1. Introduction. In a previous paper [2] the writer defined a set of rational functions \( \eta_m \) of the indeterminate \( q \) by means of

\[
(q \eta + 1)^m = \eta^m \quad (m > 1), \quad \eta_0 = 1, \quad \eta_1 = 0,
\]

and a set of polynomials

\[
\eta_m(x) = \eta_m(x, q)
\]

in \( q^x \) by

\[
\eta_m(x) = \left( [x] + q^x \eta \right)^m, \quad \eta_m(0) = \eta_m,
\]

where \([x] = (q^x - 1)/(q - 1)\); also

\[
q^x \beta_m(x) = \eta_m(x) + (q - 1)\eta_{m+1}(x), \quad \beta_m(0) = \beta_m.
\]

For \( q = 1 \), \( \beta_m \) reduces to the Bernoulli number \( B_m \), \( \beta_m(x) \) reduces to the Bernoulli polynomial \( B_m(x) \); \( \eta_m \) however does not remain finite for \( m > 1 \).

In the present paper we first define polynomials \( A_m = A_m(q) \) by means of

\[
[x]^m = \sum_{s=1}^{m} A_m \left( \frac{x + s - 1}{m} \right) \quad (m \geq 1),
\]

where

\[
\left[ \frac{x}{m} \right] = \frac{(q^x - 1)(q^{x-1} - 1) \cdots (q^{x-m+1} - 1)}{(q - 1)(q^2 - 1) \cdots (q^m - 1)}.
\]

Alternatively if we define the rational function \( H_m = H_m(x, q) \) by means of \( H_0 = 1, H_1 = 1/(x-q) \),

\[
(qH + 1)^m = xH^m \quad (m > 1),
\]

then we have

\[
H_m(x, q) = A_m(x, q)/\prod_{s=1}^{m} (x - q^s),
\]

where

\[
A_m(x, q) = \sum_{s=1}^{m} A_m x^{s-1} \quad (m \geq 1),
\]

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and the coefficients are the same as those occurring in (1.4). For \( q = 1 \), \( A_{ms} \) and \( H_m(x) \) reduce to well known functions; some of the properties of these quantities are stated in §2 below. As Frobenius [3] showed, many of the properties of the Bernoulli and related numbers can be derived from properties of \( H_m \). We shall show that much the same is true in the case of the \( q \) analogues.

In [2] a theorem somewhat analogous to the Staudt-Clausen theorem was obtained for \( \beta_m \) (with \( q \) an indeterminate). We now show that if \( p \) is an odd prime and we put \( q = a \), where the rational number \( a \) is integral (mod \( p \)), then if \( a \equiv 1 \) (mod \( p \)),

\[
\beta_m \equiv -1 \pmod{p}
\]

provided \( p - 1 \mid m \); otherwise \( \beta_m \) is integral (mod \( p \)). If \( a \not\equiv 1 \) (mod \( p \)) the situation is more complicated. In particular, if \( a \) is a primitive root (mod \( p^2 \)), then \( \beta_m \) is integral (mod \( p \)) for \( p - 1 \mid m \), while for \( p - 1 \not\mid m \) we have

\[
p\beta_m \equiv -1 \pmod{p}, \quad (k = (a^{p-1} - 1)/p).
\]

In general the denominator of \( \beta_m \) may be divisible by arbitrarily high powers of \( p \) (see Theorem 4 below).

Finally we derive some congruences of Kummer's type for \( H_m \), etc. For example if \( q = a \) is integral (mod \( p \)) while \( x \) is an indeterminate, then

\[
H^m(H^w - 1)^r \equiv 0 \pmod{p^m, p^e} \quad (p^{e-1}(p - 1) \mid w),
\]

where after expansion of the left member \( H^k \) is replaced by \( H_k \). We also obtain simple congruences for the numbers \( A_{ms} \) defined in (1.4). The corresponding results for \( \eta_m \) and \( \beta_m \) are more complicated.

2. Eulerian numbers. To facilitate comparison we quote the following formulas from the papers of Frobenius [3] and Worpitzky [5].

\[
x^m = \sum_{s=1}^{m} A_{ms} \binom{x+s-1}{m} \quad (m \geq 1),
\]

\[
A_{m+1,s} = (m + 2 - s)A_{m,s-1} + sA_{m,s},
\]

\[
A_{ms} = \sum_{r=0}^{s} (-1)^r \binom{m+1}{r} (s-r)^m,
\]

\[
B_m = \frac{1}{m+1} \sum_{r=1}^{m} (-1)^{m-r-1} \binom{m}{r-1}^{-1} A_{mr}.
\]

In the next place if we put

\[
A_m = A_m(x) = \sum_{s=1}^{m} A_{ms} x^{s-1},
\]
and let

\[ H_m = H_m(x) = (x - 1)^{-m} R_m(x), \]

then \( H_m \) satisfies

\[ (H + 1)^m = xH^m \quad (m \geq 1), \quad H_0 = 1. \]

The connection between \( H_m \) and the Bernoulli numbers is furnished by

\[ \sum_{r=0}^{k-1} \zeta^r B_m \left( \frac{r}{k} \right) = \frac{k^{1-m} \zeta}{1 - \zeta} mH_{m-1}(\zeta), \]

where \( \zeta^k = 1, \zeta \neq 1 \). An immediate consequence of (2.7) is

\[ k^n B_m \left( \frac{r}{k} \right) - B_m = -m \sum_{\zeta \neq 1} \frac{1}{\zeta - 1} H_{m-1}(\zeta), \]

where \( \zeta \) runs through the \( k \)th roots of unity distinct from 1.

