GENERALIZED nTH PRIMITIVES

BY

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1. **Introduction.** In previous papers, [3; 4](2), a Perron second integral (P^2-integral) was defined and studied. The purpose of this paper is to define a P^{2m}-integral for $m \geqq 2$ and a P^{2m+1}-integral for $m \geqq 1$. The definitions are made directly and not inductively in terms of P^{2r}- or P^{2r+1}-integrals with $r < m$.

§2 includes the necessary notation and definitions, and §3 is concerned with properties of generalized symmetric derivatives of even order. Nothing need be assumed about generalized symmetric derivatives of odd order.

In the definition of the P^2-integral, convex functions played an important part, and for the P^{2m}-integral, it is necessary to know something about what may be called $2m$-convex functions (see §2 for the definition). §4 contains these results.

The P^{2m}-integral is defined in §5. It is shown, in particular, that, under certain conditions, an everywhere finite generalized symmetric derivative of order $2m$ is necessarily P^{2m}-integrable, and that any two P^{2m}-integrals of the same function differ by a polynomial of degree $2m-1$ at most.

§6 contains the result that the P^{2m}-integral includes and is more general than the P^{2m-2}-integral when $m > 2$.

In §7, the changes needed to adapt the results for the P^{2m}-integral to the P^{2m+1}-integral are briefly indicated.

If the symmetry in the definition of the generalized derivatives is dropped, the problem is simplified. It is then easy to define a $\mathfrak{P}^n$-integral for $n \geqq 1$. This is done in §8. It is quite possible that $\mathfrak{P}^n$-integration is equivalent to the Denjoy process of totalization [2]. This is well known if $n = 1$.

Both Burkill [1] and Miss Sargent [8] have studied the properties of generalized derivatives in connection with the C_rP- and V_rD-integrals, and it is not surprising that the $\mathfrak{P}^n$-integral is closely related to these integrals. It is, in fact, quite easy to show that, if $f(x)$ is C_rP-integrable over a closed interval $[a, b]$, then it is $\mathfrak{P}^{r+1}$-integrable over $(a_i; c)$, where $a = a_1 < \cdots < a_{r+1} = b$ (see the definitions of §5 for the meaning of $\mathfrak{P}^{r+1}$-integrability over $(a_i; c)$). This result is in §9.

Conversely, if $f(x)$ is $\mathfrak{P}^{r+1}$-integrable over $(a_i; c)$, then it is C_rP-integrable

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over any closed sub-interval of (a_1, a_{2m}). The proof of this result is in §11, but it requires some further properties of n-convex functions, which are developed in §10.

It is hoped to consider the possible applications of the P^{2m}- and P^{2m+1}-integrals to trigonometric series in another paper.

2. **Notation and definitions.** The symbol (a, b) will denote an open interval and $[a, b]$ a closed interval.

Let $F(x)$ be a function defined in $[a, b]$. If there are constants $\beta_0, \beta_2, \cdots, \beta_{2r}$, depending on x_0 but not on h, such that

$$\frac{1}{2}\{F(x_0+h)+F(x_0-h)\}-\sum_{k=0}^{r}\beta_{2k}\frac{h^{2k}}{(2k!)}=o(h^{2r}), \tag{2.1}$$

as $h\to 0$, then β_{2r} is called the generalized symmetric derivative of order $2r$ of $F(x)$ at $x=x_0$, and written $D^{2r}F(x_0)$. It is clear that the existence of $D^{2r}F(x_0)$ implies that of $D^{2k}F(x_0)$, $0\leqq k\leqq r-1$, and that $\beta_{2k}=D^{2k}F(x_0)$, $0\leqq k\leqq r$.

If $D^{2k}F(x_0)$ exists for $0\leqq k\leqq m-1$, define $\theta_{2m}(x_0, h)=\theta_{2m}(F; x_0, h)$ by

$$\frac{h^{2m}}{(2m!)}\theta_{2m}(x_0, h)=\frac{1}{2}\{F(x_0+h)+F(x_0-h)\}-\sum_{k=0}^{m-1}\frac{h^{2k}}{(2k!)}D^{2k}F(x_0), \tag{2.2}$$

and let

$$\Delta^{2m}F(x_0)=\limsup_{h\to 0}\theta_{2m}(x_0, h), \tag{2.3}$$

$$\delta^{2m}F(x_0)=\liminf_{h\to 0}\theta_{2m}(x_0, h). \tag{2.4}$$

The function $F(x)$ is said to satisfy conditions A_{2m} in (a, b) if it is continuous in $[a, b]$, if, for $1\leqq k\leqq m-1$, each $D^{2k}F(x)$ exists and is finite in (a, b), and if

$$\lim_{h\to 0} h\theta_{2m}(x, h)=0 \tag{2.5}$$

for all x in $(a, b)-E$, where E is a countable set.

The function $F(x)$ is said to satisfy conditions B_{2m-2} in (a, b) if it is continuous in $[a, b]$, if, for $1\leqq k\leqq m-1$, each $D^{2k}F(x)$ exists and is finite in (a, b), and if no $D^{2k}F(x)$ has an ordinary discontinuity in (a, b).

A point of (a, b) is called a regular point (Saks [5]) if there is an open neighborhood of the point in which $D^{2m-2}F(x)$ is convex and, for $1\leqq k\leqq m-1$, each $D^{2k}F(x)$ is continuous.

If $x_0, x_1, \cdots, x_{2m}$ are distinct points, let $V(F; x_r)=V(F; x_0, x_1, \cdots, x_{2m})$ denote the $2m$th divided difference, that is, let

$$V(F; x_r)=\sum_{r=0}^{2m}F(x_r)/\omega'(x_r), \tag{2.6}$$

where

$$\omega(x) = \prod_{r=0}^{2m} (x - x_r).$$

The function $F(x)$ is said to be $2m$-convex in (a, b) (or in $[a, b]$) if $V(F; x_r) \geqq 0$ for every set of $2m+1$ distinct points $x_0, x_1, \cdots, x_{2m}$ in (a, b) (or in $[a, b]$). If $m=1$, the definition is the usual one for a convex function.

3. **Preliminary lemmas.**

LEMMA 3.1. *If $F(x)$ satisfies conditions A_{2m} in (a, b), there is a nondecreasing sequence of closed sets Q_n such that $(a, b)-E\subset \lim Q_n$, and such that, for $1\leqq k\leqq m-1$, each $D^{2k}F(x)$ is continuous on the set Q_n for every n.*

Proof. Wolf [9, Lemma 4] shows that on Q_n each $D^{2k}F(x)$ is the limit of a uniformly convergent sequence of continuous functions.

LEMMA 3.2. *If $F(x)$ satisfies conditions A_{2m} in (a, b), then, in any closed set $P\subset(a, b)$, there is a portion $P\cdot(a_0, b_0)$ on which, for $1\leqq k\leqq m-1$, each $D^{2k}F(x)$ is continuous.*

Proof. Following Wolf [9, Lemma 6], let $E=e_1+e_2+\cdots$ and $E_n=e_1+e_2+\cdots+e_n$. The sets $P_n=P\cdot(Q_n+E_n)$ are closed and $P_n\rightarrow P$ by Lemma 3.1. By Baire's theorem [5, Chap. II, §9], at least one of the sets P_n is dense in a portion of P. Thus, there is an integer r such that the set $P_r=P\cdot(Q_r+E_r)$ is dense in a portion of P. Since E_r is a finite set, the set $P\cdot Q_r$ is also dense in a portion of P. Since the set $P\cdot Q_r$ is closed, there is a portion $P\cdot(a_0, b_0)\subset P\cdot Q_r$. By Lemma 3.1, each $D^{2k}F(x)$ is continuous on Q_r and hence on $P\cdot(a_0, b_0)$.

LEMMA 3.3. *If, for $1\leqq k\leqq m-1$, each $D^{2k}F(x)$ is continuous in an interval (α, β), then $F(x)$ has continuous derivatives $F^{(r)}(x)$, $1\leqq r\leqq 2m-2$, and $D^{2k}F(x)=F^{(2k)}(x)$, $1\leqq k\leqq m-1$.*

Proof. Wolf [9, Lemma 7].

LEMMA 3.4. *If a subinterval (α, β), with $a<\alpha<\beta<b$, contains only regular points (see §2), and if $F(x)$ satisfies conditions B_{2m-2} in (a, b), then $D^{2m-2}F(x)$ is convex and, for $1\leqq k\leqq m-1$, each $D^{2k}F(x)$ is continuous in the closed subinterval $[\alpha, \beta]$.*

Proof. It follows from the definition of a regular point that, for $1\leqq k\leqq m-1$, each $D^{2k}F(x)$ is continuous and that $\Delta^2(D^{2m-2}F(x))\geqq 0$ in (α, β). Hence, in particular, $D^{2m-2}F(x)$ is convex in (α, β). Since $F(x)$ satisfies conditions B_{2m-2} in (a, b), $D^{2m-2}F(x)$ does not have an ordinary discontinuity at either $x=\alpha$ or $x=\beta$, and a convex function cannot have a discontinuity of the second kind. Hence $D^{2m-2}F(x)$ is continuous in the closed subinterval $[\alpha, \beta]$.

Now, for convenience, let $G(x)=D^{2m-4}F(x)$, $g(x)=D^{2m-2}F(x)$, so that, by

Lemma 3.3, $G''(x)=g(x)$ in (α, β) and $g(x)$ is continuous in $[\alpha, \beta]$. Then

$$G'(x) - G'(\gamma) = \int_\gamma^x g(t)dt, \qquad \alpha < \gamma \leqq x < \beta.$$

Since the integral tends to a limit as $\gamma \to \alpha+$, so does $G'(\gamma)$. Also,

$$(3.1) \quad G(x) - G(\gamma) = \int_\gamma^x G'(t)dt = \int_\gamma^x \int_\gamma^t g(u)dudt + (x-\gamma)G'(\gamma).$$

The integral and $(x-\gamma)G'(\gamma)$ each tend to a limit as $\gamma \to \alpha+$. Hence $G(\gamma)$ tends to a limit, which must be $G(\alpha)$, since $G(x)$ does not have an ordinary discontinuity at $x=\alpha$. It then follows from (3.1) that $G(x) \to G(\beta)$ as $x \to \beta-$.

The argument can now be repeated with $G(x)=D^{2m-6}F(x)$, $g(x)=D^{2m-4}F(x)$. After a finite number of similar steps, the proof is complete.

