SINGULAR POINTS OF FUNCTIONAL EQUATIONS(1)

BY

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Let Φ be a continuous function defined on a neighborhood of the origin in the product $\mathfrak{X} \times \mathfrak{Y}$ of two real or complex Banach spaces with values in $\mathfrak{X}$, and suppose that $\Phi(0, 0) = 0$. For y near the origin in $\mathfrak{Y}$, we seek solutions $x \in \mathfrak{X}$ of the equation

$$\Phi(x, y) = 0.$$

Hildebrandt and Graves [9](2) showed that if the partial differential $d_x\Phi(0, 0; h)$, considered as a linear transformation of $\mathfrak{X}$, has a continuous everywhere-defined inverse, then there exists a unique continuous single-valued function ϕ defined on a neighborhood of the origin in $\mathfrak{Y}$ with values in $\mathfrak{X}$ such that $\phi(0) = 0$ and $\Phi[\phi(y), y] = 0$ for all y in this neighborhood. Graves [8; 2, p. 408] showed that if $d_x\Phi(0, 0; h)$ maps onto $\mathfrak{X}$, then there will be at least one solution corresponding to sufficiently small y.

Cronin [3] recently considered a case in which $d_x\Phi$ need not map onto $\mathfrak{X}$ and obtained, under suitable restrictions, theorems concerning the existence of solutions in terms of the topological degree theory.

While our methods are closely related to hers, we focus our attention on the problem of studying the branching of the solutions that this situation allows. In a particular case we are able to apply Dieudonné's modification [5] of the Newton polygon method to obtain results exactly parallel to some for ordinary algebraic functions over the real or complex field. It is also seen that the work of E. Schmidt [17], L. Lichtenstein [14], and R. Iglisch [12] for a class of nonlinear integral equations hold valid for a general class of functions defined on Banach spaces. Also, in both their work and that of T. Shimizu [18], the assumption of analyticity can be replaced by that of the existence of a few continuous derivatives. Further, because of the simpler form for the equations we derive, it is possible to study particular cases in terms of initially given data.

Our final part indicates briefly how these results can be applied to nonlinear differential equations with fixed end point boundary conditions. It is possible to treat questions of existence and uniqueness of solutions in the neighborhood of a given solution for a very general type of equation.

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(2) Numbers in brackets refer to the bibliography at the end.

PART I

1. The hypotheses. In the following we shall suppose that Φ can be expressed in the form $\Phi(x, y) = L(x) + F(x, y)$, where

(H1) *L is a linear operator mapping $\mathfrak{X}$ into $\mathfrak{X}$ with closed range and such that the null spaces of L and L^* have finite and equal dimension.*

(H2) *The function*(3) $F_y(x) = F(x, y)$ *is defined and continuous for small x and y, $F(0, 0) = 0$, and F satisfies the condition $\|F(x_1, y) - F(x_2, y)\| \leq M(x_1, x_2, y)\|x_1 - x_2\|$ where M is a non-negative real-valued function which goes to zero with its arguments.*

We seek solutions for the equation

$$(1) \qquad L(x) + F(x, y) = 0$$

which lie in a neighborhood of the initial solution (0, 0).

It is known that (H1) is satisfied if $L = \lambda I - K$, where some power of K is completely continuous [1, chap. X], or if 0 is a pole of the resolvent of L and the dimension of the null space of L is finite [6, p. 208]. Since hypothesis (H2) is rather unusual in form, it is appropriate that we show it can be satisfied under reasonable circumstances. We choose to connect this hypothesis with the concept of Fréchet differentiability [7; 9; 11].

THEOREM 1.1. (a) *If $\Phi(x, y) = L(x) + F(x, y)$, where L is linear and F satisfies* (H2), *then the partial Fréchet differential $d_x\Phi(0, 0; h)$ exists and equals $L(h)$.*

(b) *Conversely, if Φ has a partial Fréchet differential $d_x\Phi(x, y; h)$ which is continuous for (x, y) near $(0, 0)$ uniformly for $\|h\| = 1$, then $F(x, y) \equiv \Phi(x, y) - d_x\Phi(0, 0; x)$ satisfies* (H2).

Proof. The first part follows from (H2) and the definition of the Fréchet differential. To prove (b) we apply the mean-value theorem [7, p. 173] to conclude that

$$\begin{aligned} F(x_1, y) - F(x_2, y) &= \Phi(x_1, y) - \Phi(x_2, y) - d_x\Phi(0, 0; x_1 - x_2) \\ &= \int_0^1 \{d_x\Phi[x_2 + \theta(x_1 - x_2), y; x_1 - x_2] \\ &\qquad - d_x\Phi(0, 0; x_1 - x_2)\}\, d\theta. \end{aligned}$$

The conclusion then follows from the continuity of $d_x\Phi$.

The condition in (b) is immediately implied by the hypothesis that Φ is in Class $\mathfrak{C}'$ in a neighborhood of the origin in $\mathfrak{X} \times \mathfrak{Y}$. We show that it is also implied by the continuity of the Fréchet second differentials. Let $\mathfrak{U}$ and $\mathfrak{V}$ be two Banach spaces and $\mathfrak{B}(\mathfrak{U}, \mathfrak{V})$ be the space of bounded operators from

(3) Here and later it will sometimes be convenient to use the operator notation $F_y(x)$ for $F(x, y)$, when y is considered to be a fixed element.

$\mathfrak{U}$ to $\mathfrak{V}$ with the *strong* operator topology. Let Λ be any metric space and λ_0 one of its points. With this notation we can use the theorem of uniform boundedness to prove:

LEMMA. *If* $\lambda \rightarrow T_\lambda$ *is a continuous mapping of a neighborhood of* λ_0 *to* $\mathfrak{B}(\mathfrak{U}, \mathfrak{V})$ *in the strong topology, then there exist two positive numbers r and R such that if* dist $(\lambda_0, \lambda) < r$, *then* $\|T_\lambda\| < R$.

THEOREM 1.2. *If* $d_{xx}f(x, y; h, h')$ *and* $d_{xy}f(x,y; h, k)$ *are continuous for* (x, y) *near* $(0, 0)$ *for each fixed* (h, h') *and* (h, k), *then* $d_x f(x, y; h)$ *is continuous for* (x, y) *near* $(0, 0)$ *uniformly for* $\|h\| = 1$.

Proof. Take a fixed h' and let $\Lambda' = \mathfrak{X} \times \mathfrak{Y}$. The lemma implies that there exist positive numbers r' and R' depending on h' such that if x and y have norms less than r', then $\|d_{xx}f(x, y; h, h')\| \leqq R' \cdot \|h\|$ for all $h \in \mathfrak{X}$. This shows that $d_{xx}f$ is continuous in h at 0 *uniformly* for (4) $(x, y) \in \mathfrak{S}_x(r') \times \mathfrak{S}_y(r')$. Hence, for each fixed h' the function $d_{xx}f$ is continuous for (x, y, h) near $(0, 0, 0)$. Now allow h' to vary and set $\Lambda = \mathfrak{S}_x(r') \times \mathfrak{S}_y(r') \times \mathfrak{S}_x$. A second application of the lemma and the homogeneity of $d_{xx}f$ in the argument h show that there exist positive numbers r and R such that if x and y have norms less than r, then $\|d_{xx}f(x, y; h, h')\| \leqq R\|h\| \cdot \|h'\|$. An exactly similar argument can be applied to $d_{xy}f$. The two inequalities thus obtained enable us to complete the proof of the theorem by applying the mean-value theorem.

