

A GENERALIZED NOTION OF BOUNDARY⁽¹⁾

BY

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This paper reports some recent results in the theory of generalized parametric surfaces. The object of this theory, introduced by Young in [18], is to generalize the elementary notions of parametric surface in such a way that a general principle of minimum holds for surface integral variational problems in parametric form. Since nonregular problems are not excluded, the notion of generalized parametric surface is of necessity more general than the usual concepts of surface, involving in many instances both a track (surface in the ordinary sense, possibly mildly discontinuous) and a linear average over a space of Jacobian matrices at each point of the track (for ordinary surfaces the average is the trivial one in which the normal has weight one and all others weight zero) ⁽²⁾.

In this paper, we first introduce abstract notions of boundary and track, defined for all non-negative linear functionals L on a certain linear space. Since, in particular, every generalized parametric surface is such a linear functional, these definitions apply to all generalized parametric surfaces. We next specialize to admit only boundaries which correspond in a certain way to boundary curves in the sense of classical analysis, and characterize the set of all L with boundary in this class. Finally, we specialize further by imposing a condition of indecomposibility with respect to the boundary.

As an application, some conditions are established for equivalence of indefinite variation problems which have fixed boundary conditions with positive definite problems.

1. Introduction. The main definitions and results of the paper are outlined in this section. Details and auxiliary results are left to later sections.

Let E^m ($m \geq 3$) denote the space of all continuous functions $f(x, J)$ of an m -dimensional vector $x = [x^1, \dots, x^m]$ and an $m \times m$ skew-symmetric matrix $J = [j^{rs}]$ of rank 2 which satisfy the homogeneity condition $f(x, kJ) = kf(x, J)$ for $k \geq 0$. A mapping $x(u, v)$ from the square $U: [0 \leq u \leq 1; 0 \leq v \leq 1]$ to m -space R^m is termed a *Dirichlet (parametric) representation* if:

- (1) $x(u, v)$ is absolutely continuous in Tonelli's sense; and
- (2) $\iint_U [x_u^2 + x_v^2] dudv < \infty$.

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⁽¹⁾ A number of the results presented here also appear in the first author's Ph.D. Thesis at the University of Wisconsin, August 1951.

⁽²⁾ The notion of generalized parametric surface is in many respects analogous to that of generalized curve and generalized nonparametric surface introduced a few years ago by Young [14; 15; 16] (cf. also McShane [10]).

Let $J(u, v)$ denote the matrix of Jacobians $x_u^r x_v^s - x_u^s x_v^r$ of the partial transformations $x^r = x^r(u, v)$, $x^s = x^s(u, v)$ of $x(u, v)$. For convenience we refer to $J(u, v)$ simply as the *Jacobian* of $x(u, v)$.

Since it is variational problems involving surface integrals which motivate our theory, we begin, as in [18], with a notion of surface expressed directly in terms of these integrals, rather than the more familiar notion of Fréchet surface: A *Dirichlet (parametric) surface* with Dirichlet representation $x(u, v)$ is the non-negative linear functional L defined, for all f in E^m , by the surface integral

$$(L, f) = \int \int_U f[x(u, v), J(u, v)] du dv.$$

In this paper the symbol L represents in general an arbitrary *non-negative* linear functional on E^m , of which every Dirichlet surface and, more generally, every classical parametric surface is a particular case. As notion of convergence of linear functionals L we use as in [18] w^* convergence, i.e., convergence of the numbers (L, f) for all f . As used in [18], the notion of *generalized parametric surface* is equivalent to that of non-negative linear L .

We call *exact integrands* those ϕ in E^m for which the surface integral (L, ϕ) vanishes whenever L is a closed polyhedron. We define the *generalized boundary* (abbreviated *g-boundary*) for an arbitrary non-negative linear L to be the linear functional $\lambda(L)$ obtained by restricting the domain of L to exact integrands⁽³⁾. The *g-boundary* is a generalization of line integration around a boundary curve, paths traversed in opposite senses being allowed to cancel. Similarly, we define the *generalized track* of L to be the restriction of L to integrands f which are linear in the variable J .

A *g-boundary* λ is called *admissible* if $\lambda = \lambda(L)$ for some Dirichlet surface L .

(1.1) THEOREM. *Let λ_0 be admissible. Then a necessary and sufficient condition that L_0 have g-boundary $\lambda(L_0) = \lambda_0$ is that L_0 be the w^* limit of a sequence of linear combinations $\sum k_j L_j$ with positive coefficients k_j of Dirichlet surfaces L_j with $\lambda(L_j) = \nu_j \lambda_0$, where each ν_j is a positive integer and $\sum k_j \nu_j = 1$. In the particular case when the dimension m of x -space R^m is 3, one can suppose further that all $\nu_j = 1$.*

We term L *closed* if $\lambda(L) = 0$. $L \neq 0$ is *basic closed* if $L = L_1 + L_2$ with L_1, L_2 closed implies $L_1 = kL$ for some k , $0 \leq k \leq 1$. If $\lambda(L) = \lambda \neq 0$, L is termed *basic* (λ) if L is an extreme point of the set of all functionals with *g-boundary* λ .

(1.2) THEOREM. *Suppose $m = 3$. Then if $\lambda \neq 0$ is admissible and L_0 is basic (λ), L_0 is the w^* limit of a sequence of Dirichlet surfaces with *g-boundary* λ .*

In stating the analogue of (1.2) for $\lambda = 0$, one must take account of the

⁽³⁾ In other words, $\lambda(L)$ is defined by the equation $(\lambda(L), \phi) = (L, \phi)$ for all exact ϕ .

fact that kL is basic closed whenever L is basic closed and $k > 0$:

(1.3) THEOREM. *Suppose $m = 3$. Then if L_0 is basic closed, kL_0 is the w^* limit of a sequence of closed Dirichlet surfaces for some $k > 0$.*

A non-negative linear L is said to be *situated* in a subset of A of R^m if a closed (nonempty) subset A_1 of A exists such that $(L, f) = 0$ whenever $f(x, J) = 0$ for all x in A_1 and all J . L is termed *concentrated* if L is situated in a point x_0 .

The methods used to prove (1.1) also yield:

(1.4) THEOREM. *Let f_0 be in E^m and A a compact, convex subset of R^m . Then a necessary and sufficient condition that an exact integrand ϕ and an $\epsilon > 0$ exist such that $f_0(x, J) \geq \phi(x, J) + \epsilon |J|$ for all x in A and all J is that either of the following hold:*

(a) $(L, f_0) > 0$ for every basic closed L situated in A ;

(b) *There exists an open set Ω containing A and an $\epsilon_1 > 0$ such that $(L, f_0) \geq \epsilon_1(\text{area of } L)$ for every closed polyhedron L situated in Ω .*

From (1.4) follow conditions for a variational problem with fixed boundary conditions to be equivalent to a positive definite problem. In particular, (1.4) allows the hypothesis of positive definiteness to be weakened in the statement of known theorems [3; 6; 13] insuring the existence of Dirichlet surface solutions for certain semi-regular problems with fixed boundary conditions (cf. §13).

2. Notation; the norm and modulus. In applying the ideas of topology and linear space theory in the calculus of variations, some unfortunate conflicts arise between the prevailing terminology in the respective fields. Let us state the position taken here with regard to two such conflicts. First, we apply the word *closed* both to surfaces (as well as to more general non-negative linear functionals L) and to subsets of topological spaces. Second, we use the term *extreme point* in the linear space sense and avoid use of *extremum* and its variants as they appear in the calculus of variations.

In addition to that introduced in §1, the following notation will be used throughout: S^m denotes the set of skew symmetric $m \times m$ matrices of rank 0 or 2. Integrands $f(x, J)$ are assumed to be defined for (x, J) in $R^m \times S^m$. For set sum and intersection, \cup and \cap are used. $|X|_k$ denotes the k -dimensional Lebesgue measure of a subset X of R^m . The adjoint of a matrix T is denoted by T' . All sums are finite; the range of summation is omitted when the meaning is clear.

For an $m \times m$ matrix $J = [j^{rs}]$, let $|J| = |\sum_{r,s} (j^{rs})^2|^{1/2}$. Define f_a in E^m by the identity $f_a(x, J) = |J|$. For each non-negative linear functional L on E^m , define the *norm* $a(L) = (L, f_a)$. If L is a Dirichlet surface with Dirichlet representation $x(u, v)$ on the unit square U , and $J(u, v)$ is the Jacobian of $x(u, v)$, then

$$a(L) = \int \int_U |J(u, v)| \, dudv$$

coincides with the *Lebesgue area* of the Fréchet surface with representation $x(u, v)$. If $x(u, v)$ satisfies a Lipschitz condition, the value for the area according to all reasonable definitions of area agrees with $a(L)$.

The *modulus* (or *essential bound*) $b(L)$ of L is defined to be the smallest b such that L is situated in the sphere $|x| \leq b$.

POLYHEDRA; EXACT INTEGRANDS

3. Polyhedra; elementary boundary. Having introduced notions of surface and boundary in terms of linear functionals in §1, we next define in these same terms an “orientation reversing” operation (*). For each f , define f^* by the identity $f^*(x, J) = f(x, -J)$. For each L , define L^* by:

$$(L^*, f) = (L, f^*).$$

If L is a Dirichlet surface with Dirichlet representation $x(u, v)$, then L^* has Dirichlet representation $x(1-u, v)$; i.e., L^* is “oriented in the sense opposite to that of L .”

