ON RADEMACHER’S EXTENSION OF THE GOLDBACH-VINOGRADOFF THEOREM

BY
RAYMOND AYOUB

1. Introduction. The Goldbach problem for odd numbers $n$ seeks to prove that the equation

$$n = p_1 + p_2 + p_3,$$

where the $p_i$ are prime numbers, is always solvable. Hardy and Littlewood, using the now classical “circle” method, proved that if $B(n)$ is the number of solutions of (1), then under certain assumptions on the zeros of Dirichlet $L$-functions,

$$B(n) = \varepsilon'(n) \frac{n^2}{2(\log n)^3} + o\left(\frac{n^2}{(\log n)^3}\right),$$

where $\varepsilon'(n)$, the so-called “singular series,” was proved to be greater than 0 for all odd $n$.

Rademacher [1], using simplifications of his own as well as of Landau, proved under similar assumptions that if $k$ be a positive integer, $a_i \ (i=1, 2, 3)$ integers with $(a_i, k) = 1$, and $A(n)$ the number of solutions of (1) with the restriction that the primes $p_i$ belong to the residue classes $a_i$ modulo $k$, then

$$A(n) = \varepsilon(n) I(n) + o\left(\frac{n^2}{(\log n)^3}\right),$$

where

$$I(n) = \int \int \frac{dudv}{\log u \log v \log (n - u - v)},$$

and the range of integration is defined by the inequalities $u \geq 2, v \geq 2, u + v \leq n - 2$. He further proved that $\varepsilon(n) > 0$ provided certain necessary arithmetic restrictions on $n$ hold. It may be shown that

$$I(n) = \frac{n^2}{2(\log n)^3} + o\left(\frac{n^2}{(\log n)^3}\right).$$

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(1) Rademacher considers the more general problem of the number of representations of $n$ as a sum of $s \ (s \geq 3)$ primes belonging to preassigned residue classes modulo $k$, and obtains a precise error term.
Vinogradoff [2], using the theorems of his own on estimates of trigonometric sums together with a theorem on the uniformity of distribution of primes in an arithmetic progression due to Siegel and Walfisz, proved the Hardy-Littlewood result without assumptions.

Following the work of Vinogradoff and Rademacher, we prove the result of Rademacher without recourse to Dirichlet $L$-functions and the assumptions on the location of their zeros.

**2. Notations and preliminary results.** Let $n$ be an integer chosen sufficiently large, $\nu = \log n$, $\theta$ is a real number with $|\theta| < 1$, $m$ is any constant $> 3$. $f(x) \ll g(x)$ means $f(x) = O(g(x))$, and denote $e^{(2\pi i/q)x}$ by $e_q(x)$.

**Theorem 2.1 (Siegel-Walfisz).** If $\pi(n, q, t)$ denotes the number of primes $\leq n$ in the progression $qx + t$, $(q, t) = 1$, and if $0 < q \leq \nu^3$, then,

\[
\pi(n, q, t) = \frac{1}{\phi(q)} \int_2^n \frac{dx}{\log x} + O((\nu^{-1} n^{1-\epsilon})),
\]

where the constant implied by the $O$ depends only on $m$.

**Theorem 2.2.** Let $(h, q) = 1$, $d|q$, $(a, d) = 1$, and denote by $(u)$ the set of integers satisfying the conditions $1 \leq u \leq q$, $u = a \pmod{d}$, $(u, q) = 1$. If

\[
S(q) = \sum_{(u)} e_q(hu),
\]

then,

\[
S(q) = \begin{cases} 
\mu(q/d) e_d(bah) & \text{if } ((q/d), d) = 1 \text{ and } (q/d)b \equiv 1 \pmod{k}, \\
0 & \text{if } ((q/d), d) > 1.
\end{cases}
\]

