A STUDY OF $\alpha$-VARIATION. I.

by

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This paper is based on the notion of the higher variation of a function introduced by N. Wiener [9] while studying the Fourier coefficients of a function with bounded variation. L. C. Young applied this idea to derive a new existence theorem for Stieltjes integration and later collaborated with E. R. Love in publishing a number of papers on subjects related to this concept.

Preliminaries

1.1. Suppose that $f(x)$ is a real- or complex-valued function defined over $a \leq x \leq b$. For $0 < \alpha \leq 1$, we define the $\alpha$-variation of $f(x)$ over this interval as the least upper bound of the sums

$$\left\{ \sum_{n=1}^{N} \left| f(x_n) - f(x_{n-1}) \right|^{1/\alpha} \right\}^\alpha$$

taken over all subdivisions $a = x_0 < \cdots < x_N = b$, and we denote this upper bound by

$$V_\alpha\{f(x); a \leq x \leq b\} \quad \text{or} \quad V_\alpha\{f(x); x \in I\},$$

where $I$ is the interval $a \leq x \leq b$. Similarly we define the $\theta$-variation (or oscillation) of $f(x)$ over this interval as the least upper bound of the difference $\left| f(x'') - f(x') \right|$ for $a \leq x' < x'' \leq b$, and we denote this upper bound by

$$V_\theta\{f(x); a \leq x \leq b\} \quad \text{or} \quad V_\theta\{f(x); x \in I\}.$$

It is often convenient to consider the $\alpha$-variation of a function over an interval, $I$, which is open or half-open, and we can appropriately define the symbol

$$V_\alpha\{f(x); x \in I\}$$

for $0 \leq \alpha \leq 1$.

Suppose that $\{x_n\}$ is any set of real or complex numbers. For $0 < \alpha \leq 1$, we denote by

$$\left\{ \sum_n \left| \sum x_n \right|^{1/\alpha} \right\}^\alpha$$

the least upper bound of all sums
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$$\left\{ \sum_k |x_{n_k-1} + \cdots + x_{n_{k-1}}|^{1/\alpha} \right\}^\alpha,$$

where \( \{n_k\} \) is any appropriate finite sequence. When \( \alpha = 0 \), we let

$$\left\{ \sum_n |x_n|^{1/\alpha} \right\}^\alpha$$

and

$$\left\{ \sum_n \sum x_n|^{1/\alpha} \right\}^\alpha$$

denote the least upper bounds for \( |x_n| \) and \( x_n + \cdots + x_n \) respectively.

1.2. For \( 0 \leq \alpha \leq 1 \), we say that the function \( f(x) \) is in \( W_\alpha \) over the interval \( 0 \leq x \leq 1 \) if \( f(x) \) has bounded \( \alpha \)-variation over this interval; \( W_0 \) is simply the class of bounded functions. The sum of two functions in \( W_\alpha \) is also in \( W_\alpha \) since, by Minkowski’s inequality,

$$V_\alpha\{f(x) + g(x); x \in I\} \leq V_\alpha\{f(x); x \in I\} + V_\alpha\{g(x); x \in I\}.$$  

Similarly, for \( 0 \leq \alpha < \beta \leq 1 \), it follows from Jensen’s inequality that

$$V_\alpha\{f(x); x \in I\} \leq V_\beta\{f(x); x \in I\},$$

and hence a function in \( W_\beta \) is also in \( W_\alpha \).

If, for \( 0 \leq \alpha \leq 1 \), \( f(x) \) has period 1 and is in the class Lip \((\alpha)\), i.e. if for some constant \( C \)

$$|f(x'') - f(x')| < C(x'' - x')^\alpha$$

for each \( x' < x'' \), \( f(x) \) is obviously in \( W_\alpha \) and the \( \alpha \)-variation of \( f(x) \) over any interval of length 1 is less than \( C \). The converse is not true since functions in \( W_\alpha \) need not be continuous. However we can prove the following result.

**Theorem 1.2.1.** (Cf. [11, p. 455].) Suppose that \( 0 \leq \alpha \leq 1 \), that \( f(x) \) is real and continuous with period 1, and that the \( \alpha \)-variation of \( f(x) \) over any interval of length 1 is less than 1. There exists a continuous increasing function \( \phi(t) \) such that \( \phi(t+1) = \phi(t) + 1 \) and such that

$$|f(\phi(t'')) - f(\phi(t'))| < (t'' - t')^\alpha$$

for each \( t' < t'' \).

We can suppose that \( \alpha > 0 \) and that \( f(x) \) assumes its positive maximum at \( x = 0 \). Define the function \( \gamma(x) \) so that \( \{\gamma(x)\}^\alpha \) is equal to the \( \alpha \)-variation of \( f(x) \) over the interval \( 0 \leq y \leq x \). \( \gamma(x) \) is continuous, \( \gamma(x+1) \geq \gamma(x) + \gamma(1) \), and \( \gamma(1) < 1 \). For each \( x > 0 \) we can select a subdivision \( 0 = y_0 < \cdots < y_N = x + 1 \) such that

$$\gamma(x + 1) < \sum_{n=1}^N |f(y_n) - f(y_{n-1})|^{1/\alpha} + \varepsilon.$$

If \( y_n < 1 \leq y_{n+1} \), we see by periodicity that
and hence that \( \gamma(x+1) < \gamma(x) + \gamma(1) + \epsilon \). For \( x > 0 \) we conclude that \( \gamma(x+1) = \gamma(x) + \gamma(1) \) and we extend \( \gamma(x) \) so this holds for all \( x \).

Let \( \theta(x) = \gamma(x) + x\{1 - \gamma(1)\} \). \( \theta(x) \) is continuous and increasing for all \( x \); let \( \phi(t) \) be the inverse function. Then \( \phi(t+1) = \phi(t) + 1 \) and
\[
| f\{\phi(t'')\} - f\{\phi(t')\} |^{1/\alpha} \leq \gamma\{\phi(t'')\} - \gamma\{\phi(t')\} < \theta\{\phi(t'')\} - \theta\{\phi(t')\}
\]
for each \( t' < t'' \). This completes the proof.

We also have the following

**Theorem 1.2.2.** Suppose that \( 0 \leq \alpha \leq 1 \) and that \( f(x) \) is in \( W_\alpha \). Then
\[
V_\alpha\{f(y); x \leq y \leq x + h\} = O(h^\alpha)
\]
almost everywhere, and similarly on the left.

Assume that \( \alpha > 0 \) and let \( E_k \) be the set of points in \( 0 \leq x < 1 \) for which
\[
\limsup_{h \to 0} h^{-\alpha} V_\alpha\{f(y); x \leq y \leq x + h\} > k.
\]
For each \( x \) in \( E_k \) and each \( \epsilon > 0 \) there exists an interval, \( I(x) = x \leq y \leq x + h \), such that
\[
V_\alpha\{f(y); x \leq y \leq x + h\} \geq k h^\alpha
\]
and such that \( h < \epsilon \). By Vitali's covering theorem there exists a sequence, \( \{I(x_n)\} \), of nonoverlapping intervals which cover almost all of \( E_k \) and
\[
\{\text{outer meas } E_k\}^\alpha \leq \left\{ \sum_{n=1}^\infty |I(x_n)| \right\}^\alpha \leq \frac{1}{k} \cdot \left\{ \sum_{n=1}^\infty V_\alpha\{f(y); y \in I(x_n)\}^{1/\alpha} \right\}^\alpha \leq \frac{1}{k} \cdot V_\alpha\{f(x); 0 \leq x \leq 1\}.
\]
The set of points in \( 0 \leq x < 1 \) for which
\[
V_\alpha\{f(y); x \leq y \leq x + h\} \neq O(h^\alpha)
\]
is contained in \( E_k \) for each \( k \) and must have zero measure.

**Corollary 1.2.3.** Suppose that \( 0 \leq \alpha \leq 1 \) and that \( f(x) \) is in \( W_\alpha \). Then
\[
| f(x + h) - f(x) | = O(h^\alpha)
\]
almost everywhere, and similarly on the left.

Hence a function in \( W_\alpha \) satisfies a Lip (\( \alpha \)) condition almost everywhere.
1.3. We need the following lemma in order to study the relation between $W_\alpha$ and the Hardy-Littlewood integrated Lipschitz classes [4, p. 612].

