COMPOUND GROUP EXTENSIONS. II[1]

BY

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1. Introduction. Consider a system of groups and homomorphisms

$$
(1.1)\qquad
\begin{tikzcd}
\cdots \arrow[r] & R_0 \arrow[r, "\rho_0"] \arrow[d, "\beta_0"] & R_1 \arrow[d, "\beta_1"] & & R_3 \arrow[r, "\rho_3"] \arrow[d, "\beta_3"] & \cdots \\
\cdots \arrow[r] & S_0 \arrow[r, "\sigma_0"] & S_1 \arrow[r, "\sigma_1"] & S_2 \arrow[r, "\sigma_2"] & S_3 \arrow[r, "\sigma_3"] & \cdots
\end{tikzcd}
$$

satisfying the following conditions:

(1.2) each horizontal sequence is exact[2],

(1.3) commutativity holds in each rectangle,

(1.4) $\sigma_1\beta_1(R_1)=\sigma_1(S_1)$.

Diagram (1.1) may be thought of as extending to infinity in both directions, with only one gap, as shown. It will soon be apparent, however, that the "infinite part" has no bearing on our problem.

By a *completion* of (1.1) we mean a group R_2, together with three homomorphisms

$$\rho_1\colon R_1 \to R_2,\ \rho_2\colon R_2 \to R_3,\ \beta_2\colon R_2 \to S_2,$$

such that when these are inserted in (1.1), conditions (1.2) and (1.3) still hold.

It is easily seen that if (1.1) admits a completion, then

(1.5) $\rho_0(R_0)$ is a normal subgroup of R_1,

(1.6) $\beta_3[\rho_3^{-1}(0)]\subset\sigma_2(S_2)$.

Henceforth we shall assume that (1.1) satisfies conditions (1.2)–(1.6).

Two completions $(R_2, \rho_1, \rho_2, \beta_2)$ and $(R_2', \rho_1', \rho_2', \beta_2')$ will be called *isomorphic* if there exists an isomorphism $W\colon R_2' \approx R_2$ such that

$$(1.7)\qquad W\rho_1' = \rho_1, \qquad \rho_2 W = \rho_2', \qquad \beta_2 W = \beta_2'.$$

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(1) This work was partially supported by the Office of Scientific Research (ARDC). The terminology and notational conventions of [6] are freely used throughout. Numbers in brackets refer to the list of references at the end of the paper.

(2) The statement that a sequence is exact is to be understood as applying only in those groups which are both ranges and domains for indicated homomorphisms. Thus, condition (1.2) is not meant to imply that $\rho_0(R_0)=R_1$ or that $\rho_3^{-1}(0)=0$. When we wish to indicate that the initial homomorphism of an exact sequence is an isomorphism-into, or that the terminal homomorphism is onto, we attach appropriate zeros.

If $R_0=S_0=R_4=S_4=0$ and β_3 is an isomorphism of R_3 onto S_3, then conditions (1.5) and (1.6) are redundant, (1.4) becomes $\beta_1(R_1)=S_1$, and by comparing the present situation with diagram (1.2) of [6], we see that a "completion" is now virtually the same thing as a continuation of the normal homomorphism $\sigma_1\beta_1$: $R_1 \to S_2$. Thus the problem of classifying the completions of (1.1) is a generalization of the problem of classifying the continuations of a normal homomorphism. The case "$R_0=S_0=R_4=S_4=0$, β_3 iso-onto" will be called the "normal homomorphism case."

In this paper we shall show that the general problem represented by (1.1) can be reduced to the normal homomorphism case. That is, with each diagram (1.1) we shall associate a certain normal homomorphism whose continuations are in a natural 1-1 correspondence with the completions of (1.1). This, in conjunction with the results of [6], constitutes the solution of the more general problem. An application will be found in [7].

2. **The categorical case.** In this section we consider a diagram (1.1) satisfying conditions (1.2) and (1.3), and also satisfying

(2.1) $R_0=S_0=R_4=S_4=0$,

(2.2) β_1 is an isomorphism of R_1 onto S_1.

We note that (2.1) implies (1.5) and (1.6), while (2.2) implies (1.4).

THEOREM 2.3. *Let* (1.1) *satisfy conditions* (1.2), (2.1), *and* (2.2). *Then there exists one and only one isomorphism-class*(3) *of completions. Moreover, any two completions are isomorphic in only one way.*

Proof. This result has been observed (in essence) by Baer(4) and MacLane(4) in the following context: S_2 is the group $A(K)$ of all automorphisms of some group K; $R_1=S_1=$the group $I(K)$ of all inner automorphisms of K; $S_3=A(K)/I(K)$; β_1 is the identity mapping of $I(K)$ onto itself; σ_1 is the identity injection of $I(K)$ into $A(K)$; σ_2 is the natural homomorphism of $A(K)$ onto $A(K)/I(K)$. (In this context, R_3 is commonly denoted Q, and β_3 is denoted θ.) For our purposes, however, the special nature of these groups is irrelevant. The following reproduces Baer's construction in the notation of (1.1).

We may assume without loss of generality that $R_1=S_1\subset S_2$, $\beta_1=$identity mapping of S_1 onto itself, $\sigma_1=$identity injection of S_1 into S_2. Let $(\Gamma, \epsilon, \pi_1, \pi_2)$ denote the graph of β_3 rel. σ_2, in the sense of [6, §6]. That is, Γ denotes the subgroup of the direct sum(5) R_3+S_2 consisting of those ordered pairs (r_3, s_2) such that $\beta_3(r_3)=\sigma_2(s_2)$; the homomorphisms ϵ, π_1, π_2 displayed in the diagram

(3) Since an empty set is never regarded as an isomorphism-class (or as an equivalence class of any kind), there exists an isomorphism-class of completions if and only if there exists a completion.

(4) [1]; see also [4, §12], and [5, §3].

(5) We speak of the direct sum rather than direct product in keeping with our convention of writing most groups additively.

$$
(2.4)\qquad
\begin{tikzcd}
0 \arrow[r] & S_1 \arrow[r, "\epsilon"] \arrow[d, leftrightarrow, "\beta_1"] & \Gamma \arrow[r, "\pi_1"] \arrow[d, "\pi_2"] & R_3 \arrow[r] \arrow[d, "\beta_3"] & 0 \\
0 \arrow[r] & S_1 \arrow[r, "\sigma_1"] & S_2 \arrow[r, "\sigma_2"] & S_3 \arrow[r] & 0
\end{tikzcd}
$$

are defined by

$$
(2.5)\qquad \epsilon(s_1) = (0, s_1); \qquad \pi_1(r_3, s_2) = r_3; \qquad \pi_2(r_3, s_2) = s_2.
$$

It follows immediately from the definitions that both rectangles of (2.4) are commutative, and that $\epsilon^{-1}(0)=0$. Given any $r_3 \in R_3$, since $\sigma_2(S_2)=S_3$, we may choose $s_2 \in S_2$ such that $\sigma_2(s_2)=\beta_3(r_3)$. Then $(r_3, s_2)\in\Gamma$, and $\pi_1(r_3, s_2)=r_3$. Therefore $\pi_1(\Gamma)=R_3$. $\pi_1^{-1}(0)$ consists of all pairs (r_3, s_2) such that $\beta_3(r_3) =\sigma_2(s_2)$ and $r_3=0$: that is, all pairs $(0, s_2)$ such that $\sigma_2(s_2)=0$: that is, all pairs $(0, s_1)$, where $s_1\in S_1$. Clearly this equals $\epsilon(S_1)$. Therefore $(\Gamma, \epsilon, \pi_1, \pi_2)$ is a completion.

Now let $(R_2, \rho_1, \rho_2, \beta_2)$ be any completion; then conditions (1.2) and (1.3) hold in the diagram

$$
(2.6)\qquad
\begin{tikzcd}
0 \arrow[r] & S_1 \arrow[r, "\rho_1"] \arrow[d, leftrightarrow, "\beta_1"] & R_2 \arrow[r, "\rho_2"] \arrow[d, "\beta_2"] & R_3 \arrow[r] \arrow[d, "\beta_3"] & 0 \\
0 \arrow[r] & S_1 \arrow[r, "\sigma_1"] & S_2 \arrow[r, "\sigma_2"] & S_3 \arrow[r] & 0.
\end{tikzcd}
$$

From commutativity in the right-hand rectangle of (2.6), it follows that for every $r_2\in R_2$, $[\rho_2(r_2), \beta_2(r_2)]\in\Gamma$. Hence we can define $W: R_2\rightarrow\Gamma$ by

$$
(2.7)\qquad W(r_2) = [\rho_2(r_2), \beta_2(r_2)].
$$

Obviously W is a homomorphism. It follows immediately from the definition that $\pi_1 W=\rho_2$, and $\pi_2 W=\beta_2$; it follows from commutativity in the left-hand rectangle of (2.6), and from $\rho_2\rho_1=0$, that $W\rho_1=\epsilon$. Hence the appropriate versions of (1.7) are satisfied.

By the well known "five-lemma," which holds also for nonabelian groups, W is an isomorphism of R_2 onto Γ.

Formula (2.7) is "forced" by conditions (1.7b) and (1.7c), which, as we indicated above, are written $\pi_1 W=\rho_2$ and $\pi_2 W=\beta_2$, respectively, in the present notation. This proves that there is only one isomorphism[6] of the arbitrary completion $(R_2, \rho_1, \rho_2, \beta_2)$ onto the particular completion $(\Gamma, \epsilon, \pi_1, \pi_2)$. It follows by an obvious argument that any two completions are isomorphic in only one way. Q.E.D.

In view particularly of the final assertion of Theorem 2.3, we may say that for any use to which the graph $(\Gamma, \epsilon, \pi_1, \pi_2)$ may be put, an entirely

[6] In fact, there is only one "homomorphism," in the obvious sense; moreover, condition (1.7a) is redundant in the concept of a "homomorphism" of one completion into another, under the conditions of Theorem 2.3.

arbitrary completion will serve just as well. However, we shall adhere to the particular completion $(\Gamma, \epsilon, \pi_1, \pi_2)$ as a matter of convenience.

3. **The case** $R_0=S_0=R_4=S_4=0$. In this section we assume that (1.1) satisfies (1.2)–(1.4) and (2.1). In the presence of (2.1), (1.4) may be rewritten $\beta_1(R_1)=S_1$. As before, we let $(\Gamma, \epsilon, \pi_1, \pi_2)$ denote the graph of $\beta_3\colon R_3\to S_3$ rel. $\sigma_2\colon S_2\to S_3$; we assume, for simplicity of notation, that $S_1\subset S_2$ and that σ_1 is the identity injection of S_1 into S_2. Then conditions (1.2) and (1.3) hold in the diagram

$$
\begin{tikzcd}
0 \arrow[r] & R_1 \arrow[d, "\beta_1"] & & R_3 \arrow[r] & 0 \\
0 \arrow[r] & S_1 \arrow[r, "\epsilon"] \arrow[d, leftrightarrow] & \Gamma \arrow[r, "\pi_1"] \arrow[d, "\pi_2"] & R_3 \arrow[r] \arrow[u, leftrightarrow] \arrow[d, "\beta_3"] & 0 \\
0 \arrow[r] & S_1 \arrow[r, "\sigma_1"] & S_2 \arrow[r, "\sigma_2"] & S_3 \arrow[r] & 0.
\end{tikzcd}
\tag{3.1}
$$

Let (3.1′) denote diagram (3.1) with the bottom row deleted. By a *completion* of (3.1) we mean a completion of (3.1′). But as we indicated in §1, a completion of (3.1′) may be regarded as the same thing as a continuation of the normal homomorphism $\epsilon\beta_1\colon R_1\to\Gamma$. It turns out that the completions of (1.1) [assuming (2.1)] are in a natural 1-1 correspondence with the completions of (3.1). Hence we may say that the case

$$R_0 = S_0 = R_4 = S_4$$

"factors," in the manner indicated by (3.1), into the normal homomorphism case and the categorical case. Formally:

THEOREM 3.2. *Let* (1.1) *satisfy conditions* (1.2)–(1.4) *and* (2.1). *Let* $(\Gamma, \epsilon, \pi_1, \psi_2)$ *denote the graph*[7] *of* $\beta_3\colon R_3\to S_3$ rel. $\sigma_2\colon S_2\to S_3$. *Then the completions of* (1.1) *are in a natural* 1-1 *correspondence with the completions of* (3.1).

COROLLARY 3.3. *Under the hypotheses of Theorem* 3.2, *the isomorphism-classes of completions of* (1.1) *are in a natural* 1-1 *correspondence with the isomorphism-classes of continuations of the normal homomorphism*

$$\epsilon\beta_1\colon R_1 \to \Gamma.$$

By way of proof of Theorem 3.2, we shall describe the required constructions, omitting detailed verification.

Let $(R_2, \rho_1, \rho_2, \delta)$ be a completion of (3.1). Then conditions (1.2) and (1.3) hold in

(7) Instead of requiring that $(\Gamma, \epsilon, \pi_1, \pi_2)$ be the actual graph, it is sufficient, by Theorem 2.3, to demand that Γ be a group, and ϵ, π_1, π_2 be homomorphisms, such that conditions (1.2) and (1.3) hold throughout (3.1).

$$
\begin{tikzcd}
0 \arrow[r] & R_1 \arrow[r, "\rho_1"] \arrow[d, "\beta_1"] & R_2 \arrow[r, "\rho_2"] \arrow[d, "\delta"] & R_3 \arrow[r] \arrow[d, leftrightarrow] & 0 \\
0 \arrow[r] & S_1 \arrow[r, "\epsilon"] \arrow[d, leftrightarrow] & \Gamma \arrow[r, "\pi_1"] \arrow[d, "\pi_2"] & R_3 \arrow[r] \arrow[d, "\beta_3"] & 0 \\
0 \arrow[r] & S_1 \arrow[r, "\sigma_1"] & S_2 \arrow[r, "\sigma_2"] & S_3 \arrow[r] & 0.
\end{tikzcd}
\tag{3.4}
$$

It follows immediately that $(R_2, \rho_1, \rho_2, \pi_2\delta)$ is a completion of (1.1).

Conversely, let $(R_2, \rho_1, \rho_2, \beta_2)$ be a completion of (1.1). Define $\delta: R_2 \to \Gamma$ by

$$\delta(r_2) = [\rho_2(r_2), \beta_2(r_2)]. \tag{3.5}$$

Then conditions (1.2) and (1.3) may be verified throughout the top half of (3.4). Therefore $(R_2, \rho_1, \rho_2, \delta)$ is a completion of (3.1).

It is easily seen that these constructions are inverses of each other. Q.E.D.

4. **The general case.** When (1.1) satisfies only conditions (1.2)–(1.6), it is easily seen that its completions are in a natural 1-1 correspondence with the completions of the following diagram derived from (1.1):

$$
\begin{tikzcd}
0 \arrow[r] & R_1/\rho_0(R_0) \arrow[d, "(\sigma_1\beta_1)_\#"] & & \rho_3^{-1}(0) \arrow[r] \arrow[d, "\beta_3"] & 0 \\
0 \arrow[r] & \sigma_2^{-1}(0) \arrow[r] & S_2 \arrow[r, "\sigma_2"] & \sigma_3^{-1}(0) \arrow[r] & 0,
\end{tikzcd}
\tag{4.1}
$$

where $(\sigma_1\beta_1)_\#: R_1/\rho_0(R_0) \to \sigma_2^{-1}(0)$ is the homomorphism induced[8] by $\sigma_1\beta_1: R_1 \to \sigma_2^{-1}(0)$.

Let $(\Gamma, \epsilon, \pi_1, \pi_2)$ denote the graph of $\beta_3 | \rho_3^{-1}(0)$ rel. $\sigma_2: S_2 \to \sigma_3^{-1}(0)$, in the sense of [6, §6]. Then, combining the obvious reduction of (1.1) to (4.1) with Corollary 3.3, we obtain

THEOREM 4.2. *The isomorphism-classes of completions of* (1.1) *are in a natural* 1-1 *correspondence with the isomorphism-classes of continuations of the normal homomorphism* $\epsilon(\sigma_1\beta_1)_\#: R_1/\rho_0(R_0) \to \Gamma$.

5. **Modular structures.** We supplement the above results with a theorem concerning modular structures in a system (1.1). For simplicity of notation, we shall confine attention to the case $R_0 = S_0 = R_4 = S_4 = 0$; the theorem will be obviously applicable to the general case via diagram (4.1). We shall also assume, for simplicity of notation, that $S_1 \subset S_2$ and that σ_1 is the identity mapping of S_1 into S_2; then $S_1 = \beta_1(R_1) = \sigma_2^{-1}(0)$.

Let $(\Gamma, \epsilon, \pi_1, \pi_2)$ denote the graph of $\beta_3: R_3 \to S_3$ rel. $\sigma_2: S_2 \to S_3$. Let X denote the subgroup $(\epsilon\beta_1)^{-1}(0) = \beta_1^{-1}(0)$ of R_1; let $N = \epsilon\beta_1(R_1) = \epsilon(S_1) = \pi_1^{-1}(0) \subset \Gamma$. Let

---

[8] Note that $\sigma_2\sigma_1\beta_1 = 0$ and $\sigma_1\beta_1\rho_0 = \sigma_1\sigma_0\beta_0 = 0$.

$$\begin{aligned} \lambda\colon A_X(R_1) &\rightarrow A_X(R_1)/c(X), \\ (\beta_1)_\#\colon A_X(R_1) &\rightarrow A(S_1), \\ (\beta_1)_*\colon A_X(R_1)/c(X) &\rightarrow A(S_1), \\ (\epsilon\beta_1)_\#\colon A_X(R_1) &\rightarrow A(N), \\ (\epsilon\beta_1)_*\colon A_X(R_1)/c(X) &\rightarrow A(N), \\ C'\colon S_2 &\rightarrow A(S_1), \\ C''\colon \Gamma &\rightarrow A(N), \\ \epsilon_\#\colon A(S_1) &\approx A(N) \end{aligned}$$

denote the natural homomorphisms, in the manner of [6, §2]. Then the following relations are easily verified:

$$\epsilon_\#(\beta_1)_\# = (\epsilon\beta_1)_\#, \qquad \epsilon_\#(\beta_1)_* = (\epsilon\beta_1)_*, \qquad \epsilon_\# C'\pi_2 = C''. \tag{5.1}$$

Given a completion $(R_2, \rho_1, \rho_2, \beta_2)$, we let $C\colon R_2 \rightarrow A_X(R_1)$ denote the homomorphism provided by (a) conjugation of elements of $\rho_1(R_1)$ by elements of R_2, (b) the isomorphism of $A[\rho_1(R_1)]$ onto $A(R_1)$ induced by $\rho_1\colon R_1 \approx \rho_1(R_1)$. We say that a modular structure $\theta\colon \Gamma \rightarrow A_X(R_1)/c(X)$ on the normal homomorphism $\epsilon\beta_1\colon R_1 \rightarrow \Gamma$ is *associated* with the completion $(R_2, \rho_1, \rho_2, \beta_2)$ if it is associated with the continuation of $\epsilon\beta_1\colon R_1 \rightarrow \Gamma$ which corresponds to $(R_2, \rho_1, \rho_2, \beta_2)$ via Theorem 3.2: that is, if

$$\theta[\rho_2(r_2), \beta_2(r_2)] = \lambda C(r_2) \qquad \text{for every } r_2 \in R_2. \tag{5.2}$$

According to [6], the continuations of $\epsilon\beta_1\colon R_1 \rightarrow \Gamma$ divide into classes according to the modular structures with which they are associated. Hence the completions of (1.1) divide into classes according to the modular structures on $\epsilon\beta_1\colon R_1 \rightarrow \Gamma$ with which they are associated, in the sense of (5.2). In particular, we might be given a definite modular structure θ on $\epsilon\beta_1\colon R_1 \rightarrow \Gamma$, and we might wish to classify those completions which are associated with θ. By Corollary 3.3, this is equivalent to classifying the extensions of the pseudomodule $(\epsilon\beta_1, \theta)$; the answer to the latter problem is given by the results of [6, §§6–10].

However, it frequently happens (as for example in [7]) that instead of a modular structure on $\epsilon\beta_1\colon R_1 \rightarrow \Gamma$, we are given a modular structure $\bar{\theta}\colon S_2 \rightarrow A_X(R_1)/c(X)$ on the normal homomorphism $\beta_1\colon R_1 \rightarrow S_2$, and we are interested in those completions which are "associated" with $\bar{\theta}$, by which we mean those completions $(R_2, \rho_1, \rho_2, \beta_2)$ which satisfy

$$\bar{\theta}\beta_2 = \lambda C. \tag{5.3}$$

THEOREM 5.4. *Let* (1.1) *satisfy conditions* (1.2)–(1.4) *and* (2.1), *and let* $\bar{\theta}\colon S_2 \rightarrow A_X(R_1)/c(X)$ *be a modular structure on the normal homomorphism* $\beta_1\colon R_1 \rightarrow S_2$. *Then*

(a) *the homomorphism* $\bar{\theta}\pi_2\colon \Gamma \rightarrow A_X(R_1)/c(X)$ *is a modular structure on the normal homomorphism* $\epsilon\beta_1\colon R_1 \rightarrow \Gamma$;

(b) *a completion of* (1.1) *is associated with* $\bar{\theta}$, *in the sense of* (5.3), *if and only if it is associated with* $\bar{\theta}\pi_2$, *in the sense of* (5.2);

(c) *the isomorphism-classes of those completions of* (1.1) *which are associated with* $\bar{\theta}$ *are in a natural* 1-1 *correspondence with the isomorphism-classes of extensions of the pseudo-module* $(\epsilon\beta_1, \bar{\theta}\pi_2)$.

Proof. Part (b) turns out to be a tautology; part (c) follows from (a), (b), and Corollary 3.3. It only remains to prove part (a).

By hypothesis, commutativity holds throughout

$$
\begin{tikzcd}
R_1 \arrow[r, "\beta_1"] \arrow[d, "c_{R_1}"] & S_2 \arrow[d, "\bar{\theta}"] \arrow[drr, "C'"] & & \\
A_X(R_1) \arrow[r, "\lambda"] & A_X(R_1)/c(X) \arrow[rr, "(\beta_1)_*"] & & A(S_1).
\end{tikzcd}
\tag{5.5}
$$

Assertion (a) is that commutativity holds throughout

$$
\begin{tikzcd}
R_1 \arrow[r, "\epsilon\beta_1"] \arrow[d, "c_{R_1}"] & \Gamma \arrow[d, "\bar{\theta}\pi_2"] \arrow[drr, "C''"] & & \\
A_X(R_1) \arrow[r, "\lambda"] & A_X(R_1)/c(X) \arrow[rr, "(\epsilon\beta_1)_*"] & & A(N).
\end{tikzcd}
\tag{5.6}
$$

Commutativity in the rectangle of (5.6) follows immediately from commutativity in the rectangle of (5.5), since $\pi_2\epsilon$ is the identity mapping of S_1 onto itself.

Using formulae (5.1b) and (5.1c), and commutativity in the triangle of (5.5), we have $(\epsilon\beta_1)_*\bar{\theta}\pi_2 = \epsilon_\#(\beta_1)_*\bar{\theta}\pi_2 = \epsilon_\# C'\pi_2 = C''$, establishing commutativity in the triangle of (5.6). Q.E.D.

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