

REPRESENTATIONS OF SEMISIMPLE LIE GROUPS. III

BY

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The main object of this paper is to define the character of an irreducible quasi-simple⁽¹⁾ representation π of a connected semisimple Lie group G on a Hilbert space \mathfrak{H} . This will be done as follows. Let $C_c^\infty(G)$ be the class of all functions on G which are indefinitely differentiable and which vanish outside a compact set. For any $f \in C_c^\infty(G)$ we consider the operator $\int f(x)\pi(x)dx$ where dx is the Haar measure on G . It turns out that this operator has a trace (which we denote by $T_\pi(f)$) and the mapping $T_\pi: f \rightarrow T_\pi(f)$ ($f \in C_c^\infty(G)$) is a *distribution* in the sense of L. Schwartz [9] such that $T_\pi(f_a) = T_\pi(f)$ where $f_a(x) = f(axa^{-1})$ ($a, x \in G$). This distribution is defined to be the character of π . We shall see that two such representations π_1, π_2 are infinitesimally equivalent (see [6, §9]) if and only if they have the same character. Therefore in particular a unitary irreducible representation is determined within unitary equivalence by its character (cf. Theorem 8 of [6]).

In the last section we give a simple proof of a formula for "spherical functions" on a *complex* semisimple group. This formula was obtained by Gelfand and Naimark [1; 2] in some special cases by direct computation.

1. Some preliminary results. We keep to the notation of our two earlier papers [6, 7] on the same subject. G is a connected, simply connected, semisimple Lie group and \mathfrak{g}_0 is its Lie algebra over the field R of real numbers. \mathfrak{g} is the complexification of \mathfrak{g}_0 and $\mathfrak{k}, \mathfrak{p}, \mathfrak{k}_0, \mathfrak{p}_0, \mathfrak{c}, \mathfrak{k}',$ and \mathfrak{m} are defined as in [6, §2] and [7, §2]. $K, K',$ and D are the analytic subgroups of G corresponding to $\mathfrak{k}_0, \mathfrak{k}'_0 = \mathfrak{k}' \cap \mathfrak{g}_0$ and $\mathfrak{c}_0 = \mathfrak{c} \cap \mathfrak{g}_0$ respectively. Let Z be the center of G and \mathfrak{Z} the center of the enveloping algebra \mathfrak{B} of \mathfrak{g} . If π is a representation of G on a Banach space we shall say that π is quasi-simple if it maps the elements of $D \cap Z$ and \mathfrak{Z} into scalar multiples of the unit operator (see [6, §10]).

Let π be a quasi-simple irreducible representation of G on a Banach space \mathfrak{H} . We denote by Ω the set of all equivalence classes of finite-dimensional simple representations of K . Let $\mathfrak{H}_{\mathfrak{D}}$ ($\mathfrak{D} \in \Omega$) denote the set of all elements in \mathfrak{H} which transform under $\pi(K)$ according to \mathfrak{D} . We know (see [6, Lemma 33]) that $\dim \mathfrak{H}_{\mathfrak{D}} < \infty$. Let $E_{\mathfrak{D}}$ denote the canonical projection of \mathfrak{H} on $\mathfrak{H}_{\mathfrak{D}}$ (see [6, §9]). For any $x \in G$ consider the operator $E_{\mathfrak{D}}\pi(x)E_{\mathfrak{D}}$. It maps $\mathfrak{H}_{\mathfrak{D}}$ into itself and $\mathfrak{H}_{\mathfrak{D}'}$ into $\{0\}$ ($\mathfrak{D}' \neq \mathfrak{D}$). Let $\text{sp}(E_{\mathfrak{D}}\pi(x)E_{\mathfrak{D}})$ denote the trace of the restriction of $E_{\mathfrak{D}}\pi(x)E_{\mathfrak{D}}$ on $\mathfrak{H}_{\mathfrak{D}}$. Since $\dim \mathfrak{H}_{\mathfrak{D}} < \infty$ this trace is well defined. Now any given linear function α on $\mathfrak{H}_{\mathfrak{D}}$ may be extended to a continuous linear function on \mathfrak{H} by setting $\alpha(\psi) = \alpha(E_{\mathfrak{D}}\psi)$ ($\psi \in \mathfrak{H}$). In particular if $\psi_i,$

Received by the editors January 9, 1953.

(¹) See [6, §10] for the definition of a quasi-simple representation.

$1 \leq i \leq r$, is a base for $\mathfrak{G}_{\mathfrak{D}}$ and $\tilde{\psi}_i$ is the linear function on $\mathfrak{G}_{\mathfrak{D}}$ which takes the value 1 at ψ_i and zero at ψ_j ($j \neq i, 1 \leq i, j \leq r$) we may extend $\tilde{\psi}_i$ on \mathfrak{G} in the above fashion. Then it is clear that

$$\phi_{\mathfrak{D}}^{\pi}(x) = \text{sp } E_{\mathfrak{D}}\pi(x)E_{\mathfrak{D}} = \sum_{i=1}^r (\tilde{\psi}_i, \pi(x)\psi_i)$$

in the notation of [6, §10]. Hence it follows from Lemmas 19 and 34 of [6] that $\phi_{\mathfrak{D}}^{\pi}$ is an analytic function on G . X_1, \dots, X_n being a base for \mathfrak{g}_0 over R , set $X(t) = t_1X_1 + \dots + t_nX_n$ ($t_j \in R$). Then we know (cf. Theorem 2 and Lemma 34 of [6]) that if $|t| = \max_j |t_j|$ is sufficiently small we get the convergent expansions

$$\pi(\exp X(t))\psi_i = \sum_{m \geq 0} \frac{1}{m!} \pi((X(t))^m)\psi_i, \quad 1 \leq i \leq r.$$

From this it follows immediately that if z is any element in \mathfrak{B} the value of $\sum_{1 \leq i \leq r} (\tilde{\psi}_i, \pi(z)\psi_i)$ can be obtained in terms of the various partial derivatives of $\phi_{\mathfrak{D}}^{\pi}(\exp X(t))$ with respect to (t) at $t_1 = t_2 = \dots = t_n = 0$. Let σ be the representation of $\mathfrak{A} = \mathfrak{Q}\mathfrak{X}$ (see [7, §2] for notation) on $\mathfrak{G}_{\mathfrak{D}}$ defined under π . Then the knowledge of the function $\phi_{\mathfrak{D}}^{\pi}$ determines in particular $\text{sp } \sigma(z)$ for any $z \in \mathfrak{A}$. Now we know (see Theorem 5 of [6]) that π defines a quasi-simple⁽²⁾ irreducible representation of \mathfrak{B} on $\mathfrak{G}^{(0)} = \sum_{\mathfrak{D}' \in \Omega} \mathfrak{G}_{\mathfrak{D}'}$ and therefore σ is irreducible (see Corollary 2 to Theorem 2 of [7]). On the other hand a finite-dimensional simple representation of an associative algebra is completely determined within equivalence by its trace (see Lemma 16 of [7]). Hence in view of Theorem 2 of [7] we can conclude that the function $\phi_{\mathfrak{D}}^{\pi}$ determines the representation of \mathfrak{B} on $\mathfrak{G}^{(0)}$ up to equivalence and therefore the representation π of G up to infinitesimal equivalence. This result may be stated in a slightly more general form as follows.

THEOREM⁽³⁾ 1. *Let π_1, \dots, π_r be a finite set of quasi-simple irreducible representations of G on Banach spaces. Suppose no two of them are infinitesimally equivalent. Then all the nonzero functions in the set $\phi_{\mathfrak{D}_1}^{\pi_1}, \dots, \phi_{\mathfrak{D}_r}^{\pi_r}$ ($\mathfrak{D}_i \in \Omega, 1 \leq i \leq r$) are linearly independent.*

Let C be the field of complex numbers. If our assertion is false we may suppose that $c_1\phi_{\mathfrak{D}_1}^{\pi_1} + \dots + c_s\phi_{\mathfrak{D}_s}^{\pi_s} = 0$ where $c_j\phi_{\mathfrak{D}_j}^{\pi_j} \neq 0, 1 \leq j \leq s$ ($c_j \in C$). Let \mathfrak{G}_i be the representation space of π_i . Consider the representation σ_i of \mathfrak{A} on $\mathfrak{G}_i, \mathfrak{D}_i$ ($1 \leq i \leq s$) induced under π_i . Then $c_1 \text{sp } \sigma_1(a) + \dots + c_s \text{sp } \sigma_s(a) = 0$ for all $a \in \mathfrak{A}$. Since $c_j\phi_{\mathfrak{D}_j}^{\pi_j} \neq 0, \mathfrak{D}_j$ occurs in π_j . Moreover π_j, π_k ($j \neq k$) are not infinitesimally equivalent ($1 \leq j, k \leq s$). Hence it follows from Corollary 2 to Theorem 2 of [7] that the representations $\sigma_1, \dots, \sigma_s$ are irreducible and no

⁽²⁾ See [7, end of §2] for the definition of quasi-simplicity in this case.

⁽³⁾ Cf. Theorem 7 of [8(a)] and Theorem 2 of [8(b)].

two of them are equivalent. This however gives a contradiction with Lemma 16 of [7]. So the theorem is proved.

We recall that for two irreducible *unitary* representations on Hilbert spaces the notions of infinitesimal equivalence and ordinary equivalence are the same (see Theorem 8 of [6]). Hence if π_1, π_2 are two such representations which are not equivalent, the corresponding functions $\phi_{\mathfrak{D}_1}^\pi, \phi_{\mathfrak{D}_2}^\pi$ are always distinct unless they are both zero.

