REPRESENTATIONS OF PRIME RINGS

BY

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This paper is a continuation of the study of prime rings started in [2]. We recall that a prime ring is a ring having its zero ideal as a prime ideal.

A right (left) ideal $I$ of a prime ring $R$ is called prime if $ab \subseteq I$ implies that $a \subseteq I$ ($b \subseteq I$), $a$ and $b$ right (left) ideals of $R$ with $b \neq 0$ ($a \neq 0$). We denote by $\mathfrak{P}$ ($\mathfrak{P}_1$) the set of all prime right (left) ideals of $R$. For any subset $A$ of $R$, $A^r$ ($A^l$) denotes the right (left) annihilator of $A$; $A^*_r$ ($A^*_l$) is a right (left) annihilator ideal of $R$. The set of all right (left) annihilator ideals of $R$ is denoted by $\mathfrak{A}_r$ ($\mathfrak{A}_l$).

For the prime rings $R$ studied in [2], it was assumed that there existed a mapping $I \to I^*$ of the set of all right (left) ideals of $R$ onto a subset $\mathfrak{S}$ ($\mathfrak{S}_l$) of $\mathfrak{A}_r$ ($\mathfrak{A}_l$) having the following seven properties:

(P1) $I^* \supseteq I$.
(P2) $I^{**} = I^*$.
(P3) If $I \supseteq I'$, then $I^* \supseteq I'^*$.
(P4) $0^* = 0$.
(P5) If $I \cap I' = 0$, then $I^* \cap I'^* = 0$.
(P6) $aI^* \subseteq (aI)^*$ ($I^*a \subseteq (Ia)^*$), $a \in R$.
(P7) $\mathfrak{S}$ ($\mathfrak{S}_l$) is atomic.

That the above properties arise naturally may be seen by letting $I^* = p(I)$, the least prime right (left) ideal of $R$ containing $I$. Then properties (P1)–(P6) are known to hold [2]. Thus (P1)–(P7) hold for any ring having minimal prime right (left) ideals. In particular, these properties hold for a primitive ring with minimal right ideals.

A subset $\mathfrak{S}$ ($\mathfrak{S}_l$) of $\mathfrak{A}_r$ ($\mathfrak{A}_l$) satisfying (P1)–(P7) will be called a right structure (left structure) of $R$. A right (left) structure $\mathfrak{S}$ ($\mathfrak{S}_l$) of $R$ may be made into a lattice in the usual way. Thus for any $I$, $I'$ in $\mathfrak{S}$ ($\mathfrak{S}_l$), define $I \cap I'$ as the intersection of these ideals and $I \cup I'$ as $(I + I')^*$. It follows from [2] that $\mathfrak{S}$ ($\mathfrak{S}_l$) is a modular lattice under these operations. A consequence of [2, p. 803] is that $\mathfrak{S}_r \subseteq \mathfrak{S}_l$ ($\mathfrak{S}_l \subseteq \mathfrak{S}_r$). Since $(I + I')^l = (I \cap I')^l$ by (P6), it is evident that $(I \cup I')^l = I^r \cap I'^r$ for any $I$, $I'$ in $\mathfrak{S}$, and similarly for $\mathfrak{S}$.

It is assumed in this paper that the prime ring $R$ has both a right and a left structure. Some properties of structures, in addition to those given in [2], are developed in the first section. Next, atoms of these structures are used for dual representation spaces of $R$. It is shown that these structures in $R$ have isomorphic structures in their dual representation spaces. Finally, the

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given ring is shown to be an $n$-fold transitive ring of transformations on these spaces in a certain restricted sense.

1. **Right-left structure relations.** We assume that the prime ring $R$ has both a right structure $\mathfrak{R}$ and a left structure $\mathfrak{L}$. Each of the results of this section has a dual obtained by interchanging the roles of $\mathfrak{R}$ and $\mathfrak{L}$.

1.1 Lemma. If $I$ is an atom of $\mathfrak{R}$ and $x$ is any nonzero element of $I$, then $(Rx)^*\mathfrak{L}$ is an atom of $\mathfrak{L}$.

