The main object of this paper is to define the character of an irreducible quasi-simple\(^{(1)}\) representation \(\pi\) of a connected semisimple Lie group \(G\) on a Hilbert space \(\mathcal{H}\). This will be done as follows. Let \(C_c^\infty(G)\) be the class of all functions on \(G\) which are indefinitely differentiable and which vanish outside a compact set. For any \(f \in C_c^\infty(G)\) we consider the operator \(\int f(x) \pi(x) dx\) where \(dx\) is the Haar measure on \(G\). It turns out that this operator has a trace (which we denote by \(T_\pi(f)\)) and the mapping \(T_\pi: f \to T_\pi(f)\) is a distribution in the sense of L. Schwartz \([9]\) such that \(T_\pi(f_a) = T_\pi(f)\) where \(f_a(x) = f(axa^{-1})\) \(a, x \in G\). This distribution is defined to be the character of \(\pi\).

We shall see that two such representations \(\pi_1, \pi_2\) are infinitesimally equivalent (see \([6, \S 9]\)) if and only if they have the same character. Therefore in particular a unitary irreducible representation is determined within unitary equivalence by its character (cf. Theorem 8 of \([6]\)).

In the last section we give a simple proof of a formula for “spherical functions” on a complex semisimple group. This formula was obtained by Gelfand and Naimark \([1; 2]\) in some special cases by direct computation.

1. Some preliminary results. We keep to the notation of our two earlier papers \([6, 7]\) on the same subject. \(G\) is a connected, simply connected, semisimple Lie group and \(\mathfrak{g}_0\) is its Lie algebra over the field \(R\) of real numbers. \(\mathfrak{g}\) is the complexification of \(\mathfrak{g}_0\) and \(\mathfrak{f}, \mathfrak{p}, \mathfrak{f}_0, \mathfrak{p}_0, \mathfrak{c}, \mathfrak{f}', \) and \(m\) are defined as in \([6, \S 2]\) and \([7, \S 2]\). \(K, K',\) and \(D\) are the analytic subgroups of \(G\) corresponding to \(\mathfrak{f}_0, \mathfrak{f}_0' = \mathfrak{f}'f_0\mathfrak{f}_0\) and \(\mathfrak{c}_0 = \mathfrak{c}\mathfrak{g}_0\) respectively. Let \(Z\) be the center of \(G\) and \(\mathfrak{z}\) the center of the enveloping algebra \(\mathfrak{B}\) of \(\mathfrak{g}\). If \(\pi\) is a representation of \(G\) on a Banach space we shall say that \(\pi\) is quasi-simple if it maps the elements of \(D\cap Z\) and \(\mathfrak{z}\) into scalar multiples of the unit operator (see \([6, \S 10]\)).

Let \(\pi\) be a quasi-simple irreducible representation of \(G\) on a Banach space \(\mathcal{H}\). We denote by \(\Omega\) the set of all equivalence classes of finite-dimensional simple representations of \(K\). Let \(\mathcal{S}_D (\mathfrak{D} \in \Omega)\) denote the set of all elements in \(\mathcal{H}\) which transform under \(\pi(K)\) according to \(\mathfrak{D}\). We know (see \([6, \text{Lemma 33}]\)) that \(\dim \mathcal{S}_D < \infty\). Let \(E_\mathfrak{D}\) denote the canonical projection of \(\mathcal{H}\) on \(\mathcal{S}_\mathfrak{D}\) (see \([6, \S 9]\)). For any \(x \in G\) consider the operator \(E_\mathfrak{D}\pi(x)E_\mathfrak{D}\). It maps \(\mathcal{S}_\mathfrak{D}\) into itself and \(\mathcal{S}_\mathfrak{D}\) into \(\{0\}\) \((\mathfrak{D}' \neq \mathfrak{D})\). Let \(sp\) \((E_\mathfrak{D}\pi(x)E_\mathfrak{D})\) denote the trace of the restriction of \(E_\mathfrak{D}\pi(x)E_\mathfrak{D}\) on \(\mathcal{S}_\mathfrak{D}\). Since \(\dim \mathcal{S}_\mathfrak{D} < \infty\) this trace is well defined. Now any given linear function \(\alpha\) on \(\mathcal{S}_\mathfrak{D}\) may be extended to a continuous linear function on \(\mathcal{H}\) by setting \(\alpha(\psi) = \alpha(E_\mathfrak{D}\psi)\) \((\psi \in \mathcal{S})\). In particular if \(\psi_i\),

\[^{(1)}\] See \([6, \S 10]\) for the definition of a quasi-simple representation.
1 \leq i \leq r$, is a base for $\mathfrak{g}_2$ and $\tilde{\psi}_i$ is the linear function on $\mathfrak{g}_2$ which takes the value $1$ at $\psi_i$ and zero at $\psi_j$ ($j \neq i, 1 \leq i, j \leq r$) we may extend $\tilde{\psi}_i$ on $\mathfrak{g}$ in the above fashion. Then it is clear that

$$\phi^\pi_2(x) = \text{sp} \ E_2 \pi(x) E_2 = \sum_{i=1}^{r} (\tilde{\psi}_i, \pi(x) \psi_i)$$

in the notation of [6, §10]. Hence it follows from Lemmas 19 and 34 of [6] that $\phi^\pi_2$ is an analytic function on $G$. $X_1, \ldots, X_r$ being a base for $\mathfrak{g}_0$ over $R$, set $X(t) = t_1 X_1 + \cdots + t_r X_r$ ($t_j \in R$). Then we know (cf. Theorem 2 and Lemma 34 of [6]) that if $|t| = \max_j |t_j|$ is sufficiently small we get the convergent expansions

$$\pi(\exp X(t)) \psi_i = \sum_{m \geq 0} \frac{1}{m!} \pi((X(t))^m) \psi_i, \quad 1 \leq i \leq r.$$

From this it follows immediately that if $z$ is any element in $\mathfrak{g}$ the value of $\sum_{i \leq r} (\tilde{\psi}_i, \pi(z) \psi_i)$ can be obtained in terms of the various partial derivatives of $\phi^\pi_2(\exp X(t))$ with respect to $(t)$ at $t_1 = t_2 = \cdots = t_n = 0$. Let $\sigma$ be the representation of $\mathfrak{g} = \mathfrak{g}_0 \mathfrak{k}$ (see [7, §2] for notation) on $\mathfrak{g}_2$ defined under $\pi$. Then the knowledge of the function $\phi^\sigma_2$ determines in particular $\text{sp} \sigma(z)$ for any $z \in \mathfrak{g}$. Now we know (see Theorem 5 of [6]) that $\pi$ defines a quasi-simple(2) irreducible representation of $\mathfrak{g}$ on $\mathfrak{g}^{(0)} = \sum_{\mathfrak{h} \in \mathfrak{b} \mathfrak{b}} \mathfrak{g}_2$, and therefore $\sigma$ is irreducible (see Corollary 2 to Theorem 2 of [7]). On the other hand a finite-dimensional simple representation of an associative algebra is completely determined within equivalence by its trace (see Lemma 16 of [7]). Hence in view of Theorem 2 of [7] we can conclude that the function $\phi^\sigma_2$ determines the representation of $\mathfrak{g}$ on $\mathfrak{g}^{(0)}$ up to equivalence and therefore the representation $\pi$ of $G$ up to infinitesimally equivalent. This result may be stated in a slightly more general form as follows.

**Theorem(3)** 1. Let $\pi_1, \ldots, \pi_r$ be a finite set of quasi-simple irreducible representations of $G$ on Banach spaces. Suppose no two of them are infinitesimally equivalent. Then all the nonzero functions in the set $\phi^\pi_2^1, \ldots, \phi^\pi_2^r$, ($\mathfrak{d}_i \in \mathfrak{b}$, $1 \leq i \leq r$) are linearly independent.

Let $C$ be the field of complex numbers. If our assertion is false we may suppose that $c_1 \phi^\pi_2^1 + \cdots + c_r \phi^\pi_2^2 = 0$ where $c_i \phi^\pi_2^i \neq 0$, $1 \leq j \leq s$ ($c_i \in C$). Let $\mathfrak{h}_i$ be the representation space of $\pi_i$. Consider the representation $\sigma_i$ of $\mathfrak{g}$ on $\mathfrak{h}_i \mathfrak{d}_i (1 \leq i \leq s)$ induced under $\pi_i$. Then $c_1 \text{sp} \sigma_1(a) + \cdots + c_r \text{sp} \sigma_r(a) = 0$ for all $a \in \mathfrak{g}$. Since $c_i \phi^\pi_2^i \neq 0$, $\mathfrak{d}_j$ occurs in $\pi_j$. Moreover $\pi_j, \pi_k$ ($j \neq k$) are not infinitesimally equivalent ($1 \leq j, k \leq s$). Hence it follows from Corollary 2 to Theorem 2 of [7] that the representations $\sigma_1, \ldots, \sigma_r$ are irreducible and no

---

(2) See [7, end of §2] for the definition of quasi-simplicity in this case.
(3) Cf. Theorem 7 of [8(a)] and Theorem 2 of [8(b)].
two of them are equivalent. This however gives a contradiction with Lemma 16 of [7]. So the theorem is proved.

We recall that for two irreducible unitary representations on Hilbert spaces the notions of infinitesimal equivalence and ordinary equivalence are the same (see Theorem 8 of [6]). Hence if \( \pi_1, \pi_2 \) are two such representations which are not equivalent, the corresponding functions \( \phi_{\pi_1}^*, \phi_{\pi_2}^* \) are always distinct unless they are both zero.