4. **$2m$-convex functions.** Before defining the P^{2m}-integral, it is necessary to know under what conditions a function is $2m$-convex. Sufficient conditions are stated in Theorem 4.2 of this section. The proof is made by induction for $m \geqq 2$, starting with

THEOREM 4.1, $2m-2$. *If $F(x)$ satisfies conditions A_{2m-2} and B_{2m-4} in (a, b) and if $\Delta^{2m-2}F(x)>0$ in (a, b), then $D^{2m-4}F(x)$ is convex and, for $1 \leqq k \leqq m-2$, each $D^{2k}F(x)$ is continuous in (a, b).*

(Note. When $m=2$, conditions A_2 require $F(x)$ to be smooth in $(a, b)-E$, and conditions B_0 simply require $F(x)$ to be continuous in $[a, b]$. It is well known that under these hypotheses, $\Delta^2F(x)>0$ implies that $F(x)$ is convex in (a, b). Hence the theorem is true for $m=2$.)

Assume then that Theorem 4.1, $2m-2$ is true. A series of lemmas will show that Theorem 4.1, $2m$ is also true.

LEMMA 4.1, $2m-2$. *If $F(x)$ satisfies conditions A_{2m-2} and B_{2m-4} in (a, b) and if $\Delta^{2m-2}F(x) \geqq 0$ in (a, b), then*

$$(4.1) \qquad \theta_{2m-2}(x, h) \geqq 0, \qquad a < x-h < x+h < b.$$

Proof. If $m=2$, $F(x)$ is convex by Theorem 4.1, 2 and $h^2\theta_2(x, h)=F(x+h)+F(x-h)-2F(x)$ is non-negative.

If $m>2$, let $G(x)=F(x)+\epsilon x^{2m-2}/(2m-2)!$, where ϵ is an arbitrary positive number. It follows from Theorem 4.1, $2m-2$ and Lemma 3.3 that the function

$$\phi(t) = \frac{t^{2m-2}}{(2m-2)!}\theta_{2m-2}(G; x, t)$$
$$= \frac{1}{2}\{G(x+t)+G(x-t)\} - \sum_{k=0}^{m-2} \frac{t^{2k}}{(2k)!}G^{(2k)}(x)$$

has continuous derivatives $\phi^{(r)}(t)$, $1 \leqq r \leqq 2m-4$, for $0 \leqq t \leqq h$, $a < x-h < x+h < b$. Moreover, $\phi^{(r)}(0)=0$, $0 \leqq r \leqq 2m-5$. Hence

$$\begin{aligned}\phi(h) &= \frac{h^{2m-4}}{(2m-4)!}\phi^{(2m-4)}(\delta h)\\ &= \frac{h^{2m-4}}{2(2m-4)!}\left\{G^{(2m-4)}(x+\delta h)+G^{(2m-4)}(x-\delta h)-2G^{(2m-4)}(x)\right\},\end{aligned}$$

where $0<\delta<1$. Since $G^{(2m-4)}(x)=D^{2m-4}G(x)$ is convex, by Theorem 4.1, $2m-2$, it follows that $\theta_{2m-2}(G;\, x,\, h) \geqq 0$ and hence that $\theta_{2m-2}(F;\, x,\, h) \geqq -\epsilon$. Since ϵ is arbitrary, the inequality (4.1) follows.

LEMMA 4.2, $2m-2$. *If the function $G(x)$ satisfies conditions A_{2m-2} and B_{2m-2} in (a, b) and if $D^{2m-2}G(x)$ attains a maximum at a point z_0 in (a, b), then $\Delta^{2m}G(z_0) \leqq 0$.*

Proof. If h_0 is sufficiently small, then $D^{2m-2}G(x) \leqq D^{2m-2}G(z_0)$ for $z_0-h_0 \leqq x \leqq z_0+h_0$. Let $H(x)=C_0x^{2m-2}/(2m-2)!-G(x)$, where $C_0=D^{2m-2}G(z_0)$. The function $H(x)$ satisfies conditions A_{2m-2} and B_{2m-4} in (a, b), and, by Lemma 4.1, $2m-2$, θ_{2m-2} $(H;\, z_0,\, h) \geqq 0$ for $0 \leqq h < h_0$. From definition (2.2) it follows that $\theta_{2m-2}(G;\, z_0,\, h) \leqq C_0$. Also from definition (2.2),

$$h^2\theta_{2m}(G;\, z_0,\, h)=2m(2m-1)\left\{\theta_{2m-2}(G;\, z_0,\, h)-C_0\right\}.$$

Hence

$$\Delta^{2m}G(z_0) = \limsup_{h\to 0}\theta_{2m}(G;\, z_0,\, h) \leqq 0.$$

LEMMA 4.3, $2m-2$. *Let $F(x)$ be a function satisfying conditions A_{2m} and B_{2m-2} in (a, b) and such that $\Delta^{2m}F(x)>0$ in (a, b). If $D^{2m-2}F(x)$ is upper semi-continuous in (a, b), it is convex in (a, b).*

Proof. If $D^{2m-2}F(x)$ is not convex there is a subinterval $[\alpha, \beta]$ of (a, b) in which the function

$$\begin{aligned}\rho(x) &= D^{2m-2}F(x)-D^{2m-2}F(\alpha)-(x-\alpha)/(\beta-\alpha)\left\{D^{2m-2}F(\beta)-D^{2m-2}F(\alpha)\right\}\\ &= D^{2m-2}F(x)-px-q\end{aligned}$$

takes positive values. Since $\rho(x)$ is upper semi-continuous in $[\alpha, \beta]$, it attains a maximum at an interior point. In other words, the function

$$D^{2m-2}\left\{F(x)-p\frac{x^{2m-1}}{(2m-1)!}-q\frac{x^{2m-2}}{(2m-2)!}\right\}$$

attains a maximum at a point z_0 of (a, b). By Lemma 4.2, $2m-2$,

$$\Delta^{2m}F(z_0) = \Delta^{2m}\left\{F(x)-p\frac{x^{2m-1}}{(2m-1)!}-q\frac{x^{2m-2}}{(2m-2)!}\right\}_{x=z_0} \leqq 0.$$

This contradicts the fact that $\Delta^{2m}F(x)>0$ in (a, b). Hence, $D^{2m-2}F(x)$ is convex in (a, b).

It is now possible to prove

THEOREM 4.1, $2m$. *If $F(x)$ satisfies conditions A_{2m} and B_{2m-2} in (a, b) and if $\Delta^{2m}F(x)>0$ in (a, b), then $D^{2m-2}F(x)$ is convex and, for $1\leqq k\leqq m-1$, each $D^{2k}F(x)$ is continuous in (a, b).*

Proof. Let P denote the sets of points in (a, b) which are not regular. This set is closed and, following Saks [5, p. 250], it will be shown that the only nonregular points are the end points a and b.

Suppose, on the contrary, that P contains points of (a, b). By Lemma 3.2 there is a subinterval (a_0, b_0) containing points of P and such that, for $1\leqq k\leqq m-1$, each $D^{2k}F(x)$ is continuous on the set $H=P\cdot(a_0, b_0)$. Then $(a_0, b_0)=H+\sum G_n$, where the sets G_n are open intervals such that $D^{2m-2}F(x)$ is convex and each $D^{2k}F(x)$ is continuous in each G_n, and, by Lemma 3.4, at its end points as well.

It will be shown first that $D^{2m-2}F(x)$ is upper semi-continuous in (a_0, b_0) and hence, by Lemma 4.3, $2m-2$, convex. It is only necessary to prove that $D^{2m-2}F(x)$ is upper semi-continuous at each point x_0 of the set H. Let

$$\limsup_{x\to x_0} D^{2m-2}F(x) = K. \tag{4.2}$$

Then there is a sequence $\{x_\nu\}$ such that $x_\nu\to x_0$ and such that $D^{2m-2}F(x_\nu)\to K$. If only a finite number of the x_ν are in $\sum G_n$, then, since $D^{2m-2}F(x)$ is continuous on H,

$$K = \lim_{\nu\to\infty} D^{2m-2}F(x_\nu) = D^{2m-2}F(x_0). \tag{4.3}$$

If an infinite number of the x_ν are in $\sum G_n$, two cases arise (cf. Wolf [9, Lemma 9]).

(a) All the x_ν are in only a finite number of different G_n. There is then a subsequence $\{\xi_\nu\}$ such that all the $\{\xi_\nu\}$ are in one G_n, say (α_0, β_0). The point x_0 must be an end point of (α_0, β_0), and since $D^{2m-2}F(x)$ is continuous at the end points (by Lemma 3.4), it follows that

$$K = \lim_{\nu\to\infty} D^{2m-2}F(\xi_\nu) = D^{2m-2}F(x_0). \tag{4.4}$$

(b) The x_ν are in an infinite number of different G_n. It is then possible to find a subsequence $\{\eta_\nu\}$ such that each η_ν lies in a different interval (α_ν, β_ν) of $\sum G_n$. Now, $\alpha_\nu\to x_0$, $\beta_\nu\to x_0$, and, since α_ν and β_ν are in H,

$$\lim_{\nu\to\infty} D^{2m-2}F(\alpha_\nu) = \lim_{\nu\to\infty} D^{2m-2}F(\beta_\nu) = D^{2m-2}F(x_0). \tag{4.5}$$

Also, by Lemma 3.4, $D^{2m-2}F(x)$ is convex in every closed interval $[\alpha_\nu, \beta_\nu]$, so that $D^{2m-2}F(\eta_\nu)\leqq\max\{D^{2m-2}F(\alpha_\nu), D^{2m-2}F(\beta_\nu)\}$ and

$$K = \lim_{\nu\to\infty} D^{2m-2}F(\eta_\nu) \leqq D^{2m-2}F(x_0). \tag{4.6}$$

Hence, by (4.3), (4.4), and (4.6), $D^{2m-2}F(x)$ is upper semi-continuous in (a_0, b_0).

Next, as in Lemma 3.4, let $G(x)=D^{2m-4}F(x)$, $g(x)=D^{2m-2}F(x)$, where $g(x)$ is convex (and hence continuous) in (a_0, b_0). Again, it is only necessary to prove that $G(x)$ is continuous at each point x_0 of H. The arguments used above apply equally well here except in case (b).