**Lemma 1.3.1.** Suppose that $0 \leq \alpha \leq 1$, that $f(x)$ is a real-valued function with period 1, and that the $\alpha$-variation of $f(x)$ over any interval of length 1 does not exceed 1. Then the $\alpha$-variation of $f(x)$ over any interval of length $k$ does not exceed $k^\alpha$ for each positive integer $k$.

We need only show for $0 < \alpha \leq 1$ that

\[ V_\alpha \{ f(x); 0 \leq x \leq k \} \leq k^\alpha. \]

Pick a subdivision $\sigma$, $0 = x_0 < \cdots < x_N = k$, so that the left-hand side of 1.3.2 is majorized by

\[ \left\{ \sum_{\sigma} |\Delta f|^{1/\alpha} \right\}^\alpha + \epsilon = S^\alpha + \epsilon. \]

We can assume that $\sigma$ contains two points, $c$ and $d$, where $d = c + 1$ and where $f(d) = f(c) = \text{Max}_x f(x_n)$ for adding such points to $\sigma$ does not decrease $S$. (Cf. proof for 1.2.1.) We have

\[
S = \sum_{\sigma \cap [0,1]} |\Delta f|^{1/\alpha} \leq \sum_{\sigma \cap [0,c]} |\Delta f|^{1/\alpha} + \sum_{\sigma \cap [c,d]} |\Delta f|^{1/\alpha} + \sum_{\sigma \cap [d,k]} |\Delta f|^{1/\alpha}.
\]

$S_2$ is majorized by 1, $\{ S_1 + S_3 \}^\alpha$ is majorized by the $\alpha$-variation of $f(x)$ over $0 \leq x \leq k - 1$, and 1.3.2 follows by induction.

**Theorem 1.3.3.** (Cf. [10, p. 259].) Suppose that $0 \leq \alpha \leq 1$ and that $f(x)$ is a measurable real-valued function with period 1. If the $\alpha$-variation of $f(x)$ over any interval of length 1 never exceeds 1, then

\[ \left\{ \int_0^1 \left| f(x + h) - f(x) \right|^{1/\alpha} dx \right\}^\alpha \leq h^\alpha \]

for every $h > 0$.

Assume that $\alpha > 0$ and let $h = m/n$, where $m$ and $n$ are relatively prime positive integers. Then

\[ I(h) = \int_0^1 \left| f(x + h) - f(x) \right|^{1/\alpha} dx = \sum_{n=1}^\infty \int_0^{1/n} \left| f(x + \frac{m}{n}) - f(x) \right|^{1/\alpha} dx. \]

and, because $f(x)$ has period 1, this last sum is equal to

\[ \int_0^{1/n} \left\{ \sum_{n=1}^\infty \left| f(x + \frac{vm}{n}) - f(x + \frac{(v - 1)m}{n}) \right|^{1/\alpha} \right\} dx \leq \int_0^{1/n} m dx = h. \]
Hence the theorem is true for rational $h$. Since $f(x)$ is bounded, $I(h)$ is continuous and the theorem holds for all $h$.

**Corollary 1.3.4.** Suppose that $0 \leq \alpha \leq 1$ and that $f(x)$ is a measurable real-valued function with period 1. Then

$$\left\{ \int_0^1 |f(x + h) - f(x)|^{1/\alpha} dx \right\}^\alpha \leq 2^\alpha V_\alpha \{f(x); 0 \leq x \leq 1\} h^\alpha$$

for every $h > 0$.

This follows from 1.3.3 since the $\alpha$-variation of $f(x)$ over any interval of length 1 is majorized by

$$V_\alpha \{f(x); 0 \leq x \leq 1\}^{1/\alpha} + V_0 \{f(x); 0 \leq x \leq 1\}^{1/\alpha}.$$

If $f(x) = e^{2\pi ix}$ and $\alpha = 1/2$, the $\alpha$-variation of $f(x)$ over any interval of length 2 exceeds $2^\alpha$ times the $\alpha$-variation of $f(x)$ over any interval of length 1. Since 1.2.1 implies 1.3.1 when $f(x)$ is continuous, the restriction that $f(x)$ be real is essential in both of these results. The same is true for 1.3.3.

From 1.3.4 it is obvious that any measurable function with bounded $\alpha$-variation over some interval, $a \leq x \leq b$, is in the Hardy-Littlewood class Lip ($\alpha$, 1/$\alpha$) over that interval. The converse is not true. Hardy and Littlewood [4, p. 621] point out that if

$$f(x) = \log \frac{1}{|x|}, \quad x \neq 0,$$

then, for $h > 0$,

$$\left\{ \int_{-h}^x |f(x + h) - f(x)|^{1/\alpha} dx \right\}^\alpha = O(h^\alpha), \quad 0 < \alpha < 1,$$

while $f(x)$ is not even bounded in the neighborhood of $x = 0$.

1.4. The following lemma generalizes a familiar theorem on uniform continuity.

**Lemma 1.4.1.** Suppose that $0 \leq \alpha \leq 1$, that $f(x)$ has bounded $\alpha$-variation over $0 \leq x \leq 1$, and that $f(x)$ is continuous in this interval. For $\epsilon > 0$ there exists a $\delta > 0$ such that, for $0 \leq x_0 < x_0 + \delta \leq 1$, we have

$$V_\alpha \{f(x); x_0 \leq x \leq x_0 + \delta\} < \epsilon.$$

When $0 < \alpha \leq 1$, 1.4.1 is an immediate consequence of the following elementary result.

**Lemma 1.4.2.** Suppose that $0 < \alpha \leq 1$ and that $f(x)$ has bounded $\alpha$-variation in some right-handed neighborhood of the point $x = x_0$. Then

$$V_\alpha \{f(x); x_0 < x < x_0 + h\} = o(1)$$
as \( h \) approaches 0. A similar result holds on the left.

It follows from 1.4.2 that any function, with bounded \( \alpha \)-variation over an open interval, has right- and left-handed limits at each point of the interval.

For \( 0 \leq \alpha \leq 1 \), we say that \( f(x) \) is in \( V_\alpha \) over the interval \( 0 \leq x \leq 1 \) if, given \( \varepsilon > 0 \), there exists a \( \delta > 0 \) such that, for any set of disjoint intervals \( 0 \leq x_1 < \cdots < x_N < y_N \leq 1 \) for which

\[
\left\{ \sum_{n=1}^{N} |y_n - x_n|^{1/\alpha} \right\}^\alpha < \delta,
\]

we have

\[
\left\{ \sum_{n=1}^{N} |f(y_n) - f(x_n)|^{1/\alpha} \right\}^\alpha < \varepsilon.
\]

\( V_0 \) is the class of functions continuous over \( 0 \leq x \leq 1 \) and \( V_1 \) is the class of functions absolutely continuous over this interval.

The method of proof used in 1.2.2 gives us the following result.

**Theorem 1.4.3.** Suppose that \( 0 \leq \alpha \leq 1 \) and that \( f(x) \) is in \( V_\alpha \). Then

\[
V_\alpha \{f(y); x \leq y \leq x + h\} = o(h^\alpha)
\]

almost everywhere, and similarly on the left.

**Theorem 1.4.4 [6].** Suppose that \( 0 \leq \alpha \leq 1 \) and that \( f(x) \) is measurable and has period \( 1 \). \( f(x) \) is in \( V_\alpha \) if and only if

\[
V_\alpha \{f(x + h) - f(x); 0 \leq x \leq 1\} = o(1)
\]

as \( h \) approaches 0.

We see that the class \( W_\alpha \) includes \( V_\alpha \) and, for \( 0 < \alpha < 1 \), it is not difficult to show that the continuous function

\[
f(x) = \sum_{n=0}^{\infty} 2^{-n\alpha} \cos 2^n \pi x
\]

is in \( W_\alpha - V_\alpha \). Hence a continuous function with bounded \( \alpha \)-variation over \( 0 \leq x \leq 1 \) is not necessarily in \( V_\alpha \). However, we can prove, for \( 0 \leq \alpha \leq 1 \), that any continuous function in \( W_\alpha \) which possesses a finite derivative everywhere in \( 0 \leq x \leq 1 \), except perhaps on an enumerable set, is in \( V_\alpha \) [2, Theorem 2.6].