Theorem 5 of [7] can now be rephrased in terms of the function \( \phi_{\pi}^* \) as follows:

**Theorem (4)** 2. Let \( \pi \) be a quasi-simple irreducible representation of \( G \) on a Banach space \( \mathfrak{H} \). Suppose \( \mathcal{D}_0 \) is a class in \( \Omega \) occurring in \( \pi \) such that \( d(\mathcal{D}_0) = 1 \). Then \( \dim \mathfrak{H}_{\mathcal{D}_0} = 1 \) and it is possible to choose linear functions \( \Lambda \) and \( \mu \) on \( \mathfrak{h} \) and \( \mathfrak{c} \) respectively such that

\[
\phi_{\mathcal{D}_0}^*(x) = \int_{K^*} e^{\mu((x, u^*))} e^{\Lambda((x, u^*))} \, du^* \quad (x \in G)
\]

and the infinitesimal character of \( \pi \) is \( \chi_\Lambda \).

Some properties of the function \( \phi_{\pi}^* \) have been studied by R. Godement [3] (see also [1; 2]).

We shall now state a few immediate consequences of the results proved in [6].

**Theorem (5)** 3. Let \( x \) be a homomorphism of \( \mathfrak{g} \) into \( \mathbb{C} \) and \( \mathcal{D}_0 \) a class in \( \Omega \). Then apart from infinitesimal equivalence there exist only a finite number of irreducible quasi-simple representations \( \pi \) of \( G \) which have the infinitesimal character \( x \) and such that \( \mathcal{D}_0 \) occurs in \( \pi \).

This follows from Theorem 2 of [7]. Similarly the following result is obtained from Theorem 3 of [8].

**Theorem 4.** Let \( \pi \) be a quasi-simple irreducible representation of \( G \) on a Banach space \( \mathfrak{H} \). Then there exists an integer \( N \) such that

\[
\dim \mathfrak{H}_{\mathcal{D}} \leq N(d(\mathcal{D}))^2
\]

for all \( \mathcal{D} \in \Omega \).

2. **Trace of an operator.** Let \( \{ c_\alpha \}_{\alpha \in I} \) be an indexed set of complex numbers. We define the convergence of the series \( \sum_{\alpha \in I} c_\alpha \) and its sum in the usual manner (see §5 of [6]). Let \( A \) be a bounded operator on a Hilbert space \( \mathfrak{H} \) and let \( \{ \psi_\alpha \}_{\alpha \in I} \) be an orthonormal base for \( \mathfrak{H} \). We say that \( A \) has a trace.

\(--\)

\( (*) \) Cf. Theorem 3 of [8(b)]. Our notation is the same as that of Theorem 5 of [7].

\( (\#) \) Cf. Theorem 6 of [8(a)].
(or $A$ is of the trace class) if for every such base the series\(^4\) $\sum_{a \in J} (\psi_a, A\psi_a)$ converges to a sum which is independent of the choice of the base. The value of this sum is called the trace of $A$ and we shall denote it by $\text{sp } A$.

**Lemma 1.** Let $\{\psi_a\}_{a \in J}$ be an orthonormal base for a Hilbert space $\mathcal{H}$ and $T$ a bounded operator such that $\sum_{a, b \in J} |t_{ab}| < \infty$ where $t_{ab} = (\psi_a, T\psi_b)$. Then if $A$ and $B$ are any bounded operators on $\mathcal{H}$, $ATB$, $BAT$, $TBA$ are all of the trace class and

$$\text{sp } (ATB) = \text{sp } (BAT) = \text{sp } (TBA).$$

Put $a_{ab} = (\psi_a, A\psi_b)$, $b_{ab} = (\psi_a, B\psi_b)$ ($\alpha, \beta \in J$) and consider the series $\sum_{a, b, \gamma \in J} |a_{ab}b_{ab}\gamma|$. Then\(^5\)

$$\sum_a |a_{ab}b_{ab}\gamma| = |t_{\beta\gamma}| \sum_a |a_{ab}b_{ab}\gamma|$$

$$\leq |t_{\beta\gamma}| \left( \sum_a |a_{ab}|^2 \right)^{1/2} \left( \sum_a |b_{ab}|^2 \right)^{1/2} \leq |t_{\beta\gamma}| |A| |B|$$

since

$$\sum_a |a_{ab}|^2 = \sum_a (\psi_a, A\psi_a)^2 = |A\psi_a|^2 \leq |A|^2$$

and similarly for $B$. Hence

$$\sum_{a, \beta, \gamma} |a_{ab}b_{ab}\gamma| \leq |A| |B| \sum_{\beta, \gamma} |t_{\beta\gamma}| < \infty.$$  

This proves that the series $\sum_{a, \beta, \gamma} a_{ab}b_{ab}\gamma$ is absolutely convergent and so it follows in the usual way that

$$\sum_a (\psi_a, ATB\psi_a) = \sum_a (\psi_a, TBA\psi_a) = \sum_a (\psi_a, BAT\psi_a).$$

Now let $U$ be a unitary transformation on $\mathcal{H}$. Consider $U^{-1}ATBU$. Since $U^{-1}A$ and $BU$ are bounded operators, we can conclude from the above result that

$$\sum_a (U\psi_a, ATBU\psi_a) = \sum_a (\psi_a, U^{-1}ATBU\psi_a)$$

$$= \sum_a (\psi_a, TBUU^{-1}A\psi_a) = \sum_a (\psi_a, TBA\psi_a)$$

$$= \sum_a (\psi_a, ATB\psi_a).$$

Since every orthonormal base in $\mathcal{H}$ is related to the base $\{\psi_a\}_{a \in J}$ by a unitary transformation, this proves that $ATB$ is of the trace class. Since $BA$ is a

\(^4\) As usual we denote by $\langle \phi, \psi \rangle$ the scalar product of $\phi$ and $\psi$ in $\mathcal{H}$.

\(^5\) For any bounded operator $Q$ we put $|Q| = \text{sup}_{|\psi| \leq 1} |Q\psi|$.

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
bounded operator it follows from this result that $BAT$ and $TBA$ are also of the trace class. Hence in view of the above equalities we conclude that $\text{sp } ATB = \text{sp } BAT = \text{sp } TBA$.

**Corollary.** If $T$ satisfies the conditions of the above lemma and if $A$ is a regular operator, then $T$ and $ATA^{-1}$ are both of the trace class and $\text{sp } ATA^{-1} = \text{sp } T$.

3. **An auxiliary lemma.** In order to prove that certain given operators are of the trace class we shall frequently need the following result.

**Lemma 2.** Let $I$ be a semisimple Lie algebra over $C$ of rank $l$. Then the series $\sum d(\Xi)^{-(l+1)}$ is convergent. Here $\Xi$ runs over all equivalence classes of finite-dimensional simple representations of $I$ and $d(\Xi)$ is the degree of any representation in $\Xi$.

Let $\mathfrak{h}$ be a Cartan subalgebra of $I$. Choose a fundamental system of roots and let $\Lambda_1, \cdots, \Lambda_l$ be a fundamental set of dominant integral functions on $\mathfrak{h}$ with respect to this system (see [5, Part I]). Then every such function can be written as $m_1\Lambda_1 + \cdots + m_l\Lambda_l$ where $m_i$ are all nonnegative integers. Let $H_1, \cdots, H_l$ be a base for $\mathfrak{h}$. Extend this to a base $X_1, \cdots, X_n$ ($n\geq l$) for $I$ so that $X_i = H_i$, $1 \leq i \leq l$. Put $g_{ij} = \text{sp } (\text{ad } X_i \text{ ad } X_j)$, $1 \leq i, j \leq n$, where $X \rightarrow \text{ad } X$ is the adjoint representation of $I$. Since $I$ is semisimple the matrix $(g_{ij})_{1\leq i,j\leq n}$ is nonsingular. Let $(g^{ij})_{1\leq i,j\leq n}$ denote its inverse. Let $U$ be the enveloping algebra of $I$. Put $\omega = \sum_{1\leq i,j\leq n} g^{ij}X_iX_j \in U$. $\omega$ is called the Casimir operator of $I$ and it is well known that $\omega$ lies in the center of $U$. For any dominant integral function $\Lambda$ on $\mathfrak{h}$ put

$$|\Lambda|^2 = \sum_{1\leq i,j \leq l} g^{ij} \Lambda(H_i)\Lambda(H_j).$$

Then it is known (see for example [4]) that $|\Lambda|^2$ is a positive real number unless $\Lambda = 0$. Now let $\sigma$ be an irreducible finite-dimensional representation of $U$ and let $\Xi$ be the class of $\sigma$. We denote by $\Lambda_\Xi$ the highest weight of $\sigma$ and by $\omega_\Xi$ the number such that $\sigma(\omega) = \omega_\Xi \sigma(1)$. Then it follows from Lemma 6 of [4] that $\omega_\Xi$ is real, $\omega_\Xi \geq |\Lambda_\Xi|^2$, and there exists a real number $\kappa$ such that $\kappa d(\Xi)^2 \geq \omega_\Xi$ for every irreducible class $\Xi$. Hence

$$d(\Xi)^{-1} \leq \kappa^{1/2} |\Lambda_\Xi|^{-1}.$$

Now the base $H_1, \cdots, H_l$ can be so chosen that every root of $I$ takes real values at $H_1, \cdots, H_l$. For such a base the quadratic form $\sum_{j=1}^l g^{ij}x_ix_j$ ($x_i \in R$) is real and positive definite. We can therefore select $H_1, \cdots, H_l$ in such a way that this form reduces to $x_1^2 + \cdots + x_l^2$. For any dominant integral function $\Lambda$ let $\epsilon_\Lambda$ denote the vector in the $l$-dimensional real Euclidean space with the components $\Lambda(H_i)$. Then the set of all points $\epsilon_\Lambda$ form one "octant" of a lattice whose generators are $\epsilon_i = \epsilon_{\Lambda_i}$, $1 \leq i \leq l$. Now
where \( \sum D \) denotes the sum over all irreducible classes \( \mathcal{D} \) except the one corresponding to the zero representation of degree 1. Since each class is completely determined by its highest weight, it follows that

\[
\sum' |\mathcal{D}|^{-(l+1)} \leq k^{(l+1)/2} \sum' |\mathcal{D}|^{-(l+1)}
\]

where \( |e| \) is the Euclidean length of the vector \( e \) and \( \sum_{(m) \geq 0} \) denotes summation over all sets of nonnegative integers \( (m_1, \cdots, m_i) \) such that \( m_1 + \cdots + m_i > 0 \). Since the series on the right is well known to be convergent, the lemma follows.

