ON RADEMACHER'S EXTENSION OF THE GOLDBACH-VINOGRADOFF THEOREM

BY

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1. Introduction. The Goldbach problem for odd numbers n seeks to prove that the equation

$$n = p_1 + p_2 + p_3, \tag{1}$$

where the p_i are prime numbers, is always solvable. Hardy and Littlewood, using the now classical "circle" method, proved that if $B(n)$ is the number of solutions of (1), then under certain assumptions on the zeros of Dirichlet L-functions,

$$B(n) = \mathfrak{S}'(n) \frac{n^2}{2(\log n)^3} + o\left(\frac{n^2}{(\log n)^3}\right)$$

where $\mathfrak{S}'(n)$, the so-called "singular series," was proved to be greater than 0 for all odd n.

Rademacher [1], using simplifications of his own as well as of Landau, proved under similar assumptions that if k be a positive integer, a_i $(i=1, 2, 3)$ integers with $(a_i, k)=1$, and $A(n)$ the number of solutions of (1) with the restriction that the primes p_i belong to the residue classes a_i modulo k, then[1]

$$A(n) = \mathfrak{S}(n)I(n) + o\left(\frac{n^2}{(\log n)^3}\right)$$

where

$$I(n) = \iint \frac{du dv}{\log u \log v \log (n - u - v)},$$

and the range of integration is defined by the inequalities $u \geqq 2$, $v \geqq 2$, $u+v \leqq n-2$. He further proved that $\mathfrak{S}(n)>0$ provided certain necessary arithmetic restrictions on n hold. It may be shown that

$$I(n) = \frac{n^2}{2(\log n)^3} + o\left(\frac{n^2}{(\log n)^3}\right).$$

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(1) Rademacher considers the more general problem of the number of representations of n as a sum of s $(s \geqq 3)$ primes belonging to preassigned residue classes modulo k, and obtains a precise error term.

Vinogradoff [2], using the theorems of his own on estimates of trigonometric sums together with a theorem on the uniformity of distribution of primes in an arithmetic progression due to Siegel and Walfisz, proved the Hardy-Littlewood result without assumptions.

Following the work of Vinogradoff and Rademacher, we prove the result of Rademacher without recourse to Dirichlet L-functions and the assumptions on the location of their zeros.

2. **Notations and preliminary results.** Let n be an integer chosen sufficiently large, $\nu = \log n$, θ is a real number with $|\theta| < 1$, m is any constant > 3. $f(x) \ll g(x)$ means $f(x) = O(g(x))$, and denote $e^{(2\pi i/q)x}$ by $e_q(x)$.

THEOREM 2.1 (SIEGEL-WALFISZ). *If $\pi(n, q, t)$ denotes the number of primes $\leqq n$ in the progression $qx+t$, $(q, t) = 1$, and if $0 < q \leqq \nu^{3m}$, then,*

$$\pi(n, q, t) = (1/\phi(q)) \int_2^n \frac{dx}{\log x} + O((\phi(q))^{-1} n \nu^{-15m-1}), \tag{2}$$

where the constant implied by the O depends only on m.

THEOREM 2.2. *Let $(h, q) = 1$, $d | q$, $(a, d) = 1$, and denote by (u) the set of integers satisfying the conditions $1 \leqq u \leqq q$, $u \equiv a$ (mod d), $(u, q) = 1$. If*

$$S(q) = \sum_{(u)} e_q(hu),$$

then,

$$S(q) = \begin{cases} \mu(q/d) e_d(bah) & \text{if } ((q/d), d) = 1 \quad \text{and} \quad (q/d)b \equiv 1 \pmod{k}, \\ 0 & \text{if } ((q/d), d) > 1. \end{cases} \tag{3}$$

Proof. The proof follows Rademacher.

Let $T(q) = \sum e_q(v)$, summed over those v such that $1 \leqq v \leqq q$, $v \equiv a$ (mod d); then

$$T(q) = \sum_{c|q} \sum_{w} e_q\left(h \frac{q}{c} w\right) = \sum_{c|q} U(c), \tag{4}$$

where, for given c, w in the inner sum ranges over those integers satisfying $(c, w) = 1$, $1 \leqq w \leqq q$, $wq/c \equiv a$ (mod d). Moreover, if $q_1 | q$ and $T(q_1) = \sum_t e_{q_1}(ht)$, where t ranges over the set $1 \leqq t \leqq q$, $t(q/q_1) \equiv a$ (mod d), then

$$T(q) = \sum_{c|q_1} U(c). \tag{5}$$

From (4) and (5), however, we get

$$\sum_{q_1|q} \mu(q/q_1) T(q_1) = \sum_{q_1|q} U(q_1) \sum_{q_2|q/q_1} \mu(q/q_1 q_2) = U(q) = S(q). \tag{6}$$