1.5. Suppose that \( f(x) \) and \( g(x) \) are defined over the interval \( 0 \leq x \leq 1 \) and that \( g(x) \) has at most discontinuities of the 1st kind. The Stieltjes integral

\[
\int_{0}^{1} f(x) dg(x)
\]

exists in the Young sense and is equal to \( I \) if, for \( \varepsilon > 0 \), there exists a finite set
$E$, contained in $0 \leq x \leq 1$, such that, for any subdivision $0 = x_0 < \cdots < x_N = 1$ which contains $E$, we have
\[
\left| \sum_{n=1}^{N} f(x_n) \left\{ g(x_n - 0) - g(x_{n-1} + 0) \right\} + \sum_{n=1}^{N-1} f(x_n) \left\{ g(x_n + 0) - g(x_n - 0) \right\} 
+ f(0) \left\{ g(0+) - g(0) \right\} + f(1) \left\{ g(1) - g(1-0) \right\} - I \right| < \varepsilon
\]
for each set $x_0 < \xi_1 < x_1 < \cdots < x_{N-1} < \xi_N < x_N$. L. C. Young [12] has proved the following

**Theorem 1.5.2.** Suppose that $\alpha + \beta > 1$ and that $f(x)$ and $g(x)$ belong to $W_\alpha$ and $W_\beta$ respectively. If $f(x)$ and $g(x)$ have no common discontinuities, 1.5.1 exists in the Riemann-Stieltjes sense. In any case, 1.5.1 exists in the Young sense.

This theorem is not true in the limiting case when $\alpha + \beta = 1$. The following result is also an immediate consequence of Young's work.

**Theorem 1.5.3.** Suppose that $\alpha + \beta > 1$ and that $f(x)$ is continuous and in $W_\alpha$. Suppose also that $\{g_n(x)\}$ is a sequence of uniformly bounded functions with uniformly bounded $\beta$-variation over $0 \leq x \leq 1$ which converges to $g(x)$, a function in $W_\beta$, on a set which includes the points $x = 0$ and $x = 1$ and which is dense in $0 \leq x \leq 1$. Then
\[
\lim_{n \to \infty} \int_{0}^{1} f(x)dg_n(x) = \int_{0}^{1} f(x)dg(x),
\]
\[
\lim_{n \to \infty} \int_{0}^{1} g_n(x)df(x) = \int_{0}^{1} g(x)df(x).
\]

**Moment problems**

2.1. In this chapter we study $W_\alpha$, the class of functions with bounded $\alpha$-variation over the interval $0 \leq x \leq 1$, by considering a moment problem. Suppose that $g(x)$ is any function in $W_\alpha$ for $0 < \alpha \leq 1$. We call the numbers
\[
\mu_n = \int_{0}^{1} x^ng(x), \quad n = 0, 1, \cdots,
\]
the Stieltjes moments of $g(x)$ and we say that $g(x)$ is normalized if
\[
g(x) = \frac{1}{2} \left\{ g(x + 0) + g(x - 0) \right\} \quad \text{for } 0 < x < 1.
\]
A uniqueness theorem follows from a known result [8, p. 60] after integration by parts.
Theorem 2.1.1. Suppose that 0 < \alpha \leq 1, that \( g(x) \) is normalized and in \( W_\alpha \), and that
\[
\int_0^1 x^n dg(x) = 0
\]
for \( n = 0, 1, \ldots \). Then \( g(x) \) is identically constant in \( 0 \leq x \leq 1 \).

For an arbitrary sequence of numbers, \( \{\mu_n\} \), we define a linear functional over the space of polynomials by letting
\[
\mu\{P\} = \mu \left\{ \sum_{n=0}^{N} a_n x^n \right\} = \sum_{n=0}^{N} a_n \mu_n.
\]
Following Hausdorff we define, for \( k = 0, 1, \ldots \) and \( n = 0, 1, \ldots, k \),
\[
\lambda_{k,n}(x) = C_n^{k-n} x^n (1 - x)^{k-n} \quad \text{and} \quad \lambda_{k,n} = \mu\{\lambda_{k,n}(x)\},
\]
where \( C_n^r \) is the binomial coefficient
\[
\binom{r + s}{r} = \frac{\Gamma(r + s + 1)}{\Gamma(r + 1)\Gamma(s + 1)}.
\]

Theorem 2.1.2. Suppose that 0 < \alpha \leq 1. A necessary and sufficient condition that the set of numbers \( \{\mu_n\} \) be the Stieltjes moments of a normalized function \( g(x) \) in \( W_\alpha \), where \( V_\alpha\{g(x); 0 \leq x \leq 1\} \leq 1 \), is that
\[
\left\{ \sum_{n=0}^{k} \sum_{i=0}^{k} |\lambda_{k,n}|^{1/\alpha} \right\}^\alpha \leq 1
\]
for all \( k \).

Hausdorff [5] has proved this theorem for the case where \( \alpha = 1 \).

2.2. In order to prove the sufficiency we derive two simple results.

Lemma 2.2.1. If \( \{\mu_n\} \) is an arbitrary sequence of numbers, then
\[
\sum_{n=0}^{k} \binom{n}{k} \lambda_{k,n} = \mu_n + O\left(\frac{1}{k}\right)
\]
for \( m = 0, 1, \ldots \).

Suppose that \( f(x) \) is any function defined over the interval \( 0 \leq x \leq 1 \) and consider
\[
B_k\{f; x\} = \sum_{n=0}^{k} f\left(\frac{n}{k}\right) \lambda_{k,n}(x),
\]
the Bernstein polynomial of order \( k \) for \( f(x) \). If \( P_m(x) \) is a polynomial of degree \( m \),
where the polynomials $Q_{m,r}(x)$ do not depend on $k$ and are identically zero when $P_m(x)$ is a constant [7, p. 8]. Setting $P_m(x) = x^m$, we have
\[
\sum_{n=0}^{k} \binom{n}{k} \lambda_{k,n} = \mu \left\{ B_k \{ x^m ; x \} \right\} = \mu_m + \sum_{r=1}^{m-1} \frac{\mu \{ Q_{m,r} \}}{k^r},
\]
and 2.2.1 follows.

**Lemma 2.2.2.** Suppose that $0 < \alpha \leq 1$, that \( \{ g_k(x) \} \) is a sequence of uniformly bounded functions, and that
\[
\liminf_{k \to \infty} V_\alpha \{ g_k(x) ; 0 \leq x \leq 1 \} \leq 1.
\]
There exists a function $g(x)$ such that
\[
V_\alpha \{ g(x) ; 0 \leq x \leq 1 \} \leq 1,
\]
and a subsequence \( \{ k_j \} \) such that $g_{k_j}(x) \to g(x)$ for every rational $x$ in $0 \leq x \leq 1$, including the end points $x = 0$ and $x = 1$.

We can use the selection principle (or diagonal process) to define $g(x)$ on the rationals. If, for irrational $x$, we let
\[
g(x) = \limsup_{r \to x} g(r), \quad r \text{ rational,}
\]
then $g(x)$ satisfies the conditions of the lemma.

To prove the sufficiency part of 2.1.2, consider the step function $g_\alpha(x)$ where $g_\alpha(0) = 0$ and
\[
g_\alpha(x) = \sum_{r=0}^{n} \lambda_{k,r} \quad \text{for} \quad \frac{n}{k+1} < x \leq \frac{n+1}{k+1}.
\]
Since
\[
| g_\alpha(x) | \leq V_\alpha \{ g_\alpha(x) ; 0 \leq x \leq 1 \} \leq 1,
\]
there exists a function $g^*(x)$ such that
\[
V_\alpha \{ g^*(x) ; 0 \leq x \leq 1 \} \leq 1,
\]
and a subsequence \( \{ k_j \} \) such that $g_{k_j}(x) \to g^*(x)$ for each rational $x$ in $0 \leq x \leq 1$. Let
\[
g(0) = g^*(0), \quad g(1) = g^*(1),
\]
2.2.3
\[
g(x) = \frac{1}{2} \left\{ g^*(x + 0) + g^*(x - 0) \right\} \quad \text{for} \quad 0 < x < 1.
\]
The function $g(x)$ is normalized and, from 1.5.3 and 2.2.1, we conclude that
\[
\mu_m = \lim_{k \to \infty} \sum_{n=0}^{k} \left(\frac{n}{k+1}\right)^m \lambda_{k,n}
\]
\[
= \lim_{k \to \infty} \int_0^1 x^m d\nu_k(x)
\]
\[
= \int_0^1 x^m d\nu(x) = \int_0^1 x^m d\nu_g(x).
\]

2.3. In order to prove the necessity we require a number of lemmas.

We say that a real function $f(x)$ is unimax in the interval $a \leq x \leq b$ if, for
$a < c < d \leq b$, $f(x) \geq \min \{ f(c), f(d) \}$.