L_i is termed a *triangle* if an oriented triangle t in R^m exists such that, for all f , (L_i, f) is the value of the surface integral of f over t . Let t^* denote t with opposite orientation. Then:

$$L_i^* = L_{t^*}.$$

L is a *polyhedron* if L is a sum of triangles. More generally, a linear combination with positive real coefficients of triangles L_i is termed a *polyhedron over the group R* of real numbers.

A mapping $x(u, v)$ from U to R^m is *quasi-linear* if $x(u, v)$ is linear in each triangle of a suitable finite subdivision of U . A quasi-linear mapping evidently defines a polyhedron L . It is easy to verify that, conversely, every polyhedron L has a quasi-linear representation. Each polyhedron L over R may be written:

$$(3.1) \quad L = \sum [a_i L_{t_i} + a_i^* L_{t_i^*}],$$

where the a_i, a_i^* are non-negative real numbers and the triangles t_i meet at most along common edges and at common vertices. Such a decomposition is called a *triangulation* of L , and those $t_i (t_i^*)$ for which $a_i > 0 (a_i^* > 0)$ are called *faces* of L . If L is a polyhedron, the a_i, a_i^* are integers, and conversely. If all $a_i = 1$ and $a_i^* = 0$, then L is said to be *realized geometrically* by the geometric polyhedron $P = \cup_i t_i$.

We next define an elementary notion of boundary as a weighted combination of bounding line segments, segments weighted equally but with opposite directions being allowed to cancel. This notion is made precise as follows:

For each triangle t , let β_t denote the linear functional on the space of forms $q = \sum q^r(x)dx^r$ with continuously differentiable coefficients q^r whose value (β_t, q) is, for each q , the value of the line integral of q around the perimeter of t . Given a triangulation (3.1) of a polyhedron L over R , define the *elementary boundary (e-boundary)* $\beta(L)$ of L to be:

$$\beta(L) = \sum [a_i\beta_{t_i} + a_i^*\beta_{t_i}^*].$$

By replacing any pair of triangulations by one finer than both, one finds that $\beta(L)$ is independent of the triangulation of L . Furthermore:

(3.2) *The mapping $L \rightarrow \beta(L)$ preserves sums and positive scalar multiples. $\beta(L^*) = -\beta(L)$.*

In other words, $\beta(L)$ behaves like the notion of boundary used in combinatorial topology. If $\beta = \beta(L)$ for some polyhedron L , then β is called a *closed polygonal path*.

A polyhedron L over R is termed *closed* if $\beta(L) = 0$ (this definition will be shown to agree with the definition $\lambda(L) = 0$ of closed given in §1). A polyhedron of the form $L_t + L_t^*$ is called a *double triangle*, since it corresponds geometrically to a pair of triangles back to back. A double triangle is evidently closed.

(3.3) LEMMA. *If a polyhedron L over R has e-boundary a closed polygonal path β_0 , then L is a linear combination with positive coefficients of polyhedra each of which has e-boundary a positive integral multiple of β_0 .*

Proof. Given a triangulation (3.1) of L , let K denote the finite euclidean 2-complex (in the terminology of Lefschetz [9]) generated by the triangles t_i , i.e., which has as simplexes the t_i plus their edges and vertices and orientation induced by that of the t_i . For $s = 1, 2$ and $G = R$ or I , the additive group of integers, let $C^s(K, G)$, $Z^s(K, G)$ denote, respectively, the groups of s -chains and s -cycles over K with coefficients in G . Let $\gamma(L)$ in $C^2(K, R)$ be defined by:

$$(3.4) \quad \gamma(L) = \sum [a_i t_i + a_i^* t_i^*] = \sum [a_i - a_i^*] t_i$$

where a_i, a_i^* are as in (3.1).

Under the correspondence induced by identifying β_t with the chain boundary ∂t of a 2-simplex t of K , $\beta_0 = \beta(L)$ corresponds to the chain boundary $\partial \gamma(L)$ of $\gamma(L)$. Since β_0 is a closed polygonal path, $\partial \gamma(L)$ is in $Z^1(K, I)$. Then since $\partial \gamma(L)$ bounds in $C^2(K, R)$, a positive integral multiple $\nu \partial \gamma(L)$ bounds some γ_1 in $C^2(K, I)$ (cf. Lefschetz [9, p. 106, (16.6)]). Since $\nu \gamma(L) - \gamma_1$ is in $Z^2(K, R)$, real coefficients p_j and z_j in $Z^2(K, I)$ exist [9, p. 105, (16.4)] such that:

$$(3.5) \quad \nu \gamma(L) - \gamma_1 = \sum p_j z_j.$$

We may suppose all p_j positive by changing signs of the z_j as required. For

each $\gamma = \sum \alpha_i t_i$, let:

$$L_\gamma = \sum [\max (\alpha_i, 0)L_{t_i} + \max (-\alpha_i, 0)L_{t_i}^*].$$

Then from (3.4):

$$(3.6) \quad L = L_{\gamma(L)} + \sum [\min (a_i, a_i^*)][L_{t_i} + L_{t_i}^*].$$

In other words, $L_{\gamma(L)}$ is obtained by eliminating the double triangles from L . Since by (3.5)

$$\gamma(L) = \nu^{-1}\gamma_1 + \sum \nu^{-1}p_j z_j,$$

non-negative k_i exist such that:

$$(3.7) \quad L_{\gamma(L)} + \sum k_i(L_{t_i} + L_{t_i}^*) = \nu^{-1}L_{\gamma_1} + \sum \nu^{-1}p_j L_{z_j}.$$

The double triangle terms on the left side of (3.7) appear because triangles with opposite orientation cancel when added in the combinatorial sense but add to form a double triangle when regarded as non-negative linear functionals.

We next obtain:

$$(3.8) \quad L_{\gamma(L)} = \sum h_j L_j,$$

where the h_j are positive and the L_j are polyhedra bounded by non-negative integral multiples of β_0 . To get (3.8), we eliminate the linear combination of double triangles on the left side of (3.7) by means of a suitable finite sequence of operations of the following type: Suppose that L' and L'' are polyhedra with $\beta(L') = \nu'\beta_0$, $\beta(L'') = \nu''\beta_0$, where ν' , ν'' are non-negative integers, and that t_i , t_i^* are faces, respectively, of L' , L'' . Then if $0 \leq h' \leq h''$,

$$(3.9) \quad \begin{aligned} h'L' + h''L'' - h'(L_{t_i} + L_{t_i}^*) \\ = (h'' - h')L'' + h'[L' + L'' - (L_{t_i} + L_{t_i}^*)]. \end{aligned}$$

The two polyhedra appearing on the right side of (3.9) are bounded by non-negative integral multiples of β_0 , and the coefficients $h'' - h'$, h' are non-negative.

Having obtained (3.8) in this fashion, note next that, by (3.6), L is likewise a positive linear combination of polyhedra bounded by non-negative multiples of β_0 . By easy manipulations this sum may be rewritten so that no closed polyhedra appear, since closed polyhedra can be added to arbitrary polyhedra without changing the e -boundaries of the latter. This establishes (3.3).

4. Exact integrands. We use the term *plane* to denote an oriented, 2-dimensional, linear subset of R^m . Each plane π has various linear representations $x(u, v) = x_1u + x_2v + x_3$. The Jacobian of $x(u, v)$ is the $m \times m$ matrix

$$J_x = [x_1^r x_2^s - x_1^s x_2^r].$$

$J = |J_x|^{-1}J_x$ is independent of the linear representation $x(u, v)$ of π ; we term J the *Jacobian of the plane* π .

For $r, s = 1, 2, \dots, m$, let

$$J_{rs} = [\delta_{kr}\delta_{ls} - \delta_{ks}\delta_{lr}]$$

denote the Jacobian of the $x^r x^s$ plane π_{rs} (δ_{ij} is Kronecker's delta). Under an affine transformation $x_T = Tx + x_0$, the Jacobian J changes according to $J_T = TJJ'$. Furthermore, it is well known that a necessary and sufficient condition that J be skew-symmetric of rank 2 is that T exist such that $J = TJ_{12}T'$, i.e., such that $|J|^{-1}J$ is the Jacobian of the plane $T\pi_{12}$. We have then:

(4.1) *A necessary and sufficient condition that an $m \times m$ matrix J be Jacobian of some plane π is that J be skew-symmetric of rank 2 and $|J| = 1$. Two planes have the same Jacobian if and only if they differ by a translation in R^m .*

REMARK. For $m = 3$, the vector $N = [j^{23}, j^{31}, j^{12}]$ is the positive unit normal to π , and J may be identified with N .

Given a Dirichlet representation $x(u, v) = [x^1(u, v), \dots, x^m(u, v)]$ with Jacobian $J(u, v)$, let (u_0, v_0) be a point at which $J(u, v)$ exists and is not zero. Then the plane π with linear representation $x_u(u_0, v_0)u + x_v(u_0, v_0)v + x(u_0, v_0)$ has Jacobian $|J(u_0, v_0)|^{-1}J(u_0, v_0)$. By (4.1), $J(u_0, v_0)$ is of rank 2. Thus:

(4.2) *The range of the Jacobian $J(u, v)$ of a Dirichlet representation $x(u, v)$ lies in the set S^m of skew-symmetric $m \times m$ matrices of rank 0 or 2.*

It also follows from (4.1) that every J in S^m is assumed by the Jacobian of some $x(u, v)$. This justifies our previous assumption that integrands f are defined for (x, J) in $R^m \times S^m$.