2. **An implicit function theorem.** We shall make essential use of the following theorem which can be easily derived. For convenience in our future reference we state the result in the form we shall need.

THEOREM 1.3. *Let* Λ *be a metric space and suppose that* $A_\lambda(x)$ *is continuous on* $\Lambda \times \mathfrak{S}_x(r)$ *to* $\mathfrak{X}$ *uniformly for* $x \in \mathfrak{S}_x(r)$. *Suppose that if* $x, x' \in \mathfrak{S}_x(r)$ *and* $\lambda \in \Lambda$, *then* $\|A_\lambda(x) - A_\lambda(x')\| \leqq \|x - x'\|/2$.

(a) *Then the equation* $x - A_\lambda(x) = u$ *has a unique solution* $x = f_\lambda(u)$ *for all* $u \in \mathfrak{S}_x(r/4)$ *and* $\lambda \in \Lambda_0 = \{\lambda; \|A_\lambda(0)\| \leqq r/4\}$.

(b) *The function* $f_\lambda(u)$ *is continuous on* $\Lambda_0 \times \mathfrak{S}_x(r/4)$ *to* $\mathfrak{S}_x(r)$ *uniformly on* $\mathfrak{S}_x(r/4)$.

(c) *If* $r < \infty$, *then* $f_\lambda(u)$ *is the uniform limit of the sequence of functions defined by*

$$f_\lambda^{(1)}(u) = u, \cdots, f_\lambda^{(n+1)}(u) = u + A_\lambda[f_\lambda^{(n)}(u)], \cdots.$$

For the sake of completeness, we state the following result, a special case of a theorem of Hildebrandt and Graves [9], which gives a complete answer to the question of existence and uniqueness of local solutions in the case that the transformation L is invertible.

(4) We denote solid closed spheres around the origin by $\mathfrak{S}_x(r) = \{x \in \mathfrak{X}; \|x\| \leqq r\}$, and spherical surfaces by $\Sigma_x(r) = \{x \in \mathfrak{X}; \|x\| = r\}$, with similar notations for other spaces. If the radius $r = 1$, it will be omitted.

THEOREM 1.4. *If L is invertible and* (H1) *and* (H2) *are satisfied, there exist two positive numbers d_1 and d_2 such that if $\|y\| \leqq d_1$, then there exists a unique solution with $\|x\| \leqq d_2$. Moreover the function $x = f(y)$ defined in this way is a continious function on $\mathfrak{S}_y(d_1)$ to $\mathfrak{S}_x(d_2)$.*

This result follows readily from Theorem 1.3.

3. **Reduction of the singular case.** We define $\mathfrak{W}$ to be the range of L and $\mathfrak{U}$ its null space, with $\mathfrak{W}^*$ and $\mathfrak{U}^*$ playing similar rôles for L^*. Then since $\mathfrak{W}$ is assumed to be closed we have(5)

LEMMA 1. *The subspaces $\mathfrak{W}$ and $\mathfrak{U}^*$ are orthogonal complements. The same is true for $\mathfrak{U}$ and $\mathfrak{W}^*$.*

Let $\{u_i: i=1, \cdots, n\}$ and $\{f_i: i=1, \cdots, n\}$ be fixed bases for $\mathfrak{U}$ and $\mathfrak{U}^*$ respectively, then there exist $\{g_i: i=1, \cdots, n\} \in \mathfrak{X}^*$ and $\{z_i: i=1, \cdots, n\} \in \mathfrak{X}$ such that $g_i(u_j) = f_i(z_j) = \delta_{ij}$. Let $\mathfrak{Z}$ be the subspace spanned by the $\{z_i\}$, and define the projection operators $U(x) = \sum_{i=1}^n g_i(x)u_i$ and $Z(x) = \sum_{i=1}^n f_i(x)z_i$. Set $L'(x) = L(x) - \sum_{i=1}^n g_i(x)z_i$.

LEMMA 2. *The operator L' has a continuous everywhere-defined inverse R such that*

$$L \cdot R = I - Z, \qquad R \cdot L = I - U.$$

Using these two lemmas we can easily prove

LEMMA 3. *If (x, y) is a solution of* (1), *then $Z \cdot F_y(x) = 0$ and $x + R \cdot F_y(x) \in \mathfrak{U}$.*

Guided by Lemma 3, we seek a condition on an element $u \in \mathfrak{U}$ in order that a solution of

$$x + R \cdot F_y(x) = u \tag{2}$$

will satisfy equation (1). By virtue of (H2) and Theorem 1.3, it follows that for any $u \in \mathfrak{S}_u(d_4)$ there exists a *unique* solution

$$x = V_y(u). \tag{3}$$

Moreover $V_y(u)$ is continuous on $\mathfrak{S}_y(d_3) \times \mathfrak{S}_u(d_4)$ to $\mathfrak{X}$ uniformly on $\mathfrak{S}_u(d_4)$, and can be obtained by a uniformly convergent iteration:

$$V_y^{(1)}(u) = u, \cdots, V_y^{(r+1)}(u) = u - RF_yV_y^{(r)}(u), \cdots. \tag{4}$$

From Lemma 3 we know that a necessary condition that this solution $V_y(u)$ of (2) be a solution of (1) is that

$$ZF_yV_y(u) = 0; \tag{5}$$

(5) Lemmas 1 and 2 are results from the abstract Fredholm theory due to Riesz [15] and Hildebrandt [10]; see also [1, Chap. X] and [16]. The proof of Lemma 1 is essentially in [1, p. 149–150; 16] and that of Lemma 2 in [1, p. 155; 10].

that is, $f_iF_yV_y(u)=0$ for $i=1, \cdots, n$.

Conversely, if we have a solution $x=V_y(u)$ of (2) which satisfies (5), it is not difficult to see that $V_y(u)$ gives a solution of (1) for that value of y. We state this formally:

THEOREM 1.5. *If* (H1) *and* (H2) *are satisfied, there exist positive numbers* d_3 *and* d_4 *such that if* $y\in\mathfrak{S}_y(d_3)$ *and* $u\in\mathfrak{S}_u(d_4)$, *then equation* (2) *has a unique solution* $x=V_y(u)$. *The function* $V_y(u)$ *can be computed by the uniformly convergent sequence* (4) *and is a continuous function on* $\mathfrak{S}_y(d_3)\times\mathfrak{S}_u(d_4)$ *to* $\mathfrak{S}_x(4d_4)$ *uniformly on* $\mathfrak{S}_u(d_4)$. *Moreover the pair* $(V_y(u), y)$ *is a solution of* (1) *if and only if* $V_y(u)$ *satisfies* (5).

COROLLARY. *If* $y_0\in\mathfrak{S}_y(d_3)$ *is such that* F_{y_0} *maps some sphere* $\mathfrak{S}_x(d_0)$ *with* $d_0\leqq d_4$ *into* $\mathfrak{W}$, *then equation* (1) *has a continuous family of solutions* $(V_{y_0}(u), y_0)$.