**Lemma 2.3.1.** Suppose that $0 \leq \alpha \leq 1$ and that $X_n = \sum_{m=1}^{\infty} a_{n,m} x_m$ for $n = 1, \ldots, N$, where $a_{n,m}$ is a non-negative unimax function of $m$ for each $n$, and where $\sum_{m=1}^{\infty} a_{n,m} = 1$ for each $m$. Then
\[
\left\{ \sum_{n=1}^{N} \left| X_n \right|^{1/\alpha} \right\}^{\alpha} \leq \left\{ \sum_{m=1}^{M} \left| x_m \right|^{1/\alpha} \right\}^{\alpha}.
\]

We consider 2.3.1 in a slightly different form.

**Lemma 2.3.2.** Suppose that $0 \leq \alpha \leq 1$ and that $X_n = \sum_{m=1}^{\infty} a_{n,m} x_m$ for $n = 1, \ldots, N$, where $a_{n,m}$ is a non-negative integer and is unimax as a function of $m$ for each $n$, and where $\sum_{m=1}^{\infty} a_{n,m} = R$ for each $m$. Then
\[
\left\{ \sum_{n=1}^{N} \left| X_n \right|^{1/\alpha} \right\}^{\alpha} \leq R \left\{ \sum_{m=1}^{M} \left| x_m \right|^{1/\alpha} \right\}^{\alpha}.
\]

**Lemma 2.3.3.** With the hypotheses of 2.3.2 we can write
\[
X_n = \sum_{r=1}^{R} y_{n,r} = \sum_{r=1}^{R} \left\{ \sum_{m=1}^{M} a_{n,m,r} x_m \right\},
\]
where $a_{n,m,r}$ is either 0 or 1 and is unimax in $m$ for each $n$ and $r$, and where $\sum_{m=1}^{\infty} a_{n,m,r} = 1$ for each $m$ and $r$.

Now 2.3.3 is true when $R = 1$. Suppose it true for $R = k - 1$ and consider the case where $R = k$.

A. There exists $n_1$ such that $a_{n_1,1} \geq 1$. Let $m_1$ be the largest $m$ for which $a_{n_1,m} \geq 1$ and define
\[
a_{n_1,m,1} = \begin{cases} 1 & \text{for } 1 \leq m \leq m_1, \\ 0 & \text{everywhere else.} \end{cases}
\]

It is not difficult to see that $a_{n_1,m} - a_{n_1,m,1}$ is a non-negative integer and is unimax as a function of $m$. 
B. If $m_1 < M$, there exists $n_2 \neq n_1$ such that $a_{n_2,m_1} < a_{n_2,m_1+1}$. Let $m_2$ be the largest $m$ for which $a_{n_2,m} \geq 1$ and define

$$a_{n_2,m_1} = \begin{cases} 1 & \text{for } m_1 < m \leq m_2, \\ 0 & \text{everywhere else.} \end{cases}$$

Again $a_{n_2,m} - a_{n_2,m_1}$ is a non-negative integer and is unimax in $m$.

C. If $m_2 < M$, we can find $n_3 \neq n_2, n_1$ such that $a_{n_2,m_2} < a_{n_2,m_2+1}$, and we can define $m_3$ and the set $\{a_{n,m,1}\}$ for $m = 1, \cdots, M$. After a finite number of steps we arrive at the place where $m_j = M$. We shall have defined a sequence of distinct integers, $n_1, n_2, \cdots, n_j$, and a set of coefficients, $\{a_{n,m,1}\}$, for $n = n_1, \cdots, n_j$ and $m = 1, \cdots, M$. Let $a_{n,m,1} = 0$ for $n \neq n_1, \cdots, n_j$ and $m = 1, \cdots, M$.

The set $\{a_{n,m,1}\}$ satisfies the conditions in 2.3.3, the set $\{a_{n,m} - a_{n,m,1}\}$ satisfies the hypotheses of 2.3.2 with $R = k - 1$, and 2.3.3 follows by applying the induction hypothesis to

$$X'_n = \sum_{m=1}^{M} \{a_{n,m} - a_{n,m,1}\} x_m.$$

For a fixed $r$, consider the set $\{y_{n,r}\}$. Since each element here is simply the sum of consecutive $x_m$, we see from 2.3.3 and Minkowski's inequality that

$$\left\{ \sum_{n=1}^{N} |X'_n|^{1/\alpha} \right\}^{\alpha} \leq \sum_{r=1}^{R} \left\{ \sum_{n=1}^{N} |y_{n,r}|^{1/\alpha} \right\}^{\alpha} \leq R \left\{ \sum_{m=1}^{M} |x_m|^{1/\alpha} \right\}^{\alpha}.$$

Now 2.3.2 implies 2.3.1 when all the $a_{n,m}$ are rational and $\sum_{n=1}^{N} a_{n,m} = c_m = 1$. A simple limiting process removes the restriction that the $a_{n,m}$ be rational. When $0 \leq c_m \leq 1$, we can apply our results to the set of linear forms

$$\sum_{m=1}^{M} a_{n,m} x_m, \quad n = 1, \cdots, N,$$

$$(1 - c_m) x_m, \quad m = 1, \cdots, M,$$

and show that

$$\left\{ \sum_{n=1}^{N} |X'_n|^{1/\alpha} + \sum_{m=1}^{M} |(1 - c_m) x_m|^{1/\alpha} \right\}^{\alpha} \leq \left\{ \sum_{m=1}^{M} |x_m|^{1/\alpha} \right\}^{\alpha}.$$

This completes the proof for 2.3.1. We can assume that no $x_m$ is zero. Hence when $\alpha > 0$, we get strict inequality if, for any $m$, $c_m < 1$. 

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Lemma 2.3.4. Suppose that \( k = 0, 1, \cdots \), and that \( 0 \leq m \leq n \leq k \). The function \( f(x) = \sum_{r=m}^{n} \lambda_{k,r}(x) \) is unimax in \( 0 \leq x \leq 1 \).

We can assume that \( k \geq 1 \) and 2.3.4 follows immediately from the identity

\[
\frac{d}{dx} \lambda_{k,n}(x) = \begin{cases} 
-k\lambda_{k-1,n}(x), & n = 0, \\
k \{ \lambda_{k-1,n-1}(x) - \lambda_{k-1,n}(x) \}, & 0 < n < k, \\
\lambda_{k-1,n-1}(x), & n = k.
\end{cases}
\]

To complete the proof for 2.1.2, fix \( k \) and consider any finite sequence \( 0 = \nu_0 < \cdots < \nu_N = k + 1 \). If

\[
\theta_n = \sum_{r_{n-1} \leq r < r_n} \lambda_{k,r} \quad \text{and} \quad \theta_n(x) = \sum_{r_{n-1} \leq r < r_n} \lambda_{k,r}(x),
\]

then

\[
\theta_n = \int_0^1 \theta_n(x) dg(x)
\]

and this integral exists in the Riemann-Stieltjes sense for each \( n \). For each subdivision \( 0 = y_0 < \cdots < y_M = 1 \), let

\[
X_n = \sum_{m=1}^{M} \theta_n(y_m) \{ g(y_m) - g(y_{m-1}) \}.
\]

\( \theta_n(y_m) \) is a non-negative unimax function of \( m \) and

\[
\sum_{n=1}^{N} \theta_n(y_m) = \sum_{n=0}^{k} \lambda_{k,r}(y_m) = 1.
\]

Applying 2.3.1 we get

\[
\left\{ \sum_{n=1}^{N} |X_n|^{1/\alpha} \right\}^{\alpha} \leq \left\{ \sum_{m=1}^{M} \left| \sum_{n=0}^{k} \lambda_{k,r}(y_m) \right|^{1/\alpha} \right\}^{\alpha}
\]

which completes the argument.