**Proof.** The proof follows Rademacher.

Let $T(q) = \sum e_q(v)$, summed over those $v$ such that $1 \leq v \leq q$, $v \equiv a \pmod{d}$; then

\[
T(q) = \sum_{c|q} \sum_{w} e_q \left( h \frac{q}{c} w \right) = \sum_{c|q} U(c),
\]

where, for given $c$, $w$ in the inner sum ranges over those integers satisfying $(c, w) = 1$, $1 \leq w \leq q$, $wq/c \equiv a \pmod{d}$. Moreover, if $q|g$ and $T(q_g) = \sum t e_q(ht)$, where $t$ ranges over the set $1 \leq t \leq q$, $t(q_g/q) \equiv a \pmod{d}$, then

\[
T(q) = \sum_{c|q} U(c).
\]

From (4) and (5), however, we get

\[
\sum_{q|q_1} \mu(q/q_1) T(q_1) = \sum_{q_1|q} U(q_1) \sum_{q_2|q_1} \mu(q/q_1q_2) = U(q) = S(q).
\]

If $((q/q_1), d) = 1$, then $T(q_1) = 0$; moreover $T(q_1) = e_{q_1}(hd)T(q_1)$. Conse-
quently $T(q_1) = 0$ for $q_1|\text{hd}$. On the other hand for $q_1|\text{hd}$, $((q/q_1), d) = 1$, determine $b$ in such a way that $bq/q_1 = 1 (\mod d)$. Then

$$T(q_1) = \sum_{1 \leq r \leq q_1/d} e_{q_1}(h(rd + ba)) = (q_1/d)e_{q_1}(ba h).$$

Since $q_1|\text{d}$, $T(q_1) = 0$ for $q_1 < d$, and the theorem is proved.

Suppose now that

$$S(x) = S_i(x) = \sum_{(p)} e(xp),$$

where $(p)$ denotes the set of primes satisfying the conditions $p \leq n$, $p \equiv a (\mod k)$ with $a = a_1, a_2, a_3$, and $e_i(x) = e(x)$, then

$$(7) \quad S(x) = S_i(x) = \sum_{(p)} e(xp).$$

$$(8) \quad A(n) = \int_{0}^{1} S_1(x)S_2(x)S_3(x)e(-nx)dx = \int_{0}^{1} f(x)dx \quad \text{(say).}$$

As usual, we divide the unit interval into Farey “arcs.” Let $h/q = r$ be a rational point of the unit interval with $(h, q) = 1$ and $1 \leq q \leq \nu^{3m}$. The major “arc” $B_r$ belonging to $r$ is the set of points $x$ in $(0, 1)$ with

$$|x - r| \leq n^{-1}\nu^{3m} = \tau^{-1}.$$ 

It is proved that no two major arcs intersect, and if $E$ denotes the set of points not belonging to any $B_r$, then $x$ in $E$ has the form

$$x = h/q + \theta/q\tau \quad \text{with} \quad \nu^{3m} < q \leq \tau.$$

Since $f(x)$ has period 1, (8) can be written as

$$(9) \quad A(n) = \sum_{r} \int_{B_r} f(x)dx + \int_{E} f(x)dx.$$

3. Estimate on the major arcs.

**Theorem 3.1.** Let $d = (k, q)$; then if $x$ belongs to $B_r$,

$$S(x) = \frac{1}{\phi(k)} \frac{\mu(q/d)}{\phi(q/d)} e_{d}(h/ab) \int_{2}^{n} \frac{e(xz)}{\log x} dx + O(n^{n-\theta m-1}).$$