Theorem 1.5 shows that the problem of finding solutions of (1) is reduced to that of finding solutions of the equation $ZF_yV_y(u)=0$ in the neighborhood of the n-dimensional space $\mathfrak{U}$. In fact, *there is a one-to-one correspondence between local solutions* x *of* (1) *and local solutions* u *of* (5) *and this correspondence is given by* $x=V_y(u)$. Equation (5) plays the rôle of the "Verzweigungsgleichung" which appears in many applied problems [14]. For fixed bases of $\mathfrak{U}$ and $\mathfrak{U}^*$, we see that our problem can be formulated as that of finding solutions of the system of n real- or complex-valued continuous functions in the n scalar variables $(\xi_1, \cdots, \xi_n)$ given by

$$\phi_i(\xi_1, \cdots, \xi_n; y) = f_iF_yV_y\left(\sum_{j=1}^{n}\xi_ju_j\right) = 0, \qquad i = 1, \cdots, n. \tag{6}$$

In general it is not possible to give an explicit solution to this problem—however, in the important special case that $\mathfrak{U}$ is known to be one-dimensional the system reduces to a single equation in one scalar unknown and is more readily treated. We shall consider this case at length in later sections.

4. **A method of computation.** We have seen that the equation $x=u-RF(x, y)$ has the unique solution $x=V_y(u)$, for sufficiently small y. Substituting this into F, we have

$$F_yV_y(u) = F_y[u - RF_y(x)]. \tag{7}$$

By repeated substitution of $u-RF(x, y)$ for x, we can compute the expression $F_yV_y(u)$ or the solution $V_y(u)$ to as many terms as desired, since F contains no linear terms in x.

At this stage it is convenient to add the following hypothesis:

(H3) *There exists a positive number* m *such that if* x *and* y *are sufficiently small, then* $\|F(x, y)-F(x, 0)\|\leqq m\|y\|$.

With this hypothesis we can now prove:

THEOREM 1.6. *If* (H1), (H2), *and* (H3) *are satisfied and* $F_yV_y(u)$ *is written* *as* $F_0V_0(u)+B(u, y)$, *then there exists a positive* m' *such that* $\|B(u, y)\| \leqq m'\|y\|$ *for sufficiently small* u *and* y.

Proof. To show the inequality, it suffices to show that $\|V_y(u)-V_0(u)\| \leqq m'\|y\|$ for small u and y, for then we have

$$\begin{aligned}\|B(u, y)\| &= \|F_yV_y(u) - F_0V_0(u)\| \\ &\leqq \|F_yV_y(u) - F_0V_y(u)\| + \|F_0V_y(u) - F_0V_0(u)\| \\ &\leqq m\|y\| + M[V_y(u), V_0(u), 0]\|V_y(u) - V_0(u)\| \\ &\leqq (m + Mm')\|y\|.\end{aligned}$$

To prove the inequality on $V_y(u)$, we observe that (H3) implies that $\|R[F_yV_y(u)-F_0V_y(u)]\| \leqq m\|R\|\|y\|$, and that (H2) implies that $\|R[F_0V_y(u) - F_0V_0(u)]\| \leqq \|V_y(u)-V_0(u)\|/2$. By the definition of $V_y(u)$, we have the identity $[V_y(u)-V_0(u)]+R[F_yV_y(u)-F_0V_y(u)]+R[F_0V_y(u)-F_0V_0(u)] =0$. Transposing the middle term and using the triangle inequality and the two relations already mentioned, the conclusion follows.

We shall have use of this theorem in Part II.

Part II

5. A generalization of the Schmidt theory.

In this part we restrict our attention to complex scalars and suppose in addition to hypotheses (H1), (H2), and (H3) that the function F satisfies

(HA) *For each* $y \in \mathfrak{S}_y(r)$, *the function* F *is an analytic function of* x *in the Banach space sense* [11, p. 81] *for* $\|x\| < r$.

That is, we assume that $F(x, y) = \sum_{i=0}^{\infty} E_i(x, y)$, where for each fixed $y \in \mathfrak{S}_y(r)$ the function E_i is a continuous Banach space polynomial [11, p. 66] which is homogeneous of degree i in the argument x, and the series converges absolutely for $\|x\| < r$.

It will be noted that the case handled by E. Schmidt [17] certainly satisfies these conditions; in fact, his functions satisfy the condition of analyticity in both variables. We shall show, however, that the conditions we have stated are sufficient to obtain results generalizing his principal theorem [17, p. 398]. At the same time this section should provide insight into the methods used in the next part.

From the definition of $V_y(u)$ and the assumption (HA) it follows that $V_y(u)$ is an analytic function of u for each fixed and sufficiently small y, and that it is continuous in a neighborhood of (0, 0). Hence $F_yV_y(u)$ is an analytic function of u for fixed y. From Theorem 1.6 of §4 it follows that if we write this last function as a sum of $F_0V_0(u)$ and $B(u, y)$, the latter function is such that $\|B(u, y)\| \leqq m\|y\|$ when u and y are sufficiently small.

We now examine some of the consequences when $\mathfrak{U}$ is one-dimensional and $u_1 \in \mathfrak{U}$, $f_1 \in \mathfrak{U}^*$ are such that $\|u_1\| = \|f_1\| = 1$. In this case, the system of equations (5) becomes the single equation

$$(8) \qquad 0 = f_1F_yV_y(\xi u_1) = f_1F_0V_0(\xi u_1) + f_1B(\xi u_1, y).$$

The first summand is independent of y and so gives rise to an ordinary complex-valued analytic function $\sum_{i=2}^{\infty} a_i\xi^i$ of the complex variable ξ which converges absolutely for $|\xi| < r_1$. The second summand acts similarly yielding an analytic function $\sum_{i=0}^{\infty} b_i(y)\xi^i$ where the coefficients $b_i(y)$ are complex-valued continuous functions of the Banach space variable $y \in \mathfrak{S}_y(r_1)$. Theorem 1.6 tells us that $|f_1B(u, y)| \leqq \|B(u, y)\| \leqq m'\|y\|$. Hence equation (8) becomes

$$(9) \qquad 0 = \sum_{i=2}^{\infty} a_i\xi^i + \sum_{i=0}^{\infty} b_i(y)\xi^i.$$

Suppose that not all a_i vanish. Then the first function has a zero of multiplicity $k \geqq 2$ at the origin, k being the first index for which $a_k \neq 0$. Draw a circle of radius ρ around the origin so small that except for $\xi = 0$ the function $\sum a_i\xi^i$ does not vanish for $|\xi| \leqq \rho$. Now *on the circle* $|\xi| = \rho$, we have $|\sum a_i\xi^i| \geqq \delta > 0$. But if $\|y\| < \delta/m'$, then $|\sum b_i(y)\xi^i| \leqq |f_1B(\xi u_1, y)| \leqq \delta$ for all $|\xi| < r_1$. In particular this inequality holds on the circle, so we conclude from the theorem of Rouché that:

LEMMA. *Let k be the smallest index such that $a_k \neq 0$; suppose that k is finite. Then there exists a $\rho > 0$ and a $\sigma = \sigma(\rho) > 0$ such that if $\|y\| < \sigma$, then equation* (9) *has exactly k zeros* (*counting multiplicities*) *with* $|\xi| < \rho$.