2.4. We have discussed the Stieltjes moments of a function \( g(x) \), i.e. the sequence \( \{ \mu_n \} \) where

\[
\mu_n = \int_0^1 x^n dg(x), \quad n = 0, 1, \cdots.
\]

We can also consider the moment sequence

\[
\mu_n = \int_0^1 x^n g(x) dx, \quad n = 0, 1, \cdots,
\]

where the integral is interpreted in the Lebesgue sense. We call such a se-
quence of numbers the Lebesgue moments of \( g(x) \) and we have the following

**Theorem 2.4.1.** Suppose that \( 0 < \alpha \leq 1 \). A necessary and sufficient condition that the set of numbers \( \{ \mu_n \} \) be the Lebesgue moments of a normalized function \( g(x) \) in \( W_a \), where

\[
V_a \{ g(x) ; 0 \leq x \leq 1 \} \leq 1,
\]

is that

\[
(k + 1) \left\{ \sum_{n=1}^{k} \left( \sum_{m=1}^{n} \lambda_{k,m} - \lambda_{k,n-1} \right)^{1/\alpha} \right\} \leq 1
\]

for all \( k \).

The necessity follows immediately from 2.1.2 since, with the help of 2.3.5, we can write

\[
(k + 1) \left\{ \lambda_{k,n} - \lambda_{k,n-1} \right\} = \int_{0}^{1} (k + 1) \left\{ \lambda_{k,n}(x) - \lambda_{k,n-1}(x) \right\} g(x) dx
\]

\[
= \int_{0}^{1} \lambda_{k+1,n}(x) dg(x)
\]

for \( 0 < n \leq k \).

For the sufficiency, observe that

\[
| \mu_0 - (k + 1) \lambda_{k,m} | \leq \sum_{n=0}^{k} | \lambda_{k,n} - \lambda_{k,m} | \leq 1
\]

for \( 0 \leq m \leq k \). Consider the step function \( g_k(x) \) where \( g_k(0) = (k + 1) \lambda_{k,0} \) and

\[
g_k(x) = (k + 1) \lambda_{k,n} \quad \text{for} \quad \frac{n}{k + 1} < x \leq \frac{n + 1}{k + 1}.
\]

From 2.4.2 and 2.2.2 it follows that there exists a function \( g^*(x) \) such that

\[
V_a \{ g^*(x) ; 0 \leq x \leq 1 \} \leq 1,
\]

and a subsequence \( \{ k_j \} \) such that \( g_{k_j}(x) \to g^*(x) \) for each rational \( x \) in \( 0 \leq x \leq 1 \). Define \( g(x) \) as in 2.2.3. From 1.5.3, 2.2.1, and 2.4.2 we can conclude that

\[
\mu_m = \lim_{k \to \infty} \sum_{n=0}^{b} \left( \frac{n}{k + 1} \right)^m \lambda_{k,n}
\]

\[
= \lim_{k \to \infty} \int_{0}^{1} x^m g_k(x) dx
\]

\[
= \int_{0}^{1} x^m g^*(x) dx = \int_{0}^{1} x^m g(x) dx.
\]
2.5. Turning back to 2.3.1, we deduce two alternative forms for this inequality which are useful in later work.

**Lemma 2.5.1.** Suppose that $0 \leq \alpha \leq 1$ and that $Y_n = \sum_{m=1}^{M} b_{n,m}(y_m - y_0)$ for $n = 0, \ldots, N$, where $b_{n,m}$ is non-negative for all $m$ and $n$, where $\sum_{m=1}^{M} b_{n,m}$ is nondecreasing in $n$ and bounded by 1, and where, for each $0 \leq n' < n'' \leq N$, $b_{n',m} - b_{n'',m}$ is at first nonpositive and then non-negative as $m$ increases from 1 to $M$. Then

$$\left\{ \sum_{n=1}^{N} \left| \sum_{m=1}^{M} \frac{Y_n - Y_{n-1}}{1/\alpha} \right|^\alpha \right\} ^{1/\alpha} \leq \left\{ \sum_{m=1}^{M} \left| \sum_{n=1}^{N} y_m - y_{m-1} \right|^{1/\alpha} \right\} ^{1/\alpha}.$$

**Lemma 2.5.2.** Suppose that $0 \leq \alpha \leq 1$ and that $Y_n = \sum_{m=0}^{M} b_{n,m}y_m$ for $n = 0, \ldots, N$, where $b_{n,m}$ is non-negative for all $m$ and $n$, where $\sum_{m=0}^{M} b_{n,m} = 1$, and where, for each $0 \leq n' < n'' \leq N$, $b_{n',m} - b_{n'',m}$ is at first nonpositive and then non-negative as $m$ increases from 0 to $M$. Then

$$\left\{ \sum_{n=1}^{N} \left| \sum_{m=1}^{M} \frac{Y_n - Y_{n-1}}{1/\alpha} \right|^\alpha \right\} ^{1/\alpha} \leq \left\{ \sum_{m=1}^{M} \left| \sum_{n=1}^{N} y_m - y_{m-1} \right|^{1/\alpha} \right\} ^{1/\alpha}.$$

For 2.5.1, let $x_m = y_m - y_{m-1}$ and pick any sequence of integers, $0 = k_0 < \cdots < k_N = N$. If

$$X_n = Y_{k_n} - Y_{k_{n-1}} = \sum_{m=1}^{M} a_{n,m}x_m,$$

then

$$a_{n,m} = \sum_{\mu=m}^{M} \{b_{\mu,m} - b_{\mu-1,m}\}.$$

The difference $a_{n,m+1} - a_{n,m} = -\{b_{n,m} - b_{n-1,m}\}$ is at first non-negative and then nonpositive as $m$ increases; hence $a_{n,m}$ is unimax in $m$. We see that

$$a_{n,1} = \sum_{m=1}^{M} b_{k_m,m} - \sum_{m=1}^{M} b_{k_{m-1},m} \geq 0,$$

$$a_{n,M} = b_{k_n,M} - b_{k_{n-1},M} \geq 0,$$

and the unimax property ensures that $a_{n,m}$ is non-negative for all $m$ and $n$. Finally

$$\sum_{n=1}^{N'} a_{n,m} = \sum_{\mu=m}^{M} \{b_{N',\mu} - b_{0,\mu}\} \leq 1,$$

and we can apply 2.3.1.

For 2.5.2 observe that the set $\{b_{n,m}\}$, for $n = 0, \ldots, N$ and $m = 0, \ldots, M$, satisfies the hypotheses of 2.5.1. Since the sums
differ by a constant which is independent of \( n \), the conclusion follows immediately.

From 2.5.2 we can deduce the following result concerning Bernstein polynomials.

**Theorem 2.5.3.** If \( 0 \leq \alpha \leq 1 \),

\[
V_{\alpha} \{ B_k \{ f; x \}; 0 \leq x \leq 1 \} \leq \left\{ \sum_{m=1}^{k} \left| \sum_{m=1}^{k} f \left( \frac{m}{k} \right) - f \left( \frac{m-1}{k} \right) \right|^{1/\alpha} \right\}^{\alpha}.
\]

Let \( 0 < x_0 < \cdots < x_N < 1 \) be any subdivision of the interval \( 0 < x < 1 \). We see that

\[
B_k \{ f; x_n \} = \sum_{m=0}^{k} f \left( \frac{m}{k} \right) \lambda_{k,m} (x_n)
\]

where \( \lambda_{k,m} (x_n) \) is non-negative for all \( m \) and \( n \), and where \( \sum_{m=0}^{k} \lambda_{k,m} (x_n) = 1 \). For \( 0 \leq n' < n'' \leq N \),

\[
\lambda_{k,m} (x_{n''}) - \lambda_{k,m} (x_{n'}) = \lambda_{k,m} (x_{n''}) \left\{ 1 - \left( \frac{x_{n'}}{x_{n''}} \right)^{m} \left( \frac{1 - x_{n'}}{1 - x_{n''}} \right)^{k-m} \right\}.
\]

Since \( 0 < x_{n'} < x_{n''} < 1 \), the bracketed quantity is negative for \( m = 0 \), positive for \( m = k \), and strictly increasing in \( m \). Applying 2.5.2 completes the proof.

In conclusion we add that 2.3.1, 2.5.1, and 2.5.2 are valid when \( M \) and/or \( N = \infty \).

**A Faltung Theorem**

3.1. In one of his papers \([11]\), L. C. Young considered a Stieltjes Faltung of the form

3.1.1

\[
s(x) = \int_{0}^{1} f(x, y) dg(y).
\]

We present here a theorem suggested by Young's results.