**Proof.** With Vinogradoff, we divide $S(x)$ into $O(n^{\theta m})$ sums of the form

$$(10) \quad S(x) = \sum_{u \leq \nu^{3m}} e((h/q + z)\nu^{3m}).$$

the range of summation being further restricted by the condition $\nu \equiv a (\mod k)$, $0 < u \leq \nu^{3m}$, and since $x$ is a point of $B_r$, it has the form $x = h/q + z$, with $|z| \leq \tau^{-1}$. We write (11) in the form

$$S_u(x) = \sum_{j} \sum_{u \leq \nu^{3m}} e((h/q + z)\nu^{3m}),$$
where \( p \) in the inner sum is further restricted by \( p \equiv a \pmod{k} \) and \( p \equiv j \pmod{q} \). We deduce by the Chinese remainder theorem
\[
S_u(x) = \sum_{j \equiv a \pmod{d}} \sum_{u < p < v} e((h/q + z)p),
\]
where in the inner sum \( p \equiv s \pmod{kg/d} \). Denote the inner sum by \( S_u^j(x) \).
Since \( p \equiv j \pmod{q} \), we get \( e((h/q + z)p) = e(hj/q + uz) + O(|z| n^{-9m}) \). On the other hand,
\[
S_u^j(x) = \{ e(hj/q)e(uz) + O(|z| n^{-9m}) \} \sum_p 1,
\]
with \( p \) satisfying the conditions of the above inner sum. Using Theorem 2.1, we deduce
\[
S_u^j(x) = (\phi(kq/d))^{-1}e(hj/q)e(uz)I_1 + O((\phi(kq/d))^{-1}(n^{1-15m} + I_1 | z | n^{-9m})),
\]
where
\[
I_1 = \int_u^v \frac{dx}{\log x}.
\]
Since \(|z| |x-u| \leq |z| n^{-9m} \), we get
\[
I_1 e(uz) = \int_u^v \frac{e(xz)}{\log x} dx + O(I_1 | z | n^{-9m}).
\]
Consequently,
\[
S_u^j(x) = (\phi(kq/d))^{-1} \int_u^v \frac{e(xz)}{\log x} dx + O((\phi(kq/d))^{-1}(n^{1-15m} + I_1 | z | n^{-9m})).
\]
Summing over \( j \), we get,
\[
S_u(x) = (\phi(kq/d))^{-1} \int_u^v \frac{e(xz)}{\log x} dx \sum_j e(hj/q)
+ O((\phi(kq/d))^{-1}(n^{1-15m} + I_1 | z | n^{-9m})) \sum_j 1.  
\]
Here \( j \) ranges over the set \( 1 \leq j \leq q \), \( (j, q) = 1 \), \( j \equiv a \pmod{d} \). If \((q/d), d) = 1\), and \( b \) is determined such that \((q/d)b \equiv 1 \pmod{d} \), we get, by Theorem 2.3,
\[
S_u(x) = (\phi(kq/d))^{-1} \mu(q/d)e(hba/d) \int_u^v \frac{e(xz)}{\log x} dx
+ O((\phi(q/d)(\phi(kq/d))^{-1}(n^{1-15m} + I_1 | z | n^{-9m})),
\]
while if \((q/d), d) > 1\),
\[ S_u(x) = O((\phi(q/d)(\phi(kq/d))^{-1})(nv^{-1-15m} + I_1 | z | nv^{-9m})). \]

Continuing with (12), we observe that \(1 = ((q/d), d) = ((q/d), (q, k)) = ((q/d), k).\) Hence

\[
S_u(x) = (\phi(k))^{-1}(\phi(q/d))^{-1}\mu(q/d)e(hab/d) \int_u^v \frac{e(xz)}{\log x} \, dx + O(nv^{-1-15m} + I_1 | z | nv^{-9m}).
\]

Summing over all intervals, we get

\[
S(x) = (\phi(k))^{-1}\phi(q/d))^{-1}\mu(q/d)e(hab/d) \int_2^n \frac{e(xz)}{\log x} \, dx + O(nv^{-15m-1}nv^{-9} + | z | nv^{-9}m \int_2^n dx)\).
\]

Since

\[
| z | nv^{-9}m \int_2^n \frac{dx}{\log x} = O(nv^{-6}m^{-1}),
\]

the result follows.