To prove this, let $L$ be any atom of $\mathfrak{L}$. The primeness of $R$ implies that $L\cap x\mathfrak{R}\neq 0$. Select $xa\subseteq L\cap x\mathfrak{R}$, $xa\neq 0$; since $L$ is an atom, $(xa)^t\mathfrak{L}$ is a maximal element of $\mathfrak{L}$ by [2, 4.1]. Now $I\cap (xa)^t\mathfrak{L}\neq 0$, and therefore $I\subseteq (xa)^t\mathfrak{L}$. Thus $(x)^t=(xa)^t\mathfrak{L}$ and $(Rx)^*\mathfrak{L}$ is an atom of $\mathfrak{L}$ by [2, 4.1].

The ring union of all atoms of $\mathfrak{R}$ is shown in [2, 4.2] to be an ideal of $R$. The above lemma shows that this ideal is also the ring union of all atoms of $\mathfrak{L}$.

1.2 Theorem. If $I$ is an atom of $\mathfrak{R}$, then $I^t\mathfrak{L}$ is a maximal element of $\mathfrak{L}$, while if $I$ is a maximal element of $\mathfrak{L}$ for which $L^t\mathfrak{L}\neq 0$, then $I^t\mathfrak{L}$ is an atom of $\mathfrak{L}$.

If $I$ is an atom of $\mathfrak{R}$, then $I^t=(x)^t\mathfrak{L}$ for any nonzero $x$ in $I$, and hence $I^t\mathfrak{L}$ is maximal in $\mathfrak{L}$ by the proof of the above lemma.

On the other hand, if $I$ is maximal in $\mathfrak{R}$ and $I^t\mathfrak{L}\neq 0$, then $(x)^t\mathfrak{L}=I$ for any nonzero $x$ in $I^t\mathfrak{L}$. Thus $(xR)^*\mathfrak{L}$ is an atom of $\mathfrak{L}$ by [2, 4.1]. Since $x$ is in $(xR)^*\mathfrak{L}$, we have by 1.1 that $(Rx)^*\mathfrak{L}$ is an atom of $\mathfrak{L}$ for every nonzero $x$ in $I^t\mathfrak{L}$. If $I^t\mathfrak{L}$ is not an atom of $\mathfrak{L}$, it must contain atoms $L_1$ and $L_2$ such that $L_1\cap L_2 = 0$ [2, 4.3]. Let $x_1$ be any nonzero element of $L_1$. Since $(x_1)^t\mathfrak{L}\neq 0$ due to the primeness of $R$, there must exist a nonzero element $x_2$ in $L_2$ such that $(x_1)^t\mathfrak{L}\neq (x_2)^t\mathfrak{L}$. Then $R(x_1+x_2)\cap Rx\neq 0$, $i = 1, 2$, and therefore $(R(x_1+x_2))^*\mathfrak{L}=(Rx_2)^*\mathfrak{L}$. This contradicts the assumption that $L_1\cap L_2 = 0$, and proves 1.2.

It is a corollary of 1.2 that the atoms of $\mathfrak{R}$ ($\mathfrak{L}$) are contained in $\mathfrak{R}$ ($\mathfrak{L}$).

1.3 Theorem. If $I$ is an atom of $\mathfrak{R}$ and $I^t\mathfrak{L}$ is any element of $\mathfrak{R}$, then $I\cup I^t\mathfrak{L}$ is also in $\mathfrak{R}$.

Since $(I\cup I^t)^*\mathfrak{L}=(I\cap I^t)^*\mathfrak{L}$, what we wish to prove is that $(I\cap I^t)^*\mathfrak{L}=I\cup I^t\mathfrak{L}$. Clearly $I\cup I^t\mathfrak{L}\subseteq (I\cap I^t)^*\mathfrak{L}$, so that we need only prove that $(I\cap I^t)^*\mathfrak{L}\subseteq I\cup I^t\mathfrak{L}$. In view of [2, 4.3], this can be accomplished by showing that every atom $I_1$ of $\mathfrak{R}$ contained in $(I\cap I^t)^*\mathfrak{L}$ is also contained in $I\cup I^t\mathfrak{L}$.

