SIMULTANEOUS PARTITIONINGS OF TWO SETS

BY

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1. Introduction and definitions. In [5] R. H. Bing introduced the concept of partitioning a set. This concept has been applied by him and others as a means of obtaining many notable results [1; 2; 3; 4; 5; 6; 7; 8; 9]. In this paper an extension of this concept to the case of two sets is considered.

We shall consider only subsets of a metric space, particularly continuous curves and similar sets. A set $M$ is partitionable if for each $e > 0$, there exists a finite collection $G = \{g_1, \ldots, g_k\}$ of disjoint open connected subsets of $M$, such that $\sum g_i$ is dense in $M$ and $\delta(g_i) < e$ for each $i$. The mesh of $G$ equals $\max \delta(g_i)$, $1 \leq i \leq k$. Each $g_i$ is an element of $G$, written $g_i \in G$. $G$ is called an $e$-partitioning of $M$. If each element of $G$ has property $S$, $G$ is called an $S, e$-partitioning of $M$. $\{G_i\}$ is a decreasing sequence of $S$-partitionings of $M$ if (1) for each $i$, $G_i$ is an $S$-partitioning of $M$ and each element of $G_i$ is contained in an element of $G_{i-1}$ (that is, $G_i$ is a refinement of $G_{i-1}$), and (2) the limit, as $i$ increases without limit, of the mesh of $G_i$ is 0. Suppose $H$ and $G$ are two partitionings of a set $M$, $G$ a refinement of $H$. Let $h \in H$ and let $g_1, g_2, \ldots, g_k$ be all the elements of $G$ which are contained in $h$. $g_i$ is called an interior element of $G$ if $g_i \subset h$. Otherwise $g_i$ is a border element of $G$. $G$ is a core refinement of $H$ if for each $h \in H$ the sum of the closures of all interior elements of $G$ contained in $h$ is connected and intersects the closure of each border element of $G$ contained in $h$.

Let $M$ and $N$ be two partitionable sets such that $N \subseteq M$. If $G = \{g_1, g_2, \ldots, g_k\}$ is a partitioning of $M$ such that $G' = \{g_1N, g_2N, \ldots, g_kN\}$ constitutes a partitioning of $N$, then $G$ is called a simultaneous partitioning of $M$ and $N$. If $g_i$ and $g_iN$ both have property $S$ for each $i$, $G$ is a simultaneous $S$-partitioning of $M$ and $N$. If $\delta(g_i) < e$ for each $i$, $G$ is a simultaneous $S, e$-partitioning of $M$ and $N$. $\{G_i\}$ is a decreasing sequence of simultaneous core partitionings of $M$ and $N$ if the following conditions are satisfied. (1) $\{G_i\}$ is a decreasing sequence of $S$-partitionings of $M$, and if $g \in G_i$ then $gN$ has property $S$. (2) $G_i$ is a core refinement of $G_{i-1}$ for each $i$. (3) Let $g \in G_{i-1}$ and let $g_1, g_2, \ldots, g_k, g_{k+1}, g_{k+2}, \ldots, g_{k+n}$ be the elements of $G_i$ which are contained in $g$ and which intersect $N$ where $g_{i-1} \subset g$ if $i \leq k$ and $g_{k+i} \subset g$ if $i > k$. Then $\sum g_iN$ is connected and intersects $\text{Cl}(g_iN)$ for each $i$ from $k+1$ to

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(2) Numbers in brackets refer to the Bibliography at the end of the paper.
The question arises, what results regarding partitionings have analogies in the case of simultaneous partitionings?

2. Preliminary results. First let us recall some facts which have been previously established. An important one which characterizes partitionable sets is: Theorem A. A set $M$ is partitionable if and only if it has property $S$. This means that if we wish to partition a set $M$ into elements which themselves can be further partitioned, then these elements must have property $S$. Thus a fundamental result (Theorem B) is that if $M$ is a partitionable set and $\epsilon$ is an arbitrary positive number, then there exists an $S, \epsilon$-partitioning of $M$. A consequence of this is that if $M$ is partitionable, then there exists a decreasing sequence of $S$-partitionings of $M$ (Theorem C). Of course many other results have been obtained concerning partitionings, and many applications of this concept have been found.

Lemma 1. Let $M$ be a continuous curve, $U$ an open partitionable connected subset of $M$, and $H_1, H_2, \ldots, H_k$ a finite number of connected disjoint closed subsets of $U$ with property $S$. Then there exists a finite collection $\{W_1, W_2, \ldots, W_k\}$ of disjoint connected open subsets of $U$ with property $S$, whose sum is dense in $U$, with $W_i \supseteq H_i$ for each $i$.

Proof. Let $G_1$ be an $S, 1$-partitioning of $U$. Let $D = \sum_{i=1}^{n} H_i$. Let $\{p_i\}$, $i = 1, 2, \ldots, n$, be a collection of points obtained by selecting a point from each element of $G_1$. Let $A_1, A_2, \ldots, A_n$ be $n$ arcs in $U$ such that $A_i \supseteq p_i$ for each $i$, each $A_i$ intersects exactly one component of $D$, and $A_i A_j = 0$ if $A_i$ and $A_j$ intersect different components of $D$. For each $i$ from 1 to $k$, let $S_i(H_i)$ equal $H_i$ plus all the above arcs which intersect $H_i$.