Theorem 5 of [7] can now be rephrased in terms of the function $\phi_{\mathfrak{D}}^\pi$ as follows:

THEOREM⁽⁴⁾ 2. *Let π be a quasi-simple irreducible representation of G on a Banach space \mathfrak{H} . Suppose \mathfrak{D}_0 is a class in Ω occurring in π such that $d(\mathfrak{D}_0) = 1$. Then $\dim \mathfrak{H}_{\mathfrak{D}_0} = 1$ and it is possible to choose linear functions Λ and μ on \mathfrak{h} and c respectively such that*

$$\phi_{\mathfrak{D}_0}^\pi(x) = \int_{K^*} e^{\mu(\Gamma(x, u^*))} e^{\Lambda(H(x, u^*))} du^* \quad (x \in G)$$

and the infinitesimal character of π is χ_Λ .

Some properties of the function $\phi_{\mathfrak{D}}^\pi$ have been studied by R. Godement [3] (see also [1; 2]).

We shall now state a few immediate consequences of the results proved in [6].

THEOREM⁽⁵⁾ 3. *Let χ be a homomorphism of \mathfrak{Z} into C and \mathfrak{D}_0 a class in Ω . Then apart from infinitesimal equivalence there exist only a finite number of irreducible quasi-simple representations π of G which have the infinitesimal character χ and such that \mathfrak{D}_0 occurs in π .*

This follows from Theorem 2 of [7]. Similarly the following result is obtained from Theorem 3 of [8].

THEOREM 4. *Let π be a quasi-simple irreducible representation of G on a Banach space \mathfrak{H} . Then there exists an integer N such that*

$$\dim \mathfrak{H}_{\mathfrak{D}} \leq N(d(\mathfrak{D}))^2$$

for all $\mathfrak{D} \in \Omega$.

2. Trace of an operator. Let $\{c_\alpha\}_{\alpha \in J}$ be an indexed set of complex numbers. We define the convergence of the series $\sum_{\alpha \in J} c_\alpha$ and its sum in the usual manner (see §5 of [6]). Let A be a bounded operator on a Hilbert space \mathfrak{H} and let $\{\psi_\alpha\}_{\alpha \in J}$ be an orthonormal base for \mathfrak{H} . We say that A has a trace

⁽⁴⁾ Cf. Theorem 3 of [8(b)]. Our notation is the same as that of Theorem 5 of [7].

⁽⁵⁾ Cf. Theorem 6 of [8(a)].

(or A is of the trace class) if for every such base the series⁽⁶⁾ $\sum_{\alpha \in J} (\psi_\alpha, A\psi_\alpha)$ converges to a sum which is independent of the choice of the base. The value of this sum is called the trace of A and we shall denote it by $\text{sp } A$.

LEMMA 1. Let $\{\psi_\alpha\}_{\alpha \in J}$ be an orthonormal base for a Hilbert space \mathfrak{H} and T a bounded operator such that $\sum_{\alpha, \beta \in J} |t_{\alpha\beta}| < \infty$ where $t_{\alpha\beta} = (\psi_\alpha, T\psi_\beta)$. Then if A and B are any bounded operators on \mathfrak{H} , ATB , BAT , TBA are all of the trace class and

$$\text{sp}(ATB) = \text{sp}(BAT) = \text{sp}(TBA).$$

Put $a_{\alpha\beta} = (\psi_\alpha, A\psi_\beta)$, $b_{\alpha\beta} = (\psi_\alpha, B\psi_\beta)$ ($\alpha, \beta \in J$) and consider the series $\sum_{\alpha, \beta, \gamma \in J} |a_{\alpha\beta} t_{\beta\gamma} b_{\gamma\alpha}|$. Then⁽⁷⁾

$$\begin{aligned} \sum_{\alpha} |a_{\alpha\beta} t_{\beta\gamma} b_{\gamma\alpha}| &= |t_{\beta\gamma}| \sum_{\alpha} |a_{\alpha\beta} b_{\gamma\alpha}| \\ &\leq |t_{\beta\gamma}| \left(\sum_{\alpha} |a_{\alpha\beta}|^2 \right)^{1/2} \left(\sum_{\alpha} |b_{\gamma\alpha}|^2 \right)^{1/2} \leq |t_{\beta\gamma}| |A| |B| \end{aligned}$$

since

$$\sum_{\alpha} |a_{\alpha\beta}|^2 = \sum_{\alpha} |(\psi_\alpha, A\psi_\beta)|^2 = |A\psi_\beta|^2 \leq |A|^2$$

and similarly for B . Hence

$$\sum_{\alpha, \beta, \gamma} |a_{\alpha\beta} t_{\beta\gamma} b_{\gamma\alpha}| \leq |A| |B| \sum_{\beta, \gamma} |t_{\beta\gamma}| < \infty.$$

This proves that the series $\sum_{\alpha, \beta, \gamma} a_{\alpha\beta} t_{\beta\gamma} b_{\gamma\alpha}$ is absolutely convergent and so it follows in the usual way that

$$\sum_{\alpha} (\psi_\alpha, ATB\psi_\alpha) = \sum_{\alpha} (\psi_\alpha, TBA\psi_\alpha) = \sum_{\alpha} (\psi_\alpha, BAT\psi_\alpha).$$

Now let U be a unitary transformation on \mathfrak{H} . Consider $U^{-1}ATBU$. Since $U^{-1}A$ and BU are bounded operators, we can conclude from the above result that

$$\begin{aligned} \sum_{\alpha} (U\psi_\alpha, ATBU\psi_\alpha) &= \sum_{\alpha} (\psi_\alpha, U^{-1}ATBU\psi_\alpha) \\ &= \sum_{\alpha} (\psi_\alpha, TBUU^{-1}A\psi_\alpha) = \sum_{\alpha} (\psi_\alpha, TBA\psi_\alpha) \\ &= \sum_{\alpha} (\psi_\alpha, ATB\psi_\alpha). \end{aligned}$$

Since every orthonormal base in \mathfrak{H} is related to the base $\{\psi_\alpha\}_{\alpha \in J}$ by a unitary transformation, this proves that ATB is of the trace class. Since BA is a

⁽⁶⁾ As usual we denote by (ϕ, ψ) the scalar product of ϕ and ψ in \mathfrak{H} .

⁽⁷⁾ For any bounded operator Q we put $|Q| = \sup_{|\psi| \leq 1} |Q\psi|$.

bounded operator it follows from this result that BAT and TBA are also of the trace class. Hence in view of the above equalities we conclude that $\text{sp } ATB = \text{sp } BAT = \text{sp } TBA$.

COROLLARY. *If T satisfies the conditions of the above lemma and if A is a regular operator, then T and ATA^{-1} are both of the trace class and $\text{sp } ATA^{-1} = \text{sp } T$.*

3. An auxiliary lemma. In order to prove that certain given operators are of the trace class we shall frequently need the following result.

LEMMA 2. *Let \mathfrak{l} be a semisimple Lie algebra over C of rank l . Then the series $\sum_{\mathfrak{D}} d(\mathfrak{D})^{-(l+1)}$ is convergent. Here \mathfrak{D} runs over all equivalence classes of finite-dimensional simple representations of \mathfrak{l} and $d(\mathfrak{D})$ is the degree of any representation in \mathfrak{D} .*

Let \mathfrak{h} be a Cartan subalgebra of \mathfrak{l} . Choose a fundamental system of roots and let $\Lambda_1, \dots, \Lambda_l$ be a fundamental set of dominant integral functions on \mathfrak{h} with respect to this system (see [5, Part I]). Then every such function can be written as $m_1\Lambda_1 + \dots + m_l\Lambda_l$ where m_i are all nonnegative integers. Let H_1, \dots, H_l be a base for \mathfrak{h} . Extend this to a base X_1, \dots, X_n ($n \geq l$) for \mathfrak{l} so that $X_i = H_i$, $1 \leq i \leq l$. Put $g_{ij} = \text{sp } (\text{ad } X_i \text{ ad } X_j)$, $1 \leq i, j \leq n$, where $X \rightarrow \text{ad } X$ is the adjoint representation of \mathfrak{l} . Since \mathfrak{l} is semisimple the matrix $(g_{ij})_{1 \leq i, j \leq n}$ is nonsingular. Let $(g^{ij})_{1 \leq i, j \leq n}$ denote its inverse. Let \mathfrak{u} be the enveloping algebra of \mathfrak{l} . Put $\omega = \sum_{1 \leq i, j \leq n} g^{ij} X_i X_j \in \mathfrak{u}$. ω is called the Casimir operator of \mathfrak{l} and it is well known that ω lies in the center of \mathfrak{u} . For any dominant integral function Λ on \mathfrak{h} put

$$|\Lambda|^2 = \sum_{1 \leq i, j \leq l} g^{ij} \Lambda(H_i) \Lambda(H_j).$$

Then it is known (see for example [4]) that $|\Lambda|^2$ is a positive real number unless $\Lambda = 0$. Now let σ be an irreducible finite-dimensional representation of \mathfrak{u} and let \mathfrak{D} be the class of σ . We denote by $\Lambda_{\mathfrak{D}}$ the highest weight of σ and by $\omega_{\mathfrak{D}}$ the number such that $\sigma(\omega) = \omega_{\mathfrak{D}} \sigma(1)$. Then it follows from Lemma 6 of [4] that $\omega_{\mathfrak{D}}$ is real, $\omega_{\mathfrak{D}} \geq |\Lambda_{\mathfrak{D}}|^2$, and there exists a real number κ such that $\kappa d(\mathfrak{D})^2 \geq \omega_{\mathfrak{D}}$ for every irreducible class \mathfrak{D} . Hence