So let us assume that $I_1\subseteq (I\cap I^t)^*\mathfrak{L}$, $I_1$ an atom of $\mathfrak{R}$. If either $I_1=I$ or $I_1\subseteq I^t\mathfrak{L}$, nothing remains to be proved; henceforth we shall assume that $I_1\neq I$ and $I_1\subseteq I^t\mathfrak{L}$. Then necessarily $I\subseteq I^t\mathfrak{L}$ and $I^t\subseteq I\mathfrak{L}$. Hence there exists an atom $L$ of $\mathfrak{L}$ [2, 4.3] such that $L\subseteq I^t\mathfrak{L}$, $L\cap I\neq 0$. Since $I^t\mathfrak{L}$ is maximal in $\mathfrak{L}$ by 1.2, evidently $L\cup I^t\mathfrak{L}=R$. From the modularity of $\mathfrak{L}$ we see that $L\cup (I\cap I^t)^*\mathfrak{L}=I^t\mathfrak{L}$, and therefore that $L^t\cap (I^t\cap I^t)^*\mathfrak{L}=I^t\mathfrak{L}$. Since $L^t\mathfrak{L}$ is a maximal element of $\mathfrak{R}$ and $I_1\subseteq L^t\mathfrak{L}$, clearly $L\cup L^t\mathfrak{L}=R$. Hence it follows from (PS) that $I_1\cap (I+L^t\mathfrak{L})\neq 0$, and therefore that $(I_1+I)\cap L^t\mathfrak{L}\neq 0$. Since also $I_1+I\subseteq (I^t\cap I^t)^*\mathfrak{L}$, it follows
that \((I_1+I) \cap I' \neq 0\). Thus \(I_1 \cap (I+I') \neq 0\) and \(I_1 \subseteq I \cup I'\). This proves the theorem.

1.4 Corollary. If \(I_1, \ldots, I_n\) are atoms of \(R\), then \(I_1 \cup \cdots \cup I_n\) is in \(\mathcal{A}\).

The corollary follows by mathematical induction.

2. \(R\)-modules. If \(M\) is a right (left) \(R\)-module and \(A\) is a subset of \(M\), we shall again use the notation \(A^r (A^l)\) to denote the annihilator of \(A\) in \(R\). A right (left) \(R\)-module \(M\) is called prime if \(A^r = 0 (A^l = 0)\) for every nonzero submodule \(A\) of \(M\). A submodule \(M'\) of \(M\) is called a prime submodule of \(M\) if \(M - M'\) is a prime module. If the ring \(R\) has a right (left) structure \(\mathcal{R} (\mathcal{L})\), then a right (left) \(R\)-module \(M\) is called admissible relative to \(\mathcal{R} (\mathcal{L})\) if \(M\) is prime and \((x)^r \subseteq \mathcal{R} ((x)^l \subseteq \mathcal{L})\) for every \(x \in M\). For any \(I\) in \(\mathcal{R} (\mathcal{L})\), both \(I\) and \(R - I\) are examples of admissible right (left) \(R\)-modules.

It is shown in [2, p. 804] that an admissible right \(R\)-module \(M\) has a structure much the same as \(R\) does. For any submodule \(N\) of \(M\), define

\[ N^* = \{ x; x \in M, [(N:x)]^* = R \} \]

Here \((N:x)\) denotes the annihilator in \(R\) of the element \(x + N\) in \(M - N\). Then the set \(\mathfrak{M}\) of all submodules \(N^*\) of \(M\) is a structure of \(M\) in that it possesses the properties analogous to (P1)-(P7). Naturally, similar remarks hold for admissible left \(R\)-modules.