Let $S'_i(H_i) = \{x \mid x \in S_i(H_i) \text{ and } \rho(x, \text{ bdy } U) \geq 1\}$. There exists a positive number $d_i$ such that $\rho(S'_i(H_i), S'_i(H_j)) > d_i$ if $i \neq j$. About each point of $S'_i(H_i)$ there exists an open connected set with property $S$ containing $p$ of diameter less than $\min (1, d_i/3)$. The closure of each such set thus is contained in $U$. A finite number of these sets cover $S'_i(H_i)$. Let $T'_i(H_i)$ equal the sum of their closures. Let $T_i(H_i) = T'_i(H_i) + S_i(H_i)$. Do this for each $i$ from 1 to $k$. Then each $T_i(H_i)$ is a closed connected partitionable subset of $U$ and $T_i(H_i) \cdot T_i(H_i) = 0$ if $i \neq j$.

Let $G_2$ be an $S, 1/2$-partitioning of $U$ which is a refinement of $G_1$. Let $D_1 = \sum_{i=1}^{n} T_1(H_i)$. Proceed in a manner exactly similar to that described in the above two paragraphs with only the added condition that each new arc must be contained in an element of $G_1$. Thus sets $T_2(H_i), i = 1, 2, \ldots, k$ are obtained which are each closed partitionable connected subsets of $U$ and such that $T_2(H_i) \cdot T_2(H_j) = 0$ if $i \neq j$.

Proceed similarly to obtain sets $T_n(H_i), i = 1, 2, \ldots, k; n = 3, 4, \ldots$, using $S$-partitionings $G_n$ of mesh less than $1/2^{n-1}$. Define $W_i = \sum_{n=1}^{\infty} T_n(H_i)$ for each $i$. All the conclusions are clearly satisfied except possibly that each $W_i$ has property $S$. To prove this, recall $W_i = \sum_{n=1}^{\infty} T_n(H_i)$. $T_{n+1}(H_i)$ is
obtained from $T_n(H_i)$ by adding a finite number of arcs, each of diameter less than $1/2^{n-2}$ and a finite number of connected sets with property S each of diameter less than $1/(2^{n-2})$. Hence any point in $T_{n+1}(H_i)$ can be joined to $T_n(H_i)$ by a connected set in $T_{n+1}(H_i)$ of diameter less than $1/(2^{n-2})$.

Given $\varepsilon > 0$ take $\varepsilon/3$. Since $\sum_{n=1}^{\infty} 1/2^n = 1$, there exists a positive integer $N$ such that each point of $W_i$ can be joined to $T_n(H_i)$ by a connected set in $W_i$ of diameter less than $\varepsilon/3$. Since $T_N(H_i)$ has property S it is the sum of a finite number of connected sets each of diameter less than $\varepsilon/3$, say $C_1, C_2, \cdots, C_m$.

Let $C'_i$ equal $C_i$ plus all points of $W_i$ which can be joined to it by a connected set of diameter less than $\varepsilon/3$ which lies in $W_i$. Then the diameter of $C'_i$ is less than $\varepsilon$ and $\sum_{i=1}^{n} C'_i = W_i$, thus proving that $W_i$ has property S.

**Lemma 2.** Let $U$ be an open partitionable connected subset of a continuous curve $M$, and $V$ a closed partitionable connected subset of $U$. Let $H$ and $K$ be two sets in $U$ such that $\rho(H, K) > 0$. Then $H$ and $K$ can be expanded into larger closed subsets of $U$, $H'$, and $K'$, such that $\rho(H', K') > 0$, and $H'$, $K'$, $H'V$, $K'V$ all have property S.

**Proof.** Let $\rho(H, K) = \varepsilon > 0$. Let $G$ ($G'$) be an $S, \varepsilon/6$-partitioning of $U$ ($V$).

Let $H'$ equal $H$ plus the closures (relative to $U$) of all elements of $G$ whose closures intersect $H$. Define $K'$ similarly.

Let $H''$ equal $H'$ plus the closures (relative to $V$) of all elements of $G'$ whose closures intersect $H'$. Define $K''$ similarly.

Then $\rho(H'', K'') > 0$ and $H''$, $K''$, $H''V$, and $K''V$ all have property S.

**Theorem 1.** Let $U$ be an open connected partitionable subset of a continuous curve $M$, $V$ a closed partitionable connected subset of $U$, and $H$ and $K$ two subsets of $U$ such that $\rho(H, K) > 0$. Then there exists a collection $\{W_1, W_2, \cdots, W_n\}$ of open disjoint connected subsets of $U$ whose sum is dense in $U$, such that no $W_i$ intersects both $H$ and $K$ but each $W_i$ intersects either $H$ or $K$, and each $W_iV$ is either void or an open connected subset of $V$ with property S, and $\sum_{i=1}^{n} W_iV$ is dense in $V$.