$$d(\mathfrak{D})^{-1} \leq \kappa^{1/2} |\Lambda_{\mathfrak{D}}|^{-1}.$$

Now the base H_1, \dots, H_l can be so chosen that every root of \mathfrak{l} takes real values at H_1, \dots, H_l . For such a base the quadratic form $\sum_{i,j=1}^l g^{ij} x_i \cdot x_j$ ($x_i \in R$) is real and positive definite. We can therefore select H_1, \dots, H_l in such a way that this form reduces to $x_1^2 + \dots + x_l^2$. For any dominant integral function Λ let e_{Λ} denote the vector in the l -dimensional real Euclidean space with the components $\Lambda(H_i)$. Then the set of all points e_{Λ} form one "octant" of a lattice whose generators are $e_i = e_{\Lambda_i}$, $1 \leq i \leq l$. Now

$$\sum'_{\mathfrak{D}} d(\mathfrak{D})^{-(l+1)} \leq \kappa^{(l+1)/2} \sum'_{\mathfrak{D}} |\Lambda_{\mathfrak{D}}|^{-(l+1)}$$

where $\sum'_{\mathfrak{D}}$ denotes the sum over all irreducible classes \mathfrak{D} except the one corresponding to the zero representation of degree 1. Since each class is completely determined by its highest weight, it follows that

$$\sum'_{\mathfrak{D}} |\Lambda_{\mathfrak{D}}|^{-(l+1)} \leq \sum'_{(m_i) \geq 0} |m_1 e_1 + \dots + m_l e_l|^{-(l+1)}$$

where $|e|$ is the Euclidean length of the vector e and $\sum'_{(m_i) \geq 0}$ denotes summation over all sets of nonnegative integers (m_1, \dots, m_l) such that $m_1 + \dots + m_l > 0$. Since the series on the right is well known to be convergent, the lemma follows.

4. A result on convergence. We use the terminology of [6, §9]. Let π be a permissible representation of G on a Banach space \mathfrak{S} and $E_{\mathfrak{D}}$ the canonical projection of \mathfrak{S} on the space $\mathfrak{S}_{\mathfrak{D}}$ consisting of all elements in \mathfrak{S} which transform under $\pi(K)$ according to \mathfrak{D} ($\mathfrak{D} \in \Omega$). We shall now prove the following lemma⁽⁸⁾.

LEMMA 3. *There exists an element $z \in \mathfrak{X}$ such that*

$$\sum_{\mathfrak{D} \in \Omega} |E_{\mathfrak{D}} \psi| \leq |\pi(z) \psi|$$

for any differentiable element ψ in \mathfrak{S} . Moreover the series

$$\sum_{\mathfrak{D} \in \mathfrak{p}} E_{\mathfrak{D}} \psi$$

converges to ψ .

Let $u \rightarrow u^*$ ($u \in K$) denote the natural mapping of K on $K^* = K/D \cap Z$. For any $u \in K$ we denote by $\Gamma(u)$ the unique element in \mathfrak{c}_0 such that $u \exp(-\Gamma(u)) \in K'$. Choose a base $\Gamma_1, \dots, \Gamma_r$ for \mathfrak{c}_0 over R such that $\exp \Gamma_i$, $1 \leq i \leq r$, is a set of generators for $D \cap Z$. Let μ be a linear function on \mathfrak{c} such that $\pi(\exp \Gamma_i) = e^{\mu(\Gamma_i)} \pi(1)$, $1 \leq i \leq r$. Let Ω_{π} be the set of all classes in Ω which occur in π . Then it is clear that if $\mathfrak{D} \in \Omega_{\pi}$ and σ is any representation in \mathfrak{D} , we must have

$$\sigma(\Gamma_i) = (2\pi(-1)^{1/2} n_i + \mu(\Gamma_i)) \sigma(1)$$

where n_i , $1 \leq i \leq r$, are all integers. Define a linear function $n_{\mathfrak{D}}$ on \mathfrak{c} by setting $n_{\mathfrak{D}}(\Gamma_i) = n_i$, $1 \leq i \leq r$, and put $|n_{\mathfrak{D}}| = (1 + n_1^2 + \dots + n_r^2)^{1/2}$. Then if $w = 1 - (1/4\pi^2) \sum_{i=1}^r (\Gamma_i - \mu(\Gamma_i))^2 \in \mathfrak{X}$, $\sigma(w) = |n_{\mathfrak{D}}|^2 \sigma(1)$. We note that w lies in the center of \mathfrak{X} . Let \mathfrak{X}' be the subalgebra of \mathfrak{B} generated by $(1, \mathfrak{f}')$. Since \mathfrak{f}' is semisimple we can find (see Lemma 4 of [7]) an element z_0 in the center of \mathfrak{X}' such that $\sigma(z_0) = d_{\sigma}^2 \sigma(1)$ for any simple representation σ of \mathfrak{X} of degree d_{σ} .

Put $\pi^*(u^*) = e^{-\mu(\Gamma(u))} \pi(u)$ ($u \in K$). Then π^* is a representation of K^* on

⁽⁸⁾ Cf. Lemma 31 of [6] which was stated without proof.

\mathfrak{S} and if $\mathfrak{D} \in \Omega_r$.

$$E_{\mathfrak{D}} = d(\mathfrak{D}) \int_{K^*} \text{conj} (\xi_{\mathfrak{D}}(u^*)) \pi^*(u^*) du^{*(\theta)}$$

where $\xi_{\mathfrak{D}}$ is the character (on K^*) of the class according to which every element in $\mathfrak{S}_{\mathfrak{D}}$ transforms under $\pi^*(K^*)$. Let M be an upper bound for $|\pi^*(u^*)|$ on the compact set K^* . Then it is clear that

$$|E_{\mathfrak{D}}| \leq d(\mathfrak{D})^2 M.$$

Let q and s be two integers ≥ 0 . Then

$$|E_{\mathfrak{D}} \pi(z_0^{q+1} w^s) \psi| \leq M d(\mathfrak{D})^2 |\pi(z_0^{q+1} w^s) \psi|.$$

But if $X \in \mathfrak{k}_0$,

$$\lim_{t \rightarrow 0} \frac{1}{t} (\pi(\exp tX) - 1) E_{\mathfrak{D}} \psi = \lim_{t \rightarrow 0} E_{\mathfrak{D}} \frac{1}{t} (\pi(\exp tX) - 1) \psi = E_{\mathfrak{D}} \pi(X) \psi$$

since $E_{\mathfrak{D}}$ commutes with $\pi(u)$ ($u \in K$). Hence it follows that $E_{\mathfrak{D}} \psi$ is differentiable under $\pi(K)$ and $\pi(x) E_{\mathfrak{D}} \psi = E_{\mathfrak{D}} \pi(x) \psi$ ($x \in \mathfrak{X}$). Therefore

$$E_{\mathfrak{D}} \pi(z_0^{q+1} w^s) \psi = \pi(z_0^{q+1} w^s) E_{\mathfrak{D}} \psi = d(\mathfrak{D})^{2q+2} |n_{\mathfrak{D}}|^{2s} E_{\mathfrak{D}} \psi$$

since $E_{\mathfrak{D}} \psi$ transforms under $\pi(K)$ according to \mathfrak{D} . Hence

$$d(\mathfrak{D})^{2q} |n_{\mathfrak{D}}|^{2s} |E_{\mathfrak{D}} \psi| \leq M |\pi(z_0^{q+1} w^s) \psi| \quad (\mathfrak{D} \in \Omega_r),$$

and therefore

$$\sum_{\mathfrak{D} \in \Omega_r} |E_{\mathfrak{D}} \psi| \leq \left(\sum_{\mathfrak{D} \in \Omega_r} d(\mathfrak{D})^{-2q} |n_{\mathfrak{D}}|^{-2s} \right) M |\pi(z_0^{q+1} w^s) \psi|.$$

For any $\mathfrak{D} \in \Omega_r$ let \mathfrak{D}' denote the class of representations of \mathfrak{l}' defined as follows. If $\sigma \in \mathfrak{D}$, \mathfrak{D}' is the class of the restriction of σ on \mathfrak{l}' . Clearly \mathfrak{D}' is irreducible and $d(\mathfrak{D}') = d(\mathfrak{D})$. Moreover \mathfrak{D} is completely determined by \mathfrak{D}' and $n_{\mathfrak{D}}$. Hence

$$\sum_{\mathfrak{D} \in \Omega_r} d(\mathfrak{D})^{-2q} |n_{\mathfrak{D}}|^{-2s} \leq \sum_{\mathfrak{D}'} d(\mathfrak{D}')^{-2q} \sum_{n_1, \dots, n_r} (1 + n_1^2 + \dots + n_r^2)^{-s}$$

where \mathfrak{D}' runs over all irreducible classes of finite-dimensional representations of \mathfrak{l}' . But if $2q$ exceeds the rank of \mathfrak{l}' it follows from Lemma 2 that $\sum_{\mathfrak{D}'} d(\mathfrak{D}')^{-2q} < \infty$. Similarly if $2s > r$

$$\sum_{(n)} (1 + n_1^2 + \dots + n_r^2)^{-s} \leq \sum_{(n)} (1 + n_1^2 + \dots + n_r^2)^{-(r+1)/2} < \infty.$$

(θ) Conj (x) means conjugate of x .