Let us assume now that \(R\) is a ring with a right structure \(\mathcal{R}\) and a left structure \(\mathcal{L}\), and that \(N\) is a fixed atom of \(\mathcal{R}\). Select an atom \(M\) of \(\mathcal{L}\) so that

\[ M \cdot N \neq 0. \]

Such an \(M\) must exist, since the ring union \(S\) of all atoms of \(\mathcal{L}\) is an ideal of \(R\) [2, 4.2], and \(S \cdot N \neq 0\) due to the primeness of \(R\). Let

\[ K = M \cap N, \]

a nonzero subring of \(R\). If we consider the rings \(K, M,\) and \(N\) as modules, it is evident that \(K\) is an \((N, M)\)-module, that \(M\) is an \((R, K)\)-module, and that \(N\) is an \((K, R)\)-module. Clearly \(N \cdot M \subseteq K\).

2.1 Lemma. For \(x\) in \(M\) and \(y\) in \(N\), \(xy = 0\) if and only if \(x = 0\) or \(y = 0\).

If \(x \neq 0\), then \((x)^r\) is an element of \(\mathcal{R}\) and therefore either \((x)^r \cap N = 0\), in which case the desired conclusion follows immediately, or \(N \subseteq (x)^r\). In this latter case evidently \(N' \cap M \neq 0\) and \(M \subseteq N'\), which is contrary to the choice of \(M\). This proves 2.1.

An obvious corollary of this lemma is that \(K\) is an integral domain.

2.2 Lemma. The integral domain \(K\) possesses a quotient division ring \(D\).

If \(x\) and \(y\) are nonzero elements of \(K\), then \((xN)^* = (yN)^* = N\) in view of 2.1 and the fact that \(N\) is an atom. Thus \(xN \cap yN \neq 0\) by (P5), and
hence $xN \cap yN \cap M \neq 0$. However, $xN \cap yN \cap M \subseteq K$ so that evidently $xK \cap yK \neq 0$. This proves that $K$ has a right quotient division ring $D$. That $D$ also is the left quotient of $K$ follows by duality.

2.3 Lemma. If $x$ and $y$ are nonzero elements of $N$, then $Kx \cap Ky \neq 0$ if and only if $(x)^r = (y)^r$.

If $Kx \cap Ky \neq 0$, then obviously $(x)^r = (y)^r$. Conversely, if $(x)^r = (y)^r$, then $x$ and $y$ are in $(x)^r$, an atom of $\mathfrak{R}$. Now $Mx \neq 0$ and $My \neq 0$, and since both $Mx$ and $My$ are contained in the atom $(x)^r$, necessarily $Mx \cap My \neq 0$ by (P5). Since $K$ is a right $M$-module, evidently $Kx \cap Ky \neq 0$ as desired.

We shall consider $K$ as having the trivial left and right structures, namely the structures consisting of the set $(0, K)$. In view of 2.2, which guarantees that (P5) holds, it is evident that these structures satisfy (P1)–(P7).

Now $N$, as an admissible $(K, R)$-module, has a left structure induced by $K$ and a right structure induced by $R$. It is clear that for any $K$-submodule $A$ of $N$, the closure $A^*$ of $A$ is defined as follows:

$$A^* = \{ x; x \in N, x = 0 \text{ or } Kx \cap A \neq 0 \}.$$

If $\mathfrak{M}$ denotes the set of all closed $K$-submodules of $N$, then $\mathfrak{M}$ is a left structure of $N$. Since $N$ is an atom of $\mathfrak{M}$, the right structure of $N$ induced by $R$ is the trivial one.

In an analogous way, of course, $M$ has left and right structures induced by $R$ and $K$ respectively. The left structure is trivial; the right structure of $M$ induced by $K$ will be denoted by $\mathfrak{M}^R$.

The following results, although frequently just stated for $\mathfrak{M}$, have the obvious duals relative to $\mathfrak{M}^R$.