We also mention

\[ H_m = \sum_{r=0}^{m} (x - 1)^{-\Delta} \theta^m. \]

3. Some preliminaries. We shall use the notation of [2]; see in particular §2 of that paper. In addition the following remarks will be useful. Let \( f(u) \) be a polynomial in \( q^u \) of degree \( \leq m \). Then the difference equation

\[ g(u + 1) - cg(u) = f(u) \quad (c \neq q^r) \]

has a unique polynomial solution \( g(u) \), as can easily be proved by comparison of coefficients. To put the solution in more useful form we rewrite (3.1) as

\[ (E - c)g(u) = f(u) \]

and recall that

\[ \Delta = E - 1, \quad \Delta^2 = (E - 1)(E - q), \quad \Delta^3 = (E - 1)(E - q)(E - q^2), \ldots. \]

In the identity

\[ \frac{1}{t - z} = \frac{1}{t - z_1} + \frac{z - z_1}{t - z_2} + \frac{(z - z_1)(z - z_2)}{(t - z_1)(t - z_2)} \frac{1}{t - z_3} + \cdots \]

\[ + \frac{(z - z_1) \cdots (z - z_n)}{(t - z_1) \cdots (t - z_n)} \frac{1}{t - z} \]

take \( t = c, z = E, z_s = q^{s-1} \), so that we get

\[ \frac{1}{c - E} = \frac{1}{c - 1} + \frac{\Delta}{(c - 1)(c - q)} + \frac{\Delta^2}{(c - 1)(c - q)(c - q^2)} + \cdots \]

\[ + \frac{\Delta^n}{(c - 1) \cdots (c - q^{n-1})} \frac{1}{c - E}. \]
Hence if we take $n > m$, we obtain the following formula for $g(u)$:

\[(3.2) \quad g(u) = \sum_{s=0}^{m} \frac{\Delta^s f(u)}{(c - 1)(c - q) \cdots (c - q^s)}.\]

4. **The number $A_{ms}$.** We suppose $A_{ms}$ defined by means of (1.4). Using the identity

\[(q^{m+1} - 1)(q^s - 1) = (q^{m+1-s} - 1)(q^{s+1} - 1) + q^{m+1-s}(q^s - 1)(q^{s+m-1} - 1)\]

and multiplying both members of (1.4) by $[x]$, we get

\[
[x]^{m+1} = \sum_{s} A_{ms} \left\{ [m + 1 - s]\left[ \frac{x + s}{m + 1} \right] + q^{m+1-s}[s]\left[ \frac{x + s - 1}{m + 1} \right] \right\}
\]

which implies the recursion

\[(4.1) \quad A_{m+1,s} = [m + 2 - s]A_{m,s-1} + q^{m+1-s}[s]A_{ms},\]

For $q = 1$ it is evident that (4.1) reduces to (2.2). As an immediate consequence of (4.1) we infer that $A_{ms}$ is a polynomial in $q$ with positive integral coefficients.

It is easy to show that $A_{ms}$ is divisible by $q^{(m-s)(m-s+1)/2}$. Indeed if we put

\[(4.2) \quad A_{ms} = q^{(m-s)(m-s+1)/2}A_{ms}^*,\]

then (4.1) becomes

\[(4.3) \quad A_{m+1,s}^* = [m + 2 - s]A_{m,s-1}^* + [s]A_{ms}^*,\]

which proves the stated property. Moreover it follows easily from (4.3) that

\[(4.4) \quad \deg A_{ms}^* = (s - 1)(m - s).\]

Indeed assuming the truth of (4.4), we get

\[
\deg ([m + 2 - s]A_{m,s-1}^*) = (m + 1 - s) + (s - 2)(m + 1 - s) = (s - 1)(m + 1 - s),
\]

\[
\deg ([s]A_{ms}^*) = (s - 1) + (s - 1)(m - s) = (s - 1)(m + 1 - s),
\]

so that

\[
\deg A_{m+1,s}^* = (s - 1)(m + 1 - s),
\]

which proves (4.4).
The symmetry properties

\[ A_{m,m-s+1}^* = A_{ms}^* \]  

and

\[ A_{ms}(q) = q^{(s-1)(m-s)}A_{ms}^*(q^{-1}) \]

will be proved below.

Comparing coefficients of \( q^{ms} \) on both sides of (1.4) we get

\[ \sum_{s=1}^{m} A_{ms} = [m!] = [m][m - 1] \cdots [1]. \]

More generally if we expand both sides in powers of \( qx \) and equate coefficients we get

\[ \left[ \begin{array}{c} m \\ s \end{array} \right] \sum_{r=1}^{m} A_{mr}q^{rs} = \left( m \right) q^{ms-s(s-1)/2}[m!] \quad (0 \leq s \leq m). \]

The following table of \( A_{ms}^* \), \( 1 \leq s \leq m \leq 5 \), is easily computed by means of (4.3).