**Proof.** We may clearly suppose that $H$ and $K$ are closed subsets of $U$. It is also no loss in generality to assume that $(H+K)$ intersects each component of $V$. To prove this, let $C$ be a component of $V$ which contains no point of $(H+K)$. Then there exists an arc $A$ in $U$ intersecting $C$ and one of the sets $H$ and $K$ but lying at a positive distance from the other set. $A$ may be added to the set which it intersects. This procedure may be repeated until each component of $V$ contains a point of $(H+K)$. Also by Lemma 2 we may assume that $H$ and $K$ are closed subsets of $U$ such that that $H$, $K$, $HV$, and $KV$ all have property S.

Let $\{H_i\}, i = 1, 2, \cdots, n$, $\{K_i\}, i = 1, 2, \cdots, m$, $\{A_i\}, i = 1, 2, \cdots, r$, $\{B_i\}, i = 1, 2, \cdots, s$, be the components of $H, K, HV,$ and $KV$ respectively.

Let $G_i$ be an $S, 1$-partitioning of $V$. Let $\{p_i\}, i = 1, 2, \cdots, n_i$, be a col-
lection of points obtained by selecting a point from each element of $G$. Let \( \{ Z_i, i = 1, 2, \ldots, n \} \) be a collection of points obtained by selecting a point from each element of $G$. Let $A_i$ (or $B_i$ or $H_i$ or $K_i$) equal $A_i$ (or $B_i$ or $H_i$ or $K_i$) plus all the arcs $Z_i$ which intersect $A_i$ (or $B_i$ or $H_i$ or $K_i$). Let $H' = \sum_{i=1}^n H'_i$ and $K' = \sum_{i=1}^n K'_i$.

Let $G'$ be an $S$, $1$-partitioning of $U$. Let $\{ q_i, i = 1, 2, \ldots, m \}$, be the set of points obtained by selecting a point from each of the elements of $G'$ which does not intersect $V$. Join $q_i$ to some component of $(H' + K')$ by an arc entirely in $U - V$ except possibly for an end point if this is possible. If not, join $q_i$ to $V$ by an arc intersecting $V$ at only one point, say $y$, and then join $y$ by an arc in $V$ to some component of $(\sum_{i=1}^n A_i + \sum_{i=1}^n B_i)$. This can be done in such a way that we have a finite family of arcs $N_1, N_2, \ldots, N_m$, such that each $N_i$ contains $q_i$ for each $i$, each $N_i$ intersects exactly one component of $(H' + K')$ and at most one component of $(\sum_{i=1}^n A_i + \sum_{i=1}^n B_i)$.

For each $i$ from $1$ to $r$ let $S_i(A_i)$ equal $A_i$ plus the intersection of $V$ with all the above arcs which intersect $A_i$. Similarly define $S_i(B_i)$ for each $i$ from $1$ to $s$. For each $i$ from $1$ to $r$ let $S_i(H_i)$ (or $S_i(K_i)$) equal $H_i$ (or $K_i$) plus all the above arcs which intersect $H_i$ (or $K_i$).

Let $E_i = \{ x \mid x \in (\sum_{i=1}^r S_i(A_i) + \sum_{i=1}^s S_i(B_i)) \text{ and } \rho(x, \text{ bdy } U) \geq 1 \}$. This is a compact set. Let $\{ D_i, i = 1, 2, \ldots, r+s \}$ be the components of $(\sum_{i=1}^r S_i(A_i) + \sum_{i=1}^s S_i(B_i))$. Let $d_i$ be a positive number such that $\rho(E_i, D_i) > d_i$ if $i \neq j$. Around each point $p$ of $E_i$ there exists an open connected subset of $U$ containing $p$ with property $S$, of diameter less than $\min (1/2, d_i/3)$. The closure of each such set thus lies in $V$. A finite number of such sets cover $E_i$. Let $T_i(A_i)$ equal $S_i(A_i)$ plus the closures of all these finite number of sets which intersect $S_i(A_i)$, for each $i$ from $1$ to $r$. Define $T_i(B_i)$ similarly for each $i$ from $1$ to $s$. Then each $T_i(A_i)$ and each $T_i(B_i)$ is a connected partitionable closed subset of $V$, and no two of these sets intersect.