Any $A$ of $\mathfrak{M}$ is actually a left $N$-module. For if $x \in N$ and $a \in A$ with $xa \neq 0$, then $(xa)^r = (a)^r$ since both of these right ideals are maximal elements of $\mathfrak{M}$ by [2, 4.11], and $K(xa) \cap Ka \neq 0$ by 2.3. Thus $K(xa) \cap A \neq 0$ and $xa \in A$ since $A^* = A$.

If $L$ is in $\mathfrak{L}$ and $kx$ is a nonzero element of $L \cap N$, $k \in K$ and $x \in N$, then $(Rx)^* \cap L \neq 0$ and, since $(Rx)^*$ is an atom of $\mathfrak{L}$ by 1.1, evidently $(Rx)^* \subseteq L$. Thus $x$ is in $L \cap N$ and we have proved that $L \cap N$ is in $\mathfrak{L}$. Furthermore, if $L$ is an atom of $\mathfrak{L}$, then $L \cap N$ is an atom of $\mathfrak{L}$. This is so since for any nonzero elements $x$ and $y$ of $L \cap N$, $(x)^r = (y)^r = L'$, and hence $Kx \cap Ky \neq 0$ by 2.3.

On the other hand, if $A$ is an atom of $\mathfrak{L}$, then $Kx \cap Ky \neq 0$ for any nonzero $x, y \in A$. Hence $A'^r = (x)^r = L$, an atom of $\mathfrak{L}$, and $A = L \cap N$.

The above remarks constitute part of the proof of the following theorem.

2.4 Theorem. The $K$-submodule $A$ of $N$ is in $\mathfrak{M}$ if and only if $A = L \cap N$ for some $L$ in $\mathfrak{L}$.

To complete the proof of this theorem, let $A$ be any nonzero element of $\mathfrak{M}$ and let $L = (RA)^* \subseteq \mathfrak{L}$. We shall prove that $A = L \cap N$. If $L_1 \subseteq L$, $L_1$ an atom of
If \( L \subseteq RA \neq 0 \) so that \( N \cdot (L \subseteq RA) \neq 0 \) and \( L \subseteq NRA \neq 0 \). Since \( NRA \subseteq NA \subseteq A \), we have proved that \( L \subseteq A \neq 0 \). Now \( L \subseteq N \) is an atom of \( N \) and therefore \( L \subseteq N \subseteq A \). It follows that \( L \subseteq N \subseteq A \), and the proof of the theorem is completed.

2.5 Theorem. The lattices \( \{ \mathcal{L}; \subseteq, \cup, \cap \} \) and \( \{ \mathcal{N}; \subseteq, \cup, \cap \} \) are isomorphic under the correspondence \( L \mapsto L \cap N \). Dually, the lattices \( \{ \mathcal{R}; \subseteq, \cup, \cap \} \) and \( \{ \mathcal{M}; \subseteq, \cup, \cap \} \) are isomorphic under the correspondence \( I \mapsto I \cap M \).

It is sufficient to prove that the mapping \( L \mapsto L \cap N \) of \( \mathcal{L} \) onto \( \mathcal{N} \) is a 1-1 order-preserving mapping in order to prove that these lattices are isomorphic. Clearly the mapping is order-preserving. In order to show that it is a 1-1 mapping, we need only note that if \( L_1 \not\subseteq L_2 \), \( L_1 \not\subseteq \mathcal{L} \), then there exists an atom \( L \) of \( \mathcal{L} \) such that \( L \not\subseteq L_1 \), \( L \cap L_2 = 0 \) by [2, 4.3]. Hence \( L \cap N \not\subseteq L_2 \cap N \) and therefore \( L \not\subseteq L_1 \cap N \). This proves 2.5.

In case \( R \) is a primitive ring with nonzero socle \( S \), and \( N \) and \( M \) are simple conjugate right and left \( R \)-modules respectively with common centralizer \( D \), this theorem yields the well known isomorphism existing between the lattice of left (right) ideals of \( S \) and the lattice of \( D \)-submodules of \( N \) (\( M \)). This application to primitive rings is obtained by letting \( \mathcal{N} \) (\( \mathcal{L} \)) be the set of all prime right (left) ideals of \( R \).