4. A result on convergence. We use the terminology of [6, §9]. Let \( \pi \) be a permissible representation of \( G \) on a Banach space \( \mathfrak{S} \) and \( E_\mathcal{D} \) the canonical projection of \( \mathfrak{S} \) on the space \( \mathfrak{S}_\mathcal{D} \) consisting of all elements in \( \mathfrak{S} \) which transform under \( \pi(K) \) according to \( \mathfrak{S}_\mathcal{D} \). We shall now prove the following lemma(8).

**Lemma 3.** There exists an element \( z \in \mathfrak{X} \) such that

\[
\sum_{\mathcal{D} \in \Omega} |E_\mathcal{D}\psi| \leq |\pi(z)\psi|
\]

for any differentiable element \( \psi \) in \( \mathfrak{S} \). Moreover the series

\[
\sum_{\mathcal{D} \in \Omega_\pi} E_\mathcal{D}\psi
\]

converges to \( \psi \).

Let \( u \rightarrow u^* \) \( (u \in K) \) denote the natural mapping of \( K \) on \( K^* = K/D \cap Z \). For any \( u \in K \) we denote by \( \Gamma(u) \) the unique element in \( \mathfrak{c}_0 \) such that \( u \exp(-\Gamma(u)) \in K^* \). Choose a base \( \Gamma_1, \cdots, \Gamma_r \) for \( \mathfrak{c}_0 \) over \( R \) such that \( \exp \Gamma_i, 1 \leq i \leq r \), is a set of generators for \( D \cap Z \). Let \( \mu \) be a linear function on \( \mathfrak{c} \) such that \( \pi(\exp \Gamma_i) = e^{\langle \Gamma_i \rangle} \pi(1), 1 \leq i \leq r \). Let \( \Omega_\pi \) be the set of all classes in \( \Omega \) which occur in \( \pi \). Then it is clear that if \( \mathcal{D} \in \Omega_\pi \) and \( \sigma \) is any representation in \( \mathcal{D} \), we must have

\[
\sigma(\Gamma_i) = (2\pi(-1)^{1/2}n_i + \mu(\Gamma_i))\sigma(1)
\]

where \( n_i, 1 \leq i \leq r \), are all integers. Define a linear function \( n_\mathfrak{D}(\Gamma_i) = n_i, 1 \leq i \leq r \), and put \( |n_\mathfrak{D}| = (1 + n_1^2 + \cdots + n_r^2)^{1/2} \). Then if \( w = 1 - (1/4\pi^2) \sum_{i=1}^r (\Gamma_i - \mu(\Gamma_i))^2 \in \mathfrak{X}, \sigma(w) = |n_\mathfrak{D}|^2\sigma(1). \) We note that \( w \) lies in the center of \( \mathfrak{X} \). Let \( \mathfrak{X}' \) be the subalgebra of \( \mathfrak{X} \) generated by \( (1, \Gamma') \). Since \( \Gamma' \) is semisimple we can find (see Lemma 4 of [7]) an element \( z_0 \) in the center of \( \mathfrak{X}' \) such that \( \sigma(z_0) = e^{\langle \Gamma'(w) \rangle} \sigma(1) \) for any simple representation \( \sigma \) of \( \mathfrak{X} \) of degree \( d_\sigma \).

Put \( \pi^*(u) = e^{-u(\Gamma'(w))}\pi(u) \) \( (u \in K) \). Then \( \pi^* \) is a representation of \( K^* \) on

---

(8) Cf. Lemma 31 of [6] which was stated without proof.
§ and if $\mathfrak{D} \in \Omega$,

$$E_\mathfrak{D} = d(\mathfrak{D}) \int_{K^*} \text{conj} (\xi_\mathfrak{D}(u^*)) \pi^*(u^*) du^*(u)$$

where $\xi_\mathfrak{D}$ is the character (on $K^*$) of the class according to which every element in $\mathfrak{S}_\mathfrak{D}$ transforms under $\pi^*(K^*)$. Let $M$ be an upper bound for $|\pi^*(u^*)|$ on the compact set $K^*$. Then it is clear that

$$|E_\mathfrak{D}| \leq d(\mathfrak{D})^2 M.$$

Let $q$ and $s$ be two integers $\geq 0$. Then

$$|E_\mathfrak{D} \pi(z_{0} w^s)\psi| \leq M d(\mathfrak{D})^2 |\pi(z_{0} w^s)\psi|.$$

But if $X \in \mathfrak{I}_0$,

$$\lim_{t \to 0} \frac{1}{t} (\pi(\exp tX) - 1) E_\mathfrak{D} \psi = \lim_{t \to 0} \frac{1}{t} (\pi(\exp tX) - 1) \psi = E_\mathfrak{D} \pi(X) \psi$$

since $E_\mathfrak{D}$ commutes with $\pi(u)$ ($u \in K$). Hence it follows that $E_\mathfrak{D} \psi$ is differentiable under $\pi(K)$ and $\pi(x) E_\mathfrak{D} \psi = E_\mathfrak{D} \pi(x) \psi$ ($x \in K$). Therefore

$$E_\mathfrak{D} \pi(z_{0} w^s) \psi = \pi(z_{0} w^s) E_\mathfrak{D} \psi = d(\mathfrak{D})^{2q+2} |n_\mathfrak{D}|^2 E_\mathfrak{D} \psi$$

since $E_\mathfrak{D} \psi$ transforms under $\pi(K)$ according to $\mathfrak{D}$. Hence

$$d(\mathfrak{D})^2 |n_\mathfrak{D}|^2 |E_\mathfrak{D} \psi| \leq M |\pi(z_{0} w^s)\psi| \quad (\mathfrak{D} \in \Omega_\mathfrak{I}),$$

and therefore

$$\sum_{\mathfrak{D} \in \Omega_\mathfrak{I}} |E_\mathfrak{D} \psi| \leq \left( \sum_{\mathfrak{D} \in \Omega_\mathfrak{I}} d(\mathfrak{D})^{-2q} |n_\mathfrak{D}|^{-2s} \right) M |\pi(z_{0} w^s)\psi|.$$

For any $\mathfrak{D} \in \Omega$, let $\mathfrak{D}'$ denote the class of representations of $\mathfrak{l}'$ defined as follows. If $\sigma \in \mathfrak{D}$, $\mathfrak{D}'$ is the class of the restriction of $\sigma$ on $\mathfrak{l}'$. Clearly $\mathfrak{D}'$ is irreducible and $d(\mathfrak{D}') = d(\mathfrak{D})$. Moreover, $\mathfrak{D}$ is completely determined by $\mathfrak{D}'$ and $n_\mathfrak{D}$. Hence

$$\sum_{\mathfrak{D} \in \Omega_\mathfrak{I}} d(\mathfrak{D})^{-2q} |n_\mathfrak{D}|^{-2s} \leq \sum_{\mathfrak{D}'} d(\mathfrak{D}')^{-2q} \sum_{n_1, \ldots, n_r} (1 + n_1^2 + \cdots + n_r^2)^{-s}$$

where $\mathfrak{D}'$ runs over all irreducible classes of finite-dimensional representations of $\mathfrak{l}'$. But if $2q$ exceeds the rank of $\mathfrak{l}'$ it follows from Lemma 2 that

$$\sum_{\mathfrak{D}'} d(\mathfrak{D}')^{-2q} < \infty.$$ 

Similarly if $2s > r$

$$\sum_{n} (1 + n_1^2 + \cdots + n_r^2)^{-s} \leq \sum_{n} (1 + n_1^2 + \cdots + n_r^2)^{-(r+1)/2} < \infty.$$

(\#) Conjugate $x$ means conjugate of $x$. 

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
Therefore if we choose \( q \) and \( s \) sufficiently large and put

\[
z = N \xi^{q+1} \psi
\]

where

\[
N = M \sum_{\mathfrak{D} \in \Omega} d(\mathfrak{D})^{-2q} \cdot n_{\mathfrak{D}}^{-2s},
\]

This proves the first assertion of the lemma. Now we come to the second part. Since \( \sum_{\mathfrak{D} \in \Omega} |E_{\mathfrak{D}}\psi| < \infty \) the series \( \sum_{\mathfrak{D} \in \Omega} E_{\mathfrak{D}}\psi \) is convergent. Let \( \phi \) denote its sum. We have to show that \( \phi = \psi \). Put \( \psi' = \psi - \phi \). Since \( E_{\mathfrak{D}}\phi = E_{\mathfrak{D}}\psi, E_{\mathfrak{D}}\psi' = 0 \). From this we shall deduce that \( \psi' = 0 \).