In general there will be multiple roots of equation (9), which will be reflected in the coincidence of solutions of (1). We say that $V_y(\xi_0 u_1)$ is a *solution of multiplicity p* of equation (1) if ξ_0 is a zero of multiplicity p of (9). (This definition coincides with that of [13, p. 66] in the case discussed there.) It is not difficult to see that this leads to an unambiguous notion of multiplicity of solutions of equation (1). In terms of it we can summarize the preceding discussion.

THEOREM 2.1. *If F satisfies* (H1), (H2), (H3), *and* (HA) *and k is* [6] *as in the lemma, then there exist positive numbers ρ_0 and $\sigma = \sigma(\rho)$ defined for $0 < \rho \leqq \rho_0$ such that if $\|y\| < \sigma$, the equation* (1) *has exactly k solutions, counting multiplicities, in the neighborhood $\mathfrak{S}_x(\rho)$.*

PART III

6. The Taylor expansion.

In this part we shall assume in addition to hypothesis (H1) that the following is satisfied:

(HT) *In the neighborhood $\mathfrak{S}_x(\rho) \times \mathfrak{S}_y(\sigma)$ of the origin, F can be expanded in a finite Taylor expansion of the form*

$$F(x, y) = \sum_{i+j=1}^{m} F_{ij}(x, y) + \sum_{i+j=m} T_{ij}(x, y), \qquad m \geqq 2,$$

[6] It can be seen that k is the first integer for which $f_1E_k(u_1, 0) \neq 0$.

where F_{ij} is a continuous Banach space polynomial which is homogeneous of degree i in x and j in y, $F_{10}\equiv 0$, and $\lim_{x,y\to 0} T_{ij}(x, y)/\|x\|^i\|y\|^j=0$. We also assume that T_{ij} has a partial Fréchet differential with respect to x which is continuous in this neighborhood.

It can be shown that this hypothesis is satisfied if F is in Class $\mathfrak{C}^{(m+1)}$ in the sense of [9]. By Theorem 1.2, (HT) implies that (H2) and (H3) are satisfied. In the case of complex scalars, (HT) implies that F is an analytic function of the two variables and generalizes the cases considered by E. Schmidt [17], L. Lichtenstein [14, Chap. I], and T. Shimizu [18]. Our primary interest, however, is in the real case when the functions are not necessarily analytic. Even so we shall be able to sharpen the results of these authors.

We shall show that hypotheses (H1) and (HT) enable us to compute the terms of low degree in F_yV_y. These coefficients are relatively simple, and since they largely determine the behavior of the function near the origin, we are able to study the nature of the solutions. We employ the method of §4 to compute these terms by repeated substitution of $u-RF(x, y)$ for x. Since F contains no linear terms in x alone, this will lead to terms containing x to successively higher degrees. Denote the polarized form of F_{r0} by P_{r0}, so that $F_{r0}(u)=P_{r0}(u, \cdots, u)$. If $k\leqq m$ is the first index i for which F_{i0} does not vanish identically and $2k\leqq m+2$, we have

$$\begin{aligned} F_yV_y(u) = {} & F_{k0}(u)+\cdots+F_{2k-2,0}(u)+F_{01}(y) \\ & + \{F_{11}(u, y) - 2P_{20}[u, RF_{01}(y)]\} \\ & + \{F_{02}(y)+F_{20}[RF_{01}(y)] - F_{11}[RF_{01}(y), y]\} \\ & + \{\text{higher terms}\}. \end{aligned} \tag{10}$$

We have written all terms containing only u with degrees less than(7) $2k-1$ and those containing y with degrees less than 3. If $k>2$, or if F_{01} vanishes identically, there will be considerable simplification in the mixed and pure y terms.

7. **The one-dimensional problem.** In the following we shall assume that $\mathfrak{U}$ is one-dimensional and that u_1 and f_1 are elements of norm equal to 1 in $\mathfrak{U}$ and $\mathfrak{U}^*$. By virtue of (HT) we can see that equation (6) in §3 corresponds to the single equation

$$0 = \sum_{i+j=1}^{m} a_{ij}(y)\xi^i + \sum_{i+j=m} \psi_{ij}(\xi u_1, y) \tag{11}$$

where $a_{ij}(y)$ is a continuous scalar-valued function which is homogeneous of degree j in y, $\psi_{ij}(\xi u_1, y)=o(|\xi|^i\|y\|^j)$ and the equation is valid for $|\xi|<\rho_1$ and $\|y\|<\sigma_1$.

It will be convenient to study the nature of the solutions of equation (11)

(7) If $m+2<2k\leqq 2m$, terms after the mth degree also involve the functions T_{ij}.

when y is restricted to a set of the form $\{\lambda y_0; y_0 \text{ fixed}, |\lambda| < \sigma_1/\|y_0\|\}$ which we shall call a *one-dimensional neighborhood* of the origin through y_0. In defining one-dimensional neighborhoods, we shall always take $y_0 \in \Sigma_y$, the surfact of the unit sphere in $\mathfrak{Y}$.

In a one-dimensional neighborhood equation (11) reduces to an expansion in the two scalar variables ξ and λ; for, substituting λy_0 for y and using the homogeneity, equation (11) becomes

$$(12)\qquad 0 = \sum_{i+j=1}^{m-1} a_{ij}\xi^i\lambda^j + \sum_{i+j=m} \xi^i\lambda^j[a_{ij} + \phi_{ij}(\xi, \lambda)], \qquad |\xi| < \rho_1, \ |\lambda| < \sigma_1.$$

We observe that a_{i0} depends only on u_1, whereas the remaining a_{ij} and ϕ_{ij} depend on the choice of y_0 as well. In any case, (HT) implies that the ϕ_{ij} are continuous and have continuous partial derivatives with respect to ξ in this neighborhood, and that $\phi_{ij}(\xi, \lambda)$ approaches zero as ξ and λ do. In §9 we shall discuss the Newton polygon method which will give us a more or less systematic method of determining the nature of the solutions of ξ in terms of y in a one-dimensional neighborhood of the origin. Then, by assuming boundedness of certain functions, we shall be able to extend these results for *spherical* (i.e., ordinary) neighborhoods of the origin.