If \( A \in \mathcal{N} \), say \( A = L \cap N \) for \( L \in \mathcal{L} \), then obviously \( A^r \supseteq L^r \). Since, however, \( A^r \subseteq \mathcal{L} \) and \( A^r \supseteq A \), evidently \( L \subseteq A^r \) and \( A^r \subseteq L^r \). Thus \( A^r = L^r \). If, in particular, \( L \in \mathcal{L} \), then \( A = A^r \cap N \). Let us denote by \( \mathcal{N}_1 \) (\( \mathcal{M}_r \)) the set of all \( K \)-submodules of \( N \) (\( M \)) that are annihilators of right (left) ideals of \( R \). Then \( \mathcal{N}_1 = \{ A; A \in \mathcal{N} \), \( A = L \cap N \) for some \( L \in \mathcal{L} \} \), and similarly for \( \mathcal{M}_r \).

In view of the isomorphism existing between \( \mathcal{L} \) and \( \mathcal{N} \), Theorem 1.3 has the following counterpart in \( \mathcal{N} \).

2.6 Theorem. If \( A \) is an atom of \( \mathcal{N} \) and \( B \) is any element of \( \mathcal{N}_1 \), then \( A \cup B \) also is in \( \mathcal{N}_1 \).

A corollary of this theorem is as follows (1.4):

2.7 Corollary. If \( A_1, \cdots, A_n \) are atoms of \( \mathcal{N} \), then \( A_1 \cup \cdots \cup A_n \) is in \( \mathcal{N}_1 \).

For a primitive ring \( R \), analogues of Theorem 2.6 and its corollary can be found in a recent paper by Artin [1, pp. 68, 69]. His results are more general than ours in that his ring \( R \) is not assumed to have minimal right ideals. Of course, they are also less general in that they are restricted to apply to primitive rather than prime rings.

3. Transitivity of \( R \) over \( N \). As usual, the elements \( x_1, \cdots, x_n \) of \( N \) are called \( K \)-linearly independent if and only if \( k_1x_1 + \cdots + k_nx_n = 0 \) implies all \( k_i = 0, k_i \in K \). An alternate lattice-theoretic definition is given by the following lemma.
3.1 Lemma. The elements \( x_1, \ldots, x_n \) of \( N \) are \( K \)-linearly independent if and only if

\[
(x_j)^r \not\subset \bigcap_{i=1, i \neq j}^n (x_i)^r, \quad j = 1, \ldots, n.
\]

To prove this lemma, note first that all \( A_j = (x_i)^r \cap N = (Kx_i)^* \) are atoms of \( \mathcal{A} \) (assuming, of course, that \( x_i \neq 0 \)). If \( k_1x_1 + \cdots + k_nx_n = 0 \) with \( k_j \neq 0 \), then

\[
A_j \subseteq \bigcup_{i=1, i \neq j}^n A_i,
\]

and, since \( A_i = (x_i)^r \),

\[
(x_j)^r \not\subset \bigcap_{i=1, i \neq j}^n (x_i)^r.
\]

Conversely, if the above inclusion relation holds for some \( j \), then

\[
A_j \subseteq \bigcup_{i=1, i \neq j}^n A_i,
\]

\[
Kx_j \cap \sum_{i=1, i \neq j}^n Kx_i \neq 0.
\]

Thus the elements \( x_1, \ldots, x_n \) are \( K \)-linearly dependent, and 3.1 follows.

3.2 Lemma. Let \( I \) be any right ideal of \( R \) and \( K' \) be any left \( N \)-submodule of \( K \). Then for any \( x \) and \( y \) in \( N \) such that \( xI \neq 0 \) and \( K'y \neq 0 \), also \( xI \cap K'y \neq 0 \).

To prove this lemma, let \( k \) be any nonzero element of \( K' \). Then, by the primeness of \( N \), \( xaky \neq 0 \) for some \( a \) in \( I \). Now \( x(aky) = (xak)y \) where \( aky \in I \) and \( xak \in K' \), and therefore the lemma is proved.

