the matrices in N' having odd integers for elements.

Bibliography

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