If $((q/q_1), d) = 1$, then $T(q_1) = 0$; moreover $T(q_1) = e_{q_1}(hd) T(q_1)$. Consequently $T(q_1)=0$ for $q_1 \nmid hd$. On the other hand for $q_1 \mid hd$, $((q/q_1), d)=1$, determine b in such a way that $bq/q_1 \equiv 1 \pmod d$. Then

$$T(q_1) = \sum_{1 \leqq v \leqq q_1/d} e_{q_1}(h(vd + ba)) = (q_1/d)e_{q_1}(bah).$$

Since $q_1 \mid d$, $T(q_1)=0$ for $q_1 < d$, and the theorem is proved.

Suppose now that

$$S(x) = S_i(x) = \sum_{(p)} e(xp), \tag{7}$$

where (p) denotes the set of primes satisfying the conditions $p \leqq n$, $p \equiv a \pmod k$ with $a = a_1, a_2, a_3$, and $e_1(x) = e(x)$, then

$$A(n) = \int_0^1 S_1(x)S_2(x)S_3(x)e(-nx)dx = \int_0^1 f(x)dx \quad \text{(say)}. \tag{8}$$

As usual, we divide the unit interval into Farey "arcs." Let $h/q = r$ be a rational point of the unit interval with $(h, q)=1$ and $1 \leqq q \leqq \nu^{3m}$. The major "arc" B_r belonging to r is the set of points x in $(0, 1)$ with

$$\left| x - r \right| \leqq n^{-1}\nu^{3m} = \tau^{-1}.$$

It is proved that no two major arcs intersect, and if E denotes the set of points not belonging to any B_r, then x in E has the form

$$x = h/q + \theta/q\tau \quad \text{with} \quad \nu^{3m} < q \leqq \tau.$$

Since $f(x)$ has period 1, (8) can be written as

$$A(n) = \sum_r \int_{B_r} f(x)dx + \int_E f(x)dx. \tag{9}$$

3. Estimate on the major arcs.

THEOREM 3.1. *Let $d=(k, q)$; then if x belongs to B_r,*

$$S(x) = \frac{1}{\phi(k)} \frac{\mu(q/d)}{\phi(q/d)} e_d(hab) \int_2^n \frac{e(xz)}{\log x} dx + O(n\nu^{-6m-1}). \tag{10}$$

Proof. With Vinogradoff, we divide $S(x)$ into $O(\nu^{9m})$ sums of the form

$$S_u(x) = \sum_{u < p \leqq v} e((h/q + z)p), \tag{11}$$

the range of summation being further restricted by the condition $p \equiv a \pmod k$, $0 < v - u \leqq n\nu^{qm}$, and since x is a point of B_r, it has the form $x = h/q + z$, with $|z| \leqq \tau^{-1}$. We write (11) in the form

$$S_u(x) = \sum_j \sum_{u < p \leqq v} e((h/q + z)p),$$

where p in the inner sum is further restricted by $p \equiv a \pmod{k}$ and $p \equiv j \pmod{q}$. We deduce by the Chinese remainder theorem

$$S_u(x) = \sum_{j \equiv a \pmod{d}} \sum_{u < p \leqq v} e((h/q + z)p),$$

where in the inner sum $p \equiv s \pmod{kq/d}$. Denote the inner sum by $S_u^j(x)$. Since $p \equiv j \pmod{q}$, we get $e((h/q+z)p) = e(hj/q+uz) + O(|z| n\nu^{-9m})$. On the other hand,

$$S_u^j(x) = \{e(hj/q)e(uz) + O(|z| n\nu^{-9m})\} \sum_p 1,$$

with p satisfying the conditions of the above inner sum. Using Theorem 2.1, we deduce

$$S_u^j(x) = (\phi(kq/d))^{-1}e(hj/q)e(uz)I_1 + O((\phi(kq/d))^{-1}(n\nu^{-1-15m} + I_1|z|\nu^{-9m})),$$

where

$$I_1 = \int_u^v \frac{dx}{\log x}\cdot$$

Since $|z|\,|x-u| \leqq |z| n\nu^{-9m}$, we get

$$I_1 e(uz) = \int_u^v \frac{e(xz)}{\log x} dx + O(I_1 |z| n\nu^{-9m}).$$

Consequently,

$$S_u^j(x) = (\phi(kq/d))^{-1} \int_u^v \frac{e(xz)}{\log x} dx + O((\phi(kq/d))^{-1}(n\nu^{-1-15m} + I_1|z|\nu^{-9m})).$$