We now are in a position to prove the main result of our paper, namely that \( R \) acts almost as an \( n \)-fold transitive ring of \( K \)-linear transformations on \( N \) for any integer \( n \) not exceeding the \( K \)-dimension of \( N \). To be more precise, we shall prove the following theorem.

3.3 Transitivity Theorem. If \( x_1, \ldots, x_n \) is any set of \( n \) \( K \)-linearly independent elements of \( N \) and if \( y_1, \ldots, y_n \) is any set of \( n \) elements of \( N \), then there exist an element \( a \) of \( R \) and a nonzero element \( k \) of \( K \) such that

\[
x_i a = ky_i, \quad i = 1, \ldots, n.
\]

To aid in the proof of this theorem, let

\[
I_j = \bigcap_{i=1, i \neq j}^n (x_i)^r.
\]
In view of 3.1, evidently \( I_j^\infty(x_j)^r \) for any \( j \). Hence, by 3.2, there exist elements \( a_j \subseteq I_j \) and \( k_j \subseteq K \), \( k_j \neq 0 \), such that \( x_ja_j = k_jy_j = 0 \) for all \( j \) such that \( y_j \neq 0 \). If \( y_j = 0 \), select \( a_j = 0 \). Now for all \( y_j \neq 0 \), \( k_jy_jK \) is a right ideal of \( K \), and \( \bigcap_j k_jy_jK \neq 0 \) by 2.2. Select \( k \subseteq \bigcap_j k_jy_jK \), \( k \neq 0 \); \( k = k_jy_jc_j \) for each \( j \) such that \( y_j \neq 0 \). Then \( x_j(a_jc_jy_j) = ky_j \), and if we let \( a = a_1c_1y_1 + \cdots + a_nc_ny_n \), evidently \( xja = ky_j \) as desired.

We give now an example of a prime ring of the type considered in this paper. Denote by \( I \) the ring of integers and by \( I_2 \) the ring of all \( 2 \times 2 \) matrices over \( I \). We use the notation \( E_{ij} \) for the matrix with 1 in its \((i, j)\) position and zeros elsewhere. Now denote by \( R \) the set of all matrices of \( I_2 \) having all even or all odd integers for components. It is easily established that \( R \) is a prime ring.

The right ideal \( N = 2IE_{11} + 2IE_{12} \) is a minimal prime right ideal and the left ideal \( M = 2IE_{11} + 2IE_{21} \) is a minimal prime left ideal. Clearly \( K = M \cap N = 2IE_{11} \) is an integral domain. The sets of prime right and left ideals of \( R \) form right and left structures of \( R \).

As an illustration of the transitivity theorem, let \( x_1 = 2E_{11}, x_2 = 4E_{12}, y_1 = 0, \) and \( y_2 = 2E_{11} \). Then for \( a = 2E_{11} \) and \( k = 4E_{11} \) we have \( x_1a = ky_1 \) and \( x_2a = ky_2 \). We note that there is no \( a \) in \( R \) such that \( x_1a = y_1 \) and \( x_2a = y_2 \).

In the case of a primitive ring \( R \), the minimal right ideals are all isomorphic as right \( R \)-modules. That such is not the case in general for a prime ring follows from this example. To show this, let \( N' = [I(E_{11} + E_{21}) + I(E_{12} + E_{22})] \cap R \). It is not too difficult to show that \( N' \) is a minimal prime right ideal of \( R \). If \( N \) and \( N' \) were isomorphic, then we would have \( 2aE_{11} \rightarrow c(E_{11} + E_{21}), 2bE_{12} \rightarrow d(E_{12} + E_{22}) \) for some integers \( a, b, c, d \) in order for the annihilators of corresponding elements of \( N \) and \( N' \) to be the same. But then \( c \) and \( d \) would have to be even integers, and nothing in \( N \) would correspond to the matrices in \( N' \) having odd integers for elements.

Bibliography


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