For future reference, we list here the first few coefficients of equation (12).

$$\begin{aligned}
a_{k0} &= f_1F_{k0}(u_1),\\
&\cdots\cdots\cdots,\\
a_{2k-2,0} &= f_1F_{2k-2,0}(u_1),\\
a_{2k-1,0} &= f_1F_{2k-1,0}(u_1) - kf_1P_{k0}[u_1, \cdots, u_1, RF_{k0}(u_1)],\\
&\cdots\cdots\cdots,\\
a_{01}(y_0) &= f_1F_{01}(y_0),\\
a_{11}(y_0) &= f_1F_{11}(u_1, y_0) - 2f_1P_{20}[u_1, RF_{01}(y_0)],\\
a_{02}(y_0) &= f_1F_{02}(y_0) - f_1F_{11}[RF_{01}(y_0), y_0] + f_1F_{20}[RF_{01}(y_0)],\\
&\cdots\cdots\cdots.
\end{aligned}$$

8. **Cases of uniqueness and nonexistence.** In the next section we shall outline a method of determining the nature of the solutions when not all of the a_{i0} and not all of the a_{0j} vanish. This method fails if all of the coefficients of either set are zero, but in this circumstance it is often possible to cancel appropriate factors of ξ or λ and reduce to an equation that can be treated by the method to be outlined. Throughout this section we suppose that all of the constants a_{i0} and the function ϕ_{m0} of equation (12) vanish. From the list above it is clear that this will certainly be true when $F_{i0}(\mathfrak{X}) \subset \mathfrak{W}$ and $T_{m0}(\mathfrak{X}, y_0) \subset \mathfrak{W}$. Additional assumptions on some other coefficients will be explicitly stated as made.

(a) If *all* of the coefficients of equation (12) vanish for some choice of y_0, then the equation is satisfied for arbitrary values of ξ. Hence, as we have seen in the Corollary to Theorem 1.5, equation (1) is solved by the pair $(V_{y_0}(\xi u_2), y_0)$ for any sufficiently small ξ. It is clear that this situation will occur when $F(\mathfrak{X}, y_0) \subset \mathfrak{W}$.

(b) Suppose that y_0 is such that $a_{01}(y_0) \neq 0$; then it is possible to divide equation (12) by λ. Since $a_{01} \neq 0$, there are no roots for ξ which go to zero with λ, and hence no local solutions for equation (1). We thus have

THEOREM 3.1. *If y_0 is such that $a_{i0}=0$, $\phi_{m0}=0$, and $a_{01}\neq 0$, then corresponding to this y_0 there are no local solutions of* (1).

REMARK. The hypotheses are satisfied if $f_1F_{i0}(\mathfrak{X})=0$ and $f_1T_{m0}(\mathfrak{X}, y_0)=0$, but $f_1F_{01}(y_0)\neq 0$.

(c) On the other hand, if *only* the linear term does not vanish identically, then equation (1) is the nonhomogeneous linear equation

$$L(x) + F_{01}(y_0) = 0. \tag{13}$$

From Theorem 3.1 we conclude that if y_0 is such that $a_{01}(y_0) = f_1F_{01}(y_0) \neq 0$, there will be no solutions of equation (13), whereas if y_0 is such that $a_{01}(y_0)=0$, the considerations of (a) show that a general solution of (13) is given by $x = V_{y_0}(\xi u_1) = \xi u_1 - RF_{01}(y_0)$, for any complex number ξ. This coincides with the usual condition of linear integral equations [14, p. 191] stating that equation (13) has a one-parameter (since $\mathfrak{U}$ is one-dimensional) family of solutions or no solution according as $F_{01}(y_0)$ is, or is not, orthogonal to the solution f_1 of the homogeneous adjoint equation.

(d) Now suppose that $a_{i0}=\phi_{m0}=a_{01}(y_0)=0$, but $a_{11}(y_0)\neq 0$. We shall show that this case gives rise to a unique local solution. As before we can disregard a factor of λ. Since $a_{11}\neq 0$, we can divide our equation by it and use the ordinary implicit function theorem to solve this expression for ξ and obtain a *unique* solution.

The preceding discussion has been for a fixed y_0; that is, in a one-dimensional neighborhood of the origin through y_0. We have shown that for each $y\in\Sigma_y$ with $a_{01}(y)\neq 0$ and $a_{11}(y)\neq 0$ there is a solution for λy when λ is sufficiently small, say $|\lambda| < s(y)$. In general, $s(y)$ will depend on y, and it is quite possible that it may get arbitrarily close to zero. If this happens, we cannot hope to find an open sphere around the origin in $\mathfrak{Y}$ such that there is a unique solution for each y in this sphere. The following theorem gives a condition which insures that the phenomenon just described does not occur.

THEOREM 3.2. *Suppose that a_{i0}, ϕ_{m0}, and a_{01} vanish for $y\in\mathfrak{Y}$ and $i=1, \cdots, m$, and assume that $a_{11}(y)$ is bounded away from zero on Σ_y. Then there exists a positive number σ_0 which is independent of y, such that if $\|y\| < \sigma_0$, then equation* (1) *has a unique local solution. Moreover, if the scalars are real, the solution will be real.*

REMARK. The hypotheses are satisfied if $f_1F_{i0}(\mathfrak{X})=0$, $f_1T_{m0}(\mathfrak{X}, y)=0$, and $f_1F_{01}(y)=0$ for all y, and $f_1F_{11}(u_1, y)$ is bounded away from zero for $y\in\Sigma_y$.

Proof. In equation (12) cancel λ and transpose the first term to the left side. This gives

$$(14)\qquad \begin{aligned} -a_{11}\xi = {} & [a_{02}\lambda + \cdots + \lambda^{m-1}(a_{0m}+\phi_{0m})] \\ & + \xi[a_{12}\lambda + \cdots + \lambda^{m-2}(a_{1,m-1}+\phi_{1,m-1})] + \cdots \\ & + \xi^{m-1}(a_{m-1,1}+\phi_{m-1,1}). \end{aligned}$$

Denoting the right side by $A_\lambda(\xi)$, we see that given $\delta>0$, if ξ and λ are sufficiently small, then for each fixed λ, $A_\lambda(\xi)$ will satisfy a Lipschitz condition in ξ with constant less than δ, uniformly on Σ_y. The uniformity is present because the a's and ϕ's are bounded on Σ_y. Since a_{11} is bounded away from zero, we can divide through by $-a_{11}$ and see that the function $(-1/a_{11})A_\lambda(\xi)$ possesses the same Lipschitz property. After further restricting λ so that this Lipschitz constant and the norm of $(-1/a_{11})A_\lambda(0)$ are sufficiently small, and putting $u=0$, we can apply Theorem 1.3. We conclude that for λ less than σ_0 in absolute value, no matter what $y\in\Sigma_y$ we choose, the equation has a unique solution for λy. This proves the theorem.

(e) A case which is similar to (b) is one in which in addition to all the a_{i0} and ϕ_{m0} vanishing, we know that all the a_{i1} and $\phi_{m-1,1}$ vanish as well, but $a_{02}\neq 0$. Here we cancel a factor of λ^2 and obtain the same conclusion as in Theorem 3.1 with a_{01} replaced by a_{02}.

(f) If, in (e), $a_{02}=0$ but $a_{12}\neq 0$, we can proceed as in (d) to obtain a unique solution. A result paralleling Theorem 3.2 is obtained if $a_{12}(y)$ is bounded away from zero on Σ_y.

9. **The Newton polygon method.** We now describe a technique of solving equation (12) for ξ in terms of λ for a fixed choice of y. The recent work of J. Dieudonné [5] allows us to employ this method in our present case and to determine which expansions will give rise to *real* solutions.

We assume that there exist integers k and p such that $a_{k0}\neq 0$ and $a_{0p}\neq 0$, and use these letters for the minimal such. We first plot, in the (i, j)-plane, the points for which $a_{ij}\neq 0$. Let $P_0=(k, 0)$ and rotate the i-axis around P_0 in the clockwise sense until it strikes some of the plotted lattice points in the first quadrant. Let $-\mu_1$ be the slope of the line obtained, and P_1 be the plotted point on this line with minimum i-coordinate. If P_1 lies on the j-axis we are through; if not, we continue this process obtaining a polygon with a finite number of sides $S_1, \cdots, S_g$. The point (i, j) is on S_1 if and only if $j+\mu_1 i=\mu_1 k$, all other plotted points satisfying $j+\mu_1 i>\mu_1 k$. We introduce the change of variable $\xi=\eta\lambda^{\mu_1}$ in equation (12) and observe that by further restricting the modulus of λ we can allow η to take values in as large a disk as we please without violating the requirement that $|\xi|<\rho_1$. Splitting equation (12) into a sum taken over S_1 and the remainder, and dividing by $\lambda^{\mu_1 k}$, we get

$$\sum_{S_1} a_{ij}\eta^i = -\sum \lambda^{j+\mu_1 i-\mu_1 k}\eta^i[a_{ij} + \phi_{ij}(\eta\lambda^{\mu_1}, \lambda)]. \tag{15}$$

Let $r=\inf\{j+\mu_1 i-\mu_1 k;\ (i,j)\notin S_1\}$ and note that the (i,j) with $j+\mu_1 i-\mu_1 k=r$ lie on the first line L_1 parallel to S_1 that contains any plotted points. Rearranging the sum in (15) we have

$$\sum_{S_1} a_{ij}\eta^i = -\lambda^r \sum_{L_1} a_{ij}\eta^i + \lambda^{r+s}(\cdots), \qquad s > 0. \tag{16}$$

Let γ_0 be a q-fold nonzero root of the polynomial

$$\sum_{S_1} a_{ij}\eta^i = 0, \tag{17}$$

so that $\sum_{S_1} a_{ij}\eta^i = (\eta-\gamma_0)^q p(\eta)$, where $p(\eta)$ is a polynomial in η which does not vanish in a neighborhood of γ_0. If the scalars are real we shall be interested in only the real roots of (17), but in both real and complex cases we discard the root $\eta=0$ as extraneous. Substituting the above identity in (16) and dividing by $p(\eta)$, which may require us to restrict η to a smaller neighborhood of γ_0, and setting $\beta(\eta)=\sum_{L_1} a_{ij}\eta^i/p(\eta)$, we obtain

$$(\eta - \gamma_0)^q = \lambda^r\{-\beta(\eta) + \lambda^s(\cdots)\}. \tag{18}$$

If $q=1$, we can apply directly the implicit function theorem to solve for η. If $q>1$, we suppose that $\beta(\gamma_0)\neq 0$, and attempt to extract the qth roots of this equation. If the scalars are complex, this can be done by restricting η so that $\beta(\eta)$ does not vanish and then restricting λ so that the binomial theorem can be applied. Allowing all q roots for $\lambda^{1/q}$, we can write

$$\eta - \gamma_0 = \lambda^{r/q}\{[-\beta(\eta)]^{1/q} + \lambda^s(\cdots)\} \tag{19}$$

and this expression is valid for (η, λ) in a suitably small neighborhood of $(\gamma_0, 0)$.

If the scalars are real and q is odd, then exactly *one* of these expansions will yield a real solution for η. On the other hand if q is even, then there will be either *no* real solution or *two* distinct real solutions according as $\lambda^r\beta(\gamma_0)$ is positive or negative. In case $\beta(\gamma_0)=0$, the above analysis is not valid, but similar considerations can be applied to higher order terms.

Since (HT) implies that the ϕ_{ij} have continuous partial derivatives with respect to ξ, it is clear that the function obtained from (19) by transposing $\eta-\gamma_0$ has a continuous partial derivative with respect to η in a neighborhood of the point $(\gamma_0, 0)$ which is equal to -1 at that point. Hence we can apply the ordinary implicit function theorem and solve for η in terms of λ, and hence evaluate ξ in terms of λ. We thus get q complex, or one, no, or two real solutions for ξ corresponding to this root γ_0 according to the possibilities we have mentioned.

We can then apply the same analysis to the other nonzero roots of (17) and then proceed to the other sides of the polygon. After a finite number of such steps this process comes to an end. It can be proved [5] that all solutions of (12) will be obtained by this program.

Throughout this discussion we have supposed that y_0 was given and held fixed. It does not appear to be possible to modify the argument so that one can treat equation (11) directly, since almost all of the coefficients depend upon y_0. However, in particular cases when some of the coefficients are bounded away from zero on Σ_y, this process can be applied along each one-dimensional neighborhood in such a way that the results hold true for spherical neighborhoods. We shall now consider a few such cases.

10. **The case of a simple vertical tangent.** We now employ the polygon method in the case that $a_{20}\neq 0$ and $a_{01}\neq 0$. When the functions are analytic it follows from Theorem 2.1 that there are two (possibly coincident) solutions for sufficiently small y. We shall show that if $a_{01}(y)\neq 0$, these two solutions are in fact distinct, and give conditions for these solutions to be real.

In this case the Newton polygon has a single side, and equation (17) is $a_{20}\eta^2+a_{01}=0$. If $a_{01}\neq 0$, this equation has two distinct nonzero roots γ^+ and γ^-, which are real if and only if a_{20} and a_{01} have opposite signs. Since only real roots give rise to real solutions, and $a_{01}(-y_0)=f_1F_{01}(-y_0)=-a_{01}(y_0)$, while a_{20} is of constant sign, in the real case there are no real solutions on one side of the one-dimensional neighborhood and two distinct real solutions on the other side. In the complex case there are always two distinct complex solutions under these conditions.

The remarks just made hold in one-dimensional neighborhoods. If we assume that $a_{01}(y)$ is bounded away from zero on Σ_y, the distinct roots $\gamma^+(y)$ and $\gamma^-(y)$ are both bounded away from zero. Since $\gamma^+(y)=-\gamma^-(y)$, there exists a positive number ρ_2 independent of y such that if $|\eta-\gamma^+|<\rho_2$, then $p(\eta)=a_{20}(\eta-\gamma^-)$ is bounded away from zero. Choose σ_2 independent of y with $\sigma_2\leqq\sigma_1$ and such that if $|\lambda|<\sigma_2$ and $|\eta-\gamma^+|<\rho_2$, then $|\xi|=|\eta\lambda^{1/2}|<\rho_1$ so that equations (15) and (16) are well-defined. Divide (16) by $p(\eta)$ and obtain an equation which corresponds to equation (18); and, since $q=1$ in this case, also corresponds to (19). The coefficients on the right side of (19) are bounded above on Σ_y (since $p(\eta)$ is bounded away from zero), so a further restriction that $|\lambda|<\sigma_3$ will insure that the function obtained from (19) by transposing $\eta-\gamma^+$ has a partial derivative with respect to η which is continuous and different from zero on this set. Hence we conclude from the implicit function theorem that if $|\lambda|<\sigma_3$, there exists a unique solution for $\eta-\gamma^+$ in terms of λy, for an arbitrary $y\in\Sigma_y$. An exactly similar argument can be applied to the root γ^-. We thus conclude the following

THEOREM 3.3. *Let $f_1F_{20}(u_1)\neq 0$ and suppose that $f_1F_{01}(y)$ is bounded away from zero on Σ_y. Then there exists a positive σ_0, independent of y, such that if $0<\|y\|<\sigma_0$ and*

(C) *the scalars are complex, then equation* (1) *has exactly two distinct local solutions;*

(R) *the scalars are real, then equation* (1) *has exactly no or two distinct real local solutions according as* $f_1F_{01}(y)$ *has the same or opposite sign as* $f_1F_{20}(u_1)$.

The same type of analysis given above leads to the more general

THEOREM 3.4. *Let* k *be the first integer such that* $f_1F_{k0}(u_1)\neq 0$ *and suppose that* $f_1F_{01}(y)$ *is bounded away from zero on* Σ_y. *Then there is a positive* σ_0, *independent of* y, *such that if* $0<\|y\|<\sigma_0$ *and*

(C) *the scalars are complex, then equation* (1) *has exactly* k *distinct local solutions;*

(R_O) *the scalars are real and* k *is odd, then equation* (1) *has a unique real local solution;*

(R_E) *the scalars are real and* k *is even, then the conclusion is the same as in Theorem* 3.3 (R) *with* F_{k0} *in place of* F_{20}.

In the complex case we can think of the solutions as arranged on k spheres $\mathfrak{S}_x(\rho_0)$, joined together in the same way as the Riemann surface for $x^k=y$ at the singular point (0, 0).

11. **A theorem of R. Iglisch.** We are now prepared to obtain, as a corollary of what we have done, a result which generalizes the principal theorem of [12] in two ways. In this paper, Iglisch considered a special case of the form $F(x, y)=G(x)-y$, where G is an analytic function containing no terms of degree less than two. Since he was interested only in real solutions, it is appropriate that we replace his assumption of analyticity with that of hypothesis (HT) where the only term containing y is the linear term $-y$. We shall suppose with him that $\mathfrak{U}$ is one-dimensional and that k is the first index such that $a_{k0}\neq 0$. His main result may now be stated.

THEOREM. *With these hypotheses, if* k *is odd, then for sufficiently small* y *there will always be real solutions for equation* (1). *If* k *is even, then there may or may not be a real solution. In particular, if* $y_0\in\mathfrak{Z}$ *is sufficiently small, then one of the elements* y_0 *and* $-y_0$ *will have no real solution and the other one will have exactly two real solutions.*

Proof. From §7 we have that $a_{01}(y)=-f_1(y)$. Now for $y_0\in\mathfrak{Z}$ with $y_0\neq 0$, we have $f_1(y_0)\neq 0$, for otherwise y_0 would be in the annihilator of f_1 which we know to be $\mathfrak{W}$. The final part of the theorem now follows from Theorem 3.4 (R_E). It remains to show that if k is odd, there will always be a real solution. If $a_{01}(y)\neq 0$ this follows from (R_O) of Theorem 3.4, while if $y\in\mathfrak{W}$ it follows from the Newton polygon method and the fact that a polynomial of odd degree and real coefficients has at least one real root.

12. **The case of a double point with distinct tangents.** In contrast to the case treated in §10, we now suppose that $a_{20}\neq 0$ but that $a_{01}=0$. We shall again give conditions that the solutions be real and distinct.

The polygon for this case has one side, and equation (17) is $a_{20}\eta^2+a_{11}\eta+a_{02}=0$, which has two distinct roots γ^+ and γ^- when $\Delta(y)=a_{11}^2-4a_{20}a_{02}$ is not zero. Both of these roots are real when the scalars are real and $\Delta(y)>0$. Hence, if these roots are nonzero, we can apply the process of §9 and obtain two distinct solutions. Note that if $\Delta\neq 0$ the roots are distinct so that $q=1$.

In order to pass to spherical neighborhoods, we assume that $\Delta(y)$ is bounded away from zero on Σ_y so that the roots can be uniformly separated. Select ρ_2 such that $p(\eta)=a_{20}(\eta-\gamma^-)$ is bounded away from zero for $|\eta-\gamma^+|<\rho_2$, and choose $\sigma_2\leqq\sigma_1$ such that if $|\lambda|<\sigma_2$ then $|\xi|=|\eta\lambda|<\rho_1$ for these η. The reasoning can now be completed as in §10.

In the parabolic case (when $\Delta(y_0)=0$) the roots coincide, so that we must examine the nature of the coefficients lying on the line L_1 with equation $i+j=3$. From §9, we conclude that, in the real case, there will be two distinct or no real solutions according as the quantity $\lambda^3\beta(\gamma^+)$, where $\beta(\eta)=(1/a_{20})(a_{30}\eta^3+a_{21}\eta^2+a_{12}\eta+a_{03})$, is negative or positive. If this quantity vanishes, still higher terms must be examined. We can summarize the results for the non-parabolic case in a theorem.

THEOREM 3.5. *Suppose that $a_{20}\neq 0$ and $a_{01}=0$, and the $\Delta(y)$ is bounded away from zero on Σ_y. Then there exists a positive σ_0, independent of y, such that if $0<\|y\|<\sigma_0$ and*

(C) *the scalars are complex, then equation* (1) *has two distinct local solutions;*

(R_H) *the scalars are real and $\Delta(y)>0$, then equation* (1) *has two distinct real local solutions;*

(R_E) *the scalars are real and $\Delta(y)<0$, then equation* (1) *has no real local solutions.*

REMARK. It follows from §7 that $a_{20}=f_1F_{20}(u_1)$ and if $F_{01}\equiv 0$ then $\Delta(y)=[f_1F_{11}(u_1, y)]^2-4[f_1F_{20}(u_1)][f_1F_{02}(y)]$. If F_{01} is not identically zero, the form of $\Delta(y)$ is somewhat more complicated.

13. **Another special case.** The last case we consider is that in which $k>2$, $a_{01}=0$, but a_{02} and a_{11} are nonzero. The polygon for this case has two sides, the first side giving rise to the equation $a_{k0}\eta^k+a_{11}\eta=0$, which has $k-1$ distinct nonzero roots when a_{11} is different from zero. If the scalars are real this side gives only one real root when k is even, and either no or two real roots when k is odd according as the signs of a_{k0} and a_{11} are the same or opposed. The other side of the polygon leads to the equation $a_{11}\eta+a_{02}=0$, which has a single real or complex root which is nonzero under these conditions.

Passage to spherical neighborhoods of the origin can be guaranteed by the assumption that $a_{11}(y)$ is bounded away from zero on Σ_y. We shall not go through the details of the arguments. It can also be seen that the solutions given by the two sides of the polygon will be distinct when y is small enough. We state formally:

THEOREM 3.6. *Let $a_{i0}=0$ for $2\leqq i<k$, but $a_{k0}\neq 0$; let $a_{01}(y)=0$ and $a_{02}(y)\neq 0$ for $y\in\Sigma_y$. Suppose that $a_{11}(y)$ is bounded away from zero on Σ_y. Then there exists a positive σ_0, independent of y, such that if $0<\|y\|<\sigma_0$ and*

(C) *the scalars are complex, then equation* (1) *has k distinct local solutions;*

(R_E) *the scalars are real and k is even, then equation* (1) *has two distinct real local solutions;*

(R_O) *the scalars are real and k is odd, then equation* (1) *has one or three solutions according as* a_{11} *and* a_{k0} *have the same or opposite signs.*

REMARK. The hypotheses are satisfied if $f_1F_{i0}(u_1)=0$ for $2\leqq i<k$, but is nonzero for k; if $F_{01}\equiv 0$ and $f_1F_{02}(y)\neq 0$, and if $f_1F_{11}(u_1, y)$ is bounded away from zero for $y\in\Sigma_y$. It is clear that in the complex case we can view the solutions as arranged on a cycle of order one and one of order $k-1$.