<table>
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<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
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</tr>
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<td>1</td>
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<td>3q^2+5q+3</td>
<td>1</td>
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<td>1</td>
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<td>6q^4+16q^3+22q^2+16q+6</td>
<td>4q^2+9q^2+9q+4</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>4q^3+9q^2+9q+4</td>
<td>6q^4+16q^3+22q^2+16q+6</td>
<td>4q^2+9q^2+9q+4</td>
<td>1</td>
</tr>
</tbody>
</table>

5. A formula for \( A_{ms}^* \). It is easy to show that if \( f(x) \) is a polynomial in \( q^x \) of degree \( \leq m \),

\[ f(x) = \sum_{s=0}^{m} C_{ms} \left[ \begin{array}{c} x + s - 1 \\ m \end{array} \right] \quad (m \geq 1), \]

then

\[ C_{m0} = (-1)^m q^{m(m+1)/2} f(0), \]

\[ C_{m,m-r} = \sum_{s=0}^{m} (-1)^s \left[ \begin{array}{c} m + 1 \\ s \end{array} \right] f(r + 1 - s)q^{(s-1)/2}. \]

Since

\[ \sum_{s=0}^{m+1} (-1)^s q^{(s-1)/2} \left[ \begin{array}{c} m + 1 \\ s \end{array} \right] f(x + m + 1 - s) = 0, \]

we have in particular
and (5.3) yields

\[ C_{mr} = \sum_{s=0}^{r} (-1)^{m-s} q^{(m-s) (m+1-s)/2} \binom{m+1}{s} f(s-r), \]

which includes (5.2) also. Thus the coefficients in (5.1) are determined.

If we take \( f(x) = [x]^m \), then \( C_{mr} = A_{mr} \) and we get after a little manipulation

\[ A_{mr}^* = q^{r(r-1)/2} \sum_{s=0}^{r} (-1)^{m-s} q^{s(s-1)/2} \binom{m+1}{s} [r-s]^m; \]

for \( q = 1 \), (5.5) reduces to (2.3).

Replacing \( q \) by \( q^{-1} \), (5.5) becomes

\[ q^{(r-1)(m-r)} A_{mr}(q^{-1}) = q^{(r-1)m+r(r-1)/2} \sum_{s=0}^{r} (-1)^{s} q^{s(s+1)/2 + e} \binom{m+1}{s} [r-s]^m, \]

where

\[ e = \frac{s(s-1) - (m+1)(m+2)}{2} + \frac{(m+1-s)(m+2-s)}{2} + \frac{s(s+1)}{2} - (r-s+1)m = -(r-1)m. \]

Hence

\[ A_{mr}(q) = q^{(r-1)(m-r)} A_{mr}(q^{-1}), \]

which is identical with (4.6).

In the next place we observe that exactly as in the proof of (6.2) of [2] we have

\[ \sum_{i=0}^{m} \binom{m}{i} [x]^{i+1} q^{(m-i)x} \frac{\eta_{m-i}}{i+1} + (q^{(m+1)x} - 1) \frac{\eta_{m+1}}{m+1} \]

\[ = \sum_{s=1}^{m} A_{ms} q^{m-s+1} \binom{x+s-1}{m+1}. \]

Divide both sides of this identity by \([x]\) and then put \( x = 0 \). We find that

\[ \beta_m = \frac{1}{[m+1]} \sum_{s=1}^{m} (-1)^{m-s} q^{-(m-s)(m-s+1)/2} \binom{m}{s-1}^{-1} A_{ms}. \]

Using (4.2) and (4.5) this becomes
the first of which may be compared with (2.4).

6. The polynomial $A_m(x)$. The polynomial $A_m(x) = A_m(x, q)$ is defined in (1.7) for $m \geq 1$; we put $A_0(x) = 1$. Put

$$(6.1) \quad \phi_m(x) = \prod_{s=0}^{m} (x - q^s)$$

and apply the Lagrange interpolation formula at the points $x = q^s$, $s = 0, 1, \ldots, m$. Since

$$\begin{align*}
\phi'(q^s) &= \prod_{i=0}^{s-1} (q^s - q^i) \prod_{j=s+1}^{m} (q^s - q^j) \\
&= (-1)^{m-s} q^{s(s-1)/2} (q - 1)^m [s]! [m - s]!,
\end{align*}$$

we get using (4.8)

$$(6.2) \quad A_m(x) = \frac{\phi_m(x)}{(q - 1)^m} \sum_{s=0}^{m} (-1)^{m-s} \binom{m}{s} \frac{1}{x - q^s}.$$

As a first application of (6.2) consider

$$\begin{align*}
A_m(x^{-1}, q^{-1}) &= \frac{x^{-m-1} q^{-m(m+1)/2} \phi_m(x)}{q^{-m(q - 1)^m}} \sum_{s=0}^{m} (-1)^{m-s} \binom{m}{s} \frac{xq^s}{x - q^s},
\end{align*}$$

which gives

$$(6.3) \quad x^{-m-1} q^{-m(m+1)/2} A_m(x^{-1}, q^{-1}) = A_m(x, q) \quad (m \geq 1).$$

Substituting from (1.7) in (6.3) we get

$$q^{m(m-1)/2} \sum_{s=1}^{m} A_{ms}(q^{-1}) x^{m-s} = \sum_{s=1}^{m} A_{ms}(q) x^{m-s},$$

which implies

$$(6.4) \quad q^{m(m-1)/2} A_{ms}(q^{-1}) = A_{m, m-s+1}(q).$$

Hence by (4.2) and (4.6), (6.4) becomes

$$q^{m(m-1)/2 - (m-s)(m-s+1)/2 - (s-1)(m-s)} A_{ms}(q^{-1}) = q^{s(s-1)/2} A_{m, m-s+1}(q),$$

which is the same as (4.5).

