ON BIEBERBACH-EILENBERG FUNCTIONS

BY

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1. Let us denote by C the class of functions $f(z)$ regular for $|z|<1$ which have power series expansion about $z=0$ beginning

$$f(z) = a_1 z + a_2 z^2 + \cdots$$

and which satisfy the condition

$$f(z_1)f(z_2) \neq 1, \qquad |z_1| < 1, \quad |z_2| < 1.$$

Bieberbach [1] first considered such functions with the additional assumption that they be univalent. He showed that for them $|a_1| \leqq 1$, equality occurring only for $f(z) \equiv e^{i\theta}z$, θ real, this result being equivalent to the Koebe 1/4 Theorem. Eilenberg [2] showed that the assumption of univalence was unnecessary. Rogosinski [8] also studied this class of functions, proving in particular the important result that any function in C is subordinate to a univalent function in C. Let us mention that the Bieberbach-Eilenberg theorem can easily be proved using the methods discussed in [5]. In this paper we shall use related methods to prove deeper theorems for functions of the class C.

2. THEOREM 1. *Let* $f_r(z) = (1-r^2)^{1/2}z/(1+irz)$ $(r<1)$. *Then* $f_r(z) \in C$ *and for* $|z|=r$, $f(z) \in C$, $|f(z)| \leqq r/(1-r^2)^{1/2}$, *equality being attained only for* $\pm f_r(ze^{i\theta})$ *at the point* $z = ie^{-i\theta}r$.

The mapping $w = f_r(z)$ carries $|z|<1$ into the interior of a circle K passing through $w=+1$ and $w=-1$. The segment of the imaginary axis joining 0 and ir goes into the segment S_1 of the imaginary axis joining 0 and $ir/(1-r^2)^{1/2}$. Set $a = r/(1-r^2)^{1/2}$. The quadratic differential $-idw^2/w(w-ia)(w+ia^{-1})$ is positive on S_1, K and on S_2, the image of S_1 under the transformation $w^* = 1/w$. Since K is invariant under this transformation we see at once that the module of the doubly-connected domain bounded by S_1 and S_2 is equal to twice the module of the doubly-connected domain bounded by K and S_i $(i=1, 2)$. The latter is equal to the module M of the unit circle slit rectilinearly from 0 to ir. In each case we mean the module for the family of curves separating the two boundaries [6].

Let now $f(z) \in C$, $f(z)$ univalent, and take any point on $|z|=r$, say $re^{i\theta}$. Let $f(z)$ map the rectilinear segment from 0 to $re^{i\theta}$ on an arc S_1^* in the w-plane from 0 to $f(re^{i\theta})$. Let S_2^* be the image of S_1^* under the transformation $w^* = 1/w$. By a well known result of Teichmüller [9] the module of the domain D bounded by S_1^* and S_2^* is at most equal to the module of the domain D^*

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obtained by slitting the w-plane rectilinearly from 0 to $f(re^{i\theta})$ and from $-1/\bar{f}(re^{i\theta})$ rectilinearly in the opposite direction to infinity. Equality is possible only if D and D^* coincide. On the other hand the module of D is at least $2M$ by Grötzsch's Lemma. Comparing D^* with the domain bounded by S_1 and S_2 we see that $|f(re^{i\theta})| \leqq a$. Using the equality conditions in the above result of Teichmüller and in Grötzsch's Lemma we see we can have $|f(re^{i\theta})| = a$ only for the functions and points stated in the theorem. Using Rogosinski's subordination result we obtain the theorem for all $f(z) \in C$.

The main result of this theorem can be obtained using instead the method developed in [6]. It has the disadvantage, however, that it does not in general provide uniqueness theorems due to the intervention of a symmetrization. That method will be used in the remainder of this paper.

3. We wish now to prove similar results for the maximum and minimum values of $|f(z)|$ for $|z| = r$ within the subclass of functions of C which have a given value of $|a_1|$. Naturally we can hope for reasonable minimum results only under the further assumption that $f(z)$ is univalent. In the case of maximum results we can also assume we are dealing with univalent functions in view of Rogosinski's subordination theorem.

We shall now construct a family of functions which will provide the extremals in this problem. Consider the unit circle $|z| < 1$ and the points $0, p, r, 1, -1$ $(-1 \leqq p \leqq 1,\ 0 < r < 1)$ on its diameter along the real axis. Regard in the upper semicircle the function

$$\zeta = \int^z \frac{((z-p)(pz-1))^{1/2}}{z((z-r)(z-r^{-1}))^{1/2}}\,dz.$$

It is understood that this function is defined by continuity in case p assumes either of the positions $0, r$. Further the radicals are to have their positive determinations on the real axis just to the right of 0. The lower limit of the integral may be taken as any suitable point. Let the images of $0, p, r, 1, -1$ be denoted by A, B, C, D, E. We shall assume r to be fixed and describe the image of the upper semicircle in various cases, depending on the value of p.

Case I: $0 < p < r$. A is situated at ∞. AB is a horizontal half-infinite segment on which $\Re\zeta$ increases as we go from A to B. BC is a vertical segment on which $\Im\zeta$ decreases as we go from B to C. CD is a horizontal segment on which $\Re\zeta$ increases as we go from C to D. DE is a vertical segment on which $\Im\zeta$ increases as we go from D to E. EA is a horizontal half-infinite segment on which $\Re\zeta$ decreases as we go from E to A. The value of $\Im\zeta$ on it is greater than the value on AB.

Case II: $-1 < p < 0$. A is situated at ∞. AC is a horizontal half-infinite segment on which $\Re\zeta$ increases as we go from A to C. CD is a vertical segment on which $\Im\zeta$ increases as we go from C to D. DE is a horizontal segment on which $\Re\zeta$ decreases as we go from D to E. EB is a vertical segment on which $\Im\zeta$ decreases as we go from E to B. BA is a horizontal half-infinite segment on which $\Re\zeta$ decreases as we go from B to A. The value of $\Im\zeta$ on it is greater than the value on AC.

Case III: $p=-1$. This is the limiting situation in Case II when B and E come into coincidence. BA becomes a continuation of DE.

Case IV: $r<p<1$. A is situated at ∞. AC is a horizontal half-infinite segment on which $\Re\zeta$ increases as we go from A to C. CB is a vertical segment on which $\Im\zeta$ increases as we go from C to B. BD is a horizontal segment on which $\Re\zeta$ increases as we go from B to D. DE is a vertical segment on which $\Im\zeta$ increases as we go from D to E. EA is a horizontal half-infinite segment on which $\Re\zeta$ decreases as we go from E to A. The value of $\Im\zeta$ on it is greater than the value on AC.

Case V: $p=1$. This is the limiting situation in Case IV when B and D come into coincidence. BC and DE are then collinear.

Case VI: $p=r$. This is the limiting case between Cases I and IV when B and C coincide.

Case VII: $p=0$. In this case A and B coincide and C, D, E form the other three corners of a rectangle. This case corresponds to the problem solved by Theorem 1. It can be treated in a manner entirely parallel to the others but because we already have a complete solution in this case we shall not give it special explicit mention.

In each case we rotate the domain so obtained through 180° about the midpoint P of DE, denoting the images of A, B, C, D, E by A', B', C', D', E'. Identifying the points of DE and $D'E'$ which coincide in pairs we obtain a new domain. The latter domain we map conformally on the left-hand half-plane $\Im w<0$ in the w-plane in such a way that A goes into $w=0$ and A' goes into $w=\infty$. Rotation in the ζ-plane through 180° about P corresponds in the w-plane to a linear transformation of $\Im w<0$ onto itself interchanging 0 and ∞. This transformation has the form $w^*=a/w$, a real and positive. The fixed point $-a^{1/2}$ (positive root) is the image of P. We adjust the original conformal mapping in each case so that this becomes the point -1 and the linear transformation becomes $w^*=1/w$.

Let the images of B, C, D, E be il, im, if, ig where $m>0$, $f>0$, $g<0$ and we may have $l=m, f$, or g. Then the images of B', C', D', E' are $-i/l$, $-i/m$, $-i/f$, $-i/g$, i.e. $f=-1/g$. If we extend ζ as a (non-single-valued) function of w to the whole w-plane by reflection in various segments of the imaginary axis we see at once that $d\zeta^2$ is a quadratic differential on the w-sphere with double poles at 0, ∞, simple poles at im, $-i/m$, and simple zeros at il, $-i/l$ (except for Case VI where the latter cancel). Indeed we can write

$$d\zeta^2 = K\,\frac{(w-il)(w+i/l)}{w^2(w-im)(w+i/m)}\,dw^2$$

with K a suitable positive constant. This is understood to be defined by continuity in the limiting cases. The curves on which $d\zeta^2>0$ will be called trajectories, those on which $d\zeta^2<0$ will be called orthogonal trajectories.

Regard now the combined mapping from the upper unit semicircle in the z-plane into the w-plane. Reflection across the segments joining -1 and 1 in the z-plane and $-i/f$ and if in the w-plane leads to a function $f(iz; r, p)$ regular and univalent for $|z|<1$. It maps $|z|<1$ on a domain in the w-plane which we denote by $D(r, p)$ and which is bounded by a trajectory or an orthogonal trajectory of $d\zeta^2$ (or by several such meeting at critical points in the limiting cases). Moreover $f(z; r, p)\in C$ in view of the fact that the mapping $w^*=1/w$ carries $D(r, p)$ into its exterior. We observe that $f(z; r, r)\equiv z$.

In Cases III and V we can obtain, however, not just one function but a whole continuum of functions which will have the extremal property. We shall discuss more completely the construction in Case III, that in Case V being entirely analogous. Now, instead of rotating the domain about P we rotate it through 180° about any point Q lying between P and D (using the same notation as before). We again identify the points of DE and $D'E'$ which coincide in pairs to obtain a new domain. However we do not now have $D=E'$ or $E=D'$. The remainder of the construction is the same except that Q goes into $w=-1$ instead of P and we do not have $f=-1/g$, indeed $f<-1/g$. We denote the continuum of functions so obtained by $f(z; r, -1, \lambda)$ where λ runs over $0<\lambda<1$ as Q runs over the open segment PD. $f(z; r, -1, \lambda)$ maps $|z|<1$ on a domain $D(r, -1, \lambda)$ bounded by two trajectories joining if and $-i/f$ (the zeros of $d\zeta^2$) and slit along the imaginary axis and from $-i/f$ to ig. As before $f(z; r, -1, \lambda)\in C$. In Case V we obtain similarly functions $f(z; r, 1, \lambda)\in C$. Here we rotate about a point R between P and E and λ runs over $0<\lambda<1$ as R runs over the open segment PE. The image of $|z|<1$ by $f(z; r, 1, \lambda)$ will be denoted by $D(r, 1, \lambda)$.

Naturally the quantities l, m, f, g depend on the particular situation and will be denoted by $m(r, p)$, $m(r, -1, \lambda)$, $m(r, 1, \lambda)$, etc., when appropriate.

4. THEOREM 2. *Let $g(w)$ be regular and univalent in $D(r, p)$ $(D(r, -1, \lambda)$, $D(r, 1, \lambda))$ with $g(0)=0$, $|g'(0)|=1$ and such that $g(w_1)g(w_2)\neq 1$, $w_1, w_2\in D(r, p)$ $(D(r, -1, \lambda), D(r, 1, \lambda))$. Then for $m=m(r, p)$ $(m(r, -1, \lambda), m(r, 1, \lambda))$*

$$|g(im)| \leqq m$$

when

$$-1 \leqq p \leqq r, \qquad 0 < \lambda < 1$$

and

$$|g(im)| \geqq m$$

when

$$r \leqq p \leqq 1, \qquad 0 < \lambda < 1.$$

When $p=r$ the result is an immediate consequence of the Bieberbach theorem. For $p=0$ it follows by our Theorem 1. Otherwise the details of the proof are somewhat different according as we are dealing with Case I, II, III, IV, or V (although the same for the whole continuum of domains corresponding to Case III or Case V). We shall give here complete details only in Case I and indicate briefly the necessary modifications for the others at the end.

We consider the orthogonal trajectories of the associated quadratic differential $d\zeta^2$ which we denote by $Q(w)dw^2$. Near $w=0$ these are Jordan curves which tend to circular form as they shrink down to $w=0$. In this case $0<l<m$ and from il emerge three arcs of orthogonal trajectories. One proceeds along the imaginary axis to im. This segment we denote by T_1. The other two join to form a Jordan curve which we denote by T_2. The latter bounds a simply-connected domain containing $w=0$. Let L be a small orthogonal trajectory in the neighborhood of $w=0$. L and T_2 bound a doubly-connected domain which we denote by D_1. Between the boundary H of $D(r, p)$ and $T_1 \cup T_2$ lies a second doubly-connected domain which we denote by D_2.

Let us regard now the image of $D(r, p)$ by $g(w)$. We may, without loss of generality, assume that the image domain is bounded by a smooth curve S since otherwise we should apply the following argument to an approximating sequence of such domains. Suppose the image to lie in the w'-plane, $w'=\xi+i\eta$. Let L', T_1', T_2' be the images of L, T_1, T_2. L' and T_2' bound D_1', the image of D_1, and $T_1' \cup T_2'$ and S bound D_2', the image of D_2. By conformal equivalence D_i' $(i=1, 2)$ has the same module as D_i, this being understood as taken in each case for the class of curves separating the boundaries [6]. Let $\omega_1(\xi, \eta)$ be the function harmonic in D_1' taking the value 2 on L' and the value 1 on T_2'. Let $\omega_2(\xi, \eta)$ be the function harmonic in D_2' taking the value 1 on $T_1' \cup T_2'$ and the value 0 on S. Define $\omega(\xi, \eta)$ to equal 2 on the interior of L', to equal $\omega_1(\xi, \eta)$ on $\overline{D}_1'$, and to equal $\omega_2(\xi, \eta)$ on $\overline{D}_2'$.

Now consider the surface $\omega=\omega(\xi, \eta)$ lying in 3-space over the (ξ, η) plane. We apply to it circular symmetrization with respect to the half-plane through the positive η-axis and perpendicular to the (ξ, η) plane. In this way we obtain a function $\tilde{\omega}(\xi, \eta)$ defined over a symmetrized domain. The set on which $1<\tilde{\omega}<2$ is a doubly-connected domain $\boldsymbol{D}_1$ bounded by curves $\boldsymbol{L}$ and $\boldsymbol{T}_2$ corresponding to L and T_2. The set on which $0<\tilde{\omega}<1$ is a doubly-connected domain $\boldsymbol{D}_2$ bounded outside by $\boldsymbol{S}$ corresponding to S and inside by $\boldsymbol{T}_2$ plus a segment $\boldsymbol{T}_1$ on the imaginary axis.

Let D_i have the module M_i, $\boldsymbol{D}_i$ the module $\boldsymbol{M}_i$ $(i=1, 2)$. Then by a standard form of argument [7, pp. 185, 194; 6] it follows that $\boldsymbol{M}_i \geqq M_i$ $(i=1, 2)$.

Let L^* be the image of L under the transformation $w^*=1/w$ and let $\mathfrak{D}$ be the doubly-connected domain bounded by L and L^*. Let F be a point on the positive imaginary axis in $\mathfrak{D}$ and F^* its image under the transformation $w^*=1/w$. In $\mathfrak{D}$ we regard the following module problem. Let C_1 denote the class of rectifiable Jordan curves lying in $\mathfrak{D}$ and separating L from F, F^*, and L^*. Let C_2 denote the class of rectifiable Jordan curves lying in $\mathfrak{D}$ and separating L and F from L^* and F^*, these curves being further restricted to be homotopic to such a curve symmetric in the imaginary axis. Let C_3 denote the class of rectifiable Jordan curves lying in $\mathfrak{D}$ and separating L^* from L, F, and F^*. Let ρ be a real-valued non-negative function of integrable square over $\mathfrak{D}$ and such that for $c_i \in C_i$ $(i=1, 2, 3)$, $\int_{c_i} \rho |dw|$ exists and that

$$\int_{c_1} \rho |dw| \geqq a, \qquad \int_{c_2} \rho |dw| \geqq b, \qquad \int_{c_3} \rho |dw| \geqq a$$

with $0 < a < b$. Then let the greatest lower bound of $\iint_{\mathfrak{D}} \rho^2 du dv$ for all such functions ρ be denoted by $M(a, b, L, F)$. This actually is a minimum attained for a function ρ for all choices as above. That this is so can be shown by a general construction method but the general result will not be needed here.

Let now F coincide with the point im. In the metric $|Q(w)|^{1/2}|dw|$, which we call for short the Q-metric, all orthogonal trajectories of $Q(w)dw^2$ in the classes C_1 and C_3 have a given length, say a, and those in the class C_2 have a given length, say b, with $b > a$. We then verify at once that $|Q(w)|^{1/2}|dw|$ provides the extremal metric in the corresponding module problem and this whatever the choice of L.

Now for a point G in $\mathfrak{D}$ with affix ik, $k > m$, we have $M(a, b, L, F) \geqq M(a, b, L, G) + d$ where a, b are as above and $d > 0$ is independent of the choice of L but depends on G. This can be seen in various ways, perhaps most easily by observing that it is possible to modify the function $|Q(w)|^{1/2}$ by setting it equal to zero in a sufficiently small neighborhood of F to obtain a function admissible in the competition for the greatest lower bound $M(a, b, L, G)$, independent of the choice of L.

In the domain $\mathfrak{D}$ let there lie four doubly-connected domains E_1, E_2, E_3, E_4 which do not overlap, have modules N_1, N_2, N_3, N_4, and are situated as follows: E_1 separates L from G, G^*, and L^*; E_2 and E_3 separate L and G from L^* and G^*, having the same topological situation as curves of the class C_2, E_4 separates L^* from L, G, and G^*. Then

$$M(a, b, L, G) \geqq a^2(N_1 + N_4) + b^2(N_2 + N_3). \tag{1}$$

Indeed if the function ρ is admissible in the module problem defining $M(a, b, L, G)$, the function ρ/a is admissible in the problem defining the module N_1 of E_1. Thus

$$a^{-2} \iint_{E_1} \rho^2 du dv \geqq N_1.$$

Similar results hold for the other domains and thus

$$\iint_{\mathfrak{D}} \rho^2 du dv \geqq a^2(N_1 + N_4) + b^2(N_2 + N_3).$$

Since $M(a, b, L, G)$ is the greatest lower bound of the left-hand side, the inequality (1) follows.

Suppose now that we had, contrary to the statement of Theorem 2, $|g(im)| > m$. Then in the configuration obtained above by symmetrization the end of $\boldsymbol{T}_1$ not on $\boldsymbol{T}_2$ would have an affix ik, $k > m$. We let this point play the role of G. Let $E_1 = \boldsymbol{D}_1 \cap \mathfrak{D}$. By the manner in which L tends to circular form as it shrinks down to $w=0$ it follows that the module N_1 of E_1 is such that $|\boldsymbol{M}_1 - N_1|$ tends to zero in this limit. Let $E_2 = \boldsymbol{D}_2$. Let E_3 and E_4 be the images of E_2 and E_1 under the transformation $w^* = 1/w$. These domains do not overlap the preceding. Indeed, using the property that $g(w_1)g(w_2) \neq 1$, $w_1, w_2 \in D(r, p)$, we see that the domain bounded by S, and thus the domain bounded by $\boldsymbol{S}$, does not overlap its image under the above transformation. Now E_1, E_2, E_3, E_4 have modules $N_1, \boldsymbol{M}_2, \boldsymbol{M}_2, N_1$. On the other hand from the fact that the Q-metric provides the minimum $M(a, b, L, F)$ we deduce at once $M(a, b, L, F) = 2a^2M_1 + 2b^2M_2$. Combining this with the preceding inequalities we have

$$2a^2M_1 + 2b^2M_2 \geqq 2a^2N_1 + 2b^2\boldsymbol{M}_2 + d.$$

Using the fact that we can make $|N_1 - \boldsymbol{M}_1|$ as small as we please by making L small enough, this contradicts the fact that $\boldsymbol{M}_i \geqq M_i$ $(i = 1, 2)$. This proves Theorem 2 for Case I.

We shall now indicate very briefly the modifications necessary for the proof in the other cases.

Case II: We work with trajectories instead of orthogonal trajectories. Instead of the two doubly-connected domains D_1 and D_2 we have a quadrangle with a pair of opposite sides on L (chosen as before) and an enclosing doubly-connected domain. The details of the proof are much as in [6].

Case III: In this limiting case the doubly-connected domain drops out. The same method works for the whole continuum of functions.

Case IV: Here we work with orthogonal trajectories again. We have two doubly-connected domains and the details are very much as in Case I.

Case V: In this limiting case one doubly-connected domain drops out. The same method works for the whole continuum of functions.

5. Corollary. *If, in addition to the conditions of Theorem* 2, $g(w)$ *satisfies the condition of being purely imaginary on the imaginary axis, then equality can be attained in the inequalities of that theorem only if* $g(w) \equiv \pm w$ *in* $D(r, p)$ $(D(r, -1, \lambda), D(r, 1, \lambda))$.

Indeed in this situation it is not necessary to apply a symmetrization and we then obtain the uniqueness result by a standard argument as in [4].

From this it follows at once that no two values of $f'(0; r, p)$, $-1 \leqq p \leqq r$, and $f'(0; r, -1, \lambda)$, $0 < \lambda < 1$, can be equal. We observe that all these values are real and positive. Also a straightforward argument in the theory of normal families shows that $f'(0; r, -1, \lambda)$ tends to zero as λ approaches 1. Since further $f'(0; r, r)=1$ it follows by continuity that for fixed r there is just one function in the above set with its derivative at $z=0$ equal to c for each c in $0<c\leqq 1$. This function we denote by $F(z; r, c)$. Similarly there is for fixed r just one function among $f(z; r, p)$, $r\leqq p\leqq 1$, and $f(z; r, 1, \lambda)$, $0<\lambda<1$, with its derivative equal to c for each c in $0<c\leqq 1$. This function we denote by $H(z; r, c)$. We are then in a position to state our main result.

THEOREM 3. *If* $f(z)\in C$ *and* $|f'(0)|=c$, *then* $|f(re^{i\theta})|\leqq \Im F(ir; r, c)$ *for each* r, $0<r<1$, θ *real.*

If $f(z)\in C$ *and is univalent and* $|f'(0)|=c$, *then* $|f(re^{i\theta})|\geqq \Im H(ir; r, c)$ *for each* r, $0<r<1$, θ *real.*

This follows at once from Theorem 2 making use of Rogosinski's subordination result.

Finally it should be remarked that the methods of this paper serve to treat various other problems for the class C.

BIBLIOGRAPHY

1. L. Bieberbach, *Über einige Extremalprobleme im Gebiete der konformen Abbildung*, Math. Ann. vol. 77 (1916) pp. 153–172.

2. S. Eilenberg, *Sur quelques propriétés topologiques de la surface de sphère*, Fund. Math. vol. 25 (1935) pp. 267–272.

3. J. A. Jenkins, *Some problems in conformal mapping*, Trans. Amer. Math. Soc. vol. 67 (1949) pp. 327–350.

4. ———, *Remarks on "Some problems in conformal mapping,"* Proc. Amer. Math. Soc. vol. 3 (1952) pp. 147–151.

5. ———, *Some results related to extremal length*, Contributions to the Theory of Riemann Surfaces, Annals of Mathematics Studies, no. 30, pp. 87–94.

6. ———, *Symmetrization results for some conformal invariants*, Amer. J. Math. vol. 75 (1953) pp. 510–522.

7. G. Pólya and G. Szegö, *Isoperimetric inequalities in mathematical physics*, Annals of Mathematics Studies, no. 27, Princeton University Press, 1951.

8. W. Rogosinski, *On a theorem of Bieberbach-Eilenberg*, J. London Math. Soc. vol. 14 (1939) pp. 4–11.

9. O. Teichmüller, *Untersuchungen über konforme und quasikonforme Abbildung*, Deutsche Mathematik vol. 3 (1938) pp. 621–678.

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