In case (b) there is a subinterval $[\alpha, \beta]$ of (a_0, b_0) containing x_0 in its interior and such that $g(x)$ is continuous in $[\alpha, \beta]$. If ν is sufficiently large, each interval (α_ν, β_ν) is contained in $[\alpha, \beta]$. As in Lemma 3.4,

$$G(\beta_\nu) - G(\alpha_\nu) = \int_{\alpha_\nu}^{\beta_\nu}\int_{\alpha_\nu}^{t} g(u)dudt + (\beta_\nu - \alpha_\nu)G'(\alpha_\nu). \tag{4.7}$$

From (4.7) it follows that $(\beta_\nu-\alpha_\nu)G'(\alpha_\nu)\to 0$. But then, since (4.7) is valid with β_ν replaced by η_ν,

$$K = \lim_{\nu\to\infty} G(\eta_\nu) = \lim_{\nu\to\infty} G(\alpha_\nu) = G(x_0).$$

This proves that lim sup $G(x)=G(x_0)$ and a similar argument shows that lim inf $G(x)=G(x_0)$.

The argument can now be repeated with $G(x)=D^{2m-6}F(x)$, $g(x)=D^{2m-4}F(x)$. After a finite number of similar steps, the conclusion is reached that $D^{2m-2}F(x)$ is convex and each $D^{2k}F(x)$ is continuous in (a_0, b_0). This contradicts the fact that there are points of P in (a_0, b_0). Hence, the assumption that P contains points of (a, b) is false and the proof of Theorem 4.1, $2m$ is complete.

By induction, Theorem 4.1, $2m$ and each of the Lemmas 4.1, $2m$, 4.2, $2m$, 4.3, $2m$ are valid for all $m \geqq 1$. In future, the $2m$ will be dropped in references to the theorem and the lemmas.

THEOREM 4.2. *If $F(x)$ satisfies conditions A_{2m} and B_{2m-2} in (a, b) and if $\Delta^{2m}F(x)\geqq 0$ in (a, b), then $F(x)$ is $2m$-convex in $[a, b]$, that is, $V(F; x_r)\geqq 0$ for every set of $2m+1$ distinct points in $[a, b]$.*

Proof. Since $F(x)$ is continuous in $[a, b]$, it is sufficient to prove that $F(x)$ is $2m$-convex in (a, b). Assume first that $\Delta^{2m}F(x)>0$ in (a, b). By the Lagrange interpolation formula,

$$F(x) = \sum_{r=0}^{2m} \lambda(x; x_r)F(x_r) + \omega(x)V(F; x, x_0, \cdots, x_{2m}),$$

where

$$\omega(x) = \prod_{i=0}^{2m} (x - x_i),$$

$$\lambda(x;\, x_r) = \omega(x)/\{(x - x_r)\omega'(x_r)\} = \prod_{i\neq r} (x - x_i)/(x_r - x_i).$$

The function $F(x)$ has, by Lemma 3.3 and Theorem 4.1, continuous derivatives $F^{(s)}(x)$ in (a, b) for $1 \leqq s \leqq 2m-2$ and so has the function

$$G(x) = F(x) - \sum_{r=0}^{2m} \lambda(x;\, x_r)F(x_r) = \omega(x)V(F;\, x_r).$$

Also, $G(x)$ vanishes at least $2m+1$ times in (a, b) and hence the continuous function $D^{2m-2}G(x)$ vanishes at least three times. It therefore has at least one maximum at a point z_0 of (a, b). At this maximum point, by Lemma 4.2, $\Delta^{2m}G(z_0) \leqq 0$. But,

$$\begin{aligned} \Delta^{2m}G(z_0) &= \Delta^{2m}F(z_0) - D^{2m}\left\{\sum \lambda(x;\, x_r)F(x_r)\right\}_{x=z_0} \\ &= \Delta^{2m}F(z_0) - (2m)!V(F;\, x_r). \end{aligned}$$

This proves the theorem when $\Delta^{2m}F(x) > 0$.

If $\Delta^{2m}F(x) \geqq 0$ and $F_n(x) = F(x) + x^{2m}/n(2m)!$, it follows from the above proof that $F_n(x)$ is $2m$-convex for each n. Then, $F(x) = \lim F_n(x)$ is also $2m$-convex.

5. **Definition of the P^{2m}-integral.**

Definition 5.1. Let $f(x)$ be a function defined in $[a, b]$ and let a_i, $i = 1, \cdots, 2m$, be fixed points such that $a = a_1 < a_2 < \cdots < a_{2m} = b$. The functions $Q(x)$ and $q(x)$ are called major and minor functions, respectively, of $f(x)$ over $(a_i) = (a_1, a_2, \cdots, a_{2m})$ if

(5.1.1) $Q(x)$ and $q(x)$ satisfy conditions A_{2m} and B_{2m-2} in (a, b);

(5.1.2) $Q(a_i) = q(a_i) = 0, \qquad i = 1, \cdots, 2m;$

(5.1.3) $\delta^{2m}Q(x) \geqq f(x) \geqq \Delta^{2m}q(x)$ in (a, b);

(5.1.4) $\delta^{2m}Q(x) \neq -\infty, \quad \Delta^{2m}q(x) \neq +\infty$ in (a, b).

Definition 5.2. For each major and minor function of $f(x)$ over (a_i), the functions defined by

$$Q^*(x) = (-1)^r Q(x), \quad q^*(x) = (-1)^r q(x), \qquad a_r \leqq x < a_{r+1}, \tag{5.2}$$

are called associated major and minor functions, respectively, of $f(x)$ over (a_i).

Lemma 5.1. *For every pair $Q(x)$, $q(x)$, the difference $Q(x) - q(x)$ is $2m$-convex in $[a, b]$.*

Proof. By (5.1.3) and (5.1.4), $\Delta^{2m}\{Q(x) - q(x)\}$ is defined and non-negative in (a, b). The result now follows from Theorem 4.2.

LEMMA 5.2. *For every pair of associated major and minor functions of* $f(x)$ *over* (a_i),

$$Q^*(x) - q^*(x) \geqq 0 \tag{5.3}$$

for all x *in* $[a, b]$.

Proof. Since, by definition, $Q^*(a_i)=q^*(a_i)=0$, $i=1, \cdots, 2m$, it is only necessary to consider $x\neq a_i$. By Lemma 5.1, $V(Q-q; x, a_1, \cdots, a_{2m})\geqq 0$ and, if $a_r<x<a_{r+1}$, this reduces to

$$\{Q(x) - q(x)\} \prod_{i=1}^{2m} 1/(x - a_i) \geqq 0, \tag{5.4}$$

or

$$(-1)^r\{Q(x) - q(x)\} \prod_{i=1}^{r} 1/(x - a_i) \prod_{i=r+1}^{2m} 1/(a_i - x) \geqq 0.$$

Since each product is positive, (5.3) follows from (5.2).

DEFINITION 5.3. Let c be a point in (a_1, a_{2m}) such that $c\neq a_i$, $i=1, \cdots, 2m$. If for every $\epsilon>0$ there is a pair, $Q(x)$, $q(x)$, such that

$$|Q(c) - q(c)| < \epsilon, \tag{5.5}$$

then $f(x)$ is said to be P^{2m}-integrable over $(a_i; c)$.

LEMMA 5.3. *If the inequality* (5.5) *holds, then*

$$|Q(x) - q(x)| < \epsilon K \tag{5.6}$$

for all x *in* $[a_1, a_{2m}]$, *where* K *is independent of* x.

Proof. In view of (5.12), it is sufficient to consider $x\neq a_i$. If $a_r<x<a_{r+1}$, $a_s<c<a_{s+1}$, and $x<c$, consider $V(Q-q;\ x,\ c,\ a_2, \cdots, a_{2m})$, which is nonnegative by Lemma 5.1. Since (5.1.2) is true, then, as in the proof of Lemma 5.2,

$$(-1)^r\{Q(x) - q(x)\}\{1/(x - c)\} \prod_{i=2}^{r} 1/(x - a_i) \prod_{i=r+1}^{2m} 1/(a_i - x)$$
$$+ (-1)^s\{Q(c) - q(c)\}\{1/(c - x)\} \prod_{i=2}^{2m} 1/|c - a_i| \geqq 0.$$

Since $x<c$,

$$\begin{aligned} Q^*(x) - q^*(x) \\ &\leqq \{Q^*(c) - q^*(c)\} \prod_{i=2}^{r} (x - a_i) \prod_{i=r+1}^{2m} (a_i - x) \prod_{i=2}^{2m} 1/|c - a_i| \\ &\leqq \{Q^*(c) - q^*(c)\} K, \end{aligned} \tag{5.7}$$

where K is the constant obtained by replacing x in the first product of (5.7) by a_{2m} and, in the second, by a_1. (The first product is empty if $r=1$.) The required result now follows from (5.5).

If $c<x$, the argument is similar, but uses $V(Q-q;\, x,\, c,\, a_1, \cdots, a_{2m-1})$.

THEOREM 5.1. *If $f(x)$ is P^{2m}-integrable over $(a_i;\, c)$, there is a function $F^*(x)$ which is the inf of all associated major functions of $f(x)$ over (a_i) and the sup of all associated minor functions.*

Proof. If $q_0^*(x)$ is a given associated minor function, it follows from Lemma 5.2 that $Q^*(x)\geqq q_0^*(x)$ for every associated major function. Let $F^*(x)$ denote the inf of all $Q^*(x)$, so that $F^*(x)\geqq q_0^*(x)$. Since this inequality is valid for every associated minor function, $F^*(x)$ is an upper bound for all $q^*(x)$. By Definition 5.3 and Lemma 5.3, there is a particular pair $Q(x)$, $q(x)$, such that

$$0 \leqq F^*(x) - q^*(x) \leqq Q^*(x) - q^*(x) = \left| Q(x) - q(x) \right| < \epsilon K.$$

Since ϵ is arbitrary, $F^*(x)$ is also the sup of all $q^*(x)$.

DEFINITION 5.4. If $f(x)$ is P^{2m}-integrable over $(a_i;\, c)$ and if $F^*(x)$ is the function of Theorem 5.1, define $F(x)$ by

$$(-1)^r F(x) = F^*(x) \qquad \text{when } a_r \leqq x < a_{r+1}.$$

If $a_s<c<a_{s+1}$, the P^{2m}-integral of $f(x)$ over $(a_i;\, c)$ is defined to be $(-1)^s F(c)$. Since $(-1)^s F(a_i)=F(a_i)=0$, the integral is defined to be zero if $c=a_i$, $i=1, \cdots, 2m$. The notation is

$$(-1)^s F(c) = \int_{(a_i)}^{c} f(x)d_{2m}x.$$

(The reason for using $(-1)^s F(c)$ instead of $F(c)$ is indicated by the result of Theorem 5.4 below.)