**Theorem 3.2.**

\[
\int_{B_r} f(x) \, dx = \frac{1}{\phi(k)^3} \frac{\mu(q/d)}{\phi(q/d)^3} e((hb(a_1 + a_2 + a_3)/d) - nh/q) \int_{-r}^{r} (I_2(z))^{\mu} (-nz) \, dz + O\left(\frac{1}{\phi(q/d)^2} n^{2}v^{-6}m^{-3}\right),
\]

where

\[
I_2(z) = \begin{cases} 
\int_{2}^{n} \frac{e(xz)}{\log x} \, dx & \text{if } | z | \leq n^{-1}, \\
| z |^{-1}v^{-1} & \text{if } n^{-1} < z \leq nv^{-8}m.
\end{cases}
\]

**Proof.** An easy calculation shows that \(I_2(z) = O(\xi)\) where

\[
\xi = \begin{cases} 
nv^{-1} & \text{if } | z | \leq n^{-1}, \\
| z |^{-1}v^{-1} & \text{if } n^{-1} < z \leq nv^{-8}m.
\end{cases}
\]

We have

\[
(\phi(k))^{-1}(\phi(q/d))^{-1}\mu(q/d)e(hab/d)I_2(z) = O((\phi(q/d))^{-1}I_2(z)),
\]

and since \(\phi(q/d)nv^{-8}m^{-1}(I_2(z))^{-1} = O(1)\), we deduce

\[
S_1(x)S_2(x)S_3(x) = (\phi(k))^{-3}(\phi(q/d))^{-3}\mu(q/d)e(hab(a_1 + a_2 + a_3)/d) \cdot (I_2(z))^{3} + O(\phi(q/d)^2nv^{-8}m^{-1}(I_2(z))^3).
\]
Therefore
\[ \int_{B_r} f(x)dx = \int_{-r}^{r} \sum_{s=1}^{s_3} S_1(x)S_2(x)S_3(x)e(-\frac{h}{q} + \frac{z}{r} + \frac{n}{d})dx \]
where
\[ I = \int_{-r}^{r} (I_2(z))^2 e(-nz)dz, \]
and
\[ I_3(z) = O\left((\phi(q/d))^{-2}\frac{n\nu^{-6}m^{-1}}{2}\int_{0}^{1} z^2 dz\right) \]
\[ = O\left((\phi(q/d))^{-2}\frac{n\nu^{-6}m^{-1}}{2}\int_{0}^{1} z^2 dz + \int_{0}^{1} n^{-2}z^{-2}dz\right) \]
\[ = O((\phi(q/d))^{-2}\frac{n\nu^{-6}m^{-3}}{2}). \]

**Corollary.**
\[ \sum_{r} \int_{B_r} f(x)dx = \frac{1}{\phi(k)} \sum_{q} \frac{\mu(q/d)}{\phi(q/d)^3} \sum_{h} e(hb(a_1 + a_2 + a_3)/d - hn/q)I \]
\[ + O(n^2\nu^{-3}m^{-3}). \]

Here the inner sum ranges over the set \( i \leq h \leq q, (h, q) = 1 \), and the outer sum over the set \((d, (q/d)) = 1, q \leq \nu^m\).

**4. Estimate on the minor arc.**

**Theorem 4.1.** Let \((u)\) and \((v)\) be two increasing sequences of positive integers and \(w\) a positive integer. Let \(1 < N' < N, n_1 = \log N, 1 < U_0 < U_1 \leq N,\)
\[ 1 < r < N, x = \frac{h}{q} + \frac{\theta}{qr}, \delta = (w, q), q = \delta q_1, w = \delta w_1, \text{ and} \]
\[ T = \sum_{u} \sum_{v} e(xuv), \]
where \(u\) runs through the elements of the sequence \((u)\) satisfying the inequalities \(U_0 < u \leq U_1\) and, for given \(u, v\) ranges over those elements of the sequence \((v)\) satisfying the inequalities \(N'/u < v \leq N_1/u; \text{ then} \)
\[ T = O(N_1(n_1/U_0 + U_1/N_1 + q_1q_2/N_1 + n_1/q_1 + w_1n_1/r)^{1/2}). \]