Let $F_i = \{ x \mid x \in (\sum_{i=1}^r S_i(H_i) + \sum_{i=1}^s S_i(K_i)) \text{ and } \rho(x, \text{ bdy } U) \geq 1 \}$. Again $F_i$ is compact. Let $\{ D'_i, i = 1, 2, \ldots, n+m \}$ be the components of $(\sum_{i=1}^r S_i(H_i) + \sum_{i=1}^s S_i(K_i))$. Let $d'_i$ be a positive number such that $\rho(F_iD'_i, D'_j) > d'_i$ if $i \neq j$. About each point $p$ of $F_i$ there exists an open partitionable connected subset of $U$ containing $p$ of diameter less than $\min (1/2, d'_i/3)$, whose closure lies in $U$ and does not intersect $V - (\sum_{i=1}^r T_i(A_i) + \sum_{i=1}^s T_i(B_i))$. A finite number of such sets cover $F_i$. For each $i$ from $1$ to $n$ let $T_i(H_i)$ equal $S_i(H_i)$ plus the closures of all these finite number of sets which intersect $S_i(H_i)$, plus all components of $(\sum_{i=1}^r T_i(A_i) + \sum_{i=1}^s T_i(B_i))$ which intersect $S_i(H_i)$. Define $T_i(K_i)$ similarly for each $i$ from $1$ to $m$. Note that each $T_i(H_i)$ and each $T_i(K_i)$ is a closed connected
Let $G_2$ be an $S, 1/2$-partitioning of $V$ which is a refinement of $G_1$. Proceed in exactly the same manner as in the second paragraph of this proof to obtain sets $T_i'(A_i), i=1, 2, \ldots, r$, $T_i'(B_i), i=1, 2, \ldots, s$, $T_i'(H_i), i=1, 2, \ldots, n$, and $T_i'(K_i), i=1, 2, \ldots, m$, with only the added restriction that each arc concerned must be contained in some element of $G_1$ and hence be of diameter less than 1.

Let $G'_2$ be an $S, 1/2$-partitioning of $U$ which is a refinement of $G_1$. Proceed in a manner similar to that described in the third paragraph of this proof to obtain sets $S_i(A_i), S_i(B_i), S_i(H_i), S_i(K_i)$, with only the added condition that if $M_i$ is one of the arcs concerned then $M_i U$ is contained in the closure of some element of $G_1$ and $M_i - V$ is contained in some element of $G'_1$.

Let $E_2 = \{x \mid x \in (\sum_{i=1}^r S_i(A_i) + \sum_{i=1}^s S_i(B_i)) \text{ and } \rho(x, \text{bdy } U) \geq 1/2\}$. Let \{ $X_i$ \}, $i=1, 2, \ldots, r+s$, be the components of $(\sum_{i=1}^r S_i(A_i) + \sum_{i=1}^s S_i(B_i))$. Let $d_2$ be a positive number such that $d_2 < \rho(x_i, x_j E_2)$ if $i \neq j$. Let $\eta'_2 = \min (1/4, d_2/3)$. About each point $p$ of $E_2$ there exists an open connected partitionable subset of $V$ of diameter less than $\eta_2$ containing $p$. A finite number of these sets cover $E_2$. Let $T_2(A_i) = S_2(A_i)$ plus the closures of all these finite number of sets which intersect $S_2(A_i)$, for each $i$ from 1 to $r$. Define $T_2(B_i)$ similarly for each $i$ from 1 to $s$.

Let \{ $Y_i$ \}, $i=1, 2, \ldots, n+m$, be the components of $(\sum_{i=1}^n S_i(H_i) + \sum_{i=1}^m S_i(K_i))$ and let $d'_2$ be a positive number such that $d'_2 < \rho(Y_i, Y_j F_2)$ if $i \neq j$ where $F_2 = \{x \mid x \in (\sum_{i=1}^n S_i(H_i) + \sum_{i=1}^m S_i(K_i)) \text{ and } \rho(x, \text{bdy } U) \geq 1/2\}$. Let $\eta'_2 = \min (1/4, d'_2/3)$. About each point $p$ of $F_2$ there exists an open connected partitionable subset of $U$ which contains $p$ and is of diameter less than $\eta'_2$, whose closure lies in $U$ and does not intersect $V - (\sum_{i=1}^r T_2(A_i) + \sum_{i=1}^s T_2(B_i))$. A finite number of these sets cover $F_2$. For each $i$ from 1 to $n$, let $T_2(H_i)$ equal $S_2(H_i)$ plus the closures of all these finite number of sets which intersect $S_2(H_i)$ plus all components of $(\sum_{i=1}^n T_2(A_i) + \sum_{i=1}^s T_2(B_i))$ which intersect $S_2(H_i)$. Define $T_2(K_i)$ similarly for each $i$ from 1 to $m$.

Thus we obtain sets $T_2(A_i), i=1, 2, \ldots, r$, $T_2(B_i), i=1, 2, \ldots, s$, $T_2(H_i), i=1, 2, \ldots, n$, $T_2(K_i), i=1, 2, \ldots, m$, all of which are closed partitionable connected subsets of $U$. Moreover each $T_2(A_i)$ is a component of $(\sum_{i=1}^r T_2(A_i) + \sum_{i=1}^s T_2(B_i))$ as is each $T_2(B_i)$. Also each $T_2(H_i)$ is a component of $(\sum_{i=1}^n T_2(H_i) + \sum_{i=1}^m T_2(K_i))$, as is each $T_2(K_i)$.

This procedure is repeated using $S$-partitionings $G_3, G_4, \ldots, G_i, G'_i, \ldots$, each $G_i$ (or $G'_i$) being a refinement of $G_{i-1}$ (or $G'_{i-1}$), and each $G_i$ (or $G'_i$) being of mesh less than $1/2^{i-1}$. Define $R_i = \sum_{i=1}^n T_i(H_i), i=1, 2, \ldots, n$, $S_i = \sum_{i=1}^s T_i(K_i), i=1, 2, \ldots, m$, $Q_i = \sum_{i=1}^r T_i(A_i), i=1, 2, \ldots, r$, $L_i = \sum_{i=1}^s T_i(B_i), i=1, 2, \ldots, s$.

Then the sets $R_i$ and $S_i$ are each open subsets of $U$ since each point of $R_i$ for example is an interior point of some $T_i(H_i)$. Also each of these sets is connected and their sum $Y$ is dense in $U$ and has property S for the same
reasons that each $W_i$ in Lemma 1 has property $S$. Each $R_i$ and each $S_i$ is a component of $Y$ and hence has property $S$ also. Similarly each $Q_i$ and each $L_i$ is an open connected subset of $V$ with property $S$. Also $(\sum_{i=1}^{t} Q_i, + \sum_{i=1}^{t} L_i)$ is dense in $V$ and actually $\sum_{i=1}^{t} Q_i = (\sum_{i=1}^{t} R_i) V$ and $\sum_{i=1}^{t} L_i = (\sum_{i=1}^{t} S_i) V$ so that each $Q_i (L_i)$ is a closed subset of some $R_j (S_j)$.

By Lemma 1, there exist sets $W_1, W_2, \ldots, W_r$, $r_1 \geq r$, in $\sum_{i=1}^{r} R_i$ such that $W_i V = Q_i$ for $1 \leq i \leq r$ and $W_i V = 0$ if $i > r$, $\sum_{i=1}^{r} W_i$ is dense in $\sum_{i=1}^{r} R_i$ and each $W_i$ is an open connected subset of $U$ with property $S$. Similarly there exist sets $W_{r_1+1}, \ldots, W_{r_1+s}$, $s \geq s$, such that $W_{r_1+i} V = L_i$ for $1 \leq i \leq s$ and $W_{r_1+i} V = 0$ if $i > s$, $\sum_{i=1}^{s} W_{r_1+i}$ is dense in $\sum_{i=1}^{s} S_i$ and each $W_{r_1+i}$ is an open connected subset of $U$ with property $S$.

3. The fundamental theorems.

Theorem 2. Let $M$ be a compact partitionable set and $N$ a closed partitionable subset. Then for every $\epsilon > 0$ there exists a simultaneous $S, \epsilon$-partitioning of $M$ and $N$.

Proof. It may easily be shown that it is sufficient to prove the theorem for the case when $M$ and $N$ are both connected (and hence continuous curves).

Given $\epsilon > 0$ take $\epsilon/4$. Pick a finite set of points, $\{p_i\}$, $i = 1, 2, \ldots, n$, such that if $p \in M$ then $D(p, p_i) < \epsilon/4$ for some $i$. Define $A_i = \{x | x \in M$ and $D(x, p_i) \leq \epsilon/4\}$, and $B_i = \{x | x \in M$ and $D(x, p_i) \geq \epsilon/2\}$. Then the sets $A_i$ and $B_i$ are at positive distance apart for each $i$. In particular $\rho(A_i, B_i) > 0$. By Theorem 1 there exist disjoint open connected subsets of $M$, say $V_1, V_2, \ldots, V_m$, such that no $V_i$ intersects both $A_1$ and $B_1$ but each $V_i$ intersects $(A_1+\sum A_1)$, $\sum_{i=1}^{m} V_i$ is dense in $M$ and has property $S$, each $V_i N$ is either void or an open connected subset of $N$ with property $S$ and $\sum_{i=1}^{m} V_i N$ is dense in $N$. Clearly if $V_i A_1 \neq 0$, $\delta(V_i) < \epsilon$. Let $X_1 (Y_1)$ be the sum of those $V_i$ which intersect $A_1 (B_1)$.

Let $V_n$ be an arbitrary component of $Y_1$. Then $V_n A_1 = 0$. Consider $V_n A_2$ and $V_n B_2$. If $V_n N$ is not void these two sets satisfy the conditions of Theorem 1 for $H$ and $K$ with $U = V_n$ and $V = V_n N$ and so we can apply the theorem again. If $V_n N = 0$, it can be $S, \epsilon$-partitioned (Theorem 4, p. 1104 of [5]). The procedure can be repeated until the components of $Y_1$ are exhausted. This gives us a simultaneous $S$-partitioning $G$ of $Y_1$ and $Y_1 N$. Let $X_2$ equal the sum of all elements of $G$ which are of diameter less than $\epsilon$. Let $Y_2$ equal the sum of the remaining elements of $G$. Proceeding similarly we obtain sets $X_3, Y_3, \ldots, X_{n-1}, Y_{n-1}$. Then the components of $(Y_{n-1} + \sum_{i=1}^{n-1} X_i)$ are the elements of a simultaneous $S, \epsilon$-partitioning of $M$ and $N$.

The following results may be obtained in the same manner and will be used several times.

Theorem 3. Let $M$ and $N$ be two continuous curves such that $N \subset M$ and let $U$ be an open partitionable connected subset of $M$ such that $UN$ is connected
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and partitionable. Then for every \( \epsilon > 0 \) there exists a simultaneous \( S, \epsilon \)-partitioning of \( U \) and \( UN \).

**Theorem 4.** Let \( M \) be a compact partitionable set and \( N \) a closed partitionable subset. Then there exists a sequence \( \{G_i\} \) of simultaneous \( S \)-partitionings of \( M \) and \( N \) such that \( G_i \) is a refinement of \( G_{i-1} \) of mesh less than \( 1/i \).

4. Remarks and examples. Theorem 2 would follow immediately (using Lemma 1) from Theorem B if for every \( S, \epsilon \)-partitioning \( G \) of \( M \) and every element \( g \) of \( G \) which intersects \( N \), \( gN \) had property \( S \). However this is not the case. Very simple examples can be found where \( G \) is an \( S \)-partitioning of \( M \), \( g \in G \), and \( gN \) does not have property \( S \). These and other examples show that it is not possible to prove Theorem 2 by working with \( M \) only or with \( N \) only at first, but rather the sets \( N \) and \( M \) must be considered simultaneously as was done. These examples can easily be constructed and so are not included.

A counterexample. One might wonder if the restriction that \( N \) must be contained in \( M \) could be removed. That is, consider the following theorem.

**False statement D.** Let \( M \) and \( N \) be two arbitrary continuous curves, \( \epsilon \) an arbitrary positive number, and \( U \) an open connected partitionable subset of \( (M+N) \) such that \( UM \) and \( UN \) are both connected and partitionable. Then there exists an \( S \)-partitioning \( G = \{g_1, g_2, \ldots, g_k\} \) of \( U \) such that \( \{g_iM\}, i=1, 2, \ldots, k, \) and \( \{g_iN\}, i=1, 2, \ldots, k, \) are \( S \)-partitionings of \( UM \) and \( UN \) respectively.

It can easily be shown that a consequence of this theorem is that if \( p \) is an arbitrary point of \( (M+N) \) and \( \epsilon \) an arbitrary positive number, there exists an open connected set \( U \) such that \( p \in U \), \( \delta(U) < \epsilon \), and \( UM \) and \( UN \) are both connected and partitionable. However consider a straight line \( L \) in euclidean 3-space and points \( p \) and \( p_1 \) on \( L \). Let \( p_2 \) be the mid-point of \( [p, p_1] \) and in general \( p_i \) be the mid-point of \( [p, p_{i-1}] \). Consider a sequence \( \{q_i\} \) of points defined as follows. Points \( q_1, q_2, \ldots, q_6 \) are the points, \( p_1k, p_2k-3, p_2k-2, p_2k-1, p_2k+4, p_2k+1 \), respectively, with \( k=1 \). The points \( q_7 \) to \( q_{24} \) are the points \( p_3k, p_4k-3, p_4k-2, p_4k-1, p_4k+4, p_4k+1 \), respectively, with \( k=2 \), and so on. Let \( \{A_i\} \) be a sequence of arcs such that (1) the end points of \( A_i \) are \( q_i \) and \( q_{i+1} \), (2) each pair of these arcs are disjoint except possibly for a common end point, and (3) \( \rho(x, [q_i, q_{i+1}]) < 1/2^i \) if \( x \in A_i \). Let \( N = p + \sum_{i=1}^\infty A_i \), and let \( M = [p, p_1] \). Then there do not exist arbitrarily small open subsets \( U \) of \( (M+N) \) containing \( p \) such that \( UM \) and \( UN \) are both connected. The proof is omitted. This shows that Theorem D is indeed a false theorem and thus that the results of this paper are not valid for two arbitrary continuous curves.

5. Some additional theorems.

**Theorem 5.** Let \( p \) be an arbitrary point of a compact partitionable set \( M \) and let \( N \) be a closed partitionable subset of \( M \) and let \( \epsilon \) be an arbitrary positive
number. Then there exists in $M$ an open partitionable connected set $U$ such that $p \in U$, $\delta(U) < \epsilon$, and $UN$ is either void or connected and partitionable.

**Proof.** (a) If $p \in (M - N)$ the proof is left to the reader. (b) Suppose $p \in N$. Given $\epsilon$, take $\epsilon/2$. By Theorem 2 there exists a simultaneous $\delta/2$-partitioning $G$ of $M$ and $N$. If $p$ belongs to an element of $G$, that element serves as the set $U$. Otherwise let $\{g_i\}, i = 1, 2, \ldots, k$, be the elements of $G$ such that $p \in \text{Cl} (g_i)$. Let $V$ be the interior of $\sum_{i=1}^k \text{Cl} (g_i)$. If $VN$ is connected we are done. If not $VN$ has a finite number of components and we may apply Lemma 1.

**Theorem 6.** Let $M$ be a compact partitionable set, $N$ a closed partitionable subset, $p$ an arbitrary point of $M$, and $\delta$ and $\epsilon$ any two positive numbers such that $\delta < \epsilon$. Then there exists an open set $U$ such that $U$ and $UN$ both have property $S$ and $A \subset U$ and $\rho(U, B) > 0$, where $A = \{x|x \in M$ and $D(x, p) \leq \delta/2\}$ and $B = \{x|x \in M$ and $D(x, p) \geq \epsilon/2\}$. (If $B$ is void omit the conclusion $\rho(U, B) > 0$.)

**Proof.** If $B = 0$, let $U = M$. Otherwise let $\gamma = (\epsilon - \delta)/2$. Let $G$ be a simultaneous $\delta'$-partitioning of $M$ and $N$ of mesh less than $\gamma/3$. Let $U$ equal the interior of the sum of the closures of all elements of $G$ whose closures intersect $A$. This set is of the required type. $U$ clearly has property $S$. $UN$ also has property $S$ because it is the sum of sets $gN$ where $gA \neq 0$, plus a subset of the limit points of this sum. The other conclusions of Theorem 6 are clearly satisfied.

**Theorem 7.** Let $M$ be a compact partitionable set and $N$ a closed partitionable subset of $M$. Then there exists a decreasing sequence $\{G_i\}$ of simultaneous $S$-partitionings of $M$ and $N$ such that if $g$ and $h$ are two elements of $G_i$ for which $gN \neq 0$, $hN \neq 0$, and $\rho(gN, hN) > 0$, then $\rho(g, h) > 0$.

**Proof.** We prove the theorem by showing that if partitionings $G_1, G_2, \ldots, G_{n-1}$ have been chosen so that the conclusions of Theorem 7 are satisfied for them, then $G_n$ can be chosen of the required type.

Let $H$ be any simultaneous $S$, $1/n$-partitioning of $M$ and $N$ which is a refinement of $G_{n-1}$. Let $\{h_i\}, i = 1, 2, \ldots, k$, be the elements of $H$ which intersect $N$ and $\{h_i\}, i = k+1, \ldots, k+t_1$, be the remaining elements. Consider $h_1$. If $\rho(h_1, h_1) = 0$ only if (a) $i > k$ or (b) $i \leq k$ and $\rho(h_1, h_iN) = 0$, pass on to $h_2$. However if, for some $i \leq k$, $\rho(h_1, h_i) = 0$ but $\rho(h_1, h_iN) > 0$, let $h_m$ be one such $h_i$ for which $\rho(h_1N, h_iN)$ is a minimum. Consider a simultaneous $S$-partitioning $H'$ of $M$ and $N$ which is a refinement of $H$ of mesh less than $(1/4) \cdot \rho(h_1N, h_mN)$. For each $i$ from 1 to $k$ let $h'_i$ equal the interior of the sum of the closures of all elements of $H'$ whose closures intersect $h_iN$ and let $\{h'_i\}, i = k+1, \ldots, k+t_2$, be the remaining elements of $H'$. Then if $i \leq k$, $\rho(h'_i, h'_i) > 0$ unless $\rho(h'_i, h'_iN) = 0$. 
Now omitting $h'_i$ and considering only the remaining elements of $H'' = \{h'_i\}, i=1, 2, \ldots, k+t_2$, treat $h'_2$ in a similar manner to the way in which $h_1$ was treated. After at most $k-1$ steps a partitioning is obtained which satisfies the conditions on $G_a$.

In fact this theorem may be strengthened so that if $\delta$ equals the minimum of $\rho(g, h)$ where $g$ and $h$ vary over all elements of $G_i$ which intersect $N$ and are such that $\rho(gN, hN)>0$, then the diameter of $g'$ is less than $\delta/H$ where $H$ is an arbitrary positive integer, and $g'$ is an arbitrary element of $G_i$ which does not intersect $N$.

**Theorem 8 (Core partitionings).** Let $M$ be a compact partitionable set and $N$ a closed partitionable subset. Then there exists a decreasing sequence of simultaneous core partitionings $\{G'_i\}$ of $M$ and $N$.

**Proof.** Let $G_1$ be a simultaneous $S, 1$-partitioning of $M$ and $N$, let $g \in G_1$ and let $G'_1$ be the partitioning of $g$ consisting of the single element $g$. If $gN=0$, it has previously been shown [7 and 8, Theorem 3, p. 1119] that there exists a core refinement of $G_1$ of mesh less than $1/2$. Thus suppose $gN \neq 0$.

Let $H$ be a simultaneous $S, 1/2$-partitioning of $g$ and $gN$. Let $T$ be a connected set in $g$ such that $TN \neq 0$, $T$ intersects every element of $H$, and $\rho(M-g, T)>0$. If $TN$ is not a connected set in $gN$ which intersects each element $h$ of $H$ for which $hN \neq 0$, a set $S$ may be obtained from $T$ by adding a finite number of dendrons in $gN$ chosen in such a way that $SN$ is a connected set in $gN$ which intersects each element $h$ of $H$ for which $hN \neq 0$. Note that $\rho(M-g, S) = \delta > 0$.

Let $H'$ be a simultaneous $S$-partitioning of $g$ and $gN$ which is a refinement of $H$ of mesh less than $\delta/3$ and which satisfies the following condition. If $h' \in H'$, $h'N \neq 0$, and $\rho(h', S) = 0$, then $\rho(h'N, S) = 0$ also. To show that $H'$ exists we may consider an arbitrary simultaneous $S$-partitioning $K$ of $g$ and $gN$ which is a refinement of $H$ of mesh less than $\delta/3$. If there exists an element $k$ of $K$ such that $kN \neq 0$, $\rho(k, S) = 0$, and $\rho(kN, S) = \gamma > 0$, it is merely necessary to consider a simultaneous $S$-partitioning of $k$ and $kN$ of mesh less than $\gamma/3$. If this procedure is repeated for all elements of $K$ similar to $k$, then the collection of all sets thus obtained plus all the unchanged elements of $K$ constitutes a partitioning satisfying the conditions on $H'$.