It is convenient to relax the requirement of continuity in (x, J) originally made in the definition of exact integrand to suppose merely *continuity in x for each J* . Since each ϕ exact according to the new definition will now be proved linear in J , the two definitions are equivalent.

(4.3) *If ϕ is exact and $J_1, J_2, J_1 + J_2$ are all in S^m , then $\phi(x, J_1) + \phi(x, J_2) = \phi(x, J_1 + J_2)$ for all x .*

Proof. One may clearly suppose $J_1, J_2 \neq 0$. Given x , let π_1, π_2 be planes through x with Jacobians $|J_1|^{-1}J_1, |J_2|^{-1}J_2$, respectively. Let $x(u, v) = x_1u + x_2v + x$ be a linear representation of π_1 . For $r = 1, 2, \dots, m$, the r th row vector X_r of J_1 is a multiple of $x_1^r x_2 - x_2^r x_1$. Thus, the X_r lie in the linear 2-space $\pi_1 - x$ spanned by x_1 and x_2 (disregarding the orientation of π_1 for the present). Since J_1 is of rank 2, the X_r themselves span $\pi_1 - x$. Letting y_1, y_2, Y_r be similarly defined for π_2 , the Y_r span $\pi_2 - x$.

Now $X_r + Y_r$ is the r th row vector of $J_1 + J_2$, for each $r = 1, 2, \dots, m$. Since $J_1 + J_2$ is in S^m , it is of rank at most 2; hence, there exist at least $m - 2$ independent linear relations:

$$\sum_{r=1}^m \alpha_{jr}(X_r + Y_r) = 0.$$

Suppose $\sum \alpha_{jr} X_r = \sum \alpha_{jr} Y_r = 0$ for all j . Then the X_r, Y_r lie in a common 2-space, which must by the remarks above coincide with both $\pi_1 - x, \pi_2 - x$. Hence, π_1 and π_2 coincide except for orientation in this case. If, on the other hand, for some j

$$\sum \alpha_{jr} X_r = - \sum \alpha_{jr} Y_r = \bar{x} \neq 0,$$

then $\bar{x} + x$ is in $\pi_1 \cap \pi_2$ since the X_r span $\pi_1 - x$ and the Y_r span $\pi_2 - x$. In either case, π_1 and π_2 lie in at least one common 3-dimensional linear subset H of R^m .

We now define as follows a sequence (L_n) of closed polyhedra with geometric realizations in H and tending to x : L_n is realized by a triangular prism whose lateral faces, respectively, have Jacobians $|J_1|^{-1} J_1, |J_2|^{-1} J_2, -|J_1 + J_2|^{-1} (J_1 + J_2)$, and are of length n^{-1} and width $n^{-2} |J_1|, n^{-2} |J_2|, n^{-2} |J_1 + J_2|$. This construction is evidently valid for $m = 3$ and follows for arbitrary dimension m upon mapping H onto $x^1 x^2 x^3$ space by a rigid motion, which preserves both norms and sums of matrices J . The sum of the areas of the bases of L_n is then no more than $n^{-4} |J_1| |J_2|$. If $J_1 = k J_2$, then all L_n collapse into "double rectangles."

Since ϕ is exact and all L_n are closed, $(L_n, \phi) = 0$ for all n . Since ϕ is continuous in x for each J , it follows that:

$$0 = \lim_n n^3 (L_n, \phi) = \phi(x, J_1) + \phi(x, J_2) + \phi[x, -(J_1 + J_2)].$$

Applying this first to the case $J_2 = -J_1$, we get that $\phi(x, -J) = -\phi(x, J)$ for all J . Then in the general case:

$$\begin{aligned} 0 &= \phi(x, J_1) + \phi(x, J_2) + \phi[x, -(J_1 + J_2)] \\ &= \phi(x, J_1) + \phi(x, J_2) - \phi(x, J_1 + J_2), \end{aligned}$$

which proves (4.3).

From (4.3) follows the linearity of ϕ in the variable J :

(4.4) *If ϕ is exact, then for all x and $J = [j^{rs}]$ in S^m ,*

$$\phi(x, J) = \sum_{r < s} \phi(x, J_{rs}) j^{rs} = \frac{1}{2} \sum_{r, s=1}^m \phi(x, J_{rs}) j^{rs}.$$

(4.4) may be obtained easily by induction on m from (4.3) using decompositions of the form $J = \sum_s J_{rs} j^{rs} + J_1$, where J_1 has all zeros in the r th row and column.

If suitable differentiability conditions are satisfied, then exact integrands correspond to *exterior derivatives* of forms $q = \sum q^r(x) dx^r$:

(4.5) *If ϕ is exact and continuously differentiable, then a form $q = \sum q^r(x) dx^r$ with continuously differentiable coefficients $q^r(x)$ exists such that, for all x and $J = [j^{rs}]$,*

$$(4.6) \quad \phi(x, J) = \sum_{r < s} (q_r^s - q_s^r) j^{rs}.$$

Conversely, for each q the corresponding ϕ defined by (4.6) is exact.

The first half of (4.5) follows from a theorem of De Rham, essentially to be found in Hodge [7, Chap. II]. The second half is a corollary of Stokes' theorem.

If one sets, in particular, $q^s(x) = x^r$ for $s = s_1$ and $q^s(x) = 0$ for $s \neq s_1$, then (4.5) implies that $\phi_{rs}(x, J) \equiv j^{rs}$ is exact for each r, s . It follows by linearity that each linear combination

$$\phi(J) = \sum_{r,s} \alpha_{rs} j^{rs}$$

with constant coefficients α_{rs} is exact.

The following remark provides the basis for passing from the elementary notion of boundary introduced in §3 for polyhedra to the notion of generalized boundary:

(4.7) *Two polyhedra L_1, L_2 over R have the same e -boundary if and only if $(L_1, \phi) = (L_2, \phi)$ for all exact ϕ .*

Proof. In view of (3.2) and (4.4), it suffices to show that $L_1 + L_2^*$ is closed if and only if $(L_1 + L_2^*, \phi) = 0$ for all exact ϕ . The necessity half of the latter assertion follows at once from the definition of exact and (3.3), while the sufficiency follows easily from Stokes' theorem.

5. A fundamental lemma. Lemma (5.1) is of fundamental importance in this paper; it is the key to the proofs given for the main theorems.

For each open subset Ω of R^m , let B_Ω denote the linear space consisting of all β which bound polyhedra over R situated in Ω .

(5.1) **LEMMA.** *Let $\Omega \subset R^m$ be bounded, open, and convex. Then given β_0 in B_Ω and f_0 in E^m such that the surface integral (L, f_0) is non-negative for every closed polyhedron L situated in Ω , there exists a measurable function $\phi_0(x, J)$ and sequences (ϕ_ν) of exact integrands and (ρ_ν) tending to 0 such that:*

(1) *The inf of (L, f_0) in the set of all polyhedra L over R situated in Ω with $\beta(L) = \beta_0$ is (L_1, ϕ_0) , where L_1 denotes any member of this set;*

(2) $\phi_0(x, J) = \lim_\nu \phi_\nu(x, J)$ for almost all x ;

(3) $\lim_\nu (L, \phi_\nu) = (L, \phi_0)$ for every polyhedron L over R ; and, for each $\nu = 1, 2, \dots,$

(4) $\phi_\nu(x, J) \leq f_0(x, J) + \rho_\nu |J|$ for (x, J) in $\Omega \times S^m$ (4).

Proof. For β in B_Ω , let

$$\mu(\beta) = \inf (L, f_0), \quad \|\beta\|_\Omega = \inf a(L),$$

where the inf is in each case taken in the set of all polyhedra L over R situated in Ω with $\beta(L) = \beta$. Let $\|\beta\|$ denote the corresponding inf for $\Omega = R^m$. It is easy to verify that $\|\beta\|_\Omega$ and $\|\beta\|$ have the usual norm properties. Since the

(4) For a corresponding result for curves, see [5].

set Ω in (5.1) is convex, $\|\beta\| = \|\beta\|_\Omega$ for β in B_Ω .

By (3.3), every closed polyhedron L over R is a positive linear combination of closed polyhedra. It then follows from the hypotheses of (5.1) that $(L, f_0) \geq 0$ for every closed polyhedron L over R situated in Ω . Thus $\mu(0) \geq 0$; since there are closed polyhedra of arbitrarily small norm situated in Ω , $\mu(0) = 0$. For all β, β' in B_Ω :

$$\mu(\beta + \beta') \leq \mu(\beta) + \mu(\beta') \quad \text{and} \quad \mu(k\beta) = k\mu(\beta) \quad \text{for } k \geq 0.$$

Since $0 = \mu(0) \leq \mu(\beta) + \mu(-\beta)$, $\mu(\beta)$ is finite for all β in B_Ω . Then by the well known Hahn-Banach theorem [2, p. 29], a linear functional l on B_Ω exists such that $(l, \beta) \leq \mu(\beta)$ for all β in B_Ω and $(l, \beta_0) = \mu(\beta_0)$. Let

$$M = \sup_{x \in \Omega, |J|=1} |f_0(x, J)|.$$

For each β in B_Ω and L situated in Ω with $\beta(L) = \beta$,

$$(l, \beta) \leq \mu(\beta) \leq (L, f_0) \leq Ma(L),$$

and

$$-(l, \beta) = (l, -\beta) \leq \mu(-\beta) \leq (L^*, f_0) \leq Ma(L^*) = Ma(L).$$

It follows that for β in B_Ω , $|(l, \beta)| \leq M\|\beta\|_\Omega$, or since $\|\beta\| = \|\beta\|_\Omega$,

$$(5.2) \quad |(l, \beta)| \leq M\|\beta\|.$$

By a corollary of the Hahn-Banach theorem [2, p. 55], l has an extension to the space of all β such that (5.2) continues to hold.