Therefore if we choose q and s sufficiently large and put

$$z = Nz_0^{q+1} w^s$$

where

$$N = M \sum_{\mathfrak{D} \in \Omega_\pi} d(\mathfrak{D})^{-2q} |n_{\mathfrak{D}}|^{-2s},$$

$$\sum_{\mathfrak{D} \in \Omega} |E_{\mathfrak{D}}\psi| \leq |\pi(z)\psi|.$$

This proves the first assertion of the lemma. Now we come to the second part. Since $\sum_{\mathfrak{D} \in \Omega} |E_{\mathfrak{D}}\psi| < \infty$ the series $\sum_{\mathfrak{D} \in \Omega} E_{\mathfrak{D}}\psi$ is convergent. Let ϕ denote its sum. We have to show that $\phi = \psi$. Put $\psi' = \psi - \phi$. Since $E_{\mathfrak{D}}\phi = E_{\mathfrak{D}}\psi$, $E_{\mathfrak{D}}\psi' = 0$. From this we shall deduce that $\psi' = 0$.

Suppose $\psi' \neq 0$. Then given any real $\epsilon > 0$ choose a continuous real non-negative function f on K^* such that $f(u^*) = 0$ if $|\pi^*(u^*)\psi' - \psi'| > \epsilon|\psi'|$ ($u^* \in K^*$) and $\int_{K^*} f(u^*) du^* = 1$. Moreover choose a finite linear combination ω of the matrix coefficients of finite-dimensional simple representations of K^* such that $|f(u^*) - \omega(u^*)| \leq \epsilon$ ($u^* \in K^*$). Then if

$$\psi'' = \int \omega(u^*) \pi^*(u^*) \psi' du^*,$$

$\psi'' \in \sum_{\mathfrak{D} \in \Omega} \mathfrak{S}_{\mathfrak{D}}$ and therefore $\psi'' = \sum_{\mathfrak{D} \in \Omega} E_{\mathfrak{D}}\psi''$. But

$$E_{\mathfrak{D}}\psi'' = \int \omega(u^*) \pi^*(u^*) E_{\mathfrak{D}}\psi' du^* = 0$$

since $E_{\mathfrak{D}}\psi' = 0$. Hence $\psi'' = 0$. On the other hand

$$|\psi'' - \psi'| \leq \int |\omega(u^*) - f(u^*)| |\pi^*(u^*)\psi'| du^*$$

$$+ \int f(u^*) |\pi^*(u^*)\psi' - \psi'| du^*$$

$$\leq M\epsilon |\psi'| + \epsilon |\psi'|$$

where $M = \sup_{u^* \in K^*} |\pi^*(u^*)|$. Therefore if ϵ is sufficiently small

$$|\psi'| = |\psi'' - \psi'| \leq |\psi'| / 2$$

which contradicts our assumption that $\psi' \neq 0$. Therefore $\psi' = 0$ and so $\sum_{\mathfrak{D} \in \Omega} E_{\mathfrak{D}}\psi$ converges to ψ .

5. Characters. Let $C_c^\infty(G)$ denote the class of all complex-valued functions on G which are indefinitely differentiable everywhere and which vanish outside a compact set. Let π be a quasi-simple irreducible representation of G on a Hilbert space \mathfrak{H} . For any $f \in C_c^\infty(G)$ consider the operator

$$T_f = \int f(x)\pi(x)dx$$

where dx is the Haar measure on G . We intend to show that T_f is of the trace class.

Let \mathfrak{D}' be the Banach space of all bounded linear operators A on \mathfrak{G} with the usual norm $|A| = \sup_{|\psi| \leq 1} |A\psi|$ ($\psi \in \mathfrak{G}$). Let \mathfrak{D}_0 be the subspace of \mathfrak{D}' consisting of all operators of the form T_f ($f \in C_c^\infty(G)$). We denote by \mathfrak{D} the closure of \mathfrak{D}_0 in \mathfrak{D}' . Now if $y \in G$,

$$\pi(y)T_f = \int f(y^{-1}x)\pi(x)dx, \quad T_f\pi(y^{-1}) = \int f(xy)\pi(x)dx.$$

Hence it follows that if $A \in \mathfrak{D}$ then $\pi(y)A$ and $A\pi(y^{-1})$ are also in \mathfrak{D} . We now define two representations l and r of G on \mathfrak{D} as follows:

$$l(x)A = \pi(x)A, \quad r(x)A = A\pi(x^{-1}) \quad (x \in G, A \in \mathfrak{D}).$$

In order to verify the conditions for continuity it is sufficient to prove that $\lim_{x \rightarrow 1, y \rightarrow 1} |\pi(x)A\pi(y^{-1}) - A| = 0$ ($A \in \mathfrak{D}$). This is done as follows. Given $\epsilon > 0$, choose $f \in C_c^\infty(G)$ such that $|A - T_f| \leq \epsilon$. Let $U = U^{-1}$ be a compact neighbourhood of 1 in G and M an upper bound for $|\pi(z)|$ for $z \in U$. Then

$$|\pi(x)A\pi(y^{-1}) - \pi(x)T_f\pi(y^{-1})| \leq M^2\epsilon \quad (x, y \in U)$$

and therefore

$$\begin{aligned} |\pi(x)A\pi(y^{-1}) - A| &\leq (M^2 + 1)\epsilon + |\pi(x)T_f\pi(y^{-1}) - T_f| \\ &\leq (M^2 + 1)\epsilon + \int |f(x^{-1}zy) - f(z)| |\pi(z)| dz. \end{aligned}$$

Let C be a compact set outside which f is zero. We can choose a neighbourhood V of 1 in G ($V \subset U$) such that $|f(x^{-1}zy) - f(z)| \leq \epsilon$ if $x, y \in V$. Let F be a real nonnegative continuous function on G which is equal to 1 on C and which vanishes outside some compact set. Then if $N_0 = \sup_{z \in V} |\pi(z)|$,

$$|\pi(x)A\pi(y^{-1}) - A| \leq (M^2 + 1)\epsilon + N_0\epsilon \int F(z)dz$$

provided $x, y \in V$. This proves that $\lim_{x \rightarrow 1, y \rightarrow 1} |\pi(x)A\pi(y^{-1}) - A| = 0$.

Since $l(x)T_f = \int f(x^{-1}z)\pi(z)dz$, it follows easily that T_f is differentiable under l . Similarly we show that it is differentiable under r . It is clear that the representations l and r are permissible. For any $\mathfrak{D} \in \Omega$ let $E_{\mathfrak{D}}$, $P_{\mathfrak{D}}$, and $Q_{\mathfrak{D}}$ denote the canonical projections (see §9 of [6]) corresponding to \mathfrak{D} under π , l , and r respectively. Then it is clear $P_{\mathfrak{D}}A = E_{\mathfrak{D}}A$; $Q_{\mathfrak{D}}A = AE_{\mathfrak{D}'}$ ($\mathfrak{D} \in \Omega$) where \mathfrak{D}' is the class contragredient to \mathfrak{D} . Let λ and ρ denote the left and right regular representations of G . Then every element in $C_c^\infty(G)$ is differentiable under both λ and ρ and $C_c^\infty(G)$ is invariant under $\lambda(\mathfrak{B})$ and $\rho(\mathfrak{B})$. Moreover

since $l(x)r(y)T_f = T_{\lambda(x)\rho(y)f}(x, y \in G)$ it follows easily that $l(a)r(b)T_f = T_{\lambda(a)\rho(b)f}(a, b \in \mathfrak{B})$.

Now define a representation ϕ of the group $G \times G$ on \mathfrak{D} as follows. $\phi(x, y)A = l(x)r(y)A = \pi(x)A\pi(y^{-1})$ ($x, y \in G$). Then ϕ is a permissible representation of the semisimple group $G \times G$ and any element of \mathfrak{D}_0 is differentiable under ϕ . Moreover the canonical projections for the representation ϕ (with respect to the subgroup $K \times K$) are exactly the operators $P_{\mathfrak{D}_1\mathfrak{D}_2}$ ($\mathfrak{D}_1, \mathfrak{D}_2 \in \Omega$). Let z_0 be the element of \mathfrak{X} which was introduced in the proof of Lemma 3. Then if we apply Lemma 3 to the representation ϕ and the differentiable element $T_{\lambda(z_0)\rho(z_0)f}$ we find that

$$\sum_{\mathfrak{D}_1, \mathfrak{D}_2 \in \Omega} |P_{\mathfrak{D}_1\mathfrak{D}_2}T_{\lambda(z_0)\rho(z_0)f}| < \infty.$$

But

$$P_{\mathfrak{D}_1\mathfrak{D}_2}T_{\lambda(z_0)\rho(z_0)f} = P_{\mathfrak{D}_1\mathfrak{D}_2}l(z_0)r(z_0)T_f = d(\mathfrak{D}_1)^2d(\mathfrak{D}_2)^2P_{\mathfrak{D}_1\mathfrak{D}_2}T_f.$$

Hence

$$\sum_{\mathfrak{D}_1, \mathfrak{D}_2 \in \Omega} d(\mathfrak{D}_1)^2d(\mathfrak{D}_2)^2 |E_{\mathfrak{D}_1}T_fE_{\mathfrak{D}_2}| < \infty.$$