7. The functions $H_m(x)$ and $H_m(u, x)$. Using (6.2) and (1.6) we get
\[(q - 1)^m H_m(x) = (x - 1) \sum_{s=0}^{m} (-1)^{m-s} \binom{m}{s} \frac{1}{x - q^s} \cdot \]

We remark that (7.1) implies

\[H_m(x) = (x - 1) \sum_{r=0}^{\infty} x^{-r-1} \binom{r}{m}\]

for \(|x| > |q^*|, 0 \leq s \leq m\). It is also evident that

\[(1 + qH)^m = (x - 1) \sum_{r=0}^{m} \binom{m}{r} q^r (q - 1)^{-r} \sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} \frac{1}{x - q^s} \frac{q^s}{(q - 1)^{m-s}} \]

which implies

\[(7.3) \quad (1 + qH)^m = xH^m \quad (m > 1).\]

We have therefore proved (1.5). Alternatively taking (7.3) as definition of \(H_m\) one can work back to the earlier formulas obtained for \(A_{m \alpha}\) above.

For some purposes it is convenient to define \(H_m(u; x) = H_m(u; x, q)\), a polynomial in \(q^u\). We put

\[H_m(u; x) = (x - 1) \sum_{s=0}^{m} (-1)^{m-s} \binom{m}{s} \frac{q^s}{x - q^s} \cdot \]

so that \(H_m(0; x) = H_m(x)\). It follows at once from (7.4) that

\[(7.5) \quad H_m(1 - u; x^{-1}, q^{-1}) = (-q)^m H_m(u; x, q) \]

and that

\[(7.6) \quad xH_m(u; x) - H_m(u + 1; x) = (x - 1) [u]^m. \]

We have also

\[\sum_{r=0}^{m} \binom{m}{r} q^r H_r(u; x) = H_m(u + 1; x), \]

which becomes, using (7.6),

\[(7.7) \quad (1 + qH(u; x))^m = xH_m(u; x) - (x - 1) [u]^m. \]
For \( m = 0 \), (7.7) reduces to (7.3).

Clearly (7.6) implies

\[
\sum_{i=0}^{k-1} x^{k-i} [u + i]^m = x^k H_m(u; x) - H_m(u + k; x),
\]

which includes (7.2) as a special case.

Since \( H_m(u; x) \) is a polynomial in \( q^u \) of degree \( m \), the remarks in §3 apply to the difference equation (7.6). In particular, application of (3.2) leads to

\[
H_m(u; x) = \sum_{r=0}^{m} \frac{\Delta^r [u]^m}{\psi_r(x)} \left( \psi_s(x) = \prod_{r=1}^{s} (x - q^r) \right),
\]

provided \( x \neq q^r, r = 0, 1, \ldots, m \). To simplify the right member of (7.9), we used (2.6) and (3.1) of [2]; thus

\[
\Delta^r [u]^m = \sum_{i=0}^{r} q^{r(r-1)/2} a_{m,r} [r]_s [u]_{r-s} q^e(u-r+s)
\]

and (7.9) becomes after a little manipulation

\[
H_m(u; x) = \sum_{r=0}^{m} q^{r(r-1)/2} a_{m,r} \frac{r!}{\psi_r(x)} [r]_s [u]_{r-s}.
\]

If we let \( G_r(u) \) denote the inner sum it is clear from (3.2) that

\[
xG_r(u) - G_r(u + 1) = [u]_r.
\]

In (7.10) put \( u = 0 \), then

\[
H_m(x) = \sum_{r=0}^{m} q^{r(r-1)/2} a_{m,r} [r]_s ! \frac{r!}{\psi_r(x)},
\]

which for \( q = 1 \) reduces to (2.9).

Using (7.1) and (7.4) it is easy to verify the formula

\[
H_m(u; x) = \sum_{r=0}^{m} \binom{m}{r} q^m H_r [u]_{m-r} = (q^m H + [u]^m).
\]

Next using (7.12), (7.11), and the explicit formula [2, (6.2)] for \( a_{m,s} \) we get

\[
H_m(u; x) = \sum_{r=0}^{m} \frac{1}{\psi_r(x)} \sum_{s=0}^{r} (-1)^s q^{s(s-1)/2} \left[ \begin{array}{c} r \\ s \end{array} \right] [u + r - s]_m,
\]

which is useful later.

8. Connection with \( \eta_m(u) \). Using the formula [2, (4.7)]

\[
(q - 1)^m \eta_m(u) = \sum_{s=0}^{m} (-1)^{m-s} \binom{m}{s} q^s u^s,
\]
we find that
\[ (q^k - 1)^m \sum_{r=0}^{k-1} \zeta^{-r} \eta_m \left( u + \frac{r}{k}, q^k \right) = \sum_{s=0}^{m} (-1)^{m-s} \binom{m}{s} \frac{s q^{k s u}}{\zeta^{-1} q^s - 1} \]
\[ = m \sum_{s=0}^{m-1} (-1)^{m-s} \binom{m - 1}{s} \frac{\zeta q^{k s u + k u - 1}}{q^s - \zeta q^{-1}}, \]
where
\[ \zeta^k = 1, \quad \zeta \neq 1. \]
Comparing with (7.4) we have therefore
\[ [k]^{m-1} \sum_{r=0}^{k-1} \zeta^{-r} \eta_m \left( u + \frac{r}{k}, q^k \right) = \frac{m \zeta q^{k u - 1}}{1 - \zeta} H_{m-1}(k u; \zeta q^{-1}), \]
and in particular for \( u = 0 \)
\[ [k]^{m-1} \sum_{r=0}^{k-1} \zeta^{-r} \eta_m \left( \frac{r}{k}, q^k \right) = \frac{m \zeta q^{k - 1}}{1 - \zeta} H_{m-1}\left(\zeta q^{-1}\right), \]
which may be compared with (2.7).

Next using the multiplication formula (see [2, (4.12)]; note that a term is missing in that formula)
\[ [k]^{m-1} \sum_{r=0}^{k-1} \eta_m \left( u + \frac{r}{k}, q^k \right) = \eta_m(k u, q) + (-1)^m \frac{k - [k]}{(q - 1)^m} \]
together with (8.1) we get
\[ k [k]^{m-1} \eta_m \left( u + \frac{r}{k}, q^k \right) - \eta_m(k u, q) - (-1)^m \frac{k - [k]}{(q - 1)^m} \]
\[ = \frac{m}{q} \sum_{t=1}^{\zeta r+1} H_{m-1}(k u; \zeta q^{-1}), \]
and in particular for \( u = 0, \)
\[ k [k]^{m-1} \eta_m \left( \frac{r}{k}, q^k \right) - \eta_m - (-1)^m \frac{k - [k]}{(q - 1)^m} \]
\[ = \frac{m}{q} \sum_{t=1}^{\zeta r+1} H_{m-1}(\zeta q^{-2}). \]
By means of (1.3) it is easy to write down formulas like (8.1), \( \cdots, \) (8.5) involving \( \beta_m. \)

9. Multiplication formulas. For the polynomial \( H_m(u; x) \) we have, using (7.4),
\[
(q^k - 1) m \sum_{r=0}^{k-1} \xi^{-r} q^{r+t} H_m\left(u + \frac{r}{k}; \xi q^{-kt}, q^k\right)
\]
\[
= (\xi q^{-kt} - 1) \sum_{s=0}^{m} (-1)^{m-s} \binom{m}{s} \frac{q^{bsu}}{\xi^{s-k} q^{s+k} - q^{bs}} \sum_{r=0}^{k-1} \xi^{-r} q^{r(s+t)}
\]
\[
= (\xi - q^{kt}) \sum_{s=0}^{m} (-1)^{m-s} \binom{m}{s} \frac{q^{bsu}}{\xi - q^{s+k(t+1)}} \frac{1 - \xi^{-k} q^{k(s+t)}}{1 - \xi^{-1} q^{t+1}}.
\]
Consequently if \(\xi^{k-1} = 1, \xi \neq 1\), we get
\[
[k] m \sum_{r=0}^{k-1} \xi^{-r} q^{r+t} H_m\left(u + \frac{r}{k}; \xi q^{-kt}, q^k\right) = \frac{\xi - q^{kt}}{\xi - q^{t}} H_m(ku; \xi q^{-t}, q),
\]
analogous to (8.3).
In the special case \(x = -g^{-1}\), the polynomial \(e_m(u)\) of [2, §8] satisfies
\[
e_m(u) = H_m(u; -q^{-1}, q);
\]
in this case (9.1) becomes (\(\xi = -1, t = 1\))
\[
[k] m \sum_{r=0}^{k-1} (-q)^r e_m\left(u + \frac{r}{k}, q^k\right) = \frac{q^{k+1}}{q+1} e_m(ku, q)
\]
for \(k\) odd; note that (8.6) of [2] requires a slight correction.

10. **Staudt-Clausen theorems for \(\beta_m\).** In [2] a theorem analogous to the Staudt-Clausen theorem was proved for \(\beta_m\) with \(q\) indeterminate. Now on the other hand we replace \(q\) by a rational number \(a\) which is assumed to be integral modulo a fixed prime \(p\). We shall use the representation [2, (6.2)]
\[
\beta_m = \sum_{s=0}^{m} (-1)^s a_{m,s} [s]!/[s + 1],
\]
where
\[
a_{m,s} = \frac{q^{-s(r-1)/2}}{s!} \sum_{r=0}^{s} (-1)^r q^{r(r-1)/2} \left[\begin{array}{c}s \\ r\end{array}\right] \left[\begin{array}{c}s - r \\ m\end{array}\right];
\]
the quantity \(a_{m,s}\) is a polynomial in \(q\) and has occurred in (7.10) and (7.11) above.

Suppose first that \(a \equiv 1 \pmod{p}\). Then from (10.1) or [2, §7] it is clear that the \(s\)th term in the right member of (10.1) is of the form \(u_s = N_s(a)/F_{s+1}(a)\), where \(F_{s+1}(x)\) is the cyclotomic polynomial and \(N_s(x)\) is a polynomial with integral coefficients. If we recall that \(F_k(1) = p\) when \(k = p^e, e \geq 1\), but \(F_k(1) = 1\) otherwise, it is clear that \(u_s\) is integral \(\pmod{p}\) except possibly when \(s+1 = p^e\); the same holds also for \(F_k(a)\). Now let \(s+1 = p^e\). Then by a simple computation it is seen that \([s]!\) is divisible by exactly \(p^e\), where
While the denominator is divisible by exactly $p^e$. Since $(p^e - 1)/(p - 1) \geq 2e$ for $e \geq 2$, $p \geq 3$, it follows that $u_*$ is integral in this case. If $e = 1$, $p \geq 3$, we have first $p(a - 1)/(a^p - 1) \equiv 1 \pmod{p}$. As for the numerator of $u_{p-1}$, it follows readily from (10.2) and

$$\left[ \frac{p - 1}{r} \right] = \binom{p - 1}{r} \pmod{p}$$

that

$$[p - 1]a_{m,p-1} = \sum_{r=0}^{p-1} (-1)^r \binom{p - 1}{r} r^m$$

$$= \sum_{r=0}^{p-1} r^m = \begin{cases} (-1) \pmod{p} & \left( \begin{array}{c} p - 1 \mid m \end{array} \right), \\ 0 \pmod{p} & \left( \begin{array}{c} p - 1 \parallel m \end{array} \right). \end{cases}$$