Summing over j, we get,

$$\begin{aligned} S_u(x) &= (\phi(kq/d))^{-1} \int_u^v \frac{e(xz)}{\log x} dx \sum_j e(hj/q) \\ &\quad + O((\phi(kq/d))^{-1}(n\nu^{-1-15m} + I_1|z| n\nu^{-9m})) \sum_j 1. \end{aligned} \tag{12}$$

Here j ranges over the set $1 \leqq j \leqq q$, $(j, q) = 1$, $j \equiv a \pmod{d}$. If $((q/d), d) = 1$, and b is determined such that $(q/d)b \equiv 1 \pmod{d}$, we get, by Theorem 2.3,

$$\begin{aligned} S_u(x) &= (\phi(kq/d))^{-1}\mu(q/d)e(hba/d) \int_u^v \frac{e(xz)}{\log x} dx \\ &\quad + O((\phi(q/d)(\phi(kq/d))^{-1})(n\nu^{-1-15m} + I_1|z| n\nu^{-9m})), \end{aligned}$$

while if $((q/d), d) > 1$,

$$S_u(x) = O((\phi(q/d)(\phi(kq/d))^{-1})(n\nu^{-1-15m} + I_1 | z | n\nu^{-9m})).$$

Continuing with (12), we observe that $1=((q/d),\ d)=((q/d),\ (q,\ k)) =((q/d),\ k)$. Hence

$$S_u(x) = (\phi(k))^{-1}(\phi(q/d))^{-1}\mu(q/d)e(hab/d)\int_u^v \frac{e(xz)}{\log x}\,dx \tag{13}$$
$$+ O(n\nu^{-1-15m} + I_1 | z | n\nu^{-9m}).$$

Summing over all intervals, we get

$$S(x) = (\phi(k))^{-1}\phi(q/d))^{-1}\mu(q/d)e(hab/d) \int_2^n \frac{e(xz)}{\log x}\,dx$$
$$+ O\left(n\nu^{-15m-1}\nu^{9m} + | z | n\nu^{-9m} \int_2^n \frac{dx}{\log x}\right).$$

Since

$$| z | n\nu^{-9m}\int_2^n \frac{dx}{\log x} = O(n\nu^{-6m-1}),$$

the result follows.

THEOREM 3.2.

$$\int_{Br} f(x)dx = \frac{1}{\phi(k)^3}\,\frac{\mu(q/d)}{\phi(q/d)^3}\,e((hb(a_1 + a_2 + a_3)/d) - nh/q) \tag{14}$$
$$\cdot \int_{-\tau^{-1}}^{\tau^{-1}} (I_2(z))^3 e(-nz)dz + O\left(\frac{1}{\phi(q/d)^2}\, n^2\nu^{-6m-3}\right),$$

where

$$I_2(z) = \int_2^n \frac{e(xz)}{\log x}\,dx. \tag{15}$$

Proof. An easy calculation shows that $I_2(z)=O(\zeta)$ where

$$\zeta = \begin{cases} n\nu^{-1} & \text{if } |z| \leqq n^{-1}, \\ |z|^{-1}\nu^{-1} & \text{if } n^{-1} < z \leqq n\nu^{-3m}. \end{cases}$$

We have

$$(\phi(k))^{-1}(\phi(q/d))^{-1}\mu(q/d)e(hab/d)I_2(z) = O((\phi(q/d))^{-1}I_2(z)),$$

and since $\phi(q/d)n\nu^{-6m-1}(I_2(z))^{-1}=O(1)$, we deduce

$$S_1(x)S_2(x)S_3(x) = (\phi(k))^{-3}(\phi(q/d))^{-3}\mu(q/d)e(hb(a_1 + a_2 + a_3)/d) \tag{16}$$
$$\cdot (I_2(z))^3 + O(\phi(q/d)^2 n\nu^{-6m-1}(I_2(z))^2).$$