Suppose that \( f(x, y) \) and \( g(y) \) have period 1 in \( y \) and define \( F(x, y) \) as the integral \( \int_{0}^{1} f(x, y + t) g(t) dt \) which we assume exists in the Lebesgue sense for all \( 0 \leq x, y \leq 1 \). Let

3.1.2

\[
s_n(x) = 2^n \{ F(x, 0) - F(x, 2^{-n}) \}
\]

\[= 2^n \int_{0}^{1} f(x, t) \{ g(t) - g(t - 2^{-n}) \} dt\]

for \( n = 0, 1, \ldots \). Then \( s_0(x) = 0 \) and the following is easily verified.
Lemma 3.1.3. Suppose that \( g(y) \) is continuous and that the integral 3.1.1 exists in the Riemann-Stieltjes sense for \( x = x_0 \). Then \( s(x_0) = \lim_{n \to \infty} s_n(x_0) \) and, for \( n \geq 1 \), we have
\[
s_n(x) - s_{n-1}(x) = 2^{n-1} \int_0^1 \{ f(x, t) - f(x, t + 2^{-n}) \} \{ g(t) - g(t - 2^{-n}) \} \, dt.
\]

Our principal result is as follows.

Theorem 3.2. (Cf. [11, Theorem 6.1].) Suppose that \( 0 < \alpha, \beta, \gamma \leq 1 \), \( 0 < \lambda = \beta + \gamma - 1 \), and \( \mu = \alpha \lambda / \beta \). Suppose also that \( f(x, y) \) and \( g(y) \) have period 1 in \( y \), that \( g(y) \) is continuous, and that
\[
\begin{align*}
3.2.1 & \quad \forall \{ f(x, y); 0 \leq x \leq 1 \} \leq A \\
3.2.2 & \quad \forall \{ f(x, y); 0 \leq y \leq 1 \} \leq B \\
3.2.3 & \quad \forall \{ g(y); 0 \leq y \leq 1 \} \leq C.
\end{align*}
\]
If \( s(x) \) is the Stieltjes Faltung 3.1.1, then
\[
\begin{align*}
3.2.4 & \quad V_\mu \{ s(x); 0 \leq x \leq 1 \} \leq k(\lambda) A^{\lambda/\beta} B^{1-\lambda/\beta} C,
\end{align*}
\]
where \( k(\lambda) \) is a finite constant.

If \( B = 0 \) and/or \( C = 0 \), \( s(x) \equiv 0 \) and 3.2.4 follows immediately. Hence we can assume that \( B = C = 1 \).

Obviously we can suppose that \( f(x, y) \) and \( g(y) \) are real. By an argument similar to that used in 1.2.1, we can find a strictly increasing continuous function \( \phi(t) \) such that
\[
\phi(0) = 0 \quad \text{and} \quad \phi(t+1) = \phi(t) + 1
\]
for all \( t \), and such that
\[
| g\{ \phi(t') \} - g\{ \phi(t'' \} | < 2(t'' - t')^\alpha
\]
for each \( t' < t'' \). Furthermore, for each \( 0 \leq x \leq 1 \),
\[
V_\beta \{ f(x, \phi(t)); 0 \leq t \leq 1 \} = V_\beta \{ f(x, y); 0 \leq y \leq 1 \},
\]
\[
\int_0^1 f\{ x, \phi(t) \} \, dg\{ \phi(t) \} = \int_0^1 f(x, y) \, dg(y),
\]
and, by performing a change of variable, we replace condition 3.2.3 by the condition
\[
3.2.5 \quad | g(y'') - g(y') | < 2(y'' - y')^\alpha
\]
for each \( y' < y'' \).

Since \( g(y) \) is continuous and \( \beta + \gamma > 1 \), the Faltung \( s(x) \) exists in the Riemann-Stieltjes sense for each \( x \) and is equal to \( \lim_{n \to \infty} s_n(x) \). Using 3.1.3, 3.2.5, Jensen’s inequality, 3.2.2, and 1.3.4 we have
\[ |s_n(x) - s_{n-1}(x)| \leq 2^{n-1} \int_0^1 |f(x, t + 2^{-n}) - f(x, t)| \, |g(t) - g(t - 2^{-n})| \, dt \]
\[
\leq 2^n(1-\gamma) \left\{ \int_0^1 |f(x, t + 2^{-n}) - f(x, t)|^{1/\beta} \, dt \right\}^\beta 
\leq 2^\beta : 2^{-n\lambda},
\]
and summing on \( n \) we get

\[ 3.2.6 \quad |s(x) - s_n(x)| \leq 2^\beta \sum_{n=0}^{\infty} 2^{-n\lambda} = \sigma(\lambda) 2^{-n\lambda} \]

for \( n = 0, 1, \cdots \). Fix \( n \) and consider \( 0 \leq x' < x'' \leq 1 \). From 3.1.2, 3.2.5, and Jensen's inequality we have

\[ 3.2.7 \quad |s_n(x'') - s_n(x')| \leq 2^n \int_0^1 |f(x'', t) - f(x', t)| \, |g(t) - g(t - 2^{-n})| \, dt \]
\[
\leq 2^n(1-\gamma) \cdot 2\Delta
\]

where
\[
\Delta = \left\{ \int_0^1 |f(x'', t) - f(x', t)|^{1/\alpha} \, dt \right\}^\alpha.
\]

By considering three different cases we prove that

\[ 3.2.8 \quad |s(x'') - s(x')| \leq k(\lambda) \Delta^{1/\alpha}, \]

where \( k(\lambda) \) is a finite constant.

A. Suppose that \( 1 < \Delta < \infty \). Setting \( n = 0 \) in 3.2.6 gives us 3.2.8 if we choose \( k(\lambda) \geq 2\sigma(\lambda) \).

B. Suppose that \( 0 < \Delta \leq 1 \). Choose \( n \geq 1 \) so that \( 2^{-n\beta} < \Delta \leq 2^{-(n-1)\beta} \), and with 3.2.6 we have

\[ |s(x) - s_n(x)| < c(\lambda) \Delta^{1/\alpha} \]

for \( 0 \leq x \leq 1 \). From 3.2.7 we get

\[ |s_n(x'') - s_n(x')| \leq 4\Delta^{1/\alpha}, \]

and 3.2.8 follows if we choose \( k(\lambda) \geq 2(\lambda) + 4 \).

C. Suppose that \( \Delta = 0 \). We see from 3.1.3 and 3.2.7 that

\[ |s(x'') - s(x')| = \lim_{n \to \infty} |s_n(x'') - s_n(x')| = 0, \]

and 3.2.8 follows if we choose \( k(\lambda) \geq 0 \).

To complete the proof for 3.2, take any subdivision \( 0 = x_0 < \cdots < x_N = 1 \). From 3.2.8 we have
The proof introduces unnecessary restrictions. For example, Young’s argument [11, p. 459] allows us to consider the case where \( g(y) \) is not continuous. We conclude this chapter by simply stating the following generalization of 3.2.

**Theorem 3.3.** Suppose that \( 0 < \alpha, \beta, \gamma \leq 1, \) \( 0 < \lambda = \beta + \gamma - 1, \) and \( u = \alpha \lambda / \beta. \) Suppose also that 3.2.1, 3.2.2, and 3.2.3 hold. If \( s(x) \) is the Stieltjes Faltung 3.1.1, then

\[
V^\mu \{ s(x) - \delta f(x, 0); 0 \leq x \leq 1 \} \leq k(\lambda) A^{\lambda / \beta},
\]

where \( \delta = g(1) - g(0) \) and \( k(\lambda) \) is a finite constant.

**Applications to infinite series**

4.1. In this chapter we apply the notion of \( \alpha \)-variation to the study of infinite series. We say that the series

\[
\sum_{n=0}^{\infty} a_n
\]

is \( \alpha \)-convergent if, given \( \epsilon > 0, \) there exists \( N(\epsilon) \) such that

\[
\left\{ \sum_{r=m}^{n} \left| \sum_{r=m}^{n} a_r \right|^{1/\alpha} \right\}^\alpha < \epsilon
\]

for \( N(\epsilon) \leq m < n. \) \( 0 \)-convergence is ordinary convergence and 1-convergence is absolute convergence. If \( 0 \leq \alpha < \beta \leq 1, \) we have by Jensen’s inequality

\[
\left\{ \sum_{r=m}^{n} \left| \sum_{r=m}^{n} a_r \right|^{1/\alpha} \right\}^\alpha \leq \left\{ \sum_{r=m}^{n} \left| \sum_{r=m}^{n} a_r \right|^{1/\beta} \right\}^\beta,
\]

and thus a \( \beta \)-convergent series is always \( \alpha \)-convergent.

We can extend the notion of \( \alpha \)-convergence to sequences. We call \( \{ S_n \} \) an \( \alpha \)-convergent sequence if \( S_n \) is the \( n \)th partial sum of an \( \alpha \)-convergent series. From Minkowski’s inequality we see that any finite linear combination of \( \alpha \)-convergent series (sequences) is itself an \( \alpha \)-convergent series (sequence). We also have the following result.

**Lemma 4.1.2.** (Cf. Lemma 1.4.2.) Suppose that \( 0 < \alpha \leq 1. \) The series 4.1.1 is \( \alpha \)-convergent if and only if

\[
\left\{ \sum_{n=0}^{\infty} \left| \sum_{n=0}^{\infty} a_n \right|^{1/\alpha} \right\}^\alpha < \infty.
\]
The same type of result is true for sequences.

Any series derived from a 1-convergent series by a rearrangement of terms is convergent to the sum of the original series. However, when \( \alpha < 1 \), an \( \alpha \)-convergent series is "conditionally convergent" and little can be said about rearrangement.

A second important property of 1-convergent series is found in multiplication theorems. For example it is well known that the Cauchy product of a 1-convergent series by a 0-convergent series is 0-convergent to the product of the sums of the series. We have the following extension of this result.

**Theorem 4.1.3.** Suppose that \( 0 < \alpha, \beta \leq 1 \), and that \( 0 < \gamma = \alpha + \beta - 1 \). Then the Cauchy product of an \( \alpha \)-convergent series by a \( \beta \)-convergent series is \( \gamma \)-convergent to the product of the sums.

This theorem follows easily from the following specialization of 3.3.

**Theorem 4.1.4.** Suppose that \( 0 < \alpha, \beta \leq 1 \), and that \( 0 < \gamma = \alpha + \beta - 1 \). If \( f(x) \) has bounded \( \alpha \)-variation over \( 0 \leq x < \infty \) and if \( g(x) \) has bounded \( \beta \)-variation over \( 0 \leq x < \infty \), then the Stieltjes Faltung

\[
s(x) = \int_0^x f(x - y) dg(y)
\]

exists in the Young sense for each \( x \) and has bounded \( \gamma \)-variation over \( 0 \leq x < \infty \).

Theorem 4.1.3 holds in the limiting case where \( \alpha + \beta = 1 \) if and only if \( \alpha = 0 \) or \( 1 \); 4.1.3 is also true when one considers the more general Dirichlet product [3, p. 239] instead of the Cauchy product.

4.2. We can apply our scale to the study of Cesaro and Abel summability. Suppose that \( S_n^* \) is the \( n \)th Cesaro mean of order \( \lambda \) for the series \( 4.1.1 \). We say that \( 4.1.1 \) is summable \((C, k; \alpha)\) to \( S \) if the sequence \( \{S_n^*\} \) is \( \alpha \)-convergent to \( S \). Thus \((C, k; 0)\) summability is ordinary Cesaro summability and \((C, k; 1)\) summability is absolute Cesaro summability. Consider the function

\[
f(x) = \sum_{n=0}^{\infty} a_n x^n
\]

and assume that this series converges for \( 0 \leq x < 1 \). We say that \( 4.1.1 \) is summable \((A; \alpha)\) to \( S \) if \( f(x) \) has bounded \( \alpha \)-variation over \( 0 \leq x < 1 \) and if \( \lim_{x \to 1^-} f(x) = S \). \((A; 0)\) summability is ordinary Abel summability and \((A; 1)\) summability is absolute Abel summability. (See [1, p. 11] for references on absolute summability.) A linear combination of series, summable \((C, k; \alpha)\) for some \( k \) and \( 0 \leq \alpha \leq 1 \), is itself a series summable \((C, k; \alpha)\) and similarly for the \((A; \alpha)\) method.

When \( 0 \leq \alpha < \beta \leq 1 \), a series summable \((C, k; \beta)\) to \( S \) is summable \((C, k; \alpha)\) to \( S \) and a series summable \((A; \beta)\) to some limit is summable \((A; \alpha)\) to the
same limit. We can also establish the following consistency result.

**Theorem 4.2.1.** Suppose that \(0 \leq \alpha \leq 1\) and that \(k > j > -1\). A series summable \((C, j; \alpha)\) to \(S\) is summable \((C, k; \alpha)\) and \((A; \alpha)\) to \(S\). If \(S_n^j\) and \(S_n^k\) are the \(n\)th Cesaro means of order \(j\) and \(k\) respectively for 4.1.1, we have

\[
4.2.2 \quad \left\{ \sum_{n=1}^{\infty} \left| \sum_{i=1}^{n} S_i^k - S_n^k \right|^{1/\alpha} \right\}^{\alpha} \leq \left\{ \sum_{n=1}^{\infty} \left| \sum_{i=1}^{n} S_i^j - S_n^j \right|^{1/\alpha} \right\}^{\alpha},
\]

\[
4.2.3 \quad V_{\alpha}\left\{ \sum_{n=0}^{\infty} a_n x^n ; 0 \leq x \leq 1 \right\} \leq \left\{ \sum_{n=1}^{\infty} \left| \sum_{i=1}^{n} S_i^k - S_n^k \right|^{1/\alpha} \right\}^{\alpha}.
\]

For the first of these inequalities let \(k - j = i > 0\) and write

\[
S_n^k = \sum_{m=0}^{n} \frac{C_{n-m}^{i}}{C_n^k} S_m^i, \quad S_n^k = \sum_{m=0}^{\infty} b_{n,m} S_m^i,
\]

where \(C_{n}^{i}\) is the binomial coefficient

\[
\binom{n}{i}.
\]

\(b_{n,m}\) is non-negative for all \(m\) and \(n\) and

\[
\sum_{m=0}^{\infty} b_{n,m} = \frac{1}{C_n^k} \sum_{m=0}^{n} C_{n-m}^{i-1} C_m^i = 1.
\]

When \(0 \leq n' < n'' < \infty\),

\[
b_{n'',m} - b_{n',m} = b_{n',m} \left\{ \frac{(n'' - m + i - 1) \cdots (n' - m + i)}{(n'' - m) \cdots (n' - m + 1)} \frac{n'' \cdots (n' + 1)}{(n'' + k) \cdots (n' + k + 1)} - 1 \right\}
\]

for \(0 \leq m \leq n'\). If \(0 < i < 1\), the bracketed quantity is negative for \(0 \leq m \leq n'\) and, since \(b_{n',m} = 0\) for \(m > n'\), we can apply 2.5.2.

For any subdivision \(0 < x_0 < \cdots < x_N < 1\) let

\[
\sum_{m=0}^{\infty} a_m(x_n)^m = \sum_{m=0}^{\infty} C_m^k(1 - x_n)^{k+1}(x_n)^m S_m^k = \sum_{m=0}^{\infty} b_{n,m} S_m^k.
\]

Again \(b_{n,m}\) is non-negative for all \(m\) and \(n\),

\[
\sum_{m=0}^{\infty} b_{n,m} = (1 - x_n)^{k+1} \sum_{m=0}^{\infty} C_m^k(x_n)^m = 1,
\]

and, for \(0 \leq n' < n'' \leq N\),
The bracketed factor here is an increasing function of \( m \) and 4.2.3 follows from 2.5.2.

The rest of the theorem follows from these two inequalities and a classical result.

4.3. The direct converse of 4.2.1 is not true. The coefficients in the power series expansion for

\[
f(x) = e^{1/(1+x)}
\]

consist of a series which is summable \((A; 1)\) but which is not summable \((C, k; 0)\) for any finite \( k \). However, we can prove some "corrected converses" and the following theorems extend two results due to A. Tauber.

**Lemma 4.3.1.** 1. \( \alpha_n = \sum_{m=1}^{n-1} \frac{(1 - (1 - 1/n)^m)}{m} \) is a positive, increasing sequence bounded by \( 1 \) for \( n \geq 2 \).

2. \( \beta_n = (1 - (1 - 1/n)^n)/n \) is a positive, decreasing sequence bounded by \( 1 \) for \( n \geq 1 \).

3. \( \gamma_n = \sum_{m=n+1}^{n} ((1 - 1/n)^m)/m \) is a positive, increasing sequence bounded by \( 1 \) for \( n \geq 1 \).

Consider the first sequence. If \( r > 1 \), \( x^r - y^r > ry^{r-1}(x - y) \) for any two positive and unequal \( x \) and \( y \). Hence for \( n \geq 2 \) we have

\[
\alpha_{n+1} - \alpha_n = \sum_{m=1}^{n-1} \frac{(1 - 1/n)^m - (1 - 1/(n + 1))^m}{m} + \frac{1 - (1 - 1/(n + 1))^n}{n} 
\]

\[
\geq - \frac{1}{n(n + 1)} \sum_{m=0}^{n-2} \left( 1 - \frac{1}{n + 1} \right)^m + \frac{1 - (1 - 1/(n + 1))^n}{n} 
\]

\[
\geq \frac{1}{n} \left( \left( 1 - \frac{1}{n + 1} \right)^{n-1} - \left( 1 - \frac{1}{n + 1} \right)^n \right) > 0. 
\]

For the boundedness we see that

\[
\alpha_n = \sum_{m=1}^{n-1} \frac{1 - (1 - 1/n)^m}{m} \leq \frac{n - 1}{n} < 1. 
\]

The proofs of 4.3.1.2 and 4.3.1.3 follow along similar lines.

We can now generalize Tauber's first theorem.

**Theorem 4.3.2.** Suppose that \( 0 \leq \alpha \leq 1 \) and that 4.1.1 is summable \((A; \alpha)\) to \( S \). If the sequence \( \{na_n\} \) is \( \alpha \)-convergent to 0, 4.1.1 is \( \alpha \)-convergent to \( S \).

We can assume that \( 0 < \alpha \leq 1 \). For \( 0 \leq x < 1 \) let
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$$f(x) = \sum_{m=0}^{\infty} a_m x^m$$

and let $S_n$ be the $n$th partial sum for 4.1.1, i.e.

$$S_n = \sum_{r=0}^{n} a_r.$$

For $n \geq 2$ we can write

$$S_n - f\left(1 - \frac{1}{n}\right) = A_n + B_n - C_n,$$

where

$$A_n = \sum_{m=1}^{n-1} \frac{1 - (1 - 1/n)^m}{m} (ma_m) = \sum_{m=1}^{\infty} \alpha_{n,m}(ma_m),$$

$$B_n = \frac{1 - (1 - 1/n)^n}{n} (na_n) = \beta_n(na_n),$$

$$C_n = \sum_{m=n+1}^{\infty} \frac{(1 - 1/n)^m}{m} (ma_m) = \sum_{m=1}^{\infty} \gamma_{n,m}(ma_m).$$

$\alpha_{n,m}$ and $\gamma_{n,m}$ are non-negative for all $m$ and $n$ and, by 4.3.1, both $\sum_{m=1}^{\infty} \alpha_{n,m}$ and $\sum_{m=1}^{\infty} \gamma_{n,m}$ are increasing in $n$ and bounded by 1. These two sets of coefficients satisfy the hypotheses of 2.5.1. Hence

$$\left\{ \sum_{n=3}^{\infty} |A_n - A_{n-1}|^{1/\alpha} \right\}^\alpha$$

and

$$\left\{ \sum_{n=3}^{\infty} |C_n - C_{n-1}|^{1/\alpha} \right\}^\alpha$$

are majorized by

$$\left\{ \sum_{m=1}^{\infty} |ma_m - (m - 1)a_{m-1}|^{1/\alpha} \right\}^\alpha,$$

and $\{A_n\}$ and $\{C_n\}$ are $\alpha$-convergent sequences. The sequence $\{B_n\}$ is also $\alpha$-convergent since $\{\beta_n\}$ is 1-convergent to 0. Because 4.1.1 is $(A; \alpha)$ summable, the sequence $\{f(1-1/n)\}$ is $\alpha$-convergent, and we conclude that $\{S_n\}$ is $\alpha$-convergent to $S$.

**Theorem 4.3.3.** Suppose that $0 \leq \alpha \leq 1$ and that the series 4.1.1 is summable $(A; \alpha)$ to $S$. 4.1.1 is $\alpha$-convergent to $S$ if and only if the sequence

$$\left\{ \frac{a_1 + \cdots + na_n}{n} \right\}$$

is $\alpha$-convergent to 0.
For the necessity let $S_n^1$ be the $n$th Cesaro mean of order 1 for 4.1.1 and we see from 4.2.1 that the sequence

$$S_n - S_{n-1}^1 = \frac{a_1 + \cdots + na_n}{n}$$

is $\alpha$-convergent to 0.

For the sufficiency set $b_0 = 0$ and let

$$\sum_{m=0}^{n} b_m = \frac{a_1 + \cdots + na_n}{n}$$

for $n \geq 1$. If $a_n = b_n + c_n$ for all $n$, then

$$c_n = \frac{a_1 + \cdots + (n-1)a_{n-1}}{n(n-1)}$$

for $n \geq 2$. $\sum_{n=0}^{\infty} a_n$ is $(A; \alpha)$ summable to $S$, $\sum_{n=0}^{\infty} b_n$ is $\alpha$-convergent to 0, and hence the series $\sum_{n=0}^{\infty} c_n$ is also $(A; \alpha)$ summable to $S$. By 4.3.2 we see that this series is then $\alpha$-convergent to $S$ and this completes the proof.

We saw in 4.3.2 how the Tauberian condition

$$\{na_n\} \text{ is } \alpha\text{-convergent to } 0$$

allowed us to pass from summability $(A; \alpha)$ to summability $(C, 0; \alpha)$ or $\alpha$-convergence. Actually more is true and we have the following result.

**Theorem 4.3.4.** Suppose that $0 \leq \alpha \leq 1$ and that 4.1.1 is $\alpha$-convergent. If the sequence $\{na_n\}$ is $\alpha$-convergent to 0, 4.1.1 is summable $(C, k; \alpha)$ to its sum for every $k > -1$.

Pick $\delta > 0$. Let $S_n^\delta$ and $S_n^{\delta-1}$ be the $n$th Cesaro means of order $\delta$ and $\delta - 1$ respectively for 4.1.1, and let $T_n^\delta$ be the $n$th Cesaro mean of order $\delta$ for the series whose $n$th partial sum is $na_n$. From the identity $(n+\delta)C_n^{\delta-1}_n - \delta C_n^{\delta-1}_{n-1} = \nu C_n^{\delta-1}_{n-\nu}$, we see that

$$S_n^{\delta-1} - S_n^\delta = \frac{1}{\delta} T_n^\delta.$$

By 4.2.1, $\{S_n^\delta\}$ and $\{T_n^\delta\}$ are $\alpha$-convergent to $S$ and 0 respectively and thus $\{S_n^{\delta-1}\}$ is $\alpha$-convergent to $S$.

In conclusion we construct, for $0 \leq \alpha < 1$, a series which is summable $(C, k; \alpha)$ for every $k > -1$ and which is not summable $(A; \beta)$ for any $\beta > \alpha$.

Let $\{b_k\}$ be any positive decreasing sequence of numbers which approach zero such that

$$\left\{ \sum_{k=1}^{\infty} b_k^{1/\alpha} \right\}^\alpha < \infty \quad \text{and} \quad \left\{ \sum_{k=1}^{\infty} b_k^{1/\beta} \right\}^\beta = \infty$$
for each $\beta > \alpha$. Define a sequence of integers, $1 = n_0 < n_1 < \cdots$, and a set of positive numbers, $\{c_k\}$, such that

$$\sum_{n_{k-1} \leq n < n_k} \frac{1}{n} > 2^k b_k = c_k \sum_{n_{k-1} \leq n < n_k} \frac{1}{n}$$

for $k = 1, 2, \cdots$. Set $a_0 = 0$ and $a_n = ((-1)^k/n)2^{-k}c_k$ for $n_{k-1} \leq n < n_k$. Then

$$\sum_{n=n_k}^{\infty} \left| \sum_{n=n_k}^{n} a_n \right|^{1/\rho} = \left( \sum_{j=k+1}^{\infty} b_j^{1/\rho} \right)^\rho$$

for each $0 \leq \rho \leq 1$, and the series is $\alpha$-convergent but not $\beta$-convergent for any $\beta > \alpha$. Since

$$\sum_{n=1}^{\infty} |na_n - (n - 1)a_n| = 2 \sum_{k=1}^{\infty} 2^{-k}c_k < 2,$$

the sequence $\{na_n\}$ is 1-convergent to 0, and we see from 4.3.2 and 4.3.4 that this series has the desired properties.

**References**


Cambridge University, Cambridge, England.

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