We have therefore proved

**Theorem 1.** Let $p \geq 3$, $q = a \equiv 1 \pmod{p}$. Then

(10.3) $p^m = \begin{cases} -1 \pmod{p} & \left( \begin{array}{c} p - 1 \mid m \end{array} \right), \\ 0 \pmod{p} & \left( \begin{array}{c} p - 1 \parallel m \end{array} \right). \end{cases}$

For $p = 2$, the preceding argument shows that all terms in (10.1) are integral (mod 2) except perhaps $u_1$ and $u_3$. Now

$$u_1 = \frac{a_{m,1}}{[2]} = \frac{1}{a + 1},$$

while

$$u_3 = \frac{[3]a_{m,3}}{[4]} = \frac{a^3}{(a + 1)(a^2 + 1)} \sum_{r=0}^{3} (-1)^r a^{r(r-1)/2} \left[ \begin{array}{c} 3 \\ r \end{array} \right] [3 - r]^m.$$

Let $2^e | (a+1)$, $2^{e+1} | (a+1)$; then $(a^2+a+1)^2 \equiv 1 \pmod{2^{e+1}}$ and

$$\sum_{r=0}^{3} (-1)^r a^{r(r-1)/2} \left[ \begin{array}{c} 3 \\ r \end{array} \right] [3 - r]^m$$

$$= (a^2 + a + 1)^m - (a^2 + a + 1)(a + 1)^m + a(a^2 + a + 1)$$

$$= \begin{cases} 0 \pmod{2^{e+1}} & \text{(m even)}, \\ a + 1 \pmod{2^{e+1}} & \text{(m odd)}. \end{cases}$$

Consequently $u_3$ is integral (mod 2) for $m$ even while for $m$ odd $2u_3 \equiv 1 \pmod{2}$. This yields the following supplement to Theorem 1.

**Theorem 2.** Let $p = 2$, $q = a \equiv 1 \pmod{2}$; also let $2^e | (a+1)$, $2^{e+1} | (a+1).
Then if \( e = 1 \) we have \( 2\beta_m = 1 \pmod{2} \) for \( m \) even \( \geq 2 \), \( 2\beta_1 = 1 \pmod{2} \), while \( \beta_m \) is integral \( \pmod{2} \) for \( m \) odd \( \geq 3 \). If \( e > 1 \) then

\[ (10.4) \quad 2\beta_m \equiv 1 \pmod{2} \]

for all \( m \geq 1 \).

In particular it is evident from (10.4) that the denominator of \( \beta_m \) may be divisible by arbitrarily high powers of 2.

In the next place we suppose \( q = a \not\equiv 1 \pmod{p} \), \( p > 2 \). It is now convenient to use \([2, \text{ (5.3)}]\]

\[ (10.5) \quad (q - 1)^m \beta_m = \sum_{s=0}^{m} (-1)^{m-s} \binom{m}{s} \frac{s+1}{s+1} . \]

We shall assume first that \( a \) is a primitive root \( \pmod{p^r} \). Clearly in the right member of (10.5) we need consider only those terms in which \( p - 1 | s + 1 \). Put \( a^{p-1} = 1 + kp \), \( p \nmid k \). Then

\[
\frac{a^{(p-1)r} - 1}{rp} = k + \frac{1}{2} \left( r - 1 \right) k^2 p + \cdots = k \pmod{p}.
\]

Thus (10.5) implies

\[ (10.6) \quad (a - 1)^m p \beta_m \equiv (-1)^m \frac{1}{k} \sum_{r>0} \binom{m}{r(p-1) - 1} \pmod{p} . \]

But it is known \([4, \text{ p. 255}]\) that

\[
\sum_{0 < r(p-1) \leq m} \binom{m}{r(p-1) - 1} = \begin{cases} -1 \pmod{p} & \text{if} \ (p-1 \nmid m), \\ 0 \pmod{p} & \text{if} \ (p-1 \mid m) . \end{cases}
\]

Hence (10.6) implies that \( p \) is integral when \( m \equiv 0 \pmod{p-1} \). This proves

**Theorem 3.** Let \( p \geq 3 \), \( q = a \), a primitive root \( \pmod{p^2} \); then \( \beta_m \) is integral \( \pmod{p} \) for \( p - 1 \nmid m \), while

\[ (10.7) \quad p \beta_m \equiv -\frac{1}{k} \pmod{p} \quad (p - 1 \mid m), \]

where \( k = (a^{p-1} - 1)/p \).

It is now clear how to handle the general situation. We may state

**Theorem 4.** Let \( p \geq 3 \), \( q = a \), where \( a \) belongs to the exponent \( e \pmod{p} \), \( e > 1 \). Put
(10.8) \[ a^e = 1 + \rho^1k \]

Then

(10.9) \[ (a - 1)^m\rho^1\beta_m \equiv \frac{e}{k} \sum_{r \geq 0} (-1)^m \binom{m}{re} (re - 1) \pmod{p} \]

In particular if \( e = p - 1 \), then

(10.10) \[ (a - 1)^m\rho^1\beta \equiv \begin{cases} 0 \pmod{p} & (p - 1 \mid m), \\ \frac{1}{k} \pmod{p} & (p - 1 \nmid m). \end{cases} \]

To prove (10.9) it is only necessary to observe that (10.8) implies

\[ \rho^1 \frac{re}{a^re - 1} = \rho^1 \frac{re}{(1 + \rho^1k)^r - 1} \equiv \frac{e}{k} \pmod{p}. \]

It is clear from (10.10) that the denominator of \( \beta_m \) may be divisible by arbitrarily high powers of \( p \). We also remark that theorems like Theorems 3 and 4 can be framed for \( \eta_m \).

When \( p^s \mid a \), it is evident from (10.5) that

(10.11) \[ \beta_m \equiv \sum_{s=0}^{m} (-1)^s \binom{m}{s} (s + 1) \equiv 0 \pmod{p^s} \quad (m > 1). \]

11. Congruences. The formula (7.11) together with (10.2) makes it possible to derive certain congruences satisfied by \( H_m(x, q) \). We observe, to begin with, that if \( q = a \) is integral \( (\pmod{p}) \) then \( \eta_m = [s]!a_{m,s} \) satisfies, for \( s \) fixed and \((p - 1)p^{s-1} \mid u_m \),

(11.1) \[ u_m^m(u^w - 1)^r \equiv 0 \pmod{p^m, p^{re}}, \]

where after expansion of the left member, \( u^n \) is replaced by \( u_n \). To prove (11.1) we need only remark that

\[ u_m(u^w - 1)^r = a^{-e(s-1)/2} \sum_{r=0}^{e} (-1)^r a^{(r-1)/2} \binom{s}{r} [s - r]^m([s - r]^w - 1)^r. \]

If we look on \( x \) in (7.11) as an indeterminate and apply (11.1), we can assert that

(11.2) \[ H_m(u^w - 1)^r \equiv 0 \pmod{p^m, p^{re}}. \]

We interpret this congruence in the following manner. The left member of (11.2) is a rational function of \( x \) such that the coefficient of each term in the numerator \( \equiv 0 \pmod{p^m, p^r} \). We may call (11.2) Kummer's congruence for \( H_m \). Using (7.13) we can prove like results for \( H_m(u; x) \), where \( u \) is now an integer.
In view of (1.6) the result (11.2) can be restated in terms of $A_m(x)$:

\[(11.3) \sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} A_m + w(x) \prod_{i=m + w + 1}^{m + rw} (x - a^i) \equiv 0 \pmod{p^m, p^r}.
\]

We may state

**Theorem 5.** Let $q = a$ be integral $(\mod p)$, $x$ an indeterminate, and $r \geq 1$; then (11.3) holds.

In the next place (11.3) implies congruences for the $A_{m,a}$ of (1.7). (For the case $q = 1$, compare [1].) Since

\[
\prod_{i=m + w + 1}^{m + rw} (x - a^i) = \prod_{i=0}^{(r-s)w} (-1)^{(r-s)w-i} (i+1)/2 + i \sum_{i=m + w}^{m + rw} \left[ (r - s)w \right] \]

examination of the coefficient of $x^{k-1}$ in (11.3) implies

\[(11.4) \sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} \sum_{i} (-1)^{(r-s)w-i} A_{m+w+k-i}^{(i+1)/2 + i} \left[ (r - s)w \right] \equiv 0 \pmod{p^m, p^r}.
\]

In order to obtain simpler results we consider some special values of the parameters. In the first place we take $r = 1$, so that (11.4) becomes

\[(11.5) A_{m+w,k} = \sum_{i} (-1)^{w-i} A_{m+w+k-i}^{(i+1)/2 + i} \left[ w \right] \equiv 0 \pmod{p^m, p^e}.
\]

Now suppose first that $a \equiv 1 \pmod{p}$. It is necessary to examine

\[(11.6) \left[ w \right] = \frac{(a^w - 1) \cdots (a^{w-i+1} - 1)}{(a - 1) \cdots (a^i - 1)}.
\]

We assume from now on that $p > 2$. If we put $a = 1 + p^i h$, $p \nmid h$, then as in the proof of Theorem 4 we find that

\[\left[ w \right] \text{ and } \left[ w \right]
\]

are divisible by exactly the same power of $p$. But if $i < p^j$, it is clear from the identity

\[\left[ w \right] = \frac{w}{i} \left[ w - 1 \right]
\]

that

\[\left[ w \right] \equiv 0 \pmod{p^{e-j}} \quad (j \leq e).
\]
Consequently if $p^{i-1} \leq k < p^i$ and $j < e$, we see that (11.5) implies
\[(11.7) \quad A_{m+w,k} \equiv A_{mk} \pmod{p^m, p^{e-i}}.\]

This proves

**Theorem 6.** Let $p \geq 3$, $q = a \equiv 1 \pmod{p}$, $p^{e-1}(p-1) \mid w$, and $p^{i-1} \leq k < p^i$, where $j < e$. Then (11.7) holds.

When $a \not\equiv 1 \pmod{p}$ let $a$ belong to the exponent $t \pmod{p}$. Then it is clear from (11.6) that we need only consider those factors in the right member with exponents divisible by $t$. Thus if $i_0$ is the greatest integer $\leq i/t$ we need only examine
\[
\left[ \frac{w}{t} \right]_{i_0}
\]
with $a$ replaced by $a^t$. The preceding discussion therefore applies and we obtain the following theorem which includes Theorem 6.

**Theorem 7.** Let $p \geq 3$, let $q = a$ belong to the exponent $t \pmod{p}$, $p^{e-1}(p-1) \mid w$ and $k < tp^i$, where $j < e$. Then (11.7) holds.

The case $k = w$ is not covered by the theorem. We find for example that if $w = t = p - 1$ (so that $a$ is a primitive root $\pmod{p}$), then
\[
A_{m+p-1,k} \equiv \begin{cases} 
A_{m,k} \pmod{p} & (k < p - 1), \\
A_{m,p-1} + \left( \frac{a}{p} \right) A_{m,0} \pmod{p} & (k = p - 1),
\end{cases}
\]
where $(a/p)$ is Legendre’s symbol.

Returning to (11.2) we can also consider the case in which $x$ is put equal to an integer $\pmod{p}$, provided the resulting denominators are not divisible by $p$. Now the least common denominator is evidently
\[
\psi_{m+r}(x) = \prod_{s=1}^{m+r} (x - a^s).
\]
It will therefore suffice to assume that $x \not\equiv a^s \pmod{p}$ for any $s$. We may therefore state

**Theorem 8.** Let $a$ and $x$ be rational numbers that are integral $\pmod{p}$ and suppose that $x \not\equiv a^s \pmod{p}$ for any $s$. Let
\[
p^{e-1}(p - 1) \mid w \quad \text{and} \quad r \geq 1.
\]

Then
\[(11.8) \quad H^m(x)(H^m(x) - 1)^r \equiv 0 \pmod{p^m, p^e}.
\]
In particular the theorem may be applied with slight changes to $\epsilon_m(u) = \epsilon_m(u, a)$ defined in (9.2); we have explicitly [2, (8.18)]

$$
\epsilon_m(u) = \sum_{s=0}^{m} \frac{(-1)^s a^s}{(a+1)(a^2+1) \cdots (a^{s+1}+1)} \\
\cdot \sum_{r=0}^{s} (-1)^r a^{r(r-1)/2} \left[ \binom{s}{r} \right] [u + s - r]^m,
$$

which is included in (7.13). If $u$ is an integer we have

$$\epsilon^m(u) (\epsilon^u(u) - 1)^r = 0 \pmod{p^{m-1}(p-1)},$$

provided $p \equiv 3 \pmod{4}$ and $a$ is a quadratic residue (mod $p$). For in this case $-1$ is a nonresidue (mod $p$) and therefore $-1 \not\equiv a^s$ for any $s$.

12. Congruences involving $\eta_m$ and $\beta_m$. Let

$$
\omega_m = \omega_{m,k,r} = \frac{1}{m} \left\{ k[k]^{m-1} \eta_m \left( \frac{r}{k}, q^k \right) - \eta_m - (-1)^m \frac{k - [k]}{(q-1)^m} \right\}
$$

so that by (8.5)

$$
\omega_m = \frac{1}{q} \sum_{\tau=1}^{\zeta r+1} \frac{H_{m-1}(\tau q^{-1})}{1 - \tau},
$$

where $\zeta$ runs through the $k$th roots of unity distinct from 1. As for the denominators in the right member of (12.2), note that

$$
\prod_{\tau=1}^{\zeta} (a^s - \tau) = \frac{a^{k(s-1)} - 1}{a^s - 1} = a^{k(s-1)} + \cdots + a^s + 1,
$$

which is prime to $p$ for all $s$ provided $p \nmid k$. We may therefore state

**Theorem 9.** If $a$ is integral (mod $p$) and $p \nmid k$, then

$$
\omega^m(\omega^u - 1)^r \equiv 0 \pmod{p^{m-1}(p-1)},
$$

where $\omega_m$ is defined by (12.1) and $p^{m-1}(p-1) \mid w$.

As for $\beta_m$ we have

$$
\eta_k[k]^{m-1} q^r \beta_m \left( \frac{r}{k}, q^k \right) - \beta_m
$$

$$
= \frac{(m+1)(q-1)}{q} \sum_{\tau=1}^{\zeta r+1} \frac{H_{m-1}(\tau q^{-1})}{1 - \tau} + \frac{m}{q} \sum_{\tau=1}^{\zeta r+1} \frac{H_{m-1}(\tau q^{-1})}{1 - \tau}
$$

analogous to (8.5). In much the same way as above (12.4) implies

**Theorem 10.** If $a$ is integral (mod $p$) and $p \nmid k$ then

$$
\Omega^m(\Omega^u - 1)^r \equiv 0 \pmod{p^{m-1}(p-1)}.
$$
where \( \Omega_m \) stands for the left member of (12.4) and \( p^{e-1}(p-1)|w \).

Unfortunately we seem unable to obtain simpler congruences for \( \beta_m \) and \( \eta_m \).

13. Combinatorial interpretation of \( a_{m\sigma} \). Put

\[
A^*_m = \sum_{r=0}^{(s-1)(m-s)} a_{m\sigma} q^r \quad (r_0 = s(s-1)/2 + r).
\]

The following combinatorial interpretation of the coefficients \( a_{m\sigma} \) was kindly suggested by J. Riordan. The number \( a_{m\sigma} \) is the number of permutations of \( m \) things requiring \( s \) readings and such that \( r = r_2 + 2r_3 + \cdots + (s-1)r_s \), where \( r_k \) is the number of elements read on the \( k \)th reading. The following numerical illustration for \( m = 4 \) was also supplied by Riordan.

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References


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