**Proof.** The proof may be found in Vinogradoff [2].

Denote by \(H\) the product of all primes \(\leq n^{1/2}\), and by \((d)\) the sequence of integers satisfying the condition \(d \mid H, d \leq n\). Using a reasoning similar to that used in the proof of Theorem 2.2, we derive the following expression for \(S(x)\)
\[
S(x) = \sum_{(d)} \mu(d)S_d + O(n^{1/2}),
\]

where
\[
S_d = \sum_{u} e(xdu).
\]

Here \(u\) ranges over the sequence satisfying the conditions \(du \leq n\), \(du \equiv a \pmod{k}\). We have
\[
\sum_{(d)} \mu(d)S_d = \sum_{(d_0)} S_d - \sum_{(d_1)} S_d = S_0 - S_1 \quad \text{(say)},
\]

where \((d_0)\) is the sequence of elements of \((d)\) having an even number of divisors and \((d_1)\) those elements of \((d)\) having an odd number of divisors. We estimate \(S_0; S_1\) can be estimated in exactly the same way. Write \(\lambda = \nu^{2(m+1)}\), and divide \(S_0\) into three sums,
\[
(20) \quad S_0 = \sum_{d \leq \lambda} S_d + \sum_{\lambda < d \leq n\lambda^{-1}} S_d + \sum_{n\lambda^{-1} \leq d \leq n} S_d = T_1 + T_2 + T_3.
\]

It is understood of course that the index \(d\) ranges over the set \((d_0)\) satisfying the given inequalities.

To estimate \(T_1\), we observe that if \(d' = d/(k, d)\) and \(a'\) is a solution of the congruence \(dx \equiv a \pmod{k}\) and \(n_2 = n(k, d)/kd\), then
\[
S_d = \sum_{u \leq n_2} e(xd'(ku + a')).
\]

Consequently, \(|S_d| \leq q\); it follows that
\[
(21) \quad T_1 \ll n\nu^{-m+2}.
\]

To estimate \(T_2\), we apply Theorem 4.1. We have
\[
T = \sum_{d} \sum_{u} e(xdu)
\]
with the prescribed ranges of summation. We have here \(N_1 = n\), \(U_0 = \lambda\), \(U_1 = n\lambda^{-1}\), \(w = 1\). Theorem 4.1 yields
\[
(22) \quad T_2 \ll n(\nu^{-2m-1})^{1/2} \ll n\nu^{-m+2}.
\]

We turn now to the estimate of \(T_3\). We have
\[
T_3 = \sum_{d} \sum_{u} e(xdu),
\]

summed over the prescribed ranges for \(d\) and \(u\). Interchange the order summation, then
\[
T_3 = \sum_{u \leq \lambda} \sum_{n\lambda^{-1} \leq d \leq n/u} e(xdu) = \sum_{u} T(u)
\]
with the inner sum further restricted by the condition \( du \equiv a \pmod{k} \). We divide the sequence \((d)\) into two sequences \((d')\) and \((d'')\) where \((d')\) is the set of \((d)\) having all prime divisors \( \leq \nu^{3m} \) and \((d'')\) those elements of \((d)\) having at least one prime divisor \( > \nu^{3m} \). \((d_0)\) is then divided into two corresponding sets \((d'_0)\) and \((d''_0)\). We get \( T(u) = T'(u) + T''(u) \) where the right-hand summands correspond to the sets \((d'_0)\) and \((d''_0)\) respectively. We estimate now the number of terms \( D \) of the set \((d')\) which satisfy the conditions \( d \leq n/u \) and \( 1 \leq u \leq \lambda \). To this end suppose that an element \( d \) of \((d')\) have \( j \) prime divisors. Then \((\nu^{3m})^j \geq n\lambda^{-1} \), and hence if \( n \) be chosen sufficiently large \( j > \nu/6m \log v \). If then \( \tau(d) \) be the number of divisors of \( d \), we get

\[
\tau(d) = 2^j > 2^{v/6m \log v} > n^{1/9m \log v},
\]

and since

\[
\sum_{1 \leq v \leq n_1} \tau(v) \ll n_1(v + 1),
\]

where \( n_1 = n/u \), we conclude that

\[
Dn^{1/9m \log v} \ll n_1(v + 1) \ll n_1n^{1/9m \log \nu} n^{-1/9m \log \nu} v^{-m} (v + 1).
\]