Now let us first suppose that the subspaces $\mathfrak{H}_{\mathfrak{D}} = E_{\mathfrak{D}}\mathfrak{H}$ ($\mathfrak{D} \in \Omega$) are mutually orthogonal. Choose an orthonormal base for each $\mathfrak{H}_{\mathfrak{D}}$. All these put together form an orthonormal base $\{\psi_\alpha\}_{\alpha \in J}$ for \mathfrak{H} . In accordance with Theorem 4 we choose an integer N such that $\dim \mathfrak{H}_{\mathfrak{D}} \leq Nd(\mathfrak{D})^2$ ($\mathfrak{D} \in \Omega$). Then

$$\sum_{\alpha, \beta \in J} |(\psi_\alpha, T_f\psi_\beta)| = \sum_{\mathfrak{D}_1, \mathfrak{D}_2 \in \Omega} \sum_{\alpha \in J_{\mathfrak{D}_1}} \sum_{\beta \in J_{\mathfrak{D}_2}} |(\psi_\alpha, T_f\psi_\beta)|$$

where $J_{\mathfrak{D}}$ is the subset of J such that $\{\psi_\alpha\}_{\alpha \in J_{\mathfrak{D}}}$ is a base for $\mathfrak{H}_{\mathfrak{D}}$. But it is clear that

$$\begin{aligned} \sum_{\alpha \in J_{\mathfrak{D}_1}} \sum_{\beta \in J_{\mathfrak{D}_2}} |(\psi_\alpha, T_f\psi_\beta)| &\leq \dim \mathfrak{H}_{\mathfrak{D}_1} \dim \mathfrak{H}_{\mathfrak{D}_2} |E_{\mathfrak{D}_1}T_fE_{\mathfrak{D}_2}| \\ &\leq N^2d(\mathfrak{D}_1)^2d(\mathfrak{D}_2)^2 |E_{\mathfrak{D}_1}T_fE_{\mathfrak{D}_2}|. \end{aligned}$$

Hence

$$\sum_{\alpha, \beta \in J} |\psi_\alpha, T_f\psi_\beta| \leq N^2 \sum_{\mathfrak{D}_1, \mathfrak{D}_2 \in \Omega} d(\mathfrak{D}_1)^2d(\mathfrak{D}_2)^2 |E_{\mathfrak{D}_1}T_fE_{\mathfrak{D}_2}| < \infty$$

and therefore, from Lemma 1, T_f is of the trace class.

Now we discard the assumption about the mutual orthogonality of the spaces $\mathfrak{H}_{\mathfrak{D}}$. Let $x \rightarrow x^*$ denote the natural mapping of G on $G^* = G/D \cap Z$. Define a representation π^* of K^* on \mathfrak{H} as in the proof of Lemma 3. Since K^* is a compact group, π^* is equivalent to a unitary representation. Hence there exists a regular operator B on \mathfrak{H} such that the representation $\pi'^*: u \rightarrow B\pi^*(u^*)B^{-1}$ ($u^* \in K^*$) is unitary. Now put $\pi'(x) = B\pi(x)B^{-1}$. Then π' is a

representation of G on \mathfrak{S} . Let $\mathfrak{S}'_{\mathfrak{D}}$ be the subspace of \mathfrak{S} consisting of all elements which transform under $\pi'(K)$ according to \mathfrak{D} ($\mathfrak{D} \in \Omega$). Then since π'^* is unitary the spaces $\mathfrak{S}'_{\mathfrak{D}}$ are mutually orthogonal. Therefore the above proof is applicable to

$$T'_f = \int f(x)\pi'(x)dx = BT_fB^{-1}.$$

Hence T'_f fulfills the condition of Lemma 1 and therefore $T_f = B^{-1}T'_fB$ is of the trace class. Moreover if P and Q are two bounded operators on \mathfrak{S} , then PB^{-1} and BQ are also bounded and $PT'_fQ = (PB^{-1})T'_f(BQ)$. Therefore, from Lemma 1, PT'_fQ , T'_fQP , QPT'_f are all of the trace class and their traces are equal. Therefore in particular $\text{sp}(AT_fA^{-1}) = \text{sp} T_f$ if A is a regular operator.

Put $T_{\pi}(f) = \text{sp}(T_f)$ for any $f \in C_c^{\infty}(G)$. Then T_{π} is a linear function on the vector space $C_c^{\infty}(G)$. Furthermore if a is a fixed element in G and g is the function $g(x) = f(axa^{-1})$ ($x \in G$) then

$$T_g = \pi(a^{-1})T_f\pi(a)$$

and therefore $T_{\pi}(g) = T_{\pi}(f)$. Hence we may say that T_{π} is invariant under the inner automorphisms of G . We prove similarly that if π and π' are equivalent representations, $T_{\pi} = T_{\pi'}$.

Our next object is to show that T_{π} is actually a distribution in the sense of L. Schwartz [9]. By going over to an equivalent representation, if necessary, we may assume that the spaces $\mathfrak{S}_{\mathfrak{D}}$ ($\mathfrak{D} \in \Omega$) are mutually orthogonal. Then it is clear that

$$\text{sp} T_f = \sum_{\mathfrak{D} \in \Omega} \text{sp} (E_{\mathfrak{D}}T_fE_{\mathfrak{D}})$$

and

$$\text{sp} (E_{\mathfrak{D}}T_fE_{\mathfrak{D}}) = \sum_{i=1}^d (\psi_i, E_{\mathfrak{D}}T_f\psi_i)$$

where (ψ_1, \dots, ψ_d) is an orthonormal base of $\mathfrak{S}_{\mathfrak{D}}$. Therefore

$$|\text{sp} (E_{\mathfrak{D}}T_fE_{\mathfrak{D}})| \leq \dim \mathfrak{S}_{\mathfrak{D}} |E_{\mathfrak{D}}T_f| \leq Nd(\mathfrak{D})^2 |E_{\mathfrak{D}}T_f|$$

and

$$|T_{\pi}(f)| \leq N \sum_{\mathfrak{D} \in \Omega} d(\mathfrak{D})^2 |E_{\mathfrak{D}}T_f| = N \sum_{\mathfrak{D} \in \Omega} |E_{\mathfrak{D}}T_{\lambda(z_0)f}|$$

where z_0 has the same meaning as above. Now by applying Lemma 3 to the representation l of G on \mathfrak{D} and the differentiable element $T_{\lambda(z_0)f}$ we conclude that

$$\sum_{\mathfrak{D} \in \Omega} |E_{\mathfrak{D}}T_{\lambda(z_0)f}| \leq |l(z)T_{\lambda(z_0)f}| = |T_{\lambda(z_0)f}|$$

where z is an element of \mathfrak{X} (which does not depend on f). Hence

$$|T_\pi(f)| \leq N |T_{\lambda(z_0)f}|.$$

Now suppose C is a compact set in G and f_n is a sequence of functions in $C_c^\infty(G)$ such that f_n vanishes outside C and for any $b \in \mathfrak{B}$, $\lambda(b)f_n \rightarrow 0$ uniformly on C . Then

$$|T_{\lambda(b)f_n}| \leq \int |\lambda(b)f_n(x)| |\pi(x)| dx \rightarrow 0$$

and therefore in particular $|T_\pi(f_n)| \leq N |T_{\lambda(z_0)f_n}| \rightarrow 0$. This proves that T_π is a distribution.

6. Operators of the Hilbert-Schmidt class. Let B be a bounded operator on the Hilbert space \mathfrak{H} and let B^* be the adjoint of B . We say that B is of the Hilbert-Schmidt (H.S.) class if B^*B has a trace. Let $\{\psi_\alpha\}_{\alpha \in J}$ be an orthonormal base for \mathfrak{H} . Then it is well known that $\|B\|^2 = \sum_{\alpha \in J} |B\psi_\alpha|^2$ is independent of the choice of this base and B is of the H.S. class if and only if $\|B\| < \infty$. Moreover $\text{sp } BB^* = \|B\|^2 = \|B^*\|^2$ if $\|B\| < \infty$ and $\|A_1BA_2\| \leq \|A_1\| \|B\| \|A_2\|$ for any two bounded operators A_1, A_2 .

Let π be a quasi-simple irreducible representation of G on \mathfrak{H} . Let f be a complex-valued measurable function on G which vanishes outside a compact set and such that $\int |f(x)|^2 dx < \infty$. It follows from the Schwartz inequality that $\int |f(x)| dx < \infty$ and therefore the operator $\int f(x)\pi(x)dx$ is a well-defined bounded operator. We intend to prove that this operator is of the H.S. class. Let S be a regular operator on \mathfrak{H} . Put $\pi'(x) = S\pi(x)S^{-1}$ ($x \in G$). Then

$$\left\| \int f(x)\pi(x)dx \right\| = \left\| S^{-1} \int f(x)\pi'(x)dx S \right\| \leq |S^{-1}| \left\| \int f(x)\pi(x)dx \right\| |S|.$$

Therefore it is enough to show that the corresponding operator for an equivalent representation is of the H.S. class.

Let $x \rightarrow x^*$ denote the natural mapping of G on $G^* = G/D \cap Z$. For any $x \in G$ we define $\Gamma(x)$ to be the unique element in \mathfrak{c}_0 such that $x = u(\exp \Gamma(x))s$ where $u \in K'$ and s lies in the solvable subgroup of G corresponding to the subalgebra $\mathfrak{g}_0 \cap (\mathfrak{h}_p + \mathfrak{n})$ of \mathfrak{g}_0 (see [6, §9]). Then if μ is the linear function on \mathfrak{c} which was introduced in the proof of Lemma 3, it is clear that $\pi(x)e^{-\mu(\Gamma(x))}$ depends only on x^* . Put $\pi^*(x^*) = \pi(x)e^{-\mu(\Gamma(x))}$. Then we verify immediately that $\pi^*(u^*x^*) = \pi^*(u^*)\pi^*(x^*)$ ($u^* \in K^*, x^* \in G^*$) and therefore $u^* \rightarrow \pi^*(u^*)$ is a representation of K^* . In view of the preceding remarks we may assume without loss of generality that this representation is unitary.