THEOREM 5.2. *If $f(x)$ is P^{2m}-integrable over $(a_i;\, c)$, it is also P^{2m}-integrable over $(a_i;\, x)$ for every x in $[a_1, a_{2m}]$. If $F(x)$ is the function of Definition 5.4, then, for $a_r\leqq x<a_{r+1}$,*

$$(-1)^r F(x) = \int_{(a_i)}^{x} f(x)d_{2m}x. \tag{5.8}$$

Proof. The fact that $f(x)$ is P^{2m}-integrable over $(a_i;\, x)$ follows at once from Lemma 5.3 and Definition 5.3. By Definition 5.4, the right-hand side of (5.8) is the sup of all $q^*(x)$ and the inf of all $Q^*(x)$. In particular, there is a pair, $q(x)$, $Q(x)$, such that

$$q^*(x) \leqq \int_{(a_i)}^{x} f(x)d_{2m}x \leqq Q^*(x) < q^*(x) + \epsilon K.$$

But, by Theorem 5.1, $q^*(x) \leqq F^*(x) \leqq Q^*(x)$, so that

$$\left| F^*(x) - \int_{(a_i)}^{x} f(x) d_{2m} x \right| < \epsilon K.$$

Since ϵ is arbitrary, it follows that (5.8) is true.

THEOREM 5.3. *The function $F(x)$ of Definition 5.4 is continuous in $[a_1, a_{2m}]$.*

Proof. For every positive integer n, there is, by Lemma 5.3, a pair $Q_n(x)$, $q_n(x)$, such that

$$0 \leqq \overset{*}{Q}_n(x) - \overset{*}{q}_n(x) = | Q_n(x) - q_n(x) | < K/n.$$

Then,

$$| F(x) - q_n(x) | = \overset{*}{F}(x) - \overset{*}{q}_n(x) \leqq \overset{*}{Q}_n(x) - \overset{*}{q}_n(x) < K/n.$$

Hence $F(x)$ is the limit of a uniformly convergent sequence of functions $q_n(x)$ which are continuous in $[a_1, a_{2m}]$.

COROLLARY. *For every major and minor function, $Q(x) - F(x)$ and $F(x) - q(x)$ are $2m$-convex in $[a_1, a_{2m}]$.*

Proof. If $Q_n(x)$ and $q_n(x)$ are the functions defined in the proof of Theorem 5.3, then by Lemma 5.1, $Q(x) - q_n(x)$ is $2m$-convex for each n. Therefore, so is $Q(x) - F(x) = \lim \{Q(x) - q_n(x)\}$. The proof that $F(x) - q(x)$ is $2m$-convex is similar.

THEOREM 5.4. *If $G(x)$ satisfies conditions B_{2m} in (a, b), then $D^{2m}G(x)$ is P^{2m}-integrable over $(a_i; x)$. If $a_r \leqq x < a_{r+1}$, then*

$$(5.9) \qquad \begin{aligned} (-1)^r \int_{(a_i)}^{x} D^{2m}G(x) d_{2m} x &= \omega(x) V(G; x, a_1, \cdots, a_{2m}) \\ &= G(x) - \sum_{i=1}^{2m} \lambda(x; a_i) G(a_i), \end{aligned}$$

where

$$\omega(x) = \prod_{i=1}^{2m} (x - a_i), \qquad \lambda(x; a_i) = \prod_{j \neq i} (x - a_j)/(a_i - a_j).$$

Proof. Since $D^{2m}G(x)$ is finite in (a, b), conditions A_{2m} are also satisfied by $G(x)$. The function

$$(5.10) \qquad Q(x) = \omega(x) V(G; x, a_i) = G(x) - \sum_{i=1}^{2m} \lambda(x; a_i) G(a_i)$$

vanishes at $x=a_i$, $i=1, \cdots, 2m$, and $D^{2m}Q(x)=D^{2m}G(x)$. Therefore, $Q(x)$ is both a major and minor function of $D^{2m}G(x)$ over (a_i) and (5.9) follows from (5.10).

THEOREM 5.5. *If $f(x)$ is P^{2m}-integrable over $(a_i; x)$, it is also P^{2m}-integrable over $(b_j; x)$, where $a_1 \leqq b_1 < \cdots < b_{2m} \leqq a_{2m}$. Moreover, if $F(x)$ is the function of Definition 5.4, and $b_s \leqq x < b_{s+1}$, then*

$$(5.11)\qquad (-1)^s \int_{(b_j)}^{x} f(x)d_{2m}x = F(x) - \sum_{j=1}^{2m} \lambda(x; b_j)F(b_j),$$

where

$$\lambda(x; b_j) = \prod_{k \neq j} (x - b_k)/(b_j - b_k)$$

is a polynomial of degree $2m-1$ at most. In other words, any two P^{2m}-integrals of $f(x)$ differ by a polynomial of degree $2m-1$ at most.

Proof. If $Q(x)$ and $q(x)$ are major and minor functions, respectively, of $f(x)$ over (a_i), then

$$S(x) = Q(x) - \sum_{j=1}^{2m} \lambda(x; b_j)Q(b_j)$$

and

$$s(x) = q(x) - \sum_{j=1}^{2m} \lambda(x; b_j)q(b_j)$$

are major and minor functions, respectively, of $f(x)$ over (b_j). It is clear that $|S(x)-s(x)| < \epsilon C$, where C is independent of x, when $|Q(x)-q(x)| < \epsilon K$, so that $f(x)$ is P^{2m}-integrable over $(b_j; x)$.

As in the proof of Theorem 5.3, there is a sequence $q_n(x)$ of minor functions of $f(x)$ over (a_i) such that $q_n(x) \to F(x)$. The corresponding minor function $s_n(x)$ therefore tends to the right-hand side of (5.11). But, by the definition of the P^{2m}-integral over $(b_j; x)$, it also tends to the left-hand side.

6. Consistency of the definition.

THEOREM 6.1. *If $f(x)$ is P^{2m-2}-integrable over $(b_j; c)$, where $a=b_1 < \cdots < b_{2m-2}=b$, then it is also P^{2m}-integrable over $(a_i; c)$.*

(Note. The theorem may not be true for $m=2$ if the P^2-integral is taken as the one defined in [4], since there the inequalities (5.1.3) and (5.1.4) were allowed to fail on a countable set. However, if the P^2-integral is interpreted as the one defined by Definitions 5.1 and 5.3 when $m=2$, the theorem remains true.)

Proof. Let $Q(x; 2m-2)$ be any P^{2m-2}-major function of $f(x)$ over (b_j), and let

$$R(x; 2m) = \int_{a_1}^{x}\int_{a_1}^{t} Q(u; 2m-2)dudt, \tag{6.1}$$

$$Q(x; 2m) = R(x; 2m) - \sum_{i=1}^{2m} \lambda(x; a_i)R(a_i; 2m), \tag{6.2}$$

where $\lambda(x; a_i)$ was defined in Theorem 5.4. Then

$$Q(x; 2m) = R(x; 2m) - P(x; 2m-1), \tag{6.3}$$

where $P(x; 2m-1)$ is a polynomial of degree $2m-1$ at most.

It follows from (6.1), (6.2), and (6.3) that $D^2Q(x; 2m) = Q''(x; 2m) = R''(x; 2m) - P''(x; 2m-1) = Q(x; 2m-2) - P''(x; 2m-1)$. Also,

$$\begin{aligned} D^4Q(x; 2m) &= \lim_{h\to 0} \theta_4(Q; x, h) = \lim_{h\to 0} \theta_2(Q''; x, h) \\ &= D^2Q(x; 2m-2) - P^{(4)}(x; 2m-1). \end{aligned}$$

Repeated application of a similar argument leads to $D^{2k}Q(x; 2m) = D^{2k-2}Q(x; 2m-2) - P^{(2k)}(x; 2m-1)$, for $1 \leqq k \leqq m-1$, and it can be shown in the same way that (2.5) holds for $Q(x; 2m)$. Hence $Q(x; 2m)$ satisfies conditions A_{2m} and B_{2m-2} because $Q(x; 2m-2)$ satisfies conditions A_{2m-2} and B_{2m-4}. In addition, (6.2) shows that $Q(a_i; 2m) = 0$, $i = 1, \cdots, 2m$.

Finally, by the argument used above, $\delta^{2m}Q(x; 2m) = \delta^{2m-2}Q(x; 2m-2) - P^{(2m)}(x; 2m-1) = \delta^{2m-2}Q(x; 2m-2) \geqq f(x)$. Hence $Q(x; 2m)$ is a P^{2m}-major function of $f(x)$ over (a_i).

In the same way, if $q(x; 2m-2)$ is a P^{2m-2}-minor function, let

$$r(x; 2m) = \int_{a_1}^{x}\int_{a_1}^{t} q(u; 2m-2)dudt \tag{6.4}$$

and

$$q(x; 2m) = r(x; 2m) - \sum_{i=1}^{2m} \lambda(x; a_i)r(a_i; 2m). \tag{6.5}$$

Then $q(x; 2m)$ is a P^{2m}-minor function of $f(x)$.

Since $f(x)$ is P^{2m-2}-integrable, there exists a particular pair $Q(x; 2m-2)$, $q(x; 2m-2)$, such that $|Q(x; 2m-2) - q(x; 2m-2)| < \epsilon K_{2m-2}$. Then from (6.1) and (6.4), $|R(x; 2m) - r(x; 2m)| < (1/2)\epsilon(a_{2m} - a_1)^2 K_{2m-2}$. The definitions (6.2) and (6.5) now show that $|Q(x; 2m) - q(x; 2m)| < \epsilon K_{2m}$ and therefore $f(x)$ is P^{2m}-integrable over $(a_i; c)$.

LEMMA 6.1. *If a function $g(x)$ has a finite derivative $g'(x)$ in (a, b) such that $V(g'; z_k) \geqq 0$ for every set $z_0, z_1, \cdots, z_{n-1}$ of n distinct points, then $V(g; x_i) \geqq 0$ for every set $x_0, x_1, \cdots, x_n$ of $n+1$ distinct points in (a, b).*

Proof. Write

$$V(g;\, x_i) = \sum_{i=0}^{n} \lambda(x_i) g(x_i),$$

where

$$\lambda(x_i) = \prod_{j \neq i} 1/(x_i - x_j),$$

and let

$$\phi(t) = \sum_{i=0}^{n} \lambda(x_i) g(x_n + (x_i - x_n)t).$$

Then $\phi(1) = V(g;\, x_i)$, $\phi(0) = 0$, and

$$\phi'(t) = \sum_{i=0}^{n-1} (x_i - x_n)\lambda(x_i) g'(x_n + (x_i - x_n)t).$$

By the mean value theorem,

$$(6.6) \quad V(g;\, x_i) = \phi(1) - \phi(0) = \phi'(\delta) = \sum_{i=0}^{n-1} (x_i - x_n)\lambda(x_i) g'(x_n + (x_i - x_n)\delta),$$

where $0 < \delta < 1$. Let $z_k = x_n + (x_k - x_n)\delta$ for $k = 0, \cdots, n-1$. Then

$$(x_k - x_n)\lambda(x_k) = \prod 1/(x_k - x_j) = \prod \delta/(z_k - z_j),$$

where each product is taken over $0 \leqq j \leqq n-1, j \neq k$, and (6.6) becomes $V(g;\, x_i) = V(g';\, z_k)\delta^{n-1}$. This proves the lemma.

THEOREM 6.2. *If $f(x)$ is P^{2m-2}-integrable over $(b_j;\, x)$, let*

$$(6.7) \quad \begin{aligned} g(x) &= (-1)^s \int_{(b_j)}^{x} f(x) d_{2m-2}x, \qquad b_s \leqq x < b_{s+1}, \\ G(x) &= \int_{b_1}^{x} \int_{b_1}^{t} g(u) du dt. \end{aligned}$$

Then, if $b_1 = a_1 < \cdots < a_{2m} = b_{2m-2}$, $a_r \leqq x < a_{r+1}$,

$$(6.8) \quad \begin{aligned} F(x) &= (-1)^r \int_{(a_i)}^{x} f(x) d_{2m}x \\ &= \omega(x) V(G;\, x, a_i) = G(x) - \sum_{i=1}^{2m} \lambda(x;\, a_i) G(a_i). \end{aligned}$$

Proof. Let $Q(x;\, 2m-2)$, $q(x;\, 2m-2)$, $Q(x;\, 2m)$, and $q(x;\, 2m)$ be the functions defined in the proof of Theorem 6.1. By the corollary to Theorem 5.3, $Q(x;\, 2m-2) - g(x)$ and $g(x) - q(x;\, 2m-2)$ are $(2m-2)$-convex functions. By Lemma 6.1, applied twice, $R(x;\, 2m) - G(x)$ and $G(x) - r(x;\, 2m)$ are $2m$-convex functions. Since $Q(x; 2m)$ and $q(x; 2m)$ differ from $R(x; 2m)$ and $r(x; 2m)$ by a polynomial of degree $2m-1$ at most, it follows that $Q(x; 2m)-G(x)$ and $G(x)-q(x; 2m)$ are also $2m$-convex functions. Hence

$$V(q; x, a_i) \leqq V(G; x, a_i) \leqq V(Q; x, a_i). \tag{6.9}$$

But, by the same reasoning that led to (5.4),

$$q(x; 2m) = \omega(x)V(q; x, a_i), \qquad Q(x; 2m) = \omega(x)V(Q; x, a_i).$$

When $a_r < x < a_{r+1}$, $(-1)^r\omega(x) \geqq 0$ and (6.9) becomes $q^*(x; 2m) = (-1)^r q(x; 2m) \leqq (-1)^r\omega(x)V(G; x, a_i) \leqq (-1)^r Q(x; 2m) = Q^*(x; 2m)$.

By the method of proof used in Theorem 5.2, it then follows that

$$(-1)^r\omega(x)V(G; x, a_i) = \int_{(a_i)}^{x} f(x)d_{2m}x.$$

This proves the first part of (6.8), and the second is simply the expanded form of $\omega(x)V(G; x, a_i)$.

COROLLARY 1. *If $f(x)$ is P^{2m-2}-integrable over $(b_j; x)$, then $F'(x)$ and $F''(x)$ exist for all x in $[b_1, b_{2m-2}]$. If $g(x)$ and $G(x)$ are the functions defined in Theorem 6.2, then*

$$F''(x) = g(x) - \sum_{i=1}^{2m} \lambda''(x; a_i)G(a_i) = g(x) - p(x),$$

where $p(x)$ is a polynomial of degree $2m-3$ at most.

Proof. The corollary is an obvious consequence of (6.7) and (6.8).

COROLLARY 2. *There exists a function which is P^{2m}-integrable, but not P^{2m-2}-integrable.*

Proof. Let

$$G(x) = x\cos(1/x), \qquad x \neq 0, \qquad G(0) = 0;$$
$$g(x) = G^{(2m)}(x), \qquad x \neq 0, \qquad g(0) = 0.$$

It is clear that $D^{2m}G(x) = g(x)$ for all values of x, including $x=0$, and that all the conditions of Theorem 5.4 are satisfied. Hence $g(x)$ is P^{2m}-integrable over $(a_i; x)$ for every x, including $x=0$.

On the other hand, if $g(x)$ were P^{2m-2}-integrable over $(b_j; 0)$, it would follow from Corollary 1 that $G'(0)$ and $G''(0)$ exist. But not even $G'(0)$ exists, so that $g(x)$ is not P^{2m-2}-integrable over $(b_j; 0)$.

7. **The P^{2m+1}-integral.** The methods of the preceding sections apply equally well to odd-numbered generalized symmetric derivatives.

If there are constants $\beta_1, \beta_3, \cdots, \beta_{2r+1}$, depending on x_0 but not on h, such that

$$(7.1)\qquad \frac{1}{2}\{F(x_0+h)-F(x_0-h)\}-\sum_{k=0}^{r}\beta_{2k+1}\frac{h^{2k+1}}{(2k+1)!}=o(h^{2r+1}),$$

as $h\to 0$, then β_{2r+1} is called the generalized symmetric derivative of order $2r+1$ of $F(x)$ at $x=x_0$, and written $D^{2r+1}F(x_0)$.

If $D^{2k+1}F(x_0)$ exists for $0\leqq k\leqq m-1$, define $\theta_{2m+1}(x_0, h)=\theta_{2m+1}(F; x_0, h)$ by

$$(7.2)\qquad \begin{aligned}\frac{h^{2m+1}}{(2m+1)!}&\theta_{2m+1}(x_0, h)\\ &=\frac{1}{2}\{F(x_0+h)-F(x_0-h)\}-\sum_{k=0}^{m-1}\frac{h^{2k+1}}{(2k+1)!}D^{2k+1}F(x_0),\end{aligned}$$

and $\Delta^{2m+1}F(x_0)$, $\delta^{2m+1}F(x_0)$ by equations similar to (2.3) and (2.4), respectively. All the other definitions of §2 apply with $2m$ replaced by $2m+1$.

In §3, Lemma 3.1, with $2m$ replaced by $2m+1$, is not proved by Wolf [9], but his method is applicable and the lemma remains true. The other lemmas of §3 are valid with minor modifications when $2m$ is replaced by $2m+1$.

In §4, when $2m$ is replaced by $2m+1$, the steps of the induction are valid, but it is not well known that Theorem 4.1, $2m-1$, is true for $m=2$. It is therefore necessary to prove

THEOREM 4.1, 3. *If $F(x)$ satisfies conditions A_3 and B_1 in (a, b) and if $\Delta^3F(x)>0$ in (a, b), then $D^1F(x)$ is convex (and hence continuous) in (a, b).*

Proof. This result is almost identical with Theorem 4 of Saks [5], and the proof is similar. Saks assumes the existence of the ordinary derivative $F'(x)$, but only to ensure that $F'(x)$ has no ordinary discontinuities in (a, b). This is also true for $D^1F(x)$ because $F(x)$ satisfies conditions B_1. He also requires the fact that every closed subset P of (a, b) contains a portion on which $F'(x)$ is upper semi-continuous. But $D^1F(x)$ has this property by Lemma 3.2 (with $2m$ replaced by $2m+1$ and $m=1$). With these modifications, the proof given by Saks is applicable (cf. Theorem 4.1, $2m$).

Once Theorem 4.1, $2m-1$ is established for $m=2$, the induction is completed as before and both Theorem 4.1 and 4.2 are true when $2m$ is replaced by $2m+1$.

In §§5 and 6, it is only necessary to replace $2m$ by $2m+1$ throughout, but it should be remarked that the note following Theorem 6.1 does not apply in this case.

It will now be shown that the two scales of integration fit together.

THEOREM 7.1. *If $n\geqq 2$ and $f(x)$ is P^n-integrable over $(b_j; x)$, then it is also P^{n+1}-integrable over $(a_i; x)$, where $b_1=a_1<\cdots<a_{n+1}=b_n$.*

Proof. Whether n is even or odd, the proof follows almost exactly that of

Theorem 6.1. The only difference is that if $Q(x;\, n)$ is a P^n-major function, then $R(x;\, n+1)$ is defined by

$$R(x;\, n+1) = \int_{a_1}^{x} Q(t;\, n)dt$$

instead of by (6.1). A similar change is made for $r(x;\, n+1)$.

THEOREM 7.2. *If $n \geqq 2$ and $f(x)$ is P^n-integrable over $(b_j;\, x)$, let*

$$(7.3)\qquad \begin{aligned} g(x) &= (-1)^s \int_{(b_j)}^{x} f(x)d_n x, \qquad b_s \leqq x < b_{s+1}, \\ G(x) &= \int_{b_1}^{x} g(t)dt. \end{aligned}$$

If $b_1 = a_1 < \cdots < a_{n+1} = b_n$, $a_r \leqq x < a_{r+1}$, then

$$(7.4)\qquad \begin{aligned} F(x) &= (-1)^r \int_{(a_i)}^{x} f(x)d_{n+1}x \\ &= \omega(x)V(G;\, x,\, a_i) = G(x) - \sum_{i=1}^{n+1} \lambda(x;\, a_i)G(a_i). \end{aligned}$$

Proof. The proof is almost identical with that of Theorem 6.2. Since $G(x)$ is now defined by (7.3) instead of by (6.7), only a single application of Lemma 6.1 is needed. With this change, the method of proof of Theorem 6.1 applies in this case.

COROLLARY 1. *If $n \geqq 2$ and $f(x)$ is P^n-integrable over $(b_j;\, x)$, then $F'(x)$ exists for all x in $[b_1,\, b_n]$. If $g(x)$ and $G(x)$ are the functions defined in Theorem 7.2, then*

$$F'(x) = g(x) - \sum_{i=1}^{n+1} \lambda'(x;\, a_i)G(a_i) = g(x) - p(x),$$

where $p(x)$ is a polynomial of degree $n-1$ at most.

COROLLARY 2. *If $n \geqq 2$, there is a function which is P^{n+1}-integrable, but not P^n-integrable.*

Proof. If n is odd let

$$G(x) = x \cos(1/x), \qquad x \neq 0, \qquad G(0) = 0.$$

If n is even let

$$G(x) = x \sin(1/x), \qquad x \neq 0, \qquad G(0) = 0.$$

Let $g(x) = G^{(n+1)}(x)$, $x \neq 0$, $g(0) = 0$. Then, in either case, $D^{n+1}G(x) = g(x)$ for all values of x, including $x = 0$, and the argument of Corollary 2 to Theorem 6.2 applies.

8. **The $\mathfrak{P}^n$-integral.** Let $F(x)$ be a function which is continuous in $[a, b]$. If there are constants $\alpha_1, \alpha_2, \cdots, \alpha_r$, depending on x_0 but not on h, such that

$$F(x_0 + h) - F(x_0) - \sum_{k=1}^{r} \alpha_k \frac{h^k}{k!} = o(h^r), \tag{8.1}$$

as $h \to 0$, then α_r is called the generalized derivative of order r of $F(x)$ at $x = x_0$ and written $F_{(r)}(x_0)$. It is clear that the existence of $F_{(r)}(x_0)$ implies that of $D^r F(x_0)$ and that $D^r F(x_0) = F_{(r)}(x_0)$.

If $F_{(k)}(x_0)$ exists for $1 \leqq k \leqq n-1$, define $\gamma_n(x_0, h) = \gamma_n(F; x_0, h)$ by

$$\frac{h^n}{n!} \gamma_n(x_0, h) = F(x_0 + h) - F(x_0) - \sum_{k=1}^{n-1} \frac{h^k}{k!} F_{(k)}(x_0), \tag{8.2}$$

and $\Delta F_{(n)}(x_0)$, $\delta F_{(n)}(x_0)$ by equations similar to (2.3) and (2.4), respectively.

It follows from (8.2) that

$$\frac{h^n}{n!} \gamma_n(x, h) = \frac{h^{n-1}}{(n-1)!} \left\{ \gamma_{n-1}(x, h) - F_{(n-1)}(x) \right\}$$

so that

$$\lim_{h \to 0} h\gamma_n(x, h) = 0. \tag{8.3}$$

Then, from (2.2) or (7.2),

$$\theta_n(x, h) = \frac{1}{2} \left\{ \gamma_n(x, h) + (-1)^n \gamma_n(x, -h) \right\},$$

and, by (8.3),

$$\lim_{h \to 0} h\theta_n(x, h) = 0. \tag{8.4}$$

This means that conditions A_n (whether $n = 2m$ or $n = 2m+1$) are automatically satisfied.

It can also be shown that conditions B_{n-1} (whether $n = 2m$ or $n = 2m+1$) must hold.

LEMMA 8.1. *If $F(x)$ is continuous in $[a, b]$, if, for $1 \leqq k \leqq r$, each $F_{(k)}(x)$ exists and is finite in (a, b), and if, for $1 \leqq k \leqq r-1$, no $F_{(k)}(x)$ has an ordinary discontinuity in (a, b), then $F_{(r)}(x)$ does not have an ordinary discontinuity in (a, b).*

Proof. This result is well known for $r = 1$. Let x_0 be a point in (a, b) and suppose that

$$\lim_{x \to x_0+} F_{(r)}(x) = \lambda.$$

For every $\epsilon>0$ there exists a $\delta>0$ such that

$$\lambda - \epsilon < F_{(r)}(x) < \lambda + \epsilon, \qquad x_0 < x < x_0 + \delta.$$

Let $G(x)=(\lambda+\epsilon)x^r/r!-F(x)$. Since, for $1\leqq k\leqq r-1$, $G_{(k)}(x)$ does not have an ordinary discontinuity in (a, b) and since $D^rG(x)=G_{(r)}(x)>0$ in $(x_0, x_0+\delta)$, all the conditions of Theorem 4.2 are satisfied. Hence $G(x)$ is r-convex for $x_0<x<x_0+\delta$. Since $G(x)$ is continuous at $x=x_0$, it is also r-convex for $x_0\leqq x<x_0+\delta$. This means that $V(G; x_k)\geqq 0$ for every set of $r+1$ distinct points such that $x_0<x_1<\cdots<x_r<x_0+\delta$. Let $x_k=x_0+kh$, $0\leqq k\leqq r$, where $rh<\delta$. Then

$$V(G; x_k) = \frac{1}{r!h^r}\sum_{k=0}^{r}(-1)^{r-k}{}_rC_kG(x_0+kh) \geqq 0.$$

It follows that $G_{(r)}(x_0)\geqq 0$ and this means that $F_{(r)}(x_0)\leqq\lambda+\epsilon$. Similarly, it can be shown that $\lambda-\epsilon\leqq F_{(r)}(x_0)$, and, since ϵ is arbitrary, $F_{(r)}(x_0)=\lambda$.

Thus $F_{(r)}(x)$ cannot have an ordinary discontinuity on the right, and the same method shows that it cannot have one on the left.

Because of (8.4) and Lemma 8.1, $\mathcal{P}^n$-major and -minor functions are only required to satisfy

(8.5.1) $Q(x)$ and $q(x)$ are continuous in $[a, b]$, and, for $1\leqq k\leqq n-1$, each $Q_{(k)}(x)$, $q_{(k)}(x)$ exists and is finite in (a, b);
(8.5.2) $Q(a_i)=q(a_i)=0$, $i=1, \cdots, n$, where $a=a_1<\cdots<a_n=b$;
(8.5.3) $\delta Q_{(n)}(x)\geqq f(x)\geqq\Delta q_{(n)}(x)$ in (a, b);
(8.5.4) $\delta Q_{(n)}(x)\neq-\infty$, $\Delta q_{(n)}(x)\neq+\infty$ in (a, b).

If $n=1$, these are the condition for major and minor functions for the Perron integral, except that (8.5.3) and (8.5.4) are usually required to hold only on $(a, b)-E$, where E is countable.

DEFINITION 8.1. Let c be a point of (a_1, a_n) such that $c\neq a_i$, $i=1, \cdots, n$. If, for every $\epsilon>0$, there is a pair $Q(x)$, $q(x)$ satisfying the conditions (8.5.1)–(8.5.4) and such that $|Q(c)-q(c)|<\epsilon$, then $f(x)$ is said to be $\mathcal{P}^n$-integrable over $(a_i; c)$.

The other definitions and results of §§5, 6, and 7 apply with appropriate changes to the $\mathcal{P}^n$-integral. It is clear that any $\mathcal{P}^n$-integrable function is also P^n-integrable. In particular, the analogue of Theorem 5.4 shows that if $F_{(n)}(x)$ exists and is finite in (a, b), then it is $\mathcal{P}^n$-integrable.

9. **The $\mathcal{P}^{r+1}$-integral includes the C_rP-integral.** For convenience, the definitions of C_r-continuity, C_r-derivatives, and the C_rP-integral (Burkill [1]) are given here. The C_0P-integral is the ordinary Perron integral (see, for example, Saks [6, Chap. VI]). Assuming that, for $r\geqq 1$, the $C_{r-1}P$-integral has been defined, the following definitions lead to the C_rP-integral.

DEFINITION 9.1. A function $M(x)$ defined in an interval $[a, b]$ is said to be C_r-continuous in $[a, b]$ if it is $C_{r-1}P$-integrable over $[a, b]$, and if

$$r/h^r(C_{r-1}P)\int_x^{x+h}(x+h-t)^{r-1}M(t)dt\to M(x),$$

as $h\to 0$, for every x in $[a, b]$.

DEFINITION 9.2. If $M(x)$ is $C_{r-1}P$-integrable over $[a, b]$, the upper and lower C_r-derivates of $M(x)$ are the lim sup and the lim inf, respectively, as $h\to 0$, of

$$(r+1)/h\left\{r/h^r(C_{r-1}P)\int_x^{x+h}(x+h-t)^{r-1}M(t)dt-M(x)\right\},$$

and denoted by $C_r\Delta M(x)$ and $C_r\delta M(x)$, respectively.

DEFINITION 9.3. Let $f(x)$ be a function defined in an interval $[a, b]$. The functions $M(x)$ and $m(x)$ are called C_rP-major and C_rP-minor functions, respectively, of $f(x)$ over $[a, b]$ if

$$M(x) \text{ and } m(x) \text{ are } C_r\text{-continuous in } [a, b];$$

$$M(a) = m(a) = 0;$$

$$C_r\delta M(x) \geqq f(x) \geqq C_r\Delta m(x) \qquad \text{in } [a, b];$$

$$C_r\delta M(x) \neq -\infty, \qquad C_r\Delta m(x) \neq +\infty \qquad \text{in } [a, b].$$

DEFINITION 9.4. If, for every $\epsilon>0$, there is a pair $M(x)$, $m(x)$ satisfying the conditions of Definition 9.3 and such that $|M(b)-m(b)|<\epsilon$, then $f(x)$ is said to be C_rP-integrable over $[a, b]$.

THEOREM 9.1. *If $f(x)$ is C_rP-integrable over $[a, b]$, it is also $\mathfrak{P}^{r+1}$-integrable over $(a_i; c)$, where $a=a_1<\cdots<a_{r+1}=b$. Moreover, if*

$$F_r(x) = (C_rP)\int_a^x f(t)dt,$$

$$(9.2)\qquad F_k(x) = (C_kP)\int_a^x F_{k+1}(t)dt, \qquad 0\leqq k\leqq r-1.$$

$$F(x) = F_0(x),$$

then, if $a_s\leqq x<a_{s+1}$,

$$(9.1)\qquad (-1)^s\int_{(a_i)}^x f(x)d_{r+1}x = F(x)-\sum_{i=1}^{r+1}\lambda(x; a_i)F(a_i),$$

where

$$\lambda(x; a_i) = \prod_{j\neq i}(x-a_j)/(a_i-a_j)$$

is a polynomial of degree r at most.

Proof. Let $M(x)$ be any C_rP-major function of $f(x)$ and let

$$G_{r-1}(x) = (C_{r-1}P)\int_a^x M(t)dt,$$

(9.3)
$$G_k(x) = (C_kP)\int_a^x G_{k+1}(t)dt, \qquad 0 \leqq k \leqq r-2,$$

$$G(x) = G_0(x).$$

Then, by $r-2$ integrations by parts (cf. Sargent [8]),

(9.4)
$$1/(r-1)!(C_{r-1}P)\int_x^{x+h}(x+h-t)^{r-1}M(t)dt = G(x+h) - G(x) - \sum_{k=1}^{r-1}\frac{h^k}{k!}G_k(x).$$

But, since $M(x)$ is C_r-continuous,

$$r/h^r(C_{r-1}P)\int_x^{x+h}(x+h-t)^{r-1}M(t)dt = M(x) + o(1).$$

It follows that

(9.5)
$$G(x+h) - G(x) - \sum_{k=1}^{r-1}\frac{h^k}{k!}G_k(x) = \frac{h^r}{r!}M(x) + o(h^r),$$

and, by definition (8.1), $G_{(r)}(x) = M(x)$, $G_{(k)}(x) = G_k(x)$, $0 \leqq k \leqq r-1$. Moreover,

$$\begin{aligned}\delta G_{(r+1)}(x) &= \liminf_{h\to 0}\,(r+1)!/h^{r+1}\left\{G(x+h) - \sum_{k=0}^{r}\frac{h^k}{k!}G_{(k)}(x)\right\}\\ &= \liminf_{h\to 0}\,(r+1)/h\left\{r/h^r\int_x^{x+h}(x+h-t)^{r-1}M(t)dt - M(x)\right\}\\ &= C_r\delta M(x) \geqq f(x),\end{aligned}$$

where the integral is a $C_{r-1}P$-integral.

If

(9.6)
$$Q(x) = G(x) - \sum_{i=1}^{r+1}\lambda(x;\, a_i)G(a_i),$$

then

(9.7)
$$\delta Q_{(r+1)}(x) = \delta G_{(r+1)}(x) \geqq f(x),$$

and $Q(x)$ satisfies all the requirements for a $\mathfrak{P}^{r+1}$-major function.

A corresponding result holds for any C_rP-minor function, and it is clear that C_r-P-integrability implies $\mathfrak{P}^{r+1}$-integrability.

Since $f(x)$ is C_rP-integrable, there is a sequence $M(x;\, n)$ of C_rP-major functions such that

$$0 \leqq M(x;\, n) - (C_rP)\int_a^x f(t)dt < 1/n. \tag{9.8}$$

Formulas (9.3) may be written in the form

$$G(x) = \int_a^x \int_a^{t_1} \cdots \int_a^{t_{r-1}} M(t_r)dt_r dt_{r-1} \cdots dt_1,$$

where the inner integral is a $C_{r-1}P$-integral, the next a $C_{r-2}P$-integral, $\cdots$, and the outer one C_0P- or Perron integral. Similarly, (9.1) may be written

$$F(x) = \int_a^x \int_a^{t_1} \cdots \int_a^{t_r} f(t)dt dt_r \cdots dt_1,$$

where this time the inner integral is a C_rP-integral. From (9.8) it follows that there is a sequence $G(x;\, n)$ such that $0 \leqq G(x;\, n) - F(x) < (b-a)^r/(r!n)$. Hence $G(x;\, n) \to F(x)$.

Then, by (9.6), the corresponding $\mathfrak{P}^{r+1}$-major function $Q(x;\, n)$ tends to the right-hand side of (9.2). By the definition of the $\mathfrak{P}^{r+1}$-integral, it also tends to the left-hand side.

Since Miss Sargent [7; 8] has proved that the C_rD- and V_rD-integrals are equivalent to the C_rP-integral, the $\mathfrak{P}^{r+1}$-integral also includes these integrals.

COROLLARY. *The function $F(x)$ defined in Theorem 9.1 has generalized derivatives $F_{(k)}(x)$ which are C_k-continuous in $[a, b]$ for $1 \leqq k \leqq r$. In particular,*

$$F_{(r)}(x) = (C_rP)\int_a^x f(t)dt. \tag{9.9}$$

Proof. Since $F_r(x)$ is a C_rP-integral, it is C_r-continuous. By the proof of Theorem 9.1, with $M(x)$ replaced by $F_r(x)$ and $G_k(x)$ by $F_k(x)$, it follows that

$$F(x+h) - F(x) - \sum_{k=1}^{r-1} \frac{h^k}{k!} F_k(x) = \frac{h^r}{r!} F_r(x) + o(h^r).$$

Hence $F_{(r)}(x) = F_r(x)$ and consequently $F_{(k)}(x) = F_k(x)$ for $1 \leqq k \leqq r$. Formula (9.9) now follows from (9.1).

10. **Properties of n-convex functions.** It is assumed in this section that $n \geqq 2$ and that, if $n=2$, an $(n-1)$-convex function is a monotone increasing function.

LEMMA 10.1. *If the function $\phi(x)$ is n-convex in $[a, b]$ and z is any point of $[a, b]$, then the function*

$$\rho(x) = \rho(x; z) = \{\phi(x) - \phi(z)\}/(x - z) \tag{10.1}$$

is $(n-1)$*-convex in each of the half-open intervals* $[a, z)$, $(z, b]$. *Moreover, if* $\phi(x)$ *has generalized derivatives* $\phi_{(k)}(x)$, $1 \leqq k \leqq n-1$, *in* (a, b), *so does* $\rho(x)$ *in* (a, z) *and* (z, b), *and, for* $x \neq z$,

$$\phi_{(k)}(x) = k\rho_{(k-1)}(x) + (x - z)\rho_{(k)}(x). \tag{10.2}$$

Proof. It follows by elementary algebra that, if $x_1, x_2, \cdots, x_n$ are distinct points in either $[a, z)$ or $(z, b]$, then $V(\phi; z, x_1, \cdots, x_n) = V(\rho; x_1, \cdots, x_n)$. The proof of (10.2) is also elementary.

LEMMA 10.2. *If the function* $\phi(x)$ *is* n*-convex in* $[a, b]$ *and has finite generalized derivatives* $\phi_{(k)}(x)$, $1 \leqq k \leqq n-2$, *in* (a, b), *then, for each* x_0 *in* (a, b),

$$\begin{aligned}\gamma_{n-1}(x_0, h) &= \gamma_{n-1}(\phi; x_0, h)\\ &= \frac{(n-1)!}{h^{n-1}}\left\{\phi(x_0 + h) - \phi(x_0) - \sum_{k=1}^{n-2} \frac{h^k}{k!}\phi_{(k)}(x_0)\right\}\end{aligned} \tag{10.3}$$

is monotone decreasing as $h \to 0+$ *and monotone increasing as* $h \to 0-$.

Proof. If $a < x_0 < x_1 < \cdots < x_{n-1} < x_0 + h_1 < x_0 + h_2$, it follows by elementary algebra that if

$$V(\phi; x_0, x_1, \cdots, x_{n-1}, x_0 + h_1, x_0 + h_2) \geqq 0,$$

then the determinant

$$\begin{vmatrix} \phi(x_0 + h_2) & (x_0 + h_2)^{n-1} & \cdots & (x_0 + h_2)^2 & x_0 + h_2 & 1 \\ \phi(x_0 + h_1) & (x_0 + h_1)^{n-1} & \cdots & (x_0 + h_1)^2 & x_0 + h_1 & 1 \\ \phi(x_{n-1}) & x_{n-1}^{n-1} & \cdots & x_{n-1}^2 & x_{n-1} & 1 \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ \phi(x_2) & x_2^{n-1} & \cdots & x_2^2 & x_2 & 1 \\ \phi(x_1) & x_1^{n-1} & \cdots & x_1^2 & x_1 & 1 \\ \phi(x_0) & x_0^{n-1} & \cdots & x_0^2 & x_0 & 1 \end{vmatrix} \tag{10.4}$$

is non-negative.

By subtracting the last row from the second-last row, dividing by $x_1 - x_0$, which is positive, and letting $x_1 \to x_0+$, it follows that the determinant (10.4), with the second-last row replaced by

$$\phi_{(1)}(x_0) \quad (n-1)x_0^{n-2} \quad \cdots \quad 2x_0 \quad 1 \quad 0, \tag{10.5}$$

is also non-negative.

Similarly, by subtracting the last row plus $x_2 - x_0$ times the new second-last row from the third-last row, dividing by $(x_2 - x_0)^2/2$, and letting $x_2 \to x_0+$, the determinant (10.4), with the second-last row replaced by (10.5) and the third-last row by

$$\phi_{(2)}(x_0) \quad (n-1)(n-2)x_0^{n-3} \quad \cdots \quad 2 \quad 0 \quad 0,$$

is also non-negative.

If the process is repeated a sufficient number of times, then the determinant

$$(10.6) \qquad \begin{vmatrix} \phi(x_0+h_2) & (x_0+h_2)^{n-1} & \cdots & (x_0+h_2)^2 & x_0+h_2 & 1 \\ \phi(x_0+h_1) & (x_0+h_1)^{n-1} & \cdots & (x_0+h_1)^2 & x_0+h_1 & 1 \\ \phi_{(n-2)}(x_0) & (n-1)!x_0 & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ \phi_{(2)}(x_0) & (n-1)(n-2)x_0^{n-3} & \cdots & 2 & 0 & 0 \\ \phi_{(1)}(x_0) & (n-1)x_0^{n-2} & \cdots & 2x_0 & 1 & 0 \\ \phi(x_0) & x_0^{n-1} & \cdots & x_0^2 & x_0 & 1 \end{vmatrix}$$

is non-negative.

Finally, by subtracting appropriate combinations of rows 3, 4, $\cdots$, $n+1$ from the first row and from the second, it follows that the determinant (10.6) with the first and second rows replaced by

$$\gamma_{n-1}(x_0, h_2) \quad (n-1)! \quad 0 \quad \cdots \quad 0 \quad 0 \quad 0,$$

and

$$\gamma_{n-1}(x_0, h_1) \quad (n-1)! \quad 0 \quad \cdots \quad 0 \quad 0 \quad 0,$$

respectively, is non-negative. It is then clear that

$$\begin{vmatrix} \gamma_{n-1}(x_0, h_2) & (n-1)! \\ \gamma_{n-1}(x_0, h_1) & (n-1)! \end{vmatrix} \geqq 0,$$

and this shows that $\gamma_{n-1}(x_0, h)$ is monotone decreasing as $h \to 0+$.

The proof that $\gamma_{n-1}(x_0, h)$ is monotone increasing as $h \to 0-$ is similar.

LEMMA 10.3. *If the function $\phi(x)$ is n-convex in (a, b) and has finite generalized derivatives $\phi_{(k)}(x)$, $1 \leqq k \leqq n-1$, in (a, b), then $\phi_{(n-1)}(x)$ is monotone increasing in (a, b).*

Proof. This result is well known(3) in the case $n=2$. Assume, then, that $n \geqq 3$ and that the lemma is true for $(n-1)$-convex functions. If x_1 and x_2 are in (a, b) and $x_1 < x_2$, let z be any number such that $x_1 < x_2 < z$. It follows from (8.2), (10.1), and (10.2) that

(3) G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, Cambridge, 1934, Theorem 111.

$$\frac{(z-x_2)^{n-1}}{(n-1)!}\gamma_{n-1}(x_2, z-x_2)$$
$$= (z-x_2)\rho(x_2) - \sum_{k=1}^{n-2}\frac{(z-x_2)^k}{k!}\left\{k\rho_{(k-1)}(x_2) - (z-x_2)\rho_{(k)}(x_2)\right\}$$
$$= (z-x_2)\rho(x_2) - \sum_{k=0}^{n-3}\frac{(z-x_2)^{k+1}}{k!}\rho_{(k)}(x_2) + \sum_{k=1}^{n-2}\frac{(z-x_2)^{k+1}}{k!}\rho_{(k)}(x_2)$$
$$= \frac{(z-x_2)^{n-1}}{(n-2)!}\rho_{(n-2)}(x_2),$$

or

$$\gamma_{n-1}(x_2, z-x_2) = (n-1)\rho_{(n-2)}(x_2). \tag{10.7}$$

Similarly,

$$\gamma_{n-1}(x_1, z-x_1) = (n-1)\rho_{(n-2)}(x_1). \tag{10.8}$$

Since $\rho(x)$ is $(n-1)$-convex in (a, z), by Lemma 10.1, $\rho_{(n-2)}(x)$ is monotone increasing in (a, z) by the hypothesis of the induction. Hence, by Lemma 10.2, (10.7), and (10.8), $\phi_{(n-1)}(x_1) \leqq \gamma_{n-1}(x_1, z-x_1) \leqq \gamma_{n-1}(x_2, z-x_2)$. Since this is true for any $z > x_2$,

$$\phi_{(n-1)}(x_1) \leqq \lim_{z\to x_2}\gamma_{n-1}(x_2, z-x_2) = \phi_{(n-1)}(x_2).$$

LEMMA 10.4. *Let $\phi(x)$ be a function which is n-convex in (a, b) and has finite generalized derivatives $\phi_{(k)}(x)$, $1 \leqq k \leqq n-1$, in (a, b). Let α, β, and α_i, β_i, $i=1, \cdots, n-1$, be fixed numbers such that*

$$a < \alpha_{n-1} < \alpha_{n-2} < \cdots < \alpha_1 < \alpha < \beta < \beta_1 < \cdots < \beta_{n-1} < b.$$

Then, for all x in $[\alpha, \beta]$,

$$(n-1)!V(\phi;\alpha,\alpha_i) \leqq \phi_{(n-1)}(x) \leqq (n-1)!V(\phi;\beta,\beta_i). \tag{10.9}$$

Proof. If $n=2$, (10.9) becomes

$$\left\{\phi(\alpha) - \phi(\alpha_1)\right\}/(\alpha-\alpha_1) \leqq \phi_{(1)}(x) \leqq \left\{\phi(\beta_1) - \phi(\beta)\right\}/(\beta_1-\beta),$$

and this is a well known result for ordinary convex functions.

Assume that $n \geqq 3$ and that (10.9) holds for every $(n-1)$-convex function with finite generalized derivatives. If $\rho(x)$ is the function defined in Lemma 10.1, then, by Lemmas 10.2 and 10.3 and (10.8),

$$\begin{aligned}\phi_{(n-1)}(x) &\leqq \phi_{(n-1)}(\beta)\\ &\leqq \gamma_{n-1}(\beta, \beta_{n-1}-\beta) = (n-1)\rho_{(n-2)}(\beta;\beta_{n-1}).\end{aligned} \tag{10.10}$$

But $\rho(x;\beta_{n-1})$ is $(n-1)$-convex in (a, β_{n-1}) and hence in $(\alpha_{n-1}, \beta_{n-1})$. Then, by the hypothesis of the induction,

$$\rho(\beta;\beta_{n-1}) \leqq (n-2)!V(\rho;\beta,\beta_1,\cdots,\beta_{n-2}). \tag{10.11}$$

Since (as in the proof of Lemma 10.1)

$$V(\rho;\beta,\beta_1,\cdots,\beta_{n-2}) = V(\phi;\beta,\beta_1,\cdots,\beta_{n-1}),$$

it follows from (10.10) and (10.11) that the right-hand inequality of (10.9) is valid. The proof of the validity of the left-hand inequality is similar and completes the induction.

11. The C_rP- and the $\mathfrak{P}^{r+1}$-integrals.

LEMMA 11.1. *If $\phi(x)$ is continuous and has finite generalized derivatives $\phi_{(k)}(x)$, $1\leqq k\leqq r$, in $[\alpha,\beta]$, then each $\phi_{(k)}(x)$ is C_k-continuous in $[\alpha,\beta]$.*

Proof. Since $\phi_{(1)}(x)=\phi'(x)$ is finite in $[\alpha,\beta]$, it is Perron or C_0P-integrable over $[\alpha,\beta]$ and

$$\{\phi(x+h)-\phi(x)\}/h = (1/h)\int_x^{x+h}\phi'(t)dt \to \phi'(x).$$

This shows that $\phi_{(1)}(x)$ is C_1-continuous. Also,

$$\begin{aligned} C_1D\phi_{(1)}(x) &= \lim_{h\to 0}(2/h)\left\{(1/h)\int_x^{x+h}\phi_{(1)}(t)dt - \phi_{(1)}(x)\right\} \\ &= \lim_{h\to 0}(2/h^2)\{\phi(x+h)-\phi(x)-h\phi_{(1)}(x)\} = \phi_{(2)}(x). \end{aligned}$$

Since $\phi_{(2)}(x)$ is an exact C_1D-derivative, it is C_1P-integrable (Burkill [2]). It then follows as above that it is C_2-continuous.

After a finite number of similar steps, the lemma is proved.

THEOREM 11.1. *If $f(x)$ is $\mathfrak{P}^{r+1}$-integrable over $(a_i;x)$ and $[\alpha,\beta]$ is any closed sub-interval of (a_1,a_{r+1}), then $f(x)$ is C_rP-integrable over $[\alpha,\beta]$. Moreover, if*

$$F(x) = (-1)^s\int_{(a_i)}^x f(x)d_{r+1}x, \qquad a_s \leqq x < a_{s+1},$$

then $F(x)$ has generalized derivatives $F_{(k)}(x)$, $1\leqq k\leqq r$, in $[\alpha,\beta]$ and

$$F_{(r)}(\beta) - F_{(r)}(\alpha) = (C_rP)\int_\alpha^\beta f(x)dx. \tag{11.1}$$

Proof. If $Q(x)$ is any $\mathfrak{P}^{r+1}$-major function of $f(x)$ over (a_i), it follows from Lemma 11.1 that $Q_{(r)}(x)$ is C_r-continuous. Also, by the proof of Theorem 9.1, with $M(x)$ replaced by $Q_{(r)}(x)$ and $G_{(k)}(x)$ by $Q_{(k)}(x)$ for $0\leqq k\leqq r-2$,

$$C_r\delta Q_{(r)}(x) = \delta Q_{(r+1)}(x) \geqq f(x).$$

Hence, if $M(x)=Q_{(r)}(x)-Q_{(r)}(\alpha)$, then $M(x)$ is a C_rP-major function of $f(x)$ over $[\alpha, \beta]$. Similarly, if $q(x)$ is a $\mathcal{P}^{r+1}$-minor function, then $m(x)=q_{(r)}(x)-q_{(r)}(\alpha)$ is a C_rP-minor function.

Since $Q(x)-q(x)$ is $(r+1)$-convex, by Lemma 5.1, it follows from Lemma 10.4, with $\phi(x)=Q(x)-q(x)$, that

$$
(11.2)\qquad \begin{aligned} r!V(Q-q;\alpha,\alpha_1,\cdots,\alpha_r) &\leqq Q_{(r)}(x)-q_{(r)}(x) \\ &\leqq r!V(Q-q;\beta,\beta_1,\cdots,\beta_r). \end{aligned}
$$

Also, by Lemma 10.3,

$$
(11.3)\qquad M(x)-m(x)=\{Q_{(r)}(x)-q_{(r)}(x)\}-\{Q_{(r)}(\alpha)-q_{(r)}(\alpha)\}
$$

is monotone increasing. Thus, by (11.2) and (11.3),

$$
(11.4)\qquad \begin{aligned} 0 &= M(\alpha)-m(\alpha)\leqq M(\beta)-m(\beta) \\ &\leqq r!\{V(Q-q;\beta,\beta_1,\cdots,\beta_r)-V(Q-q;\alpha,\alpha_1,\cdots,\alpha_r)\}. \end{aligned}
$$

Since $f(x)$ is $\mathcal{P}^{r+1}$-integrable, there is a pair $Q(x)$, $q(x)$, such that

$$
|Q(x)-q(x)|<\epsilon K,
$$

where K is independent of x. From (11.4) it then follows that $M(\beta)-m(\beta)\leqq r!\ \{|V(Q-q;\ \beta,\ \beta_i)|+|V(Q-q;\ \alpha,\ \alpha_i)|\}<\epsilon C$, where C is a constant. Hence $f(x)$ is C_rP-integrable over $[\alpha, \beta]$.

Now let

$$
H_r(x)=(C_rP)\int_a^x f(t)dt,
$$

$$
H_k(x)=(C_kP)\int_a^x H_{k+1}(t)dt,\qquad 0\leqq k\leqq r-1,
$$

$$
H(x)=H_0(x).
$$

It follows from Theorem 9.1 and its corollary that

$$
(11.5)\qquad H_{(r)}(\beta)-H_{(r)}(\alpha)=(C_rP)\int_\alpha^\beta f(t)dt.
$$

But, by the analogue of Theorem 5.5 for the $\mathcal{P}^{r+1}$-integral, with b_1 replaced by α and b_{r+1} by β,

$$
(11.6)\qquad H_{(r)}(x)=F_{(r)}(x)-\sum_{j=1}^{r+1}\lambda^{(r)}(x;b_j)F(b_j).
$$

Since each $\lambda(x;\ b_j)$ is a polynomial of degree r at most, each $\lambda^{(r)}(x;\ b_j)$ is a constant. Formula (11.1) then follows from (11.5) and (11.6).

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