Let J_0 in S^m with $|J_0| = 1$ be given; let π_0 denote the plane through $x = 0$ with Jacobian J_0 , and T the matrix of an orthogonal transformation carrying the x^1x^2 plane π_{12} into π_0 . Let V_1, \dots, V_m denote the column vectors of T . For each $p = [p^3, \dots, p^m]$, let

$$x_p(u, v) = V_1u + V_2v + \sum_{s=3}^m V_s p^s,$$

and π_p the plane which has linear representation $x_p(u, v)$. The planes π_p cover R^m and have Jacobian J_0 (cf. §4).

For each uv interval Δ and each p , let $L_{\Delta,p}$ denote the polyhedron realized geometrically by the rectangle $\Delta_p = x_p(\Delta)$, in π_p . Let $\beta_{\Delta,p}$ denote the ϵ -boundary of $L_{\Delta,p}$ and

$$(5.3) \quad F_p(\Delta) = (l, \beta_{\Delta,p}).$$

By (5.2) and the orthogonality of T ,

$$|F_p(\Delta)| \leq M|\Delta_p|_2 = M|\Delta|_2$$

for all Δ . Moreover, F_p is an additive function of intervals Δ , since l is an

additive function of boundaries β . Consequently, for each p the derivative $D_p(u, v)$ of F_p exists for almost all (u, v) , and for all Δ (cf. Saks [12, Chap. IV]),

$$(5.4) \quad F_p(\Delta) = \int \int_{\Delta} D_p(u, v) du dv.$$

The choice of J_0 of norm 1 was arbitrary; hence, we write $D_p(u, v)$ as a function $D_p[u, v; J]$ of J as well as (u, v) . Define $\phi_0(x, J)$ for $x = x_p(u, v)$ in π_p and $|J| = 1$ by:

$$\phi_0[x_p(u, v), J] = \begin{cases} D_p[u, v; J], & \text{if } D_p[u, v; J] \text{ exists;} \\ f_0[x_p(u, v), J], & \text{otherwise.} \end{cases}$$

The latter definition applies for each p only in a subset of π_p of measure 0; hence, by Fubini's theorem, it applies only for a subset of R^m of m -dimensional measure 0. The definition of ϕ_0 is extended by homogeneity to J of arbitrary norm.

We next observe that, for all polyhedra L over R ,

$$(5.5) \quad (L, \phi_0) = (l, \beta(L)).$$

For, (5.5) follows for triangles easily from the definition of ϕ_0 , (5.2), and (5.4). It then follows by linearity for all polyhedra over R . Assertion (1) of (5.1) is now an immediate consequence of (5.5) and the fact that $(l, \beta_0) = \mu(\beta_0)$.

To show that ϕ_0 is measurable, let (N_ν) denote a binary sequence [12, p. 191] of nets N_ν of half-open uv intervals Δ , and J_0 a fixed element of S^m with $|J_0| = 1$. For each $\nu = 1, 2, \dots$ and Δ in N_ν , let

$$h_\nu(x) = \frac{(l, \beta_{\Delta, p})}{|\Delta|_2}$$

for x in $\Delta_p = x_p(\Delta)$. Now $h_\nu(x)$ is continuous in the interior of the block $D_\Delta = \cup_p \Delta_p$. To see this, consider $x \in \Delta_p, x' \in \Delta_{p'}$ in D_Δ , and let L denote the polyhedron realized by joining corresponding corners of the rectangles $\Delta_p, \Delta_{p'}$. Since T is orthogonal,

$$a(L) = \sigma |p - p'| \leq \sigma |x - x'|$$

where σ denotes the length of the perimeter of Δ .

$$\beta(L) = \pm (\beta_{\Delta, p} - \beta_{\Delta, p'}).$$

Hence,

$$(5.6) \quad \begin{aligned} |\Delta|_2 |h_\nu(x) - h_\nu(x')| &= |(l, \beta_{\Delta, p}) - (l, \beta_{\Delta, p'})| = |(l, \beta_{\Delta, p} - \beta_{\Delta, p'})| \\ &= |(l, \beta(L))| \leq M a(L) \leq M \sigma |x - x'|, \end{aligned}$$

and $h_\nu(x)$ is Lipschitzian and therefore continuous in D_Δ .

Since $h_\nu(x)$ is thus piecewise continuous for $\nu = 1, 2, \dots$ and

$$\lim_{\nu} h_\nu(x) = \phi_0(x, J_0)$$

for almost all x , $\phi_0(x, J_0)$ is measurable. Since J_0 is arbitrary and ϕ_0 will now be proved linear in J , ϕ_0 is measurable in (x, J) .

We may now define for $\nu = 1, 2, \dots$ and all J in S^m the mean value

$$\phi_\nu(x, J) = \frac{1}{|D_\nu|_m} \int_{D_\nu} \phi_0(x + x', J) dx'$$

over a cube D_ν in R^m with center 0 and diameter ν^{-1} . By (5.2) and the definition of ϕ_0 , $|\phi_0(x, J)| \leq M|J|$ for all (x, J) ; hence, for all $\nu = 1, 2, \dots$ and all (x, J) :

$$(5.7) \quad |\phi_\nu(x, J)| \leq M|J|.$$

Let L be a closed polyhedron with quasi-linear representation $x(u, v)$. Then for every x' , $x(u, v) + x'$ is a quasi-linear representation with the same Jacobian as $x(u, v)$ of a closed polyhedron L' (the translate of L by x'). Using this remark, Fubini's theorem, and the fact that by (5.5) $(L', \phi_0) = 0$ for all such L' , one finds that $(L, \phi_\nu) = 0$ for all ν . Since each ϕ_ν is a continuous function of x and L is arbitrary, each ϕ_ν is exact (cf. the definition of exact given in §4). $\phi_\nu(x, J)$ converges to $\phi_0(x, J)$ for almost all x given J , and hence simultaneously for J in a countable dense subset of S^m except for a fixed null set N of x . Then, by (5.2), $\phi_\nu(x, J)$ converges to $\phi_0(x, J)$ for all J and x not in N . Thus, (2) of (5.1) holds.

For $\nu = 1, 2, \dots$, define linear functionals l_ν by the equation

$$(l_\nu, \beta(L)) = (L, \phi_\nu).$$

From (5.7) it follows that, for all β ,

$$(5.8) \quad |(l_\nu, \beta)| \leq M\|\beta\|.$$

Let J_0 of norm 1 be given, and x_p, π_p as before. The sequence $\phi_\nu(x, J_0)$ converges to $\phi_0(x, J_0)$ almost everywhere in R^m and hence, except for a set H of p such that $|H|_{m-2} = 0$, almost everywhere in π_p . It then follows from the uniform boundedness (5.7) of the ϕ_ν that for every triangle t in π_p for p not in H ,

$$(5.9) \quad \begin{aligned} (l_\nu, \beta_t) &= \iint_{\pi_p^{-1}(t)} \phi_\nu[x_p(u, v), J_0] dudv \\ &\rightarrow \iint_{\pi_p^{-1}(t)} \phi_0[x_p(u, v), J_0] dudv = (l, \beta_t) \end{aligned}$$

as $\nu \rightarrow \infty$. It follows from (5.8) that (5.9) holds for arbitrary p , since an arbitrary t with direction J_0 can be approximated in the $\| \cdot \|$ sense by a sequence of triangles in planes π_p for p not in H . Since J_0 is arbitrary, assertion (3) of (5.1) follows by linearity and (5.5).

By the definition of ϕ_0 , l , and μ , one has for $x = x_p(u, v)$ in Ω :

$$(5.10) \quad \phi_0[x_p(u, v), J_0] = \lim_{\Delta \rightarrow (u, v)} \frac{(l, \beta_{\Delta, p})}{|\Delta|_2} \leq \lim_{\Delta \rightarrow (u, v)} \frac{(L_{\Delta, p}, f_0)}{|\Delta|_2} = f_0[x_p(u, v), J_0],$$

whenever the first limit appearing in (5.10) exists, and is equal to f_0 otherwise. Thus, since J_0 is arbitrary, $\phi_0(x, J) \leq f_0(x, J)$ for all (x, J) in $\Omega \times S^m$. For $\nu = 1, 2, \dots$, let

$$\rho_\nu = \sup_{x' - x \in D_\nu, x \in \Omega, |J|=1} |f_0(x', J) - f_0(x, J)|.$$

Since Ω is bounded and f_0 is uniformly continuous on bounded sets, $\rho_\nu \rightarrow 0$ as $\nu \rightarrow \infty$. For all ν and (x, J) in $\Omega \times S^m$,

$$\phi_\nu(x, J) \leq \sup_{x' \in x + D_\nu} \phi_0(x', J) \leq \sup_{x' \in x + D_\nu} f_0(x', J) \leq f_0(x, J) + \rho_\nu |J|,$$

which proves (4) of (5.1).