Therefore \( D \ll n\nu^{-m} \). Consequently we deduce that

\[
T(u) = T''(u) + O(\nu^{-m}nu^{-1}).
\]

For the sum \( T''(u) \) we have evidently \( j < v \), hence

\[
T''(u) = \sum_j T_j(u)
\]

where \( T_j(u) \) is summed over those \( d \) belonging to \((d'_0)\) satisfying the inequalities \( n\lambda^{-1} < d \leq n_1 \), and having exactly \( j \) prime divisors \( > \nu^{3m} \). In order to estimate the sum \( T_j(u) \) we consider with Vinogradoff the more general sum

\[
T_j'(u) = \sum_v \sum_w e(xuvw)
\]

where \( v \) ranges over all primes \( > \nu^{3m} \) belonging to \((d)\) and, for given \( v \), \( w \) ranges over those numbers satisfying the inequalities \( n\lambda^{-1}/v < w \leq n_1/v \), the congruence \( uvw \equiv a \pmod{k} \) and containing exactly \( j - 1 \) prime divisors \( > \nu^{3m} \) and belonging to \((d_i)\). Every term \( e(xdu) \) of the sum \( T_j(u) \) is found in the sum \( T_j'(u) \) and indeed is found exactly \( j \) times. In addition, however, \( T_j'(u) \) contains terms of the form \( e(xp^2w_1) \) with \( n\lambda^{-1}/p^2 < w_1 \leq n_1/p^2 \), where \( p > \nu^{3m} \), and \( w_1 \) runs over elements of \((d_0)\) containing \( j - 2 \) prime divisors \( > \nu^{3m} \). These terms evidently occur without duplication. For given \( p \), the number of \( w_1p^2 \) satisfying \( n\lambda^{-1}/p^2 < w_1 \leq n_1/p^2 \) is \( \leq n_1/p^2 \), consequently

\[
T_j'(u) = jT_j(u) + \sum_{\nu^{3m} < p \leq n^{1/2}} n_1/p^2\]

\[
= jT_j(u) + O(\nu^{-3m}nu^{-1}).
\]
We now apply Theorem 4.1 to the sum \( T'(u) \). We take \( U_0 = \nu^{3m} \), \( U_1 = n^{1/2} \), \( N' = n\lambda^{-1} \), and conclude
\[
T(u) \ll n/u(\nu^{-3m+2})^{1/2} \ll \nu^{-3m/2+1}u^{-1/2}.
\]
Therefore, \( T'(u) \ll n^{-1/2}nu^{-3m/2+1} \), from which we deduce that
\[
T''(u) \ll nu^{-1/2}\nu^{-3m/2+1} \log \nu,
\]
and hence that
\[
T(u) \ll nu^{-1/2}\nu^{-3m/2+1} \log \nu + nu^{-1}\nu^{-m}.
\]
Summing over \( u \), we deduce that
\[
(23) \quad T \ll n\nu^{-m+1} \log \nu + n\nu^{-m} \log \nu \ll n\nu^{-m+2}.
\]
Using (19), (20), (21), (22), and (23), we conclude the following:

**Theorem 4.2.** Let \( m \) be any constant \( > 3 \),
\[
x = h/q + \theta/qr, \quad (h, q) = 1, \quad \nu^3m < q \leq \tau, \quad \tau = \nu^{-3m};
\]
then
\[
(24) \quad \sum_{(p)} e(xp) = O(n\nu^{-m+2}).
\]