Suppose \( \psi' \neq 0 \). Then given any real \( \epsilon > 0 \) choose a continuous real non-negative function \( f \) on \( K^* \) such that \( f(u^*) = 0 \) if \( |\pi^*(u^*)\psi' - \psi'| > \epsilon |\psi'| \) \((u^* \in K^*)\) and \( \int_{K^*} f(u^*) du^* = 1 \). Moreover choose a finite linear combination \( \omega \) of the matrix coefficients of finite-dimensional simple representations of \( K^* \) such that \( |f(u^*) - \omega(u^*)| \leq \epsilon \) \((u^* \in K^*)\). Then if

\[
\psi'' = \int \omega(u^*)\pi^*(u^*)\psi' du^*,
\]

\( \psi'' \in \sum_{\mathfrak{D} \in \Omega} \mathfrak{D}_{\mathfrak{D}} \) and therefore \( \psi'' = \sum_{\mathfrak{D} \in \Omega} E_{\mathfrak{D}}\psi'' \). But

\[
E_{\mathfrak{D}}\psi'' = \int \omega(u^*)\pi^*(u^*)E_{\mathfrak{D}}\psi' du^* = 0
\]

since \( E_{\mathfrak{D}}\psi' = 0 \). Hence \( \psi'' = 0 \). On the other hand

\[
|\psi'' - \psi'| \leq \int |\omega(u^*) - f(u^*)||\pi^*(u^*)\psi'| du^*
\]

\[
+ \int f(u^*)|\pi^*(u^*)\psi' - \psi'| du^*
\]

\[
\leq M \epsilon |\psi'| + \epsilon |\psi'|
\]

where \( M = \sup_{u^* \in K^*} |\pi^*(u^*)| \). Therefore if \( \epsilon \) is sufficiently small

\[
|\psi'| = |\psi'' - \psi'| \leq |\psi'| / 2
\]

which contradicts our assumption that \( \psi' \neq 0 \). Therefore \( \psi' = 0 \) and so \( \sum_{\mathfrak{D} \in \Omega} E_{\mathfrak{D}}\psi \) converges to \( \psi \).

5. Characters. Let \( C_c^\infty(G) \) denote the class of all complex-valued functions on \( G \) which are indefinitely differentiable everywhere and which vanish outside a compact set. Let \( \pi \) be a quasi-simple irreducible representation of \( G \) on a Hilbert space \( \mathfrak{E} \). For any \( f \in C_c^\infty(G) \) consider the operator
\[ T_f = \int f(x)\pi(x)\,dx \]

where \( dx \) is the Haar measure on \( G \). We intend to show that \( T_f \) is of the trace class.

Let \( \mathcal{S}' \) be the Banach space of all bounded linear operators \( A \) on \( \mathcal{S} \) with the usual norm \( |A| = \sup_{\psi \in \mathcal{S}} |A\psi| \) (\( \psi \in \mathcal{S} \)). Let \( \mathcal{S}_0 \) be the subspace of \( \mathcal{S}' \) consisting of all operators of the form \( T_f \) (\( f \in C_c^\infty(G) \)). We denote by \( \mathcal{S} \) the closure of \( \mathcal{S}_0 \) in \( \mathcal{S}' \). Now if \( y \in G \),

\[ \pi(y)T_f = \int f(y^{-1}x)\pi(x)\,dx, \quad T_f\pi(y^{-1}) = \int f(xy)\pi(x)\,dx. \]

Hence it follows that if \( A \in \mathcal{S} \) then \( \pi(y)A \) and \( A\pi(y^{-1}) \) are also in \( \mathcal{S} \). We now define two representations \( l \) and \( r \) of \( G \) on \( \mathcal{S} \) as follows:

\[ l(x)A = \pi(x)A, \quad r(x)A = A\pi(x^{-1}) \quad (x \in G, A \in \mathcal{S}). \]

In order to verify the conditions for continuity it is sufficient to prove that \( \lim_{x \to y, y \to 1} |\pi(x)A\pi(y^{-1}) - A| = 0 \) (\( A \in \mathcal{S} \)). This is done as follows. Given \( \epsilon > 0 \), choose \( f \in C_c^\infty(G) \) such that \( |A - T_f| \leq \epsilon \). Let \( U = U^{-1} \) be a compact neighbourhood of 1 in \( G \) and \( M \) an upper bound for \( |\pi(z)| \) for \( z \in U \). Then

\[ |\pi(x)A\pi(y^{-1}) - \pi(x)T_f\pi(y^{-1})| \leq M^2\epsilon \quad (x, y \in U) \] and therefore

\[ |\pi(x)A\pi(y^{-1}) - A| \leq (M^2 + 1)\epsilon + |\pi(x)T_f\pi(y^{-1}) - T_f| \]

\[ \leq (M^2 + 1)\epsilon + \int |f(x^{-1}xy) - f(x)| |\pi(z)| \,dz. \]

Let \( C \) be a compact set outside which \( f \) is zero. We can choose a neighbourhood \( V \) of 1 in \( G \) (\( V \subset U \)) such that \( |f(x^{-1}xy) - f(x)| \leq \epsilon \) if \( x, y \in V \). Let \( F \) be a real nonnegative continuous function on \( G \) which is equal to 1 on \( U \) and which vanishes outside some compact set. Then if \( N_0 = \sup_{x \in U \setminus V} |\pi(z)| \),

\[ |\pi(x)A\pi(y^{-1}) - A| \leq (M^2 + 1)\epsilon + N_0\epsilon \int F(z)\,dz \]

provided \( x, y \in V \). This proves that \( \lim_{x \to 1, y \to 1} |\pi(x)A\pi(y^{-1}) - A| = 0 \).

Since \( l(x)T_f = \int f(x^{-1}z)\pi(z)\,dz \), it follows easily that \( T_f \) is differentiable under \( l \). Similarly we show that it is differentiable under \( r \). It is clear that the representations \( l \) and \( r \) are permissible. For any \( \mathcal{D} \in \mathcal{S} \) let \( E_\mathcal{D}, P_\mathcal{D}, \) and \( Q_\mathcal{D} \) denote the canonical projections (see §9 of [6]) corresponding to \( \mathcal{D} \) under \( \pi, l, \) and \( r \) respectively. Then it is clear \( P_\mathcal{D}A = E_\mathcal{D}A; Q_\mathcal{D}A = A E_\mathcal{D} \) (\( \mathcal{D} \in \mathcal{S} \)) where \( \mathcal{S}' \) is the class contragredient to \( \mathcal{S} \). Let \( \lambda \) and \( \rho \) denote the left and right regular representations of \( G \). Then every element in \( C_c^\infty(G) \) is differentiable under both \( \lambda \) and \( \rho \) and \( C_c^\infty(G) \) is invariant under \( \lambda(\mathcal{B}) \) and \( \rho(\mathcal{B}) \). Moreover
since \( l(x)r(y)T_f = T_{\lambda(x)\rho(y)}f \) \((x, y \in G\) it follows easily that \( l(a)r(b)T_f = T_{\lambda(a)\rho(b)}f \) \((a, b \in \mathbb{B}\)).

Now define a representation \( \phi \) of the group \( G \times G \) on \( \mathcal{S} \) as follows. \( \phi(x, y)A = l(x)r(y)A = \pi(x)A\pi(y^{-1}) \) \((x, y \in G\)\). Then \( \phi \) is a permissible representation of the semisimple group \( G \times G \) and any element of \( \mathcal{S}_0 \) is differentiable under \( \phi \). Moreover the canonical projections for the representation \( \phi \) (with respect to the subgroup \( K \times K \)) are exactly the operators \( P_{\mathcal{D}_1}(Q_{\mathcal{D}_2} \{ \xi_1, \xi_2 \in \mathcal{D} \}) \). Let \( z_0 \) be the element of \( \mathcal{X} \) which was introduced in the proof of Lemma 3. Then if we apply Lemma 3 to the representation \( \phi \) and the differentiable element \( T_{\lambda(z_0)\rho(z_0)} \) we find that

\[
\sum_{\mathcal{D}_1, \mathcal{D}_2 \in \mathcal{L}} | P_{\mathcal{D}_1}Q_{\mathcal{D}_2}T_{\lambda(z_0)\rho(z_0)} | < \infty.
\]

But

\[
P_{\mathcal{D}_1}Q_{\mathcal{D}_2}T_{\lambda(z_0)\rho(z_0)} = P_{\mathcal{D}_1}Q_{\mathcal{D}_2}l(z_0)r(z_0)T_f = d(\mathcal{D}_1)^2d(\mathcal{D}_2)^2P_{\mathcal{D}_1}Q_{\mathcal{D}_2}T_f.
\]

Hence

\[
\sum_{\mathcal{D}_1, \mathcal{D}_2 \in \mathcal{L}} d(\mathcal{D}_1)^2d(\mathcal{D}_2)^2 | E_{\mathcal{D}_1}T_fE_{\mathcal{D}_2} | < \infty.
\]

Now let us first suppose that the subspaces \( \mathcal{S}_\mathcal{D} = E_\mathcal{D} \mathcal{S} \) \((\mathcal{D} \in \mathcal{L})\) are mutually orthogonal. Choose an orthonormal base for each \( \mathcal{S}_\mathcal{D} \). All these put together form an orthonormal base \( \{ \psi_x \}_{x \in J} \) for \( \mathcal{S} \). In accordance with Theorem 4 we choose an integer \( N \) such that \( \dim \mathcal{S}_\mathcal{D} \leq Nd(\mathcal{D})^2 \) \((\mathcal{D} \in \mathcal{L})\). Then

\[
\sum_{a, b \in J} | (\psi_a, T_f\psi_b) | = \sum_{\mathcal{D}_1, \mathcal{D}_2 \in \mathcal{L}} \sum_{a \in J_{\mathcal{D}_1}} \sum_{b \in J_{\mathcal{D}_2}} | (\psi_a, T_f\psi_b) |
\]

where \( J_\mathcal{D} \) is the subset of \( J \) such that \( \{ \psi_x \}_{x \in J_\mathcal{D}} \) is a base for \( \mathcal{S}_\mathcal{D} \). But it is clear that

\[
\sum_{a \in J_{\mathcal{D}_1}} \sum_{b \in J_{\mathcal{D}_2}} | (\psi_a, T_f\psi_b) | \leq \dim \mathcal{S}_{\mathcal{D}_1} \dim \mathcal{S}_{\mathcal{D}_2} | E_{\mathcal{D}_1}T_fE_{\mathcal{D}_2} | = \leq N^2d(\mathcal{D}_1)^2d(\mathcal{D}_2)^2 | E_{\mathcal{D}_1}T_fE_{\mathcal{D}_2} |.
\]

Hence

\[
\sum_{a, b \in J} | \psi_a, T_f\psi_b | \leq N^2 \sum_{\mathcal{D}_1, \mathcal{D}_2 \in \mathcal{L}} d(\mathcal{D}_1)^2d(\mathcal{D}_2)^2 | E_{\mathcal{D}_1}T_fE_{\mathcal{D}_2} | < \infty
\]

and therefore, from Lemma 1, \( T_f \) is of the trace class.

Now we discard the assumption about the mutual orthogonality of the spaces \( \mathcal{S}_\mathcal{D} \). Let \( x \to x^* \) denote the natural mapping of \( G \) on \( G^* = G/D \cap Z \). Define a representation \( \pi^* \) of \( K^* \) on \( \mathcal{S} \) as in the proof of Lemma 3. Since \( K^* \) is a compact group, \( \pi^* \) is equivalent to a unitary representation. Hence there exists a regular operator \( B \) on \( \mathcal{S} \) such that the representation \( \pi' \) : \( u \to B\pi^*(u^*)B^{-1} \) \((u^* \in K^*)\) is unitary. Now put \( \pi'(x) = B\pi(x)B^{-1} \). Then \( \pi' \) is a
representation of $G$ on $H$. Let $H_\omega$ be the subspace of $H$ consisting of all elements which transform under $\pi'(K)$ according to $\omega$ ($\omega \in \Omega$). Then since $\pi'^*$ is unitary the spaces $H_\omega$ are mutually orthogonal. Therefore the above proof is applicable to

$$T_f' = \int f(x)\pi'(x)dx = BT_fB^{-1}. $$

Hence $T_f'$ fulfills the condition of Lemma 1 and therefore $T_f = B^{-1}T_fB$ is of the trace class. Moreover if $P$ and $Q$ are two bounded operators on $H$, then $PB^{-1}$ and $BQ$ are also bounded and $PT_fQ = (PB^{-1})T_f(BQ)$. Therefore, from Lemma 1, $PT_fQ$, $T_fQP$, $QPT_f$ are all of the trace class and their traces are equal. Therefore in particular $\text{sp} (AT_fA^{-1}) = \text{sp} T_f$ if $A$ is a regular operator. 

Put $T_\omega(f) = \text{sp}(T_f)$ for any $f \in C_\omega^*(G)$. Then $T_\omega$ is a linear function on the vector space $C_\omega^*(G)$. Furthermore if $a$ is a fixed element in $G$ and $g$ is the function $g(x) = f(axa^{-1})$ ($x \in G$) then

$$T_\omega = \pi(a^{-1})T_f\pi(a) $$

and therefore $T_\omega(g) = T_\omega(f)$. Hence we may say that $T_\omega$ is invariant under the inner automorphisms of $G$. We prove similarly that if $\pi$ and $\pi'$ are equivalent representations, $T_\omega = T'_{\omega'}$. 

Our next object is to show that $T_\omega$ is actually a distribution in the sense of L. Schwartz [9]. By going over to an equivalent representation, if necessary, we may assume that the spaces $H_\omega$ ($\omega \in \Omega$) are mutually orthogonal. Then it is clear that

$$\text{sp} T_f = \sum_{\omega \in \Omega} \text{sp} (E_\omega T_f E_\omega) $$

and

$$\text{sp} (E_\omega T_f E_\omega) = \sum_{i=1}^d (\psi_i, E_\omega T_f \psi_i) $$

where $(\psi_1, \cdots, \psi_d)$ is an orthonormal base of $H_\omega$. Therefore

$$| \text{sp} (E_\omega T_f E_\omega) | \leq \text{dim } H_\omega \ | E_\omega T_f | \leq N d(\omega)^2 \ | E_\omega T_f | $$

and

$$| T_\omega(f) | \leq N \sum_{\omega \in \Omega} d(\omega)^2 \ | E_\omega T_f | = N \sum_{\omega \in \Omega} | E_\omega T_{\lambda(z_0)/f} | $$

where $z_0$ has the same meaning as above. Now by applying Lemma 3 to the representation $l$ of $G$ on $H$ and the differentiable element $T_{\lambda(z_0)/f}$ we conclude that

$$\sum_{\omega \in \Omega} | E_\omega T_{\lambda(z_0)/f} | \leq | l(z) T_{\lambda(z_0)/f} | = | T_{\lambda(z_0)/f} | .$$
where \( z \) is an element of \( \mathfrak{x} \) (which does not depend on \( f \)). Hence

\[
| T_\pi(f) | \leq N | T_{\lambda(zx_0)}f |.
\]

Now suppose \( C \) is a compact set in \( G \) and \( f_n \) is a sequence of functions in \( C_c^\infty(G) \) such that \( f_n \) vanishes outside \( C \) and for any \( b \in \mathfrak{g}, \lambda(b)f_n \to 0 \) uniformly on \( C \). Then

\[
| T_{\lambda(b)}f_n | \leq \int | (\lambda(b)f_n)(x) | | \pi(x) | dx \to 0
\]

and therefore in particular \( | T_\pi(f_n) | \leq N | T_{\lambda(zx_0)f_n} | \to 0 \). This proves that \( T_\pi \) is a distribution.

6. Operators of the Hilbert-Schmidt class. Let \( B \) be a bounded operator on the Hilbert space \( \mathcal{H} \) and let \( B^* \) be the adjoint of \( B \). We say that \( B \) is of the Hilbert-Schmidt (H.S.) class if \( B^*B \) has a trace. Let \( \{ \phi_\sigma \}_{\sigma \in J} \) be an orthonormal base for \( \mathcal{H} \). Then it is well known that \( \| B \|^2 = \sum_{\sigma \in J} \| B\phi_\sigma \|^2 \) is independent of the choice of this base and \( B \) is of the H.S. class if and only if \( \| B \| < \infty \). Moreover \( \mathrm{sp} \ BB^* = \| B \|^2 = \| B^* \|^2 \) if \( \| B \| < \infty \) and \( \| A_1BA_2 \| \leq \| A_1 \| \| B \| \| A_2 \| \) for any two bounded operators \( A_1, A_2 \).

Let \( \pi \) be a quasi-simple irreducible representation of \( G \) on \( \mathcal{H} \). Let \( f \) be a complex-valued measurable function on \( G \) which vanishes outside a compact set and such that \( \int | f(x) |^2 dx < \infty \). It follows from the Schwartz inequality that \( \int | f(x) | dx < \infty \) and therefore the operator \( \int f(x) \pi(x) dx \) is a well-defined bounded operator. We intend to prove that this operator is of the H.S. class. Let \( S \) be a regular operator on \( \mathcal{H} \). Put \( \pi'(x) = S \pi(x) S^{-1} \) \( (x \in G) \). Then

\[
\left| \int f(x) \pi(x) dx \right| = \left| S^{-1} \int f(x) \pi'(x) dx S \right| \leq | S^{-1} | \left| \int f(x) \pi(x) dx \right| | S |.
\]

Therefore it is enough to show that the corresponding operator for an equivalent representation is of the H.S. class.

Let \( x \to x^* \) denote the natural mapping of \( G \) on \( G^* = G/D \cap Z \). For any \( x \in G \) we define \( \Gamma(x) \) to be the unique element in \( \mathfrak{c}_0 \) such that \( x = u(\exp \Gamma(x))s \) where \( u \in K' \) and \( s \) lies in the solvable subgroup of \( G \) corresponding to the subalgebra \( \mathfrak{g}_0 \land (\mathfrak{h}_b + \mathfrak{n}) \) of \( \mathfrak{g}_0 \) (see [6, §9]). Then if \( \mu \) is the linear function on \( \mathfrak{c} \) which was introduced in the proof of Lemma 3, it is clear that \( \pi(x)e^{-\mu(\Gamma(x))} \) depends only on \( x^* \). Put \( \pi^*(x^*) = \pi(x)e^{-\mu(\Gamma(x))} \). Then we verify immediately that \( \pi^*(u^*x^*) = \pi^*(u^*)\pi^*(x^*) \) \( (u^* \in K^*, x^* \in G^*) \) and therefore \( u^* \to \pi^*(u^*) \) is a representation of \( K^* \). In view of the preceding remarks we may assume without loss of generality that this representation is unitary.

Let \( x^* \in G^* \) and \( y \in G \). We say that \( y \) lies above \( x^* \) and write \( y > x^* \) if \( (y)^* = x^* \). Put

\[
f^*(x^*) = \sum_{\alpha \in \mathfrak{x}} e^{\mu(\Gamma(x))} f(x) \quad (x^* \in G^*).
\]
Let $A$ be a compact set outside which $f$ is zero. Since $D \cap Z$ is discrete, $(D \cap Z) \cap A^{-1} A$ is a finite set. Let $N_0$ be the number of elements in it. Then it is clear that not more than $N_0$ distinct elements in $A$ can lie above the same element in $G^*$. Hence at most $N_0$ terms in the above sum are different from zero and therefore the function $f^*$ is well-defined. Moreover if $A^*$ is the image of $A$ in $G^*$, $f^*$ is zero outside $A^*$. Now let $x^* \in A^*$. Then

$$|f^*(x^*)| = \left| \sum_{z \in A} e^{i \phi(z)} f(x) \right| \leq \left( \sum_{z \in A} |f(x)|^2 \right)^{1/2} \left( \sum_{z \in A} |e^{i \phi(z)}|^2 \right)^{1/2}$$

$$\leq M_0 N_0^{1/2} \left( \sum_{z \in A} |f(x)|^2 \right)^{1/2}$$

where $M_0 = \sup_{z \in A} |e^{i \phi(z)}|$. Hence if the Haar measure $dx^*$ on $G^*$ is suitably normalised it follows that

$$\int f(x) \pi(x) dx = \int f^*(x^*) \pi^*(x^*) dx^*$$

and

$$\int |f^*(x^*)|^2 dx^* \leq M_0^2 N_0 \int |f(x)|^2 dx < \infty.$$ 

Let $B^*$ be a compact neighbourhood of $A^*$. Choose a real-valued non-negative function $F$ on $G^*$ such that $F=1$ on $K^*B^*$ and $F=0$ outside some compact set. Let $g$ be any continuous function on $G^*$ which vanishes outside $B^*$. Consider the operator

$$\int g(x^*) \pi(x^*) dx^* = \int du^* \int g(u^* x^*) \pi^*(u^* x^*) dx^*.$$ 

(Here $du^*$ is the normalised Haar measure on $K^*$ so that $\int du^* = 1$.) Then

$$\left\| \int g(x^*) \pi^*(x^*) dx^* \right\| \leq \int dx^* \int g(u^* x^*) \pi^*(u^* x^*) du^*.$$ 

But

$$\left\| \int g(u^* x^*) \pi^*(u^* x^*) du^* \right\| \leq \int g(u^* x^*) \pi^*(u^*) du^* \left| \pi^*(x) \right|$$

and from Theorem 4 we can find an integer $N$ such that $\dim \mathfrak{H}_D \leq N d(\mathfrak{D})^2$ for any $\mathfrak{D} \subseteq \Omega$. Let $\Omega^*$ be the set of all classes of irreducible finite-dimensional representations of $K^*$. Then no $\pi^* \in \Omega^*$ occurs more than $N d(\mathfrak{D}^*)$ times in the reduction of $\pi^*(K^*)$. Since every $\mathfrak{D}^* \subseteq \Omega^*$ occurs exactly $d(\mathfrak{D}^*)$ times in the left regular representation $\lambda$ of $K^*$ (on the Hilbert space $L_2(K^*)$ of all
square integrable functions on $K^*$ and since the representation $u^* \mapsto \pi^*(u^*)$ ($u^* \in K^*$) is unitary, we may conclude that

$$\left\| \int g(u^*x^*) \pi^*(u^*) du^* \right\|^2 \leq N \left\| \int g(u^*x^*) \lambda(u^*) du^* \right\|^2.$$

But from the Peter-Weyl theorem we know that

$$\int g(u^*x^*) \pi^*(u^*) du^* = \int g(u^*x^*)^2 du^*.$$

Hence

$$\left\| \int g(u^*x^*) \pi^*(u^*) du^* \right\|^2 \leq N^{1/2} \left( \int \left| g(u^*x^*)^2 du^* \right)^{1/2} \right)^2.$$

Now it is easy to see that $|\pi^*(x^*)|$ is bounded on the compact set $K^*B^*$. Let $M = \sup_{x^* \in K^*n^*} |\pi^*(x^*)|$. Then

$$\left\| \int g(u^*x^*) \pi^*(u^*) du^* \right\|^2 \leq MN^{1/2} \left( \int \left| g(u^*x^*)^2 du^* \right)^{1/2} \right)^2$$

and therefore

$$\left\| \int g(x^*) \pi^*(x^*) dx^* \right\|^2 \leq MN^{1/2} \left( \int \left| g(u^*x^*)^2 du^* \right)^{1/2} \right)^2$$

$$= MN^{1/2} \int F(x^*) dx^* \left( \int \left| g(u^*x^*)^2 du^* \right)^{1/2} \right)^2$$

$$\leq M_1 \left( \int \left| g(x^*)^2 dx^* \right)^{1/2} \right)^2,$$

where $M_1 = MN^{1/2} \left( \int \left| F(x^*)^2 dx^* \right)^{1/2} \right)^{1/2}$. Now choose a sequence $g_n$ of continuous functions on $G^*$ which vanish outside $B^*$ and such that $\int \left| f^*(x^*) - g_n(x^*) \right|^2 dx^* \to 0$. Then since

$$\int \left| f^*(x^*) - g_n(x^*) \right| dx^*$$

$$\leq \left( \int \left| F(x^*)^2 dx^* \right)^{1/2} \left( \int \left| f^*(x^*) - g_n(x^*) \right|^2 dx^* \right)^{1/2}$$

it follows that

$$\int (f^*(x^*) - g_n(x^*)) \pi^*(x^*) dx^* \to 0.$$

Moreover we have seen above that
and therefore the sequence of operators $T_n = \int g_n(x^*) \pi^*(x^*) dx^*$ is a Cauchy sequence with respect to the Hilbert-Schmidt norm $\| \cdot \|$. Since the space of operators of the H.S. class is complete with respect to this norm, there exists an operator $T$ of this class such that $\| T_n - T \| \to 0$. But $| T_n - T | \leq \| T_n - T \|$. Hence $| T_n - T | \to 0$. However we have seen already that

$$\left| T_n - \int f(x^*) \pi^*(x^*) dx^* \right| \to 0$$

and therefore $T = \int f(x^*) \pi^*(x^*) dx^*$. This proves that $T = \int f(x) \pi(x) dx$ is of the H.S. class. Thus we have the following theorem(10).

**Theorem 5.** Let $\pi$ be a quasi-simple irreducible representation of $G$ on a Hilbert space $\mathcal{H}$ and let $f$ be a measurable and square integrable function on $G$ which vanishes outside a compact set. Then the operator $\int f(x) \pi(x) dx$ is of the Hilbert-Schmidt class.

7. **Linear independence of characters.** Let $T_\pi$ be the character of a quasi-simple irreducible representation $\pi$ of $G$ on a Hilbert space $\mathcal{H}$. Let $E_\mathcal{D}$ denote the canonical projection of $\mathcal{H}$ on $\mathcal{H}_\mathcal{D}$ ($\mathcal{D} \subseteq \Omega$). Then it follows easily from its definition (see §5) that

$$T_\pi(f) = \text{sp} \left( \int f(x) \pi(x) dx \right) = \sum_{\mathcal{D} \subseteq \Omega} \text{sp} \left( E_\mathcal{D} \int f(x) \pi(x) dx \cdot E_\mathcal{D} \right)$$

$$= \sum_{\mathcal{D} \subseteq \Omega} \int f(x) \phi_\mathcal{D}(x) dx$$

in the notation of §1. Now if $\pi_1$, $\pi_2$ are two infinitesimally equivalent representations, we have seen in §1 that $\phi_\mathcal{D} = \phi_\mathcal{D}_2$ ($\mathcal{D} \subseteq \Omega$) and therefore $T_{\pi_1} = T_{\pi_2}$. Hence two infinitesimally equivalent quasi-simple irreducible representations (on Hilbert spaces) have the same character. Conversely we shall show that two such representations having the same character are infinitesimally equivalent(11).

**Theorem 6.** Let $\pi_1, \cdots, \pi_q$ be a finite set of quasi-simple irreducible representations of $G$ on the Hilbert spaces $\mathcal{H}_1, \cdots, \mathcal{H}_q$ respectively. Suppose no two of them are infinitesimally equivalent. Then their characters $T_{\pi_1}, \cdots, T_{\pi_q}$ are linearly independent.

For otherwise suppose $c_1 T_{\pi_1} + \cdots + c_q T_{\pi_q} = 0$ ($c_i \in \mathbb{C}$) where, say, $c_1 \neq 0$. 

(10) Cf. Theorem 4 of [8(c)].

(11) Cf. Theorem 3 of [8(c)].
Let $\eta_i$ be the homomorphism of $D \cap Z$ into $C$ such that $\pi_i(\gamma) = \eta_i(\gamma)\pi_i(1)$ ($\gamma \in D \cap Z$). Then if $f \in C^a_\pi(G)$ and $\gamma \in D \cap Z$, the function $f_\gamma: x \to f(\gamma^{-1}x)$ ($x \in G$) is also in $C^a_\pi(G)$ and it is obvious that $T_{\pi_i}(f_\gamma) = \eta_i(\gamma)T_{\pi_i}(f)$. Therefore

$$\sum_{i=1}^{q} c_iT_{\pi_i}(f_\gamma) = \sum_{i=1}^{q} c_i\eta_i(\gamma)T_{\pi_i}(f) = 0.$$

This proves that

$$\sum_{i=1}^{q} c_i\eta_i(\gamma)T_{\pi_i} = 0$$

for all $\gamma \in D \cap Z$. Now if we recall that $D \cap Z$ is a free abelian group with $r$ generators ($r = \dim_R C_0$) we can conclude that $c_iT_{\pi_i} + \cdots + c_qT_{\pi_q} = 0$ assuming that $\eta_i = \eta_1$ ($1 \leq j \leq s$) and $\eta_j \neq \eta_i$ for $s < j \leq q$.

Choose a base $\Gamma_1, \cdots, \Gamma_r$ for $C_0$ over $R$ such that $\exp \Gamma_i$, $1 \leq i \leq r$, is a set of generators for $D \cap Z$. Select a linear function $\mu$ on $C$ such that $\eta_i(\exp \Gamma_i) = e^{i\mu(\Gamma_i)}$, $1 \leq i \leq r$. Put $f_\pi(x^*) = e^{-\mu(\Gamma(x^*)))\pi_i(x)} (x \in G, 1 \leq j \leq s)$ in the notation of §6. Let $\mathfrak{D}$ be a class in $\Omega$ which occurs in $\pi_1$. We denote by $E^i_\mathfrak{D}$ the canonical projection of $\mathfrak{D}_1$ on $\mathfrak{D}_i,\mathfrak{D}$. Then

$$E^i_\mathfrak{D} = d(\mathfrak{D}) \int \text{conj } (\xi_{\mathfrak{D}_1}(u^*))\pi_i(u^*)du^*$$

where $\mathfrak{D}^*$ is the irreducible class according to which every element in $\mathfrak{D}_1,\mathfrak{D}$ transforms under $\pi_1(K^*)$ and $\xi_{\mathfrak{D}_1}$ is the character of $\mathfrak{D}_1$. Let $K_0$ be the set of all elements in $K$ of the form $(\exp \Gamma)v$ where $\Gamma = t_1\Gamma_1 + \cdots + t_r\Gamma_r$ ($t_j \in R$, $|t_j| \leq 1/2$) and $v \in K'$. Then $K_0$ is compact and if we put

$$\xi_{\mathfrak{D}_1}(u) = e^{-\text{conj } (\mu(\Gamma^*(u)))\xi_{\mathfrak{D}^*}(u^*)} \quad (u \in K),$$

we get

$$E^i_\mathfrak{D} = d(\mathfrak{D}) \int_{K_0} \text{conj } (\xi_{\mathfrak{D}_1}(u))\pi_i(u)du$$

where the Haar measure $du$ on $K$ is so normalised that $\int_{K_0}du = 1$.

Now we use the notation of Theorem 1. Put

$$\phi = c_1\phi^r_\mathfrak{D}_1 + \cdots + c_s\phi^r_\mathfrak{D}_s.$$  

It follows from Theorem 1 that $\phi \neq 0$. Since $\phi$ is continuous we can find a function $f \in C^a_\pi(G)$ such that $\int f(x)\phi(x)dx \neq 0$. Now

$$E^i_\mathfrak{D} \int f(x)\pi_i(x)dx = \int f_\mathfrak{D}(x)\pi_i(x)dx \quad (1 \leq i \leq s)$$

where

$$f_\mathfrak{D}(x) = d(\mathfrak{D}) \int_{K_0} \text{conj } (\xi_{\mathfrak{D}_1}(u))f(u^{-1}x)du.$$  

Since $K_0$ is compact it is clear that $f_\mathfrak{D} \in C^a_\pi(G)$. On the other hand

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
\[
\text{sp} \left( \int f(x) \pi_i(x) dx \right) = \text{sp} \left( E_2^i \int f(x) \pi_i(x) dx \right) = \text{sp} \left( E_2^i \int f(x) \pi_i(x) dx \cdot E_2^i \right) = \int f(x) \phi_2^i(x) dx.
\]

Therefore \( c_1 T_{\pi_1}(f_2) + \cdots + c_s T_{\pi_s}(f_2) = \int f(x) \phi(x) dx \neq 0 \). This however implies that \( c_1 T_{\pi_1} + \cdots + c_s T_{\pi_s} \neq 0 \) and so we get a contradiction.

**Corollary.** Two irreducible unitary representations are equivalent if and only if their characters are the same.

First of all every irreducible unitary representation is quasi-simple (see for example Segal [10]). Moreover infinitesimal equivalence is the same as ordinary equivalence for two such representations (see Theorem 8 of [6]). Hence the corollary is an immediate consequence of the theorem.

### 8. Complex semisimple groups

Suppose the group \( G \) is complex. Then \( K \) is semisimple and there exists a 1-1 linear mapping \( i \) of \( \mathfrak{f}_0 \) onto \( \mathfrak{p}_0 \) such that

\[
[X, iY] = i([X, Y]), \quad [iX, iY] = -[X, Y] \quad (X, Y \in \mathfrak{f}_0).
\]

Let \( \mathfrak{h}_0 \) be a maximal abelian subalgebra of \( \mathfrak{f}_0 \). Then \( i(\mathfrak{h}_0) \) is clearly a maximal abelian subspace of \( \mathfrak{p}_0 \). Hence we may take \( \mathfrak{h}_0 = i(\mathfrak{h}_0) \). Then \( \mathfrak{h}_0 + \mathfrak{h}_0 \) is a maximal abelian subalgebra of \( \mathfrak{g}_0 \). Let \( \mathfrak{h}_0 \) and \( \mathfrak{h}_0 \) be the subspaces of \( \mathfrak{g} \) spanned by \( \mathfrak{h}_0 \) and \( \mathfrak{h}_0 \) over \( C \). Then \( \mathfrak{h} = \mathfrak{h}_0 + \mathfrak{h}_0 \) is a Cartan subalgebra of \( \mathfrak{g} \). We extend \( i \) to a mapping of \( \mathfrak{f} \) into \( \mathfrak{p} \) by linearity.

Let \( \alpha_1, \cdots, \alpha_p \) be a maximal set of linearly independent roots of \( \mathfrak{f} \) (with respect to \( \mathfrak{h}_0 \)). We order all roots \( \alpha \) of \( \mathfrak{f} \) lexicographically with respect to this set (see [5, Part I]). For every root \( \alpha \) let \( H_\alpha \) be the element in \( \mathfrak{h}_0 \) such that \( \text{sp} (\text{ad}' H \text{ ad}' H_\alpha) = \alpha(H) \) \((H \in \mathfrak{h}_0)\) where \( X \to \text{ad}' X \) \((X \in \mathfrak{f})\) is the adjoint representation of \( \mathfrak{f} \). We denote by \( W \) the Weyl group of \( \mathfrak{f} \) and by \( 2\sigma \) the sum of all positive roots of \( \mathfrak{f} \). Let \( \lambda \) be a linear function on \( \mathfrak{h}_0 \). We put \( \lambda' = \lambda + \sigma \) and use the notation of [5, Part III]. We know (see [5, p. 70]) that the power series \( \sum_{\alpha \in \mathfrak{w}} e(s) e^{\lambda'(H_\alpha)} \) is divisible by \( \prod_{\alpha > 0} \lambda'(H_\alpha) \prod_{\alpha > 0} \alpha(H) \) \((H \in \mathfrak{h}_0)\) and therefore the quotient

\[
\frac{\prod_{\alpha \in \mathfrak{w}} e(s) e^{\lambda'(sH)}}{\prod_{\alpha > 0} \lambda'(H_\alpha) \prod_{\alpha > 0} \alpha(H)}
\]

is an analytic function on \( \mathfrak{h}_0 \). Similarly

\[
\frac{\prod_{\alpha > 0} \alpha(H)}{\prod_{\alpha > 0} (e^{\alpha(H)/2} - e^{-\alpha(H)/2})}
\]
is a meromorphic function on \( \mathfrak{h}_t \) all whose singularities lie on hyperplanes of the form \( \alpha(H) = 2\pi(-1)^{1/2} \) where \( \alpha \) is a root and \( n \) is some nonzero integer. Hence the function

\[
\Phi^*(\lambda, H) = \prod_{\alpha > 0} \sigma(H_{\alpha}) \frac{\sum_{t \in W} \epsilon(s) e^{\lambda'\langle sH \rangle}}{\prod_{\alpha > 0} \lambda'(H_{\alpha}) \prod_{\alpha > 0} \alpha(H)} \prod_{\alpha > 0} \left( e^{\alpha(H)/2} - e^{-\alpha(H)/2} \right)
\]

is a meromorphic function on \( \mathfrak{h}_t \) and it is analytic everywhere on \( (-1)^{1/2}\mathfrak{h}_t \) and also on a suitable neighbourhood of zero in \( \mathfrak{h}_t \). We know from [5, Part III, p. 71] that if \( H \in \mathfrak{h}_t \) and \( |t| \) is sufficiently small \( (t \in \mathbb{C}) \) then

\[
\Phi^*(\lambda, tH) = \sum_{m \geq 0} \frac{t^m}{m!} \xi_\lambda(H^m)
\]

where \( \xi_\lambda \) is the (infinitesimal) character of the algebra \( \mathfrak{g} \) corresponding to the linear function \( \lambda \) on \( \mathfrak{h}_t \).

\( \Lambda \) being any linear function on \( \mathfrak{h}_t \), consider the integral \( \int_K e^{\Lambda(H(x, u))} du \) which occurs in Theorem 2. (Notice that \( K = K^* \) in our case and therefore \( \int_K du = 1 \).) We shall now express this integral in terms of the function \( \Phi^* \).

Consider the representation \( \pi_\Lambda \) of \( G \) on \( L_2(K) \) given by

\[
\pi_\Lambda(x)f(u) = e^{\langle \Lambda(2x^2-1), H(x^{-1}, u) \rangle} f(u_x^{-1}) \quad (x \in G, u \in K, f \in L_2(K))
\]

in the notation of [6, §12]. Let \( \mathcal{S} \) be the smallest closed subspace of \( L_2(K) \) which is invariant under \( \pi_\Lambda(G) \) and which contains the constant function 1. Then we have seen in [5, Part IV] that the representation \( \pi \) of \( G \) induced on \( \mathcal{S} \) is quasi-simple and its infinitesimal character is \( \chi_\Lambda \) where \( \Lambda \) is to be extended to a linear function on \( \mathfrak{h}_t \) by putting it equal to zero on \( \mathfrak{h}_t \). Let \( \psi_0 \) denote the vector in \( \mathcal{S} \) corresponding to the constant function 1. Then if we denote the scalar product of two elements in the usual way we get

\[
(\psi_0, \pi(x)\psi_0) = \int_K e^{\langle \Lambda(H(x, u)), H(x^{-1}, u) \rangle} du.
\]

On the other hand suppose \( x = \exp tH_0 \) where \( H_0 \in \mathfrak{h}_t \) and \( t \in \mathbb{R} \). Then since \( \psi_0 \) is well-behaved under \( \pi \) (see Lemma 34 of [6]),

\[
(\psi_0, \pi(\exp tH_0)\psi_0) = \sum_{m \geq 0} \frac{t^m}{m!} (\psi_0, \pi(H_0^m)\psi_0)
\]

provided \( |t| \) is sufficiently small. Now put \( i_+(X) = (X + (-1)^{1/2}i(X))/2 \), \( i_-(X) = (X - (-1)^{1/2}i(X))/2 \) (\( X \in \mathfrak{t} \)). Then \( f_+ = i_+(f) \), \( f_- = i_-(f) \) are ideals in \( \mathfrak{g} \) and \( \mathfrak{g} \) is their direct sum. Let \( \mathfrak{x}_+ \) and \( \mathfrak{x}_- \) be the subalgebras of \( \mathfrak{B} \) generated by \( (\mathfrak{t}_+, 1) \) and \( (\mathfrak{t}_-, 1) \) respectively. Then if \( a \in \mathfrak{x}_+ \) and \( b \in \mathfrak{x}_- \), \( ab - ba = 0 \).

(\( ^{12} \)) This is easily seen by the argument used in the proof of Lemma 48 of [5].

---

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
Choose \( H_0^* \in \mathfrak{h}_0 \) such that \( H_0 = i(H_0^*) \). Then
\[
H_0 = i(H_0^*) = 2(-1)^{1/2}[i_-(H_0^*) - H_0^*].
\]
Moreover \( H_0^* \) and \( i_-(H_0^*) \) commute and \( \pi(f)\psi_0 = \{0\} \). Hence
\[
\pi(H_0^m)\psi_0 = (2(-1)^{1/2})^m \pi(H_0^m)\psi_0
\]
where \( H_- = i_-(H_0^*) \). On the other hand if \( X, Y \in \mathfrak{t} \),
\[
[X, i_-(Y)] = [i_-(X), i_-(Y)].
\]
Hence \( [X, z] = [i_-(X), z] \) \((X \in \mathfrak{t}, z \in \mathfrak{k}_-)\). Therefore we get a representation \( \nu \) of \( \mathfrak{f} \) on \( \mathfrak{k}_- \) such that \( \nu(X)z = [X, z] \) \((X \in \mathfrak{f}, z \in \mathfrak{k}_-)\). Obviously \( \nu \) is quasi semisimple (see Lemma 10 of [6]). Therefore if \( z \) is any element in \( \mathfrak{k}_- \), it follows from Lemma 7 of [6] that \( z = z_0 \mod \nu(f)\mathfrak{k}_- \) where \( z_0 \) is some element of \( \mathfrak{k}_- \) which commutes with \( \mathfrak{f} \). But then
\[
[i_-(X), z_0] = [X, z_0] = 0 \quad (X \in \mathfrak{f})
\]
and moreover \( z_0 \) commutes with \( \mathfrak{f}_+ \) since it lies in \( \mathfrak{k}_- \). Therefore \( z_0 \) is in the center of \( \mathfrak{g} \). Furthermore the representation of \( K \) induced under \( \pi \) is unitary and therefore if \( X \in \mathfrak{f}_0 \) and \( a \in \mathfrak{g} \),
\[
(\psi_0, \pi(Xa - aX)\psi_0) = (- \pi(X)\psi_0, \pi(a)\psi_0) = 0
\]
since \( \pi(f)\psi_0 = \{0\} \). This shows that
\[
(\psi_0, \pi(z)\psi_0) = (\psi_0, \pi(z_0)\psi_0) = \chi_\Lambda(z_0).
\]
Now if we extend \( \chi_\Lambda \) to a linear function on \( \mathfrak{g} \) such that \( \chi_\Lambda(ab) = \chi_\Lambda(ba) \) \((a, b \in \mathfrak{g})\) (see Part III of [5]) it follows that \( \chi_\Lambda(z_0) = \chi_\Lambda(z) \). Hence
\[
(\psi_0, \pi(z)\psi_0) = \chi_\Lambda(z) \quad (z \in \mathfrak{k}_-).
\]
This proves that
\[
(\psi_0, \pi(\exp i\mathfrak{H})\psi_0) = \sum_{m \geq 0} \frac{i^m}{m!} (2(-1)^{1/2})^m \chi_\Lambda(H_0^m).
\]
We extend \( i_- \) to an isomorphism of \( \mathfrak{k} \) with \( \mathfrak{k}_- \). Then the mapping \( z \rightarrow \chi_\Lambda(i_-(z)) \) \((z \in \mathfrak{k})\) is clearly a character of the algebra \( \mathfrak{k} \). Hence from Theorem 5 of [5] there exists a linear function \( \lambda \) on \( \mathfrak{h}_0 \) such that \( \chi_\Lambda(i_-(z)) = \xi_\Lambda(z) \) \((z \in \mathfrak{k})\). Therefore
\[
(\psi_0, \pi(\exp i\mathfrak{H})\psi_0) = \sum_{m \geq 0} \frac{i^m}{m!} (2(-1)^{1/2})^m \xi_\Lambda(H_0^m) = \Phi^*(\lambda, 2(-1)^{1/2} i\mathfrak{H}_0^*)
\]
if \(|z|\) is sufficiently small. In view of equation (25) (p. 81) of [5], \( \lambda \) may be chosen in such a way that \( \lambda(\mathfrak{H}) = \Delta(i_-(\mathfrak{H})) \) \((\mathfrak{H} \in \mathfrak{h}_0)\). Since \( \Delta \) vanishes on \( \mathfrak{h}_0 \) we conclude that \( \lambda(\mathfrak{H}) = -(1/2)\Delta(i(\mathfrak{H})) \) and therefore \( 2(-1)^{1/2}(\mathfrak{H}_0^*)^* = \Delta(H_0) \). Put \( si(\mathfrak{H}) = i(s\mathfrak{H}) \) \((s \in \mathfrak{W})\) and \( \alpha(i(\mathfrak{H})) = (-1)^{1/2}(\mathfrak{H}, \sigma(i(\mathfrak{H}))) \)
= (-1)^{1/2}\sigma(H) (H \in \mathfrak{h}_0). Then \lambda(H_\sigma) = -((-1)^{1/2}/2)\Delta(i(H_\sigma)) = \Delta(H'_\sigma) \) \) where 
\( H'_\sigma = -((-1)^{1/2}/2)i(H_\sigma) \) and \( \sigma(H'_\sigma) = \sigma(H_\sigma)/2 \). Then if we put

\[ \Phi(\Lambda, H) = \prod_{\alpha > 0} 2\sigma(H'_\alpha) \sum_{s \in W} \epsilon(s)e^{(\Lambda + 2\sigma)(sH)} \prod_{\alpha > 0} (e^\alpha(H) - e^{-\alpha(H)}) \quad (H \in \mathfrak{h}_0) \]

it is clear that \( \Phi(\Lambda, H) \) is an analytic function on \( \mathfrak{h}_0 \) and

\[ \Phi(\Lambda, 2(-1)^{1/2}tH_0^*) = \Phi(\Lambda, tH_0). \]

Hence \( \lambda_0, \pi(\exp tH_0)\psi_0) = \Phi(\Lambda, tH_0) \) for all sufficiently small values of \( |t| \). Since both sides are analytic functions of \( t \), the equality must hold for all values of \( t \). Thus we have the following result.

**Theorem 7.** Let \( \Lambda \) be a linear function on \( \mathfrak{h}_0 \). Then if \( x = \exp H \) \( (H \in \mathfrak{h}_0) \) we have the formula

\[ \int_K e^{\Lambda(H(x,u))} du = \prod_{\alpha > 0} 2\sigma(H'_\alpha) \sum_{s \in W} \epsilon(s)e^{(\Lambda + 2\sigma)(sH)} \prod_{\alpha > 0} (e^\alpha(H) - e^{-\alpha(H)}) \]

Put \( \phi(x) = \int_K e^{\Lambda(H(x,u))} du = (\psi_0, \pi(x)\psi_0) \) \( (x \in G) \). Then it is clear that \( \phi(uxv) = \phi(x) \) \( (u, v \in K) \). Since every element in \( G \) can be written in the form \( u(\exp H)v \) \( (H \in \mathfrak{h}_0); u, v \in K \), the above formula determines \( \phi \) completely.

The particular case of this formula for the complex immodular group has been obtained by Gelfand and Naimark [2, p. 77] by means of a lengthy computation (see also [1]).

**References**


Tata Institute of Fundamental Research, Bombay, India.

Columbia University, New York, N. Y.