REMARK. It probably is not true that in (5.1) ϕ_0 may be supposed continuous. Damköhler [4] has given a counterexample in a corresponding problem for curves.

G-BOUNDARIES

6. G-boundaries and Dirichlet surfaces. We are now ready to consider the general notions of boundary and track (especially boundary) introduced in §1. From the definitions and the linearity in the variable J of exact integrands, established in (4.4), it follows at once that:

(6.1) *The generalized boundary and track are preserved under addition and positive scalar multiplication, and change sign under the operation (*). If L_1 and L_2 have the same generalized track, they have the same generalized boundary. The w^* limit of a sequence with fixed generalized boundary or track has again that generalized boundary or track.*

In view of (4.7) the notion of generalized boundary (abbreviated, g -boundary) is an extension of the notion of e -boundary defined in §3 for polyhedra; i.e., *the g -boundary is a generalization of line integration around a boundary curve, paths traversed in opposite senses being allowed to cancel.*

We are interested here primarily in admissible g -boundaries. An admissible g -boundary λ is termed *rectifiable*, or a *closed polygonal path*, if it bounds some Dirichlet surface with a Dirichlet representation $x(u, v)$ which reduces on the perimeter of U to a representation of a Fréchet curve with the named property.

If $x(u, v)$ is a Dirichlet representation of a Dirichlet surface L , a new

representation of L may be found which is constant on one side of U . One gets such a representation, for instance, by choosing a Lipschitzian mapping $u = u(\xi, \eta), v = v(\xi, \eta)$ of U into itself which is constant on $\xi = 1$ and biunique otherwise, and setting $x'(\xi, \eta) = x[u(\xi, \eta), v(\xi, \eta)]$. According to [17, p. 325, (11.2)], $x'(\xi, \eta)$ is a Dirichlet representation and the partial derivatives of x' are given almost everywhere by the usual formula. It follows from a well known change of variables theorem and the homogeneity in J of integrands f that, for all f in E^m ,

$$(L, f) = \iint_U f[x(u, v), J(u, v)]dudv = \iint_U f[x'(\xi, \eta), J'(\xi, \eta)]d\xi d\eta;$$

i.e., x' is also a Dirichlet representation of L .

Using this remark, it is easy to "join two Dirichlet surfaces with a thread" and get:

(6.2) *If L_1 and L_2 are Dirichlet surfaces, so is $L_1 + L_2$.*

Dirichlet surfaces bounded in the sense of classical analysis by the same rectifiable curve C have the same g -boundary:

(6.3) *If L_1, L_2 have, respectively, Dirichlet representations $x_1(u, v), x_2(u, v)$ which reduce on the perimeter of U to representations of the same rectifiable closed Fréchet curve C , then $\lambda(L_1) = \lambda(L_2)$ ⁽⁵⁾.*

(6.3) is a trivial consequence of (4.5), Stokes' theorem, and the fact that every exact integrand ϕ is the limit uniformly on bounded sets of (x, J) of a sequence of continuously differentiable exact integrands (obtained, for instance, by approximating ϕ by mean value integrals with respect to the variable x).

If the condition of rectifiability is dropped, (6.3) does not always hold. For instance, let C be a simple closed curve in the unit square in the x^1x^2 plane of R^3 such that the set $[C]$ occupied by C has positive 2-dimensional measure μ . Let L_0 denote the Dirichlet surface realized by the surface Σ of the unit cube in R^3 (with orientation that of C); L_1 , the smaller component of $\Sigma - [C]$; and L_2 , the larger component. L_1 and L_2^* are Dirichlet surfaces with C as boundary curve in the classical sense. Consider the exact integrand $\phi_{12}(x, J) \equiv j^{12}$. Now

$$(L_0 - (L_1 + L_2), \phi_{12}) = \pm \iint_{[C]} dx^1 dx^2,$$

the sign being positive if the orientation of C is positive. Since L_0 is closed, $(L_0, \phi_{12}) = 0$; hence,

$$(L_1, \phi_{12}) = (L_2^*, \phi_{12}) \pm \mu.$$

Thus $\lambda(L_1) \neq \lambda(L_2^*)$.

⁽⁵⁾ (Added in proof.) Using recent results of Young, as yet unpublished, (6.3) can also be proved for nonrectifiable curves C whose set of points is of 2-dimensional Hausdorff measure 0.

(6.4) Given $\epsilon > 0$ and a Dirichlet surface L situated in an open set Ω , there exist a closed polygonal path λ_ϵ and a Dirichlet surface L_ϵ situated in Ω with $a(L_\epsilon) < \epsilon$ and $\lambda(L_\epsilon) = \lambda(L) - \lambda_\epsilon$.

The proof of (6.4) is elementary if $\lambda(L)$ is rectifiable. To establish the general case, it then suffices to prove (6.4) with λ_ϵ rectifiable. Let $x(u, v)$ be a Dirichlet representation of L . Since $x(u, v)$ is absolutely continuous in Tonelli's sense, $x(u_0, v)$ and $x(u, v_0)$ define rectifiable curves for sets u_0 and v_0 dense, respectively, in $(0, 1)$. Choose a rectangle U' : $[u_1 \leq u \leq u_2; v_1 \leq v \leq v_2]$, where u_1, u_2 and v_1, v_2 lie, respectively, in these sets, such that $\iint_{U-U'} |J(x, v)| < \epsilon$. Let L_ϵ be the Dirichlet surface defined by the representation $x(u, v)$ on $U - U'$; let $\lambda_\epsilon = \lambda(L) - \lambda(L_\epsilon)$.

7. Non-negative linear L with admissible g -boundary. For each $A \subset R^m$, let $E^{m,A}$ denote the linear space of restrictions of integrands $f(x, J)$ in E^m to values (x, J) in $A \times S^m$. If A is bounded, $E^{m,A}$ is a Banach space under the norm

$$\|f\|_A = \sup_{x \in A, |J|=1} |f(x, J)|.$$

The set of non-negative linear L situated in the closure of A may be identified in the obvious fashion with the set of non-negative elements of the conjugate of $E^{m,A}$, and $a(L)$ coincides with the usual conjugate space norm. We further define in this set the McShane distance [10, p. 534]:

$$\|L_1, L_2\|_A = \sup_{f \in F_A} |(L_1, f) - (L_2, f)|,$$

where F_A denotes the set of f in $E^{m,A}$ such that $\|f\|_A = 1$ and $|f(x, J) - f(x', J')| \leq (|x - x'|^2 + |J - J'|^2)^{1/2}$. In the set of non-negative linear L situated in A , the McShane distance defines a metric topology equivalent to the w^* topology. This follows from the known equivalence of convergence in the McShane distance with w^* convergence for sequences of elements of this set, and the fact that from a w^* convergent directed sequence (L_α) of non-negative linear functionals situated in a fixed bounded set one can extract a sequence in the ordinary sense converging w^* to the same limit. The latter assertion follows from the separability of $E^{m,A}$, and the fact that $a(L) = (L, f_\alpha)$ for L non-negative which implies that a bounded directed subsequence of (L_α) can be found.

We are now prepared to apply the results of the preceding sections to characterize the set of non-negative linear L which have an admissible g -boundary.

(7.1) LEMMA. Let $\Omega \subset R^m$ be bounded, open, and convex. Let λ_0 bound some polyhedron over R situated in Ω . Then every non-negative linear L_0 situated in Ω with $\lambda(L_0) = \lambda_0$ is the w^* limit of a sequence (L_n) of polyhedra over R situated in Ω such that $\lambda(L_n) = \lambda_0$ for $n = 1, 2, \dots$.

Proof. By hypothesis, $\lambda_0 = \lambda(L)$ for some polyhedron L over R situated in Ω . Then $\beta_0 = \beta(L)$ is in the set B_Ω defined in §5. The set Γ of all polyhedra over R situated in Ω and bounded by non-negative multiples of β_0 is a convex cone in the conjugate of the Banach space $E^{m,\Omega}$. We show first that L_0 is in the w^* closure of Γ .

Suppose this is not the case. Then by the well known separation theorem for w^* closed, convex subsets of the conjugate of a Banach space, L_0 can be separated from Γ by a hyperplane. This means that f_0 in $E^{m,\Omega}$ and α exist such that $(L_0, f_0) < \alpha$ and $(L, f_0) \geq \alpha$ for all L in Γ . Since Γ is a cone, $\alpha = 0$. Since f_0 is the restriction of an integrand in E^m , we regard f_0 itself as being in E^m .

Every closed polyhedron situated in Ω is in Γ ; hence, the hypotheses of Lemma (5.1) are satisfied. Let $\phi_0, (\phi_\nu), (\rho_\nu)$, and L_1 be as in (5.1). By (1) of (5.1) and hypothesis, $(L_1, \phi_0) \geq 0$. Then using the fact that $\lambda(L_1) = \lambda(L_0)$ and (3) of (5.1):

$$\begin{aligned} \lim_{\nu} (L_0, f_0 - \phi_\nu + \rho_\nu f_0) &= (L_0, f_0) - \lim_{\nu} (L_0, \phi_\nu) \\ &= (L_0, f_0) - \lim_{\nu} (L_1, \phi_\nu) = (L_0, f_0) - (L_1, \phi_0) < 0. \end{aligned}$$

But $f_0 - \phi_\nu + \rho_\nu f_0 \geq 0$ for all ν ; hence, $(L_0, f_0 - \phi_\nu + \rho_\nu f_0) \geq 0$ for all ν since L_0 is non-negative, a contradiction.