Let $C$ equal the sum of the closures of all elements of $H'$ whose closures intersect $S$, say $C = \sum_{i=1}^k \text{Cl}(h'_i)$ and let $D = \sum_{i=1}^k \text{Cl}(h'_i N)$. Then $C$ and $D$ are both connected sets in $g$. $C$ is connected because for each $i$ from 1 to $k$, $h'_i$ has a limit point in $S$. $D$ is connected because for each $i$ from 1 to $k$ for which $h'_i N \neq 0$, $h'_i N$ has a limit point in $SN$.

Consider the border elements $h_1, h_2, \ldots, h_m$ of $H$ which are contained in $g$. Consider an arbitrary component $K$ of $(\sum_{i=1}^m h_i) - C$. Thus $K$ will be contained in some element of $H$. If $KN=0$, leave $K$ alone. If $KN \neq 0$, $KN$ has a
finite number of components, \( X_1, X_2, \ldots, X_n \), and \( K \) may be partitioned into open disjoint connected partitionable subsets \( W_1, W_2, \ldots, W_n \) such that \( W_i \supset X_i \) for each \( i \) (Lemma 1). Treat each component of \( \bigcup_{i=1}^{n} h_i - C \) in this manner. Let \( \{ R_i \}, i = 1, 2, \ldots, r \), and \( \{ S_i \}, i = 1, 2, \ldots, s \), be the sets thus obtained which have a boundary point in common with \( \text{bdy} \ g \), where \( R_i N = 0 \) and \( S_i N \) is nonvoid and connected. These sets plus all the sets obtained by the same process whose closures lie in \( g \), plus all elements of \( H' \) which are contained in \( C + (g - \sum_{i=1}^{n} h_i) \), constitute a simultaneous \( S, 1/2 \)-partitioning \( G \) of \( g \) and \( gN \). The border elements of \( G \) are \( \{ R_1, \ldots, R_r, S_1, \ldots, S_s \} \) and all the remaining elements are interior elements (border and interior elements with respect to \( G_1 \)).

Suppose \( E \) equals the sum of the closures of all the interior elements of \( G \), defined above, say \( E = \sum_{i=1}^{r} \text{Cl} (g_i) \) and let \( F = \sum_{i=1}^{s} \text{Cl} (g_iN) \). Then \( E \) and \( F \) are both connected, as we shall now prove. Suppose \( E \) is not connected. Let \( E' \) be a component of \( E \) which does not intersect \( C \). Suppose \( E' \) contains an element \( g' \) of \( G \) which is contained in a border element \( h \) of \( H \). If \( g'N = 0 \), \( g' \) is a component of \( h - C \) and hence \( g' \) has a limit point in \( C \), since \( h \) is connected. If \( g'N \neq 0 \), \( g'N \) is a component of \( (h - C)N \) and hence has a limit point in \( CN \) because \( hN \) is connected. In each case \( E' \) intersects \( C \), contrary to assumption. \( E' \) cannot contain an element of \( H' \) which is contained in an interior element \( h \) of \( H \) for in this case \( E' \) would contain \( h \) and hence again intersect \( C \), a contradiction. Hence \( E \) is connected. We may proceed in exactly the same manner to prove that \( F \) is connected. The same arguments show that if \( g'' \) is an arbitrary border element of \( G \), then \( E \) contains a limit point of \( g''N \) and \( F \) contains a limit point of \( g''N \) unless the latter set is void.

Once \( G \) has been defined, this shows how the other elements of \( G_1 \) may be treated. By dealing with each of them in the same manner, we obtain a simultaneous \( S, 1/2 \)-partitioning \( G_2 \) of \( M \) and \( N \) of the required type. The method is general and so the proof is complete.

6. Unanswered questions. Consider the following two theorems which may possibly be true, and if so, might lead to further results.

**Conjecture E.** Let \( M \) and \( N \) be two continuous curves such that \( N \subset M \), \( U \) a connected, uniformly locally connected (ulc), open subset of \( M \) such that \( UN \) is ulc and connected, and \( \varepsilon \) an arbitrary positive number. Then there exists a simultaneous \( \varepsilon \)-partitioning \( G \) of \( U \) and \( UN \), such that if \( g \in G \) then \( g \) and \( gN \) both are ulc.

This is a stronger result than Theorem 2 since for subsets of a compact metric space ulc implies property S but not conversely.

**Conjecture F.** Let \( M \) and \( N \) be two continuous curves such that \( N \subset M \), and \( \varepsilon \) an arbitrary positive number. Then there exists a simultaneous \( S, \varepsilon \)-partitioning \( G \) of \( M \) and \( N \) such that if \( g \in G \) then \( gN = \text{Cl} (gN) \).
Bibliography


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