Let $x^* \in G^*$ and $y \in G$. We say that y lies above x^* and write $y > x^*$ if $(y)^* = x^*$. Put

$$f^*(x^*) = \sum_{y > x^*} e^{\mu(\Gamma(y))} f(y) \quad (x^* \in G^*).$$

Let A be a compact set outside which f is zero. Since $D \cap Z$ is discrete, $(D \cap Z) \cap A^{-1}A$ is a finite set. Let N_0 be the number of elements in it. Then it is clear that not more than N_0 distinct elements in A can lie above the same element in G^* . Hence at most N_0 terms in the above sum are different from zero and therefore the function f^* is well-defined. Moreover if A^* is the image of A in G^* , f^* is zero outside A^* . Now let $x^* \in A^*$. Then

$$\begin{aligned} |f^*(x^*)| &= \left| \sum_{x > x^*} e^{\mu(\Gamma(x))} f(x) \right| \leq \left(\sum_{x > x^*} |f(x)|^2 \right)^{1/2} \left(\sum_{y > x^*} |e^{\mu(\Gamma(y))}|^2 \right)^{1/2} \\ &\leq M_0 N_0^{1/2} \left(\sum_{x > x^*} |f(x)|^2 \right)^{1/2} \end{aligned}$$

where $M_0 = \sup_{y \in A} |e^{\mu(\Gamma(y))}|$. Hence if the Haar measure dx^* on G^* is suitably normalised it follows that

$$\int f(x) \pi(x) dx = \int f^*(x^*) \pi^*(x^*) dx^*$$

and

$$\int |f^*(x^*)|^2 dx^* \leq M_0^2 N \int |f(x)|^2 dx < \infty.$$

Let B^* be a compact neighbourhood of A^* . Choose a real-valued non-negative function F on G^* such that $F=1$ on K^*B^* and $F=0$ outside some compact set. Let g be any continuous function on G^* which vanishes outside B^* . Consider the operator

$$\int g(x^*) \pi(x^*) dx^* = \int du^* \int g(u^*x^*) \pi^*(u^*x^*) dx^*.$$

(Here du^* is the normalised Haar measure on K^* so that $\int du^* = 1$.) Then

$$\left\| \int g(x^*) \pi^*(x^*) dx^* \right\| \leq \int dx^* \left\| \int g(u^*x^*) \pi^*(u^*x^*) du^* \right\|.$$

But

$$\left\| \int g(u^*x^*) \pi^*(u^*x^*) du^* \right\| \leq \left\| \int g(u^*x^*) \pi^*(u^*) du^* \right\| |\pi^*(x)|$$

and from Theorem 4 we can find an integer N such that $\dim \mathfrak{H}_{\mathfrak{D}} \leq Nd(\mathfrak{D})^2$ for any $\mathfrak{D} \in \Omega$. Let Ω^* be the set of all classes of irreducible finite-dimensional representations of K^* . Then no $\mathfrak{D}^* \in \Omega^*$ occurs more than $Nd(\mathfrak{D}^*)$ times in the reduction of $\pi^*(K^*)$. Since every $\mathfrak{D}^* \in \Omega^*$ occurs exactly $d(\mathfrak{D}^*)$ times in the left regular representation λ of K^* (on the Hilbert space $L_2(K^*)$ of all

square integrable functions on K^*) and since the representation $u^* \rightarrow \pi^*(u^*)$ ($u^* \in K^*$) is unitary, we may conclude that

$$\left\| \int g(u^*x^*)\pi^*(u^*)du^* \right\|^2 \leq N \left\| \int g(u^*x^*)\lambda(u^*)du^* \right\|^2.$$

But from the Peter-Weyl theorem we know that

$$\left\| \int g(u^*x^*)\lambda(u^*)du^* \right\|^2 = \int |g(u^*x^*)|^2 du^*.$$

Hence

$$\left\| \int g(u^*x^*)\pi^*(u^*)du^* \right\| \leq N^{1/2} \left(\int |g(u^*x^*)|^2 du^* \right)^{1/2}.$$

Now it is easy to see that $|\pi^*(x^*)|$ is bounded on the compact set K^*B^* . Let $M = \sup_{x^* \in K^*B^*} |\pi^*(x^*)|$. Then

$$\left\| \int g(u^*x^*)\pi^*(u^*x^*)du^* \right\| \leq MN^{1/2} \left(\int |g(u^*x^*)|^2 du^* \right)^{1/2}$$

and therefore

$$\begin{aligned} \left\| \int g(x^*)\pi^*(x^*)dx^* \right\| &\leq MN^{1/2} \int dx^* \left(\int |g(u^*x^*)|^2 du^* \right)^{1/2} \\ &= MN^{1/2} \int F(x^*)dx^* \left(\int |g(u^*x^*)|^2 du^* \right)^{1/2} \\ &\leq M_1 \left(\int |g(x^*)|^2 dx^* \right)^{1/2} \end{aligned}$$

where $M_1 = MN^{1/2} \int |F(x^*)|^2 dx^*$. Now choose a sequence g_n of continuous functions on G^* which vanish outside B^* and such that $\int |f^*(x^*) - g_n(x^*)|^2 dx^* \rightarrow 0$. Then since

$$\begin{aligned} \int |f^*(x^*) - g_n(x^*)| dx^* &\leq \left(\int |F(x^*)|^2 dx^* \right)^{1/2} \left(\int |f^*(x^*) - g_n(x^*)|^2 dx^* \right)^{1/2} \end{aligned}$$

it follows that

$$\left| \int (f^*(x^*) - g_n(x^*))\pi^*(x^*)dx^* \right| \rightarrow 0.$$

Moreover we have seen above that

$$\left\| \int (g_m(x^*) - g_n(x^*))\pi^*(x^*)dx^* \right\| \leq M_1 \left(\int |g_m(x^*) - g_n(x^*)|^2 dx^* \right)^{1/2}$$

and therefore the sequence of operators $T_n = \int g_n(x^*)\pi^*(x^*)dx^*$ is a Cauchy sequence with respect to the Hilbert-Schmidt norm $\| \cdot \|$. Since the space of operators of the H.S. class is complete with respect to this norm, there exists an operator T of this class such that $\|T_n - T\| \rightarrow 0$. But $|T_n - T| \leq \|T_n - T\|$. Hence $|T_n - T| \rightarrow 0$. However we have seen already that

$$\left| T_n - \int f^*(x^*)\pi^*(x^*)dx^* \right| \rightarrow 0$$

and therefore $T = \int f^*(x^*)\pi^*(x^*)dx^*$. This proves that $T = \int f(x)\pi(x)dx$ is of the H.S. class. Thus we have the following theorem⁽¹⁰⁾.

THEOREM 5. *Let π be a quasi-simple irreducible representation of G on a Hilbert space \mathfrak{H} and let f be a measurable and square integrable function on G which vanishes outside a compact set. Then the operator $\int f(x)\pi(x)dx$ is of the Hilbert-Schmidt class.*

7. Linear independence of characters. Let T_τ be the character of a quasi-simple irreducible representation π of G on a Hilbert space \mathfrak{H} . Let $E_{\mathfrak{D}}$ denote the canonical projection of \mathfrak{H} on $\mathfrak{H}_{\mathfrak{D}}$ ($\mathfrak{D} \in \Omega$). Then it follows easily from its definition (see §5) that

$$\begin{aligned} T_\tau(f) &= \text{sp} \left(\int f(x)\pi(x)dx \right) = \sum_{\mathfrak{D} \in \Omega} \text{sp} \left(E_{\mathfrak{D}} \int f(x)\pi(x)dx \cdot E_{\mathfrak{D}} \right) \\ &= \sum_{\mathfrak{D} \in \Omega} \int f(x)\phi_{\mathfrak{D}}^\pi(x)dx \end{aligned}$$

in the notation of §1. Now if π_1, π_2 are two infinitesimally equivalent representations, we have seen in §1 that $\phi_{\mathfrak{D}}^{\pi_1} = \phi_{\mathfrak{D}}^{\pi_2}$ ($\mathfrak{D} \in \Omega$) and therefore $T_{\pi_1} = T_{\pi_2}$. Hence two infinitesimally equivalent quasi-simple irreducible representations (on Hilbert spaces) have the same character. Conversely we shall show that two such representations having the same character are infinitesimally equivalent⁽¹¹⁾.

THEOREM 6. *Let π_1, \dots, π_q be a finite set of quasi-simple irreducible representations of G on the Hilbert spaces $\mathfrak{H}_1, \dots, \mathfrak{H}_q$ respectively. Suppose no two of them are infinitesimally equivalent. Then their characters $T_{\pi_1}, \dots, T_{\pi_q}$ are linearly independent.*

For otherwise suppose $c_1 T_{\pi_1} + \dots + c_q T_{\pi_q} = 0$ ($c_i \in C$) where, say, $c_1 \neq 0$.

⁽¹⁰⁾ Cf. Theorem 4 of [8(c)].

⁽¹¹⁾ Cf. Theorem 3 of [8(c)].

Let η_i be the homomorphism of $D \cap Z$ into C such that $\pi_i(\gamma) = \eta_i(\gamma)\pi_i(1)$ ($\gamma \in D \cap Z$). Then if $f \in C_c^\infty(G)$ and $\gamma \in D \cap Z$, the function $f_\gamma: x \rightarrow f(\gamma^{-1}x)$ ($x \in G$) is also in $C_c^\infty(G)$ and it is obvious that $T_{\pi_i}(f_\gamma) = \eta_i(\gamma)T_{\pi_i}(f)$. Therefore $\sum_{i=1}^q c_i T_{\pi_i}(f_\gamma) = \sum_{i=1}^q c_i \eta_i(\gamma) T_{\pi_i}(f) = 0$. This proves that

$$\sum_{i=1}^q c_i \eta_i(\gamma) T_{\pi_i} = 0$$

for all $\gamma \in D \cap Z$. Now if we recall that $D \cap Z$ is a free abelian group with r generators ($r = \dim_{\mathbb{R}} \mathfrak{c}_0$) we can conclude that $c_1 T_{\pi_1} + \dots + c_s T_{\pi_s} = 0$ assuming that $\eta_j = \eta_1$ ($1 \leq j \leq s$) and $\eta_j \neq \eta_i$ for $s < j \leq q$.

Choose a base $\Gamma_1, \dots, \Gamma_r$ for \mathfrak{c}_0 over R such that $\exp \Gamma_i$, $1 \leq i \leq r$, is a set of generators for $D \cap Z$. Select a linear function μ on \mathfrak{c} such that $\eta_1(\exp \Gamma_i) = e^{\mu(\Gamma_i)}$, $1 \leq i \leq r$. Put $\pi_j^*(x^*) = e^{-\mu(\Gamma(x))} \pi_j(x)$ ($x \in G$, $1 \leq j \leq s$) in the notation of §6. Let \mathfrak{D} be a class in Ω which occurs in π_1 . We denote by $E_{\mathfrak{D}}^i$ the canonical projection of \mathfrak{S}_i on $\mathfrak{S}_{i, \mathfrak{D}}$. Then

$$E_{\mathfrak{D}}^i = d(\mathfrak{D}) \int \text{conj} (\xi_{\mathfrak{D}^*}(u^*)) \pi_i^*(u^*) du^*$$

where \mathfrak{D}^* is the irreducible class according to which every element in $\mathfrak{S}_{1, \mathfrak{D}}$ transforms under $\pi_1^*(K^*)$ and $\xi_{\mathfrak{D}^*}$ is the character of \mathfrak{D}^* . Let K_0 be the set of all elements in K of the form $(\exp \Gamma)v$ where $\Gamma = t_1 \Gamma_1 + \dots + t_r \Gamma_r$ ($t_j \in R$, $|t_j| \leq 1/2$) and $v \in K'$. Then K_0 is compact and if we put

$$\xi_{\mathfrak{D}}(u) = e^{-\text{conj}(\mu(\Gamma(u)))} \xi_{\mathfrak{D}^*}(u^*) \tag{u \in K},$$

we get

$$E_{\mathfrak{D}}^i = d(\mathfrak{D}) \int_{K_0} \text{conj} (\xi_{\mathfrak{D}}(u)) \pi_i(u) du$$

where the Haar measure du on K is so normalised that $\int_{K_0} du = 1$.

Now we use the notation of Theorem 1. Put

$$\phi = c_1 \phi_{\mathfrak{D}}^{\pi_1} + \dots + c_s \phi_{\mathfrak{D}}^{\pi_s}.$$

It follows from Theorem 1 that $\phi \neq 0$. Since ϕ is continuous we can find a function $f \in C_c^\infty(G)$ such that $\int f(x)\phi(x)dx \neq 0$. Now

$$E_{\mathfrak{D}}^i \int f(x)\pi_i(x)dx = \int f_{\mathfrak{D}}(x)\pi_i(x)dx \tag{1 \leq i \leq s}$$

where

$$f_{\mathfrak{D}}(x) = d(\mathfrak{D}) \int_{K_0} \text{conj} (\xi_{\mathfrak{D}}(u)) f(u^{-1}x) du.$$

Since K_0 is compact it is clear that $f_{\mathfrak{D}} \in C_c^\infty(G)$. On the other hand

$$\begin{aligned} \text{sp} \left(\int f_{\mathfrak{D}}(x) \pi_i(x) dx \right) &= \text{sp} \left(E_{\mathfrak{D}}^i \int f(x) \pi_i(x) dx \right) \\ &= \text{sp} \left(E_{\mathfrak{D}}^i \int f(x) \pi_i(x) dx \cdot E_{\mathfrak{D}}^i \right) = \int f(x) \phi_{\mathfrak{D}}^{\pi_i}(x) dx. \end{aligned}$$

Therefore $c_1 T_{\tau_1}(f_{\mathfrak{D}}) + \dots + c_s T_{\tau_s}(f_{\mathfrak{D}}) = \int f(x) \phi(x) dx \neq 0$. This however implies that $c_1 T_{\tau_1} + \dots + c_s T_{\tau_s} \neq 0$ and so we get a contradiction.

COROLLARY. *Two irreducible unitary representations are equivalent if and only if their characters are the same.*

First of all every irreducible unitary representation is quasi-simple (see for example Segal [10]). Moreover infinitesimal equivalence is the same as ordinary equivalence for two such representations (see Theorem 8 of [6]). Hence the corollary is an immediate consequence of the theorem.

8. Complex semisimple groups. Suppose the group G is complex. Then K is semisimple and there exists a 1-1 linear mapping i of \mathfrak{k}_0 onto \mathfrak{p}_0 such that

$$[X, i(Y)] = i([X, Y]), \quad [i(X), i(Y)] = -[X, Y] \quad (X, Y \in \mathfrak{k}_0).$$

Let $\mathfrak{h}_{\mathfrak{k}_0}$ be a maximal abelian subalgebra of \mathfrak{k}_0 . Then $i(\mathfrak{h}_{\mathfrak{k}_0})$ is clearly a maximal abelian subspace of \mathfrak{p}_0 . Hence we may take $\mathfrak{h}_{\mathfrak{p}_0} = i(\mathfrak{h}_{\mathfrak{k}_0})$. Then $\mathfrak{h}_{\mathfrak{k}_0} + \mathfrak{h}_{\mathfrak{p}_0}$ is a maximal abelian subalgebra of \mathfrak{g}_0 . Let $\mathfrak{h}_{\mathfrak{k}}$ and $\mathfrak{h}_{\mathfrak{p}}$ be the subspaces of \mathfrak{g} spanned by $\mathfrak{h}_{\mathfrak{k}_0}$ and $\mathfrak{h}_{\mathfrak{p}_0}$ over C . Then $\mathfrak{h} = \mathfrak{h}_{\mathfrak{k}} + \mathfrak{h}_{\mathfrak{p}}$ is a Cartan subalgebra of \mathfrak{g} . We extend i to a mapping of \mathfrak{k} into \mathfrak{p} by linearity.

Let $\alpha_1, \dots, \alpha_p$ be a maximal set of linearly independent roots of \mathfrak{k} (with respect to $\mathfrak{h}_{\mathfrak{k}}$). We order all roots α of \mathfrak{k} lexicographically with respect to this set (see [5, Part I]). For every root α let H_{α} be the element in $\mathfrak{h}_{\mathfrak{k}}$ such that $\text{sp}(\text{ad}' H \text{ad}' H_{\alpha}) = \alpha(H)$ ($H \in \mathfrak{h}_{\mathfrak{k}}$) where $X \rightarrow \text{ad}' X$ ($X \in \mathfrak{k}$) is the adjoint representation of \mathfrak{k} . We denote by W the Weyl group of \mathfrak{k} and by 2σ the sum of all positive roots of \mathfrak{k} . Let λ be a linear function on $\mathfrak{h}_{\mathfrak{k}}$. We put $\lambda' = \lambda + \sigma$ and use the notation of [5, Part III]. We know (see [5, p. 70]) that the power series $\sum_{s \in W} \epsilon(s) e^{\lambda'(sH)}$ is divisible by $\prod_{\alpha > 0} \lambda'(H_{\alpha}) \prod_{\alpha > 0} \alpha(H)$ ($H \in \mathfrak{h}_{\mathfrak{k}}$) and therefore the quotient

$$\frac{\prod_{s \in W} \epsilon(s) e^{\lambda'(sH)}}{\prod_{\alpha > 0} \lambda'(H_{\alpha}) \prod_{\alpha > 0} \alpha(H)}$$

is an analytic function on $\mathfrak{h}_{\mathfrak{k}}$. Similarly

$$\frac{\prod_{\alpha > 0} \alpha(H)}{\prod_{\alpha > 0} (e^{\alpha(H)/2} - e^{-\alpha(H)/2})}$$

is a meromorphic function on \mathfrak{h}_t all whose singularities lie on hyperplanes of the form $\alpha(H) = 2\pi(-1)^{1/2}n$ where α is a root and n is some nonzero integer. Hence the function

$$\Phi^*(\lambda, H) = \prod_{\alpha>0} \sigma(H_\alpha) \frac{\sum_{s \in W} \epsilon(s) e^{\lambda'(sH)}}{\prod_{\alpha>0} \lambda'(H_\alpha) \prod_{\alpha>0} \alpha(H)} \frac{\prod_{\alpha>0} \alpha(H)}{\prod_{\alpha>0} (e^{\alpha(H)/2} - e^{-\alpha(H)/2})}$$

is a meromorphic function on \mathfrak{h}_t and it is analytic everywhere on $(-1)^{1/2}\mathfrak{h}_t$, and also on a suitable neighbourhood of zero in \mathfrak{h}_t . We know from [5, Part III, p. 71] that if $H \in \mathfrak{h}_t$ and $|t|$ is sufficiently small ($t \in C$) then

$$\Phi^*(\lambda, tH) = \sum_{m \geq 0} \frac{t^m}{m!} \xi_\lambda(H^m)$$

where ξ_λ is the (infinitesimal) character of the algebra \mathfrak{X} corresponding to the linear function λ on \mathfrak{h}_t .

Λ being any linear function on \mathfrak{h}_p , consider the integral $\int_K e^{\Lambda(H(x, u))} du$ which occurs in Theorem 2. (Notice that $K = K^*$ in our case and therefore $\int_K du = 1$.) We shall now express this integral in terms of the function Φ^* .

Consider the representation π_Λ of G on $L_2(K)$ given by

$$\pi_\Lambda(x)f(u) = e^{-(\Lambda+2\rho)(H(x^{-1}, u))} f(u_{x^{-1}}) \quad (x \in G, u \in K, f \in L_2(K))$$

in the notation of [6, §12]. Let \mathfrak{S} be the smallest closed subspace of $L_2(K)$ which is invariant under $\pi_\Lambda(G)$ and which contains the constant function 1. Then we have seen in [5, Part IV] that the representation π of G induced on \mathfrak{S} is quasi-simple and its infinitesimal character⁽¹²⁾ is χ_Λ where Λ is to be extended to a linear function on \mathfrak{h} by putting it equal to zero on \mathfrak{h}_t . Let ψ_0 denote the vector in \mathfrak{S} corresponding to the constant function 1. Then if we denote the scalar product of two elements in the usual way we get

$$(\psi_0, \pi(x)\psi_0) = \int_K e^{\Lambda(H(x, u))} du.$$

On the other hand suppose $x = \exp tH_0$ where $H_0 \in \mathfrak{h}_{p_0}$ and $t \in R$. Then since ψ_0 is well-behaved under π (see Lemma 34 of [6]),

$$(\psi_0, \pi(\exp tH_0)\psi_0) = \sum_{m \geq 0} \frac{t^m}{m!} (\psi_0, \pi(H_0^m)\psi_0)$$

provided $|t|$ is sufficiently small. Now put $i_+(X) = (X + (-1)^{1/2}i(X))/2$, $i_-(X) = (X - (-1)^{1/2}i(X))/2$ ($X \in \mathfrak{f}$). Then $\mathfrak{f}_+ = i_+(\mathfrak{f})$, $\mathfrak{f}_- = i_-(\mathfrak{f})$ are ideals in \mathfrak{g} and \mathfrak{g} is their direct sum. Let \mathfrak{X}_+ and \mathfrak{X}_- be the subalgebras of \mathfrak{B} generated by $(\mathfrak{f}_+, 1)$ and $(\mathfrak{f}_-, 1)$ respectively. Then if $a \in \mathfrak{X}_+$ and $b \in \mathfrak{X}_-$, $ab - ba = 0$.

⁽¹²⁾ This is easily seen by the argument used in the proof of Lemma 48 of [5].

Choose $H_0^* \in \mathfrak{h}_\mathfrak{r}$, such that $H_0 = i(H_0^*)$. Then

$$H_0 = i(H_0^*) = 2(-1)^{1/2} [i_-(H_0^*) - H_0^*].$$

Moreover H_0^* and $i_-(H_0^*)$ commute and $\pi(\mathfrak{f})\psi_0 = \{0\}$. Hence

$$\pi(H_0^m)\psi_0 = (2(-1)^{1/2})^m \pi(H_-^m)\psi_0$$

where $H_- = i_-(H_0^*)$. On the other hand if $X, Y \in \mathfrak{f}$,

$$[X, i_-(Y)] = [i_-(X), i_-(Y)].$$

Hence $[X, z] = [i_-(X), z]$ ($X \in \mathfrak{f}, z \in \mathfrak{X}_-$). Therefore we get a representation ν of \mathfrak{f} on \mathfrak{X}_- such that $\nu(X)z = [X, z]$ ($X \in \mathfrak{f}, z \in \mathfrak{X}_-$). Obviously ν is quasi semisimple (see Lemma 10 of [6]). Therefore if z is any element in \mathfrak{X}_- , it follows from Lemma 7 of [6] that $z \equiv z_0 \pmod{\nu(\mathfrak{f})\mathfrak{X}_-}$ where z_0 is some element of \mathfrak{X}_- which commutes with \mathfrak{f} . But then

$$[i_-(X), z_0] = [X, z_0] = 0 \tag{X \in \mathfrak{f}}$$

and moreover z_0 commutes with \mathfrak{f}_+ since it lies in \mathfrak{X}_- . Therefore z_0 is in the center of \mathfrak{B} . Furthermore the representation of K induced under π is unitary and therefore if $X \in \mathfrak{f}_0$ and $a \in \mathfrak{B}$,

$$(\psi_0, \pi(Xa - aX)\psi_0) = (-\pi(X)\psi_0, \pi(a)\psi_0) = 0$$

since $\pi(\mathfrak{f})\psi_0 = \{0\}$. This shows that

$$(\psi_0, \pi(z)\psi_0) = (\psi_0, \pi(z_0)\psi_0) = \chi_\Lambda(z_0).$$

Now if we extend χ_Λ to a linear function on \mathfrak{B} such that $\chi_\Lambda(ab) = \chi_\Lambda(ba)$ ($a, b \in \mathfrak{B}$) (see Part III of [5]) it follows that $\chi_\Lambda(z_0) = \chi_\Lambda(z)$. Hence

$$(\psi_0, \pi(z)\psi_0) = \chi_\Lambda(z) \tag{z \in \mathfrak{X}_-}.$$

This proves that

$$(\psi_0, \pi(\exp tH)\psi_0) = \sum_{m \geq 0} \frac{t^m}{m!} (2(-1)^{1/2})^m \chi_\Lambda(H_-^m).$$

We extend i_- to an isomorphism of \mathfrak{X} with \mathfrak{X}_- . Then the mapping $z \rightarrow \chi_\Lambda(i_-(z))$ ($z \in \mathfrak{X}$) is clearly a character of the algebra \mathfrak{X} . Hence from Theorem 5 of [5] there exists a linear function λ on $\mathfrak{h}_\mathfrak{r}$ such that $\chi_\Lambda(i_-(z)) = \xi_\lambda(z)$ ($z \in \mathfrak{X}$). Therefore

$$(\psi_0, \pi(\exp tH_0)\psi_0) = \sum_{m \geq 0} \frac{t^m}{m!} (2(-1)^{1/2})^m \xi_\lambda(H_0^{*m}) = \Phi^*(\lambda, 2(-1)^{1/2} tH_0^*)$$

if $|t|$ is sufficiently small. In view of equation (25) (p. 81) of [5], λ may be chosen in such a way that $\lambda(H) = \Lambda(i_-(H))$ ($H \in \mathfrak{h}_\mathfrak{r}$). Since Λ vanishes on $\mathfrak{h}_\mathfrak{r}$ we conclude that $\lambda(H) = -((-1)^{1/2}/2)\Lambda(i(H))$ and therefore $2(-1)^{1/2}\lambda(H_0^*) = \Lambda(H_0)$. Put $si(H) = i(sH)$ ($s \in W$) and $\alpha(i(H)) = (-1)^{1/2}\alpha(H)$, $\sigma(i(H))$

$= (-1)^{1/2} \sigma(H)$ ($H \in \mathfrak{h}_t$). Then $\lambda(H_\alpha) = -((-1)^{1/2}/2)\Lambda(i(H_\alpha)) = \Lambda(H'_\alpha)$ where $H'_\alpha = -((-1)^{1/2}/2)i(H_\alpha)$ and $\sigma(H'_\alpha) = \sigma(H_\alpha)/2$. Then if we put

$$\Phi(\Lambda, H) = \frac{\prod_{\alpha>0} 2\sigma(H'_\alpha)}{\prod_{\alpha>0} (\Lambda + 2\sigma)(H'_\alpha)} \frac{\sum_{s \in W} \epsilon(s) e^{(\Lambda+2\sigma)(sH)}}{\prod_{\alpha>0} (e^{\alpha(H)} - e^{-\alpha(H)})} \quad (H \in \mathfrak{h}_{\mathfrak{p}_0})$$

it is clear that $\Phi(\Lambda, H)$ is an analytic function on $\mathfrak{h}_{\mathfrak{p}_0}$ and

$$\Phi^*(\lambda, 2(-1)^{1/2}tH_0^*) = \Phi(\Lambda, tH_0).$$

Hence $(\psi_0, \pi(\exp tH_0)\psi_0) = \Phi(\Lambda, tH_0)$ for all sufficiently small values of $|t|$. Since both sides are analytic functions of t , the equality must hold for all values of t . Thus we have the following result.

THEOREM 7. *Let Λ be a linear function on $\mathfrak{h}_{\mathfrak{p}}$. Then if $x = \exp H$ ($H \in \mathfrak{h}_{\mathfrak{p}_0}$) we have the formula*

$$\int_K e^{\Lambda(H(x,u))} du = \frac{\prod_{\alpha>0} 2\sigma(H'_\alpha)}{\prod_{\alpha>0} (\Lambda + 2\sigma)(H'_\alpha)} \frac{\sum_{s \in W} \epsilon(s) e^{(\Lambda+2\sigma)(sH)}}{\prod_{\alpha>0} (e^{\alpha(H)} - e^{-\alpha(H)})}.$$

Put $\phi(x) = \int_K e^{\Lambda(H(x,u))} du = (\psi_0, \pi(x)\psi_0)$ ($x \in G$). Then it is clear that $\phi(uxv) = \phi(x)$ ($u, v \in K$). Since every element in G can be written in the form $u(\exp H)v$ ($H \in \mathfrak{h}_{\mathfrak{p}_0}$; $u, v \in K$), the above formula determines ϕ completely.

The particular case of this formula for the complex immodular group has been obtained by Gelfand and Naimark [2, p. 77] by means of a lengthy computation (see also [1]).

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