Therefore, L_0 is a w^* limit point of Γ . Since Ω is bounded, the w^* topology is metrizable by the McShane distance in the set of non-negative linear L situated in Ω . Hence, a sequence (L_n) in Γ converges w^* to L_0 . For all $n = 1, 2, \dots$, $\lambda(L_n) = k_n \lambda_0$, where $k_n \geq 0$. For all exact ϕ ,

$$(\lambda_0, \phi) = (\lambda(L_0), \phi) = \lim_n (\lambda(L_n), \phi) = \lim_n k_n (\lambda_0, \phi).$$

Then either $\lambda_0 = 0$ or else $k_n \rightarrow 1$, in which case $k_n^{-1} L_n$ converges w^* to L_0 and $\lambda(k_n^{-1} L_n) = \lambda_0$ for all n . In either case, the proof of (7.1) is complete.

We now prove the first half of Theorem (1.1):

(7.2) *Let λ_0 be admissible. Then a necessary and sufficient condition that L_0 have g -boundary $\lambda(L_0) = \lambda_0$ is that L_0 be the w^* limit of a sequence of linear combinations $\sum k_j L_j$ with positive coefficients k_j of Dirichlet surfaces L_j with $\lambda(L_j) = \nu_j \lambda_0$, where each ν_j is a positive integer and $\sum k_j \nu_j = 1$.*

Proof. The sufficiency is an immediate consequence of the definitions of g -boundary and w^* convergence. In proving the necessity, suppose first that λ_0 is a closed polygonal path. By Lemma (7.1), $L_0 = w^* \lim_n L_n$, where each L_n is a polyhedron over R with $\lambda(L_n) = \lambda_0$. Applying (3.3), each L_n is of the form $\sum k_j L_j$, where the k_j are positive real coefficients and the L_j are polyhedra with each $\lambda(L_j)$ a positive integral multiple $\nu_j \lambda_0$ of λ_0 . Thus, to prove (7.2) for closed polygonal paths λ_0 it suffices to show that one may suppose $\sum k_j \nu_j = 1$.

If $\lambda_0 \neq 0$, the argument used in the last paragraph of the proof of (7.1) shows that $\eta = \sum k_j \nu_j \rightarrow 1$; replace each k_j by $\eta^{-1} k_j$.

Suppose $\lambda_0 = 0$. Then since the L_j are closed polyhedra, so are NL_j for all positive integers N and all j . Choose all $\nu_j = 1$, and $N > \sum k_j$. Replacing $\sum k_j L_j$ by $\sum (N^{-1} k_j)(NL_j) + [1 - \sum (N^{-1} k_j)]L'$, where L' is a closed polyhedron of small area, establishes that for $\lambda_0 = 0$ one may suppose $\sum k_j \nu_j = \sum k_j = 1$.

Having proved (7.2) for closed polygonal paths, now let λ_0 be an arbitrary admissible g -boundary. Let L be a Dirichlet surface with $\lambda(L) = \lambda_0$, and Ω a bounded open convex set in which both L and L_0 are situated. Given $\epsilon > 0$ let λ_ϵ and L_ϵ be as in (6.4). Then

$$\lambda(L_0 + L_\epsilon^*) = \lambda_\epsilon.$$

By what has already been proved, $L_0 + L_\epsilon^*$ is the w^* limit of sums $\sum k_j L_j$, where $\lambda(L_j) = \nu_j \lambda_\epsilon$ for all j and $\sum k_j \nu_j = 1$ as above. The L_j may be assumed to be situated in Ω . Then

$$L_0 + L_\epsilon^* + L_\epsilon = w^* \lim \sum k_j (L_j + \nu_j L_\epsilon),$$

and each $L_j + \nu_j L_\epsilon$ is a Dirichlet surface with

$$\lambda(L_j + \nu_j L_\epsilon) = \nu_j \lambda_0.$$

Hence (7.2) holds for $L_0 + L_\epsilon^* + L_\epsilon$. Since the set of non-negative linear L situated in Ω is metrizable in a way equivalent to w^* convergence and $a(L_\epsilon) + a(L_\epsilon^*) < 2\epsilon$, (7.2) follows for L_0 upon letting $\epsilon \rightarrow 0$.

8. Completion of the proof of Theorem (1.1). The first half of Theorem (1.1) was established in (7.2). It remains to show that for $m = 3$, one can suppose in (7.2) that all $\nu_j = 1$, i.e., that the set of non-negative linear L with admissible g -boundary consists of all w^* limits of convex combinations of Dirichlet surfaces with that g -boundary. To do this, we prove first:

(8.1) *Suppose $m = 3$. Let L_0 be a polyhedron with e -boundary $\beta(L_0) = \nu \beta_0$, where $\beta_0 \neq 0$ is a closed polygonal path and ν is a positive integer. Then polyhedra L_1, \dots, L_ν exist with $\beta(L_1) = \dots = \beta(L_\nu) = \beta_0$ such that $L_0 = \sum L_j$.*

Proof. It suffices to suppose that L_0 has no double triangles, i.e., that for no triangle t do both t and t^* appear as faces of L_0 , since every polyhedron reduces to such a polyhedron upon subtracting a sum of double triangles (which is closed). Then L_0 has a triangulation

$$L_0 = \sum n_i L_{i_i}$$

with all n_i positive integers. As in the proof of (3.3), let K denote the euclidean 2-complex in R^3 generated by the t_i ; let

$$\gamma(L_0) = \sum n_i t_i.$$

The closed polygonal path β_0 corresponds to $\zeta = \nu^{-1} \partial \gamma(L_0)$ in $Z^1(K, I)$. Since

K is in R^3 , $Z^1(K, I)$ contains no torsion cycles (cf. Alexandroff-Hopf [1, p. 390]); hence, $\zeta = \partial\gamma$ for some γ in $C^2(K, I)$ since $\nu\zeta$ bounds in $C^2(K, I)$. Then $\gamma(L_0) = \nu\gamma + z$, where z is in $Z^2(K, I)$. Since L_0 has no double triangles, we have as in (3.7),

$$(8.2) \quad L_0 + \sum h_i(L_{t_i} + L_{t_i}^*) = \nu L_\gamma + L_z = (\nu - 1)L_\gamma + (L_\gamma + L_z),$$

for suitable non-negative integers h_i . To complete the proof of (8.1), the sum of double triangles in (8.2) will be eliminated in $\sum h_i$ successive steps. The means for doing this will be the result that if t is a face of L_1 , t^* a face of L_2 , and $\beta(L_1) = \beta(L_2) = \beta_0$, then polyhedra L'_1, L'_2 exist with $\beta(L'_1) = \beta(L'_2) = \beta_0$ such that

$$L_1 + L_2 - (L_t + L_{t^*}) = L'_1 + L'_2.$$

To establish this result, it clearly suffices to establish (8.1) directly for $\nu = 2$. We therefore suppose $\beta(L_0) = 2\beta_0$, and proceed by induction on $N = \sum n_i$. The case $N = 1$ is vacuous. Suppose true for $N' < N$.

Case 1. $Z^2(K, I) = 0$. Then $\gamma(L_0) = 2\gamma$, since $\partial[\gamma(L_0) - 2\gamma] = 0$ ($K, \gamma(L_0)$, and γ being defined as above). It suffices for (8.1) to write $L_0 = 2L_\gamma$, where L_γ is defined as in the proof of (3.3).

Case 2. $Z^2(K, I) \neq 0$. Then since K has no torsion, $z_0 = \sum \epsilon_i t_i$ in $Z^2(K, I)$ exists such that all $\epsilon_i = 0$ or ± 1 and some $\epsilon_i = -1$ [1, pp. 277-283]. Let

$$(8.3) \quad \gamma_0 = \sum_{\epsilon_i = -1} t_i; \quad \gamma' = \gamma(L_0) - z_0 - 2\gamma_0 = \sum (n_i - |\epsilon_i|) t_i.$$

Then

$$(8.4) \quad \partial\gamma' = \partial\gamma(L_0) - 2\partial\gamma_0 = 2(\zeta - \partial\gamma_0).$$

Suppose first that $\zeta = \partial\gamma_0$. Then for (8.1) it suffices to write

$$L_0 = (L_0 - L_{\gamma_0}) + L_{\gamma_0}.$$

Note that $L_0 - L_{\gamma_0}$ is non-negative and so is a polyhedron, since all $n_i \geq 1$ and

$$L_{\gamma_0} = \sum_{\epsilon_i = -1} L_{t_i}.$$

Suppose then that $\zeta \neq \partial\gamma_0$. Then $\zeta - \partial\gamma_0$ corresponds to a closed polygonal path $\beta_1 \neq 0$ and, by (8.4), $2\beta_1$ bounds $L_{\gamma'}$. Now

$$L_{\gamma'} = \sum (n_i - |\epsilon_i|) L_{t_i}.$$

Since $\sum (n_i - |\epsilon_i|) < N$, the induction hypothesis yields

$$L_{\gamma'} = L' + L'',$$

where $\beta(L') = \beta(L'') = \beta_1$. Then for (8.1) it suffices to note that

$$(8.5) \quad L_0 = (L' + L_{z_0 + \gamma_0}) + (L'' + L_{\gamma_0}).$$

(8.5) is an immediate consequence of (8.3) as soon as it is observed that no t_i^* is a face of any polyhedron in (8.5) and hence that no double triangle terms appear in that sum. This completes the proof of (8.1) for $\nu=2$, and so (as noted above) for arbitrary ν .