5. The asymptotic formula and proof of the theorem.

**Theorem 5.1.** If \( m > 9/4 \), then
\[
(25) \quad I = \int_{-\tau^{-1}}^{\tau^{-1}} (I_2(z))^3e(-n\zeta)dz = \frac{n^2}{2\nu^3} + O\left(\frac{n^2}{\nu^{7/2}} \log \nu\right).
\]

**Proof.** Vinogradoff [2].

Using this result, we deduce readily that
\[
\sum_{q} (\phi(q/d))^{-3}\mu(q/d) \sum_{h} e(hb(a_1 + a_2 + a_3)/d - hn/q)I = O(n^2\nu^{-4}),
\]
where the inner sum ranges over the set of \( h \) such that \( 1 \leq h \leq q \), \( (h, q) = 1 \), and the outer sum over those \( q \) satisfying \( (q/d, d) = 1 \), \( q > \nu^3m \). This result, together with (17), permits us to conclude that
\[
(26) \quad \sum_{r} \int_{B_r} f(x)dx = \mathcal{S}(n) \frac{n}{2\nu^3} + O(n^2\nu^{-7/2} \log \nu),
\]
where \( \mathcal{S}(n) \), the singular series, is given by
\[
(27) \quad \mathcal{S}(n) = \frac{1}{\phi(k)^3} \sum_{\ell=1}^{\infty} \frac{\mu(\ell/d)}{\phi(q/d)^3} \sum_{1 \leq h \leq q} e(hb(a_1 + a_2 + a_3)/d - hn/q),
\]
where as above \( q \) is restricted by \( ((q/d, d) = 1 \) and \( h \) by \( (h, q) = 1 \). On the
other hand, using Theorem 4.2, we get

\[
\int_E f(x)dx \ll \int_E |S_1(x)S_2(x)S_3(x)| \, dx \\
\ll n^{\nu-m+2} \int_0^1 |S_2(x)S_3(x)| \, dx \\
\ll n^{\nu-m+2} \left( \int_0^1 |S_2(x)|^2 \, dx \right)^{1/2} \left( \int_0^1 |S_3(x)|^2 \, dx \right)^{1/2} \\
\ll n^{\nu-m+2} n^{\nu-1} \ll n^2 n^{\nu-m+1}.
\]

From (9), (26), and (28), we conclude

\[
A(n) = \mathcal{S}(n) \frac{n^2}{2(\log n)^3 \log \log n} + O\left( \frac{n^2}{(\log n)^{7/2}} \log \log n \right).
\]

On the other hand Rademacher has shown that if \( n \) is odd and \( n = a_1 + a_2 + a_3 \) (mod \( k \)), then

\[
\mathcal{S}(n) = \frac{C}{k^2} \prod_{p | k} \frac{p^3}{(p-1)^3 + 1} \prod_{p \mid n, p \mid k} \frac{(p-1)((p-1)^2 - 1)}{(p-1)^3 + 1} \prod_{p > 2} \left( 1 + \frac{1}{(p-1)^3} \right),
\]

where throughout \( p > 2 \), \( C = 2 \) for odd \( k \), and \( C = 8 \) for even \( k \). If \( n \) fails to satisfy the above conditions, then \( \mathcal{S}(n) = 0 \).

We formulate the:

**Main Theorem.** Let \( k \) be a positive integer, \( a_1, a_2, a_3 \) be residue classes modulo \( k \) with \( (a_i, k) = 1 \). If \( n \) is a sufficiently large odd integer satisfying the congruence \( n = a_1 + a_2 + a_3 \) (mod \( k \)), then \( n \) can be represented as a sum of three primes belonging respectively to the residue classes \( a_1, a_2, a_3 \), modulo \( k \). The asymptotic formula for the number of representations is given by (29).

6. **Concluding remarks.** The method of Linnik-Tchudakoff will provide another proof of this result. The corresponding question for the simultaneous Goldbach-Waring problem may be posed and solved.

**Bibliography**


Harvard University, Cambridge, Mass.