Therefore

$$\int_{B_r} f(x)dx = \int_{-\tau^{-1}}^{\tau^{-1}} S_1(x)S_2(x)S_3(x)e(-(h/q+z)n)dz$$
$$= (\phi(k)^{-3}(\phi(q/d))^{-3}\mu(q/d)e(hb(a_1+a_2+a_3)/d)I + I_3(z),$$

where

$$I = \int_{-\tau^{-1}}^{\tau^{-1}} (I_2(z))^3 e(-nz)dz,$$

$$I_3(z) = O\left((\phi(q/d))^{-2}n\nu^{-6m-1})\int_0^{n^{-1}\nu^{3m}} \zeta^2 dz\right)$$
$$= O\left((\phi(q/d))^{-2}n\nu^{-6m-1}\int_0^{n^{-1}} n^2\nu^{-2}dz + \int_{n^{-1}}^{n^{-1}\nu^{3m}} n^{-2}z^{-2}dz\right)$$
$$= O((\phi(q/d)^{-2}n^2\nu^{-6m-3})).$$

COROLLARY.

$$\sum_r \int_{B_r} f(x)dx = \frac{1}{\phi(k)^3}\sum_q \frac{\mu(q/d)}{\phi(q/d)^3}\sum_h e(bh(a_1+a_2+a_3)/d - hn/q)I$$
$$+ O(n^2\nu^{-3m-3}). \tag{17}$$

Here the inner sum ranges over the set $i \leqq h \leqq q$, $(h, q) = 1$, and the outer sum over the set $(d, (q/d)) = 1$, $q \leqq \nu^{3m}$.

4. Estimate on the minor arc.

THEOREM 4.1. *Let (u) and (v) be two increasing sequences of positive integers and w a positive integer. Let $1 < N' < N_1$, $n_1 = \log N_1$, $1 < U_0 < U_1 \leqq N_1$, $1 < \tau < N_1$, $x = h/q + \theta/q\tau$, $\delta = (w, q)$, $q = \delta q_1$, $w = \delta w_1$, and*

$$T = \sum_u \sum_v e(xwuv),$$

where u runs through the elements of the sequence (u) satisfying the inequalities $U_0 < u \leqq U_1$ and, for given u, v ranges over those elements of the sequence (v) satisfying the inequalities $N'/u < v \leqq N_1/u$; then

$$T = O(N_1(n_1/U_0 + U_1/N_1 + q_1 n_1^3/N_1 + n_1^2/q_1 + w_1 n_1^2/\tau)^{1/2}). \tag{18}$$

Proof. The proof may be found in Vinogradoff [2].

Denote by H the product of all primes $\leqq n^{1/2}$, and by (d) the sequence of integers satisfying the condition $d | H$, $d \leqq n$. Using a reasoning similar to that used in the proof of Theorem 2.2, we derive the following expression for $S(x)$

$$S(x) = \sum_{(d)} \mu(d) S_d + O(n^{1/2}), \tag{19}$$

where

$$S_d = \sum_u e(xdu).$$

Here u ranges over the sequence satisfying the conditions $du \leqq n$, $du \equiv a \pmod{k}$. We have

$$\sum_{(d)} \mu(d) S_d = \sum_{(d_0)} S_d - \sum_{(d_1)} S_d = S_0 - S_1 \quad \text{(say)},$$

where (d_0) is the sequence of elements of (d) having an even number of divisors and (d_1) those elements of (d) having an odd number of divisors. We estimate S_0; S_1 can be estimated in exactly the same way. Write $\lambda = \nu^{2(m+1)}$, and divide S_0 into three sums,

$$S_0 = \sum_{d \leqq \lambda} S_d + \sum_{\lambda < d \leqq n\lambda_{-1}} S_d + \sum_{n\lambda_{-1} \leqq d \leqq n} S_d = T_1 + T_2 + T_3. \tag{20}$$

It is understood of course that the index d ranges over the set (d_0) satisfying the given inequalities.

To estimate T_1, we observe that if $d' = d/(k, d)$ and a' is a solution of the congruence $dx \equiv a \pmod{k}$ and $n_2 = n(k, d)/kd$, then

$$S_d = \sum_{u \leqq n_2} e(xd'(ku + a')).$$

Consequently, $|S_d| \leqq q$; it follows that

$$T_1 \ll n\nu^{-m+2}. \tag{21}$$

To estimate T_2, we apply Theorem 4.1. We have

$$T = \sum_d \sum_u e(xdu)$$

with the prescribed ranges of summation. We have here $N_1 = n$, $U_0 = \lambda$, $U_1 = n\lambda^{-1}$, $w = 1$. Theorem 4.1 yields

$$T_2 \ll n(\nu^{-2m-1})^{1/2} \ll n\nu^{-m+2}. \tag{22}$$

We turn now to the estimate of T_3. We have

$$T_3 = \sum_d \sum_u e(xdu),$$

summed over the prescribed ranges for d and u. Interchange the order summation, then

$$T_3 = \sum_{u \leqq \lambda} \sum_{n\lambda^{-1} < d \leqq n/u} e(xdu) = \sum_u T(u)$$

with the inner sum further restricted by the condition $du \equiv a \pmod{k}$. We divide the sequence (d) into two sequences (d') and (d'') where (d') is the set of (d) having all prime divisors $\leqq \nu^{3m}$ and (d'') those elements of (d) having at least one prime divisor $>\nu^{3m}$. (d_0) is then divided into two corresponding sets (d_0') and (d_0''). We get $T(u)=T'(u)+T''(u)$ where the right-hand summands correspond to the sets (d_0') and (d_0'') respectively. We estimate now the number of terms D of the set (d') which satisfy the conditions $d\leqq n/u$ and $1\leqq u\leqq \lambda$. To this end suppose that an element d of (d') have j prime divisors. Then $(\nu^{3m})^j \geqq n\lambda^{-1}$, and hence if n be chosen sufficiently large $j>\nu/6m \log \nu$. If then $\tau(d)$ be the number of divisors of d, we get

$$\tau(d) = 2^j > 2^{\nu/6m \log \nu} > n^{1/9m \log \nu},$$

and since

$$\sum_{1\leqq v\leqq n_1} \tau(v) \ll n_1(\nu+1),$$

where $n_1=n/u$, we conclude that

$$Dn^{1/9m \log \nu} \ll n_1(\nu+1) \ll n_1 n^{1/9m \log \nu} n^{-1/9m \log \nu} \nu^m \nu^{-m}(\nu+1).$$

Therefore $D \ll n_1\nu^{-m}$. Consequently we deduce that

$$T(u) = T''(u) + O(\nu^{-m}nu^{-1}).$$

For the sum $T''(u)$ we have evidently $j<\nu$, hence

$$T''(u) = \sum_j T_j(u)$$

where $T_j(u)$ is summed over those d belonging to (d_0'') satisfying the inequalities $n\lambda^{-1}<d\leqq n_1$, and having exactly j prime divisors $>\nu^{3m}$. In order to estimate the sum $T_j(u)$ we consider with Vinogradoff the more general sum

$$T_j'(u) = \sum_v \sum_w e(xuvw)$$

where v ranges over all primes $>\nu^{3m}$ belonging to (d) and, for given v, w ranges over those numbers satisfying the inequalities $n\lambda^{-1}/v<w\leqq n_1/v$, the congruence $uvw\equiv a \pmod{k}$ and containing exactly $j-1$ prime divisors $>\nu^{3m}$ and belonging to (d_1). Every term $e(xdu)$ of the sum $T_j(u)$ is found in the sum $T_j'(u)$ and indeed is found exactly j times. In addition, however, $T_j'(u)$ contains terms of the form $e(xp^2w_1)$ with $n\lambda^{-1}/p^2<w_1\leqq n_1/p^2$, where $p>\nu^{3m}$, and w_1 runs over elements of (d_0) containing $j-2$ prime divisors $>\nu^{3m}$. These terms evidently occur without duplication. For given p, the number of w_1p^2 satisfying $n\lambda^{-1}/p^2<w_1\leqq n_1/p^2$ is $\leqq n_1/p^2$, consequently

$$T_j'(u) = jT_j(u) + O\Bigl(\sum_{\nu^{3m}<p\leqq n^{1/2}} n_1/p^2\Bigr) = jT_j(u) + O(\nu^{-3m}nu^{-1}).$$

We now apply Theorem 4.1 to the sum $T_j'(u)$. We take $U_0=\nu^{3m}$, $U_1=n^{1/2}$, $N'=n\lambda^{-1}$, and conclude

$$T(u) \ll n/u(u\nu^{-3m+2})^{1/2} \ll n\nu^{-3m/2+1}u^{-1/2}.$$

Therefore, $T_j(u) \ll j^{-1}nu^{-1/2}\nu^{-3m/2+1}$, from which we deduce that

$$T''(u) \ll nu^{-1/2}\nu^{-3m/2+1} \log \nu,$$

and hence that

$$T(u) \ll nu^{-1/2}\nu^{-3m/2+1} \log \nu + nu^{-1}\nu^{-m}.$$

Summing over u, we deduce that

(23) $$T_3 \ll n\nu^{-m+1} \log \nu + n\nu^{-m} \log \nu \ll n\nu^{-m+2}.$$

Using (19), (20), (21), (22), and (23), we conclude the following:

THEOREM 4.2. *Let m be any constant >3,*

$$x = h/q + \theta/q\tau, \qquad (h, q) = 1, \qquad \nu^{3m} < q \leqq \tau, \qquad \tau = n\nu^{-3m};$$

then

(24) $$\sum_{(p)} e(xp) = O(n\nu^{-m+2}).$$

5. The asymptotic formula and proof of the theorem.

THEOREM 5.1. *If $m>9/4$, then*

(25) $$I = \int_{-\tau^{-1}}^{\tau^{-1}} (I_2(z))^3 e(-nz)dz = \frac{n^2}{2\nu^3} + O\left(\frac{n^2}{\nu^{7/2}} \log \nu\right).$$

Proof. Vinogradoff [2].

Using this result, we deduce readily that

$$\sum_q (\phi(q/d))^{-3}\mu(q/d) \sum_h e(bh(a_1 + a_2 + a_3)/d - hn/q)I = O(n^2\nu^{-4}),$$

where the inner sum ranges over the set of h such that $1 \leqq h \leqq q$, $(h, q)=1$, and the outer sum over those q satisfying $((q/d), d)=1$, $q>\nu^{3m}$. This result, together with (17), permits us to conclude that

(26) $$\sum_r \int_{B_r} f(x)dx = \mathfrak{S}(n) \frac{n}{2\nu^3} + O(n^2\nu^{-7/2} \log \nu),$$

where $\mathfrak{S}(n)$, the singular series, is given by

(27) $$\mathfrak{S}(n) = \frac{1}{\phi(k)^3} \sum_{q=1}^{\infty} \frac{\mu(q/d)}{\phi(q/d)^3} \sum_{1 \leqq h \leqq q} e(hb(a_1 + a_2 + a_3)/d - hn/q),$$

where as above q is restricted by $((q/d), d)=1$ and h by $(h, q)=1$. On the other hand, using Theorem 4.2, we get

$$\begin{aligned}
\int_E f(x)dx &\ll \int_E \left| S_1(x)S_2(x)S_3(x) \right| dx \\
&\ll nv^{-m+2}\int_0^1 \left| S_2(x)S_3(x) \right| dx \\
&\ll nv^{-m+2}\left(\int_0^1 \left| S_2(x) \right|^2 dx\right)^{1/2}\left(\int_0^1 \left| S_3(x) \right|^2 dx\right)^{1/2} \\
&\ll nv^{-m+2}nv^{-1} \ll n^2v^{-m+1}.
\end{aligned} \tag{28}$$

From (9), (26), and (28), we conclude

$$A(n) = \mathfrak{S}(n)\,\frac{n^2}{2(\log n)^3} + O\left(\frac{n^2}{(\log n)^{7/2}}\log\log n\right). \tag{29}$$

On the other hand Rademacher has shown that if n is odd and $n \equiv a_1+a_2+a_3 \pmod{k}$, then

$$\mathfrak{S}(n) = \frac{C}{k^2}\prod_{p\mid k}\frac{p^3}{(p-1)^3+1}\prod_{p\mid n,\, p\nmid k}\frac{(p-1)((p-1)^2-1)}{(p-1)^3+1}\prod_{p>2}\left(1+\frac{1}{(p-1)^3}\right),$$

where throughout $p>2$, $C=2$ for odd k, and $C=8$ for even k. If n fails to satisfy the above conditions, then $\mathfrak{S}(n)=0$.

We formulate the:

MAIN THEOREM. *Let k be a positive integer, a_1, a_2, a_3 be residue classes modulo k with $(a_i, k)=1$. If n is a sufficiently large odd integer satisfying the congruence $n\equiv a_1+a_2+a_3 \pmod{k}$, then n can be represented as a sum of three primes belonging respectively to the residue classes a_1, a_2, a_3, modulo k. The asymptotic formula for the number of representations is given by* (29).

6. **Concluding remarks.** The method of Linnik-Tchudakoff will provide another proof of this result. The corresponding question for the simultaneous Goldbach-Waring problem may be posed and solved.

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