REMARK. (8.1) is not true for $m>3$. For instance, a Klein bottle has a realization in R^4 which does not intersect itself. If we let L_0 denote a polyhedron realized in R^4 by such a bottle, then $\beta(L_0)=2\beta$ where β corresponds to the rim of the bottle. L_0 is not the sum of two polyhedra with e -boundary β .

Upon applying (8.1) at the appropriate point in the first paragraph of the proof of (7.2), and noting that for the case $\lambda_0=0$ it was shown in (7.2) that one can suppose all $\nu_j=1$ (m arbitrary), we get the desired extension:

(8.6) *In case $m=3$, one can suppose in (7.2) that all $\nu_j=1$.*

BASIC (λ); BASIC CLOSED

9. Irreducible closed polyhedra. A closed polyhedron L is termed *irreducible* if it has no decomposition $L=L_1+L_2$ into closed polyhedra L_1, L_2 . Evidently:

(9.1) *Every closed polyhedron is a sum of irreducible closed polyhedra and double triangles.*

In particular, if L is irreducible closed then L has no double triangles. When the dimension m is 3, we have the following simple characterization of irreducible closed polyhedra:

(9.2) *Suppose $m=3$. A necessary and sufficient condition that L be irreducible closed is that L have a realization in R^3 as a geometric polyhedron $P(L)$ separating R^3 into precisely two components of which it is the common boundary.*

(9.2) follows from the fact that if K is a euclidean 2-complex in R^3 generated by triangles t_1, \dots, t_n and $P=t_1 \cup \dots \cup t_n$, then the number of bounded components of R^3-P is the number of independent elements of $Z^2(K, I)$ [1, p. 380].

10. Theorems (1.2) and (1.3). The notions of basic (λ) and basic closed as defined in §1 are intimately connected with the notion of extreme point. For $\lambda \neq 0$, the L basic (λ) are, by definition, the extreme points of the set of all non-negative linear L with g -boundary $\lambda(L)=\lambda$, while for $\lambda=0$ one has:

(10.1) *L_0 is basic closed if and only if $[a(L_0)]^{-1}L_0$ is an extreme point of the set of all closed non-negative linear functionals L of norm 1.*

The importance here of the notion of extreme point lies in the following two principles, valid for sets of non-negative linear L situated in a fixed compact subset of R^m : First, every extreme point of the w^* closed convex span of a set Γ is in the w^* closure of Γ ; and second, the minimum of (L, f_0) over a w^* closed, convex set Γ is attained at an extreme point of Γ if it is attained at all in Γ . These principles are discussed further in the appendix.

Proof of Theorems (1.2) and (1.3). Let $\lambda \neq 0$ be admissible and L_0 basic (λ). By Theorem (1.1), L_0 is the w^* limit of a sequence of convex combina-

tions $\sum k_j L_j$ of Dirichlet surfaces L_j bounded by λ . Since the sequence converges w^* , its terms are situated in a fixed compact subset A of R^m . By (A1) (see Appendix), L_0 is the w^* limit of a sequence (L_n) of Dirichlet surfaces situated in A such that $\lambda(L_n) = \lambda$ for all n , which proves Theorem (1.2).

The proof of Theorem (1.3) is a little more involved. Let L_0 be basic closed; we may suppose $a(L_0) = 1$. By Theorem (1.1), L_0 is the w^* limit of a sequence $\sum k_j L_j$ of positive linear combinations of closed polyhedra L_j . Using (9.1) and approximating double triangles by flat tetrahedra, one may suppose that all L_j are irreducible. As before, all L_j are situated in some fixed compact set A since the sequence converges w^* . Let Γ denote the set of all L of the form $[a(L_1)]^{-1}L_1$ where L_1 is an irreducible closed polyhedron situated in A . Now

$$\eta = \sum k_j a(L_j) = \sum k_j (L_j, f_a) \rightarrow (L_0, f_a) = a(L_0) = 1;$$

hence, by replacing all k_j by $\eta^{-1}k_j$, one can suppose all $\sum k_j a(L_j) = 1$. It follows that L_0 is the w^* limit of the convex sums of Γ :

$$\sum \{ k_j a(L_j) \} \{ [a(L_j)]^{-1}L_j \}.$$

By (10.1) and (A1), L_0 is the w^* limit of a sequence in Γ ; thus, a sequence (L_n) of irreducible closed polyhedra situated in A exists such that

$$L_0 = w^* \lim_n [a(L_n)]^{-1}L_n.$$

There are three cases:

Case 1. For a subsequence of n , $a(L_n) \rightarrow a_0 > 0$. Then $a_0 L_0 = w^* \lim_n L_n$.

Case 2: $a(L_n) \rightarrow 0$. Choose a sequence (N_n) of positive integers such that $N_n a(L_n) \rightarrow 1$. For every $n = 1, 2, \dots$, $N_n L_n$ is a closed polyhedron, and $L_0 = \lim_n N_n L_n$. L_0 can, in fact, be shown to be concentrated in this case.

Case 3: $a(L_n) \rightarrow \infty$. It is known that for $m = 3$, every concentrated closed L is the w^* limit of a sequence of closed polyhedra [18, (9.2)]. We suppose L_0 not concentrated and derive a contradiction.

For arbitrary non-negative linear L , we denote by *spt* L the smallest set in which L is situated. Since L_0 is not concentrated, *spt* L_0 contains two distinct points x_1, x_2 . Let π be a plane separating x_1 and x_2 . Let

$$L_n = L'_n + L''_n$$

where L'_n is situated on the side of π containing x_1 , and L''_n on the side containing x_2 . Since $[a(L_n)]^{-1}L'_n, [a(L_n)]^{-1}L''_n$ have bounded norms and are situated in a fixed bounded set, a subsequence of n and L', L'' exist (by a well known selection process) such that

$$(10.2) \quad L' = w^* \lim_n [a(L_n)]^{-1}L'_n; \quad L'' = w^* \lim_n [a(L_n)]^{-1}L''_n.$$

By (9.2), each L_n has a geometric realization $P(L_n)$ separating R^3 into exactly two components. L'_n and L''_n may be closed up, for each $n = 1, 2, \dots$, by adding polyhedra realized by the (suitably oriented) portion of π interior to $P(L_n)$. Since these additions have area less than $\pi b^2/4$ where b is the diameter of A and $a(L_n) \rightarrow \infty$, the additions contribute nothing to (10.2) in the limit. Thus L' and L'' , being the w^* limits of closed functionals, are closed. $\text{spt } L'$ contains x_1 and lies on the side of π containing x_1 ; $\text{spt } L''$ lies on the opposite side of π and contains x_2 . Then $L_0 = L' + L''$, and $L' = kL_0$ for no k , contradicting the assumption that L_0 is basic closed. This completes the proof of Theorem (1.3).

REMARK. Theorem (1.2) is false for $m > 3$. For example, let L correspond as in §8 to a Klein bottle in R^4 which does not intersect itself and λ to the rim of the bottle. Then $L/2$ is basic (λ); yet $L/2$ can be shown not to be a w^* limit of a sequence of Dirichlet surfaces with g -boundary λ . It is not known whether Theorem (1.3) is true for $m > 3$.

11. **Proof of Theorem (1.4).** The necessity of (a) in (1.4) is a trivial consequence of the non-negativeness of the functionals L involved and the definition of closed. We next establish that (a) implies (b). For, suppose (b) does not hold. Then there exists a sequence (L_n) of closed polyhedra such that L_n is situated in the n^{-1} neighborhood of A and

$$\limsup_n [a(L_n)]^{-1}(L_n, f_0) \leq 0.$$

Since the functionals $[a(L_n)]^{-1}L_n$ have bounded norms and A is compact, a subsequence converges w^* to L_1 situated in A . L_1 is closed, $a(L_1) = 1$, and $(L_1, f_0) \leq 0$. Hence

$$\mu = \inf (L, f_0) \leq 0,$$

where the inf is taken over the set Γ of all closed L of norm 1 situated in A . By the Krein-Milman theorem, μ is attained at an extreme point of Γ , which by (10.1) implies that (a) does not hold. Thus (a) implies (b).

To complete the proof of Theorem (1.4), it suffices to show that (b) is sufficient. To this end, choose an open, convex set Ω_1 such that $A \subset \Omega_1 \subset \Omega$ and apply Lemma (5.1) to $f'_0 = f_0 - \epsilon_1 f_a$.

Theorem (1.4) has been stated in two equivalent forms. Condition (b) has the advantage of being stated directly in terms of the elementary notion of closed polyhedron. On the other hand, condition (a) is sometimes more convenient to apply (cf. (13.2) below).