PART IV

14. Applications to differential equations.

It is easy to see how the preceding results can be applied to the nonlinear integral equations of E. Schmidt in cases where the functions are given in concrete cases so that the a_{ij} can be computed. J. Cronin [4] has also shown how this type of theory can be applied to elliptic partial differential equations. In this final section we discuss an ordinary differential equation of the form

$$\phi''(t) + \omega^2\phi(t) + f[t, \epsilon, \phi(t), \phi'(t)] = 0, \tag{20}$$

with the boundary conditions $\phi(0)=\phi(\pi)=0$. We suppose that the scalars are real and seek real solutions for ϵ near 0. For our purposes it will be appropriate to suppose that ϕ is in the Banach space of functions of Class $\mathfrak{C}''[0, \pi]$ where we take the norm $\|\phi\|=\sup|\phi(t)|+\sup|\phi'(t)|+\sup|\phi''(t)|$.

It is well known that the Green's function for the problem $\phi''(t)=0$, $\phi(0)=\phi(\pi)=0$ is

$$K(t, s) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin nt \sin ns}{n^2} = \begin{cases} t(1 - s/\pi), & 0 \leqq t \leqq s \leqq \pi, \\ s(1 - t/\pi), & 0 \leqq s \leqq t \leqq \pi, \end{cases}$$

which, as the kernel of an integral operator, has eigenvalues $\lambda_n=n^2$ and eigenfunctions $u_n(t)=(2/\pi)^{1/2}\sin nt$, for $n=1, 2, \cdots$, each eigenvalue being simple. We let $x(t)=\phi''(t)\in\mathfrak{C}[0, \pi]$ and consider x to be our unknown function. Conversely, given an x, we obtain ϕ and by ϕ' by $\phi(t)=-\int_0^\pi K(t, s)x(s)ds$ and $\phi'(t)=-\int_0^\pi K_t(t, s)x(s)ds$, so if x_1 and x_2 are two functions in $\mathfrak{C}[0, \pi]$, then the corresponding ϕ_i satisfy $\|\phi_1-\phi_2\|\leqq(1+k'+k)\|x_1-x_2\|$, where k' and k are bounds for the norms of the integral operators with kernels K_t and K. Substituting in equation (20) we obtain

$$\begin{aligned} x(t) - \omega^2\int_0^\pi K(t, s)x(s)ds& \\ + f\left[t, \epsilon, -\int_0^\pi K(t, s)x(s)ds, \ -\int_0^\pi K_t(t, s)x(s)ds\right] &= 0. \end{aligned} \tag{21}$$

If f contains no linear terms in x alone, this equation is of the same form as equation (1) with $L=I-\omega^2K$, where K is a completely continuous operator on $\mathfrak{X}=\mathfrak{C}[0, \pi]$ and the real parameter ϵ plays the rôle of y. If the function is chosen to have a certain amount of differentiability in its last three arguments, then the associated function $F(x, \epsilon)$ satisfies hypothesis (HT). In such a case we can apply our preceding results to derive information about solutions of (20) in the neighborhood of a given solution.

As a particular illustration, we consider

$$\phi''(t) + \omega^2\phi(t) + A(t)[\phi(t)]^2 + B(t)[\phi(t)]^3 + \epsilon[C(t) + D(t)\phi(t)] + \epsilon^2 E(t) = 0, \tag{22}$$

with the boundary conditions $\phi(0)=\phi(\pi)=0$, and seek solutions in a neighborhood of the trivial solution (0, 0). In the case that ω is not a positive integer there is a unique local solution, by Theorem 1.4. We suppose that $\omega=1$(8).

Bearing in mind that $u_1(t)=(2/\pi)^{1/2}\sin t$ and $f_1(x)=\int_0^\pi x(t)\cdot u_1(t)dt$, we can easily compute some of the coefficients of §7 to be

$$a_{20} = hA^*, \qquad A^* = \int_0^\pi A(t)(3 \sin t - \sin 3t)dt,$$

$$a_{30} = -hB^*, \qquad B^* = \int_0^\pi B(t)(3 - 4 \cos 2t + \cos 4t)dt,$$

$$a_{01} = hC^*, \qquad C^* = \int_0^\pi C(t) \sin t dt,$$

where h has been used to denote three different and unimportant positive constants. We have the following cases:

(1) If $A^*C^*<0$, then Theorem 3.3 implies that there are two distinct local solutions for ϵ small and positive and none for ϵ negative. If $A^*C^*>0$, this situation is reversed.

(2) If $A^*=0$, but $B^*C^*\neq 0$, then Theorem 3.4 implies that there is one local solution.

(3) If $A^*=B^*=0$, but $C^*\neq 0$, Theorem 3.1 implies that there are no local solutions.

If C vanishes identically the computation is facilitated and

$$a_{11} = -hD^*, \qquad D^* = \int_0^\pi D(t)(1 - \cos 2t)dt,$$

$$a_{02} = hE^*, \qquad E^* = \int_0^\pi E(t) \sin t dt,$$

(8) Almost all of the special results we cite are true without change if ω is an odd integer. In the case that ω is an even integer the appropriate conclusions can be determined in a similar manner.

so that if C vanishes identically we have:

(4) If $A^*E^* \neq 0$, Theorem 3.5 implies that there are either two distinct local solutions or none according as D^{*2} is greater or less than $4A^*E^*$.

(5) If $A^*=0$, but $B^*D^*E^* \neq 0$, then Theorem 3.6 implies that there are either one or three distinct solutions according as $B^*D^*\epsilon$ is positive or negative.

If both C and E vanish identically, it is readily seen that for ϵ sufficiently small, $\phi=0$ is automatically a solution. Under this hypothesis we also have:

(6) If $A^* \neq 0$, then Theorem 3.1 implies that there is also a nontrivial local solution.

(7) If $A^*=0$, then Theorem 3.3 implies that there are two nontrivial solutions when $B^*D^*\epsilon$ is negative, but none if it is positive.

It is interesting to note that if we replace (22) with

$$\phi''(t) + \phi(t) + A(t)[\phi'(t)]^2 + B(t)[\phi'(t)]^3 + \epsilon[C(t) + D(t)\phi(t)] + \epsilon^2 E(t) = 0,$$

and A^* by $A^{**}=\int_0^\pi A(t)(\sin t+\sin 3t)dt$ and B^* by $B^{**}=\int_0^\pi B(t)$ $(2 \sin 2t +\sin 4t)dt$, then all of the special results listed above remain true without change. Other modifications can be made as desired, and in this way we can obtain information about the existence of local solutions for differential equations of this form.

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