APPLICATIONS

12. **Closed with 2-acyclic support.** We again restrict attention to the case $m = 3$. One thinks intuitively of a closed surface in R^3 as being situated in a number of 2-cycles. This is not necessarily the case for more general

closed non-negative linear functionals L . In fact, a closed L may be situated in a set X which is acyclic in dimension 2; however, we shall show that such an L degenerates (under certain restrictions on X) into a limit of sums of *concentrated* closed functionals.

For $X \subset R^3$ and $\epsilon > 0$, let X_ϵ denote the ϵ -neighborhood of X .

(12.1) *Suppose $m = 3$. Let X be a compact set such that $R^3 - X_\epsilon$ is connected for sufficiently small ϵ , and such that given x_1, x_2 in X there exists a plane π separating x_1 and x_2 for which $|X \cap \pi|_2 = 0$. Then given a non-concentrated closed L_0 situated in X , there exist a plane π and closed L_1, L_2 situated on opposite sides of π (and not situated entirely in π) such that $L_0 = L_1 + L_2$.*

REMARK. If, in particular, X is a piecewise smooth 2-dimensional subset of R^3 and $R^3 - X$ is connected, then the hypotheses of (12.1) hold.

Proof of (12.1). Let $\epsilon > 0$ be small enough that $R^3 - X_\epsilon$ is connected. Let L_0 be closed, nonconcentrated, and situated in X . By (1.1), L_0 is the w^* limit of a sequence of convex combinations $\sum k_j L_j$ of closed polyhedra L_j .

We show first that all L_j may be modified so as to be situated in X_ϵ in such a way that the sums $\sum k_j L_j$ still converge w^* to L_0 . To this end, let Ω be open with piecewise linear boundary and such that $X \subset \Omega \subset X_{\epsilon/3}$. Using (9.1) and approximating double triangles by flat tetrahedra, one may suppose all L_j irreducible closed, so that each L_j has a geometric realization $P(L_j)$ as in (9.2). By trivial modifications, one can suppose no face of any L_j parallel to a face of the boundary of Ω . Let L'_j denote the portion of L_j situated in $R^3 - \Omega$. Let f_1 coincide with f_a for x in $R^3 - \Omega$ and vanish for x in X . Then

$$\sum k_j a(L'_j) = \sum k_j (L'_j, f_1) \leq \sum k_j (L_j, f_1).$$

The right side, and hence also the left side, of this expression tends to 0 since $\sum k_j L_j$ tends w^* to L_0 situated in X . Let a_0 denote the area of the boundary P of Ω . If $a(L'_j) \geq \epsilon^2/9$, replace L'_j by the portion of P interior to $P(L_j)$ with orientation chosen to agree with that of L_j . This increases $a(L'_j)$ by a factor of no more than $9a_0\epsilon^{-2}$. Suppose $a(L'_j) < \epsilon^2/9$. Given $\delta > 0$, let β_δ denote the closed polygonal path realized by the intersection of $P(L'_j)$ with the boundary of the δ -neighborhood Ω_δ of Ω . Letting $\rho(\beta_\delta)$ denote the length of β_δ , it is elementary to verify that

$$a(L'_j) \geq \int_0^\infty \rho(\beta_\delta) d\delta.$$

It follows, in particular, that

$$\rho(\beta_\delta) \leq 3a(L'_j)\epsilon^{-1} < \epsilon/3$$

for at least one δ in $(0, \epsilon/3)$. For this δ , β_δ is situated in $X_{2\epsilon/3}$ and is of length at most $\epsilon/3$. Its convex span is therefore contained in X_ϵ . Then β_δ bounds a polyhedron situated in X_ϵ whose area approximates arbitrarily closely the

inf for polyhedra bounded by β_δ , and so by the isoperimetric inequality has area arbitrarily little more than $(1/4\pi)[\rho(\beta_\delta)]^2$. But

$$\frac{1}{4\pi} [\rho(\beta_\delta)]^2 \leq \frac{9}{4\pi} [a(L'_j)]^2 \epsilon^{-2} < \frac{a(L'_j)}{4\pi} .$$

Replace the portion of L'_j exterior to Ω_δ (if such exists) by this polyhedron.

These modifications increase the sums $\sum_j k_j a(L'_j)$ by a factor of at most the fixed constant $\max [1/4\pi, 9a_0\epsilon^{-2}]$, so that these sums still converge to 0. It follows that $\sum_j k_j L_j$ still tends w^* to L_0 .

By (9.1) one can again suppose the modified L_j to be irreducible closed. Since $R^3 - X_\epsilon$ is connected and each $P(L_j)$ is contained in X_ϵ , the interior component of $R^3 - P(L_j)$ is contained in X_ϵ for each L_j .

Choose a sequence (ϵ_n) tending to 0. By the above remarks and the fact that the w^* topology is metrizable in the set of non-negative linear L situated in X_{ϵ_1} (assume $\epsilon_1 \geq \epsilon_n$ for all n), there exists a sequence of convex linear combinations $\sum_j k_{jn} L_{jn}$ tending w^* to L_0 such that, for each j and n , L_{jn} is irreducible closed and the interior component of $P(L_{jn})$ is contained in X_{ϵ_n} . Let x_1, x_2 be distinct points of $\text{spt } L_0$, and π a plane separating x_1 and x_2 such that $|X \cap \pi|_2 = 0$. Each L_{jn} may be split into closed polyhedra by adding the (suitably oriented) portions of π interior to $P(L_{jn})$, which have area at most $|X_{\epsilon_n} \cap \pi|_2$ since the interior of $P(L_{jn})$ is contained in X_{ϵ_n} . Since

$$\lim_n |X_{\epsilon_n} \cap \pi|_2 = |X \cap \pi|_2 = 0$$

and $\sum_j k_{jn} = 1$ for all n , these added portions contribute nothing in the limit. Thus, proceeding as in the proof of Theorem (1.3), $L_0 = L_1 + L_2$ where L_1 and L_2 are closed and $\text{spt } L_1$ is on one side of π and contains x_1 and $\text{spt } L_2$ is on the other side and contains x_2 .

An immediate corollary of (12.1) is:

(12.2) *If L is basic closed and situated in a set X satisfying the hypotheses of (12.1), then L is concentrated.*

Under slightly stronger restrictions on X , we prove:

(12.3) *If X is a compact set such that $R^3 - X_\epsilon$ is connected for sufficiently small ϵ , and if parallel to each coordinate plane a nondenumerable dense set of planes π exists for which $|X \cap \pi|_2 = 0$, then every closed L situated in X is representable as a Bochner integral (in the McShane distance):*

$$(12.4) \quad L = \int_X L_x d\omega(x),$$

where ω is a Borel measure vanishing outside X and each L_x is closed and concentrated at x .

Proof. From known theorems, it follows that every non-negative linear

L situated in X has the form of a Bochner integral (12.4) (in the McShane distance), where ω is a Borel measure vanishing on $R^3 - X$ and L_x is concentrated at x . Except for a set of ω measure 0, L_x is the derivative of the integral (12.4) with respect to a network of intervals D bounded by planes π for which $|X \cap \pi|_2 = \omega(X \cap \pi) = 0$. It follows from (12.1) that $\int_D L_x d\omega$ is closed for each such D ; thus, $L_x = \lim_{D \rightarrow x} [\omega(D)]^{-1} \int_D L_x d\omega$ is closed except for a set of ω measure 0, on which L_x may be defined arbitrarily. This proves (12.3).

We observe next that if L is a non-negative linear functional concentrated at a point x_0 , L is closed if and only if L has generalized track 0. This follows at once from the definitions, the linearity in J of exact integrands, and the remark that if $f(x, J)$ is an arbitrary integrand which is linear in J , then $\phi(J) = f(x_0, J)$ is exact. From this it follows at once that each integral (12.4) in which the L_x are closed and concentrated has generalized track 0. The converse is also true. To establish it write L with generalized track 0 in the form (12.4), where each L_x is concentrated at x and except for a set of ω measure 0 is the derivative of (12.4). Let π be a plane such that $\omega(\pi) = 0$, and let L_1, L_2 denote the portions of L situated on opposite sides of π . Let $f(x, J)$ in E^m be linear in J , and let (f_n) be a sequence of integrands in E^m linear in J such that f_n coincides with f on the side of π in which L_1 is situated, vanishes at all points in the other at distance more than n^{-1} of π , and has the same bounds as f . Then

$$(L_1, f) = \lim_n (L, f_n) = 0;$$

i.e., L_1 has generalized track 0. Similarly, L_2 has generalized track 0. Proceeding as in the proof of (12.3), we find that, except for a set of ω measure 0, L_x has generalized track 0, and so being concentrated is closed. We have:

(12.5) *A necessary and sufficient condition that L have generalized track 0 is that L be of the form (12.4), where ω is a Borel measure on R^m vanishing outside some compact set X and each L_x is closed and concentrated at x .*

13. An existence theorem for an ordinary minimum problem. Consider the following:

(13.1) *Minimum problem: Let $A \subset R^3$ be compact and convex. Find the minimum $\mu = \inf (L, f_0)$ in the set Γ of all Dirichlet surfaces L situated in A which have Dirichlet representations on the unit square U reducing on the perimeter of U to representations of a given simple closed Fréchet curve C .*

According to Cesari [3] (for similar results arrived at independently of [3] and each other, see Danskin [6] and Sigalov [13]), the problem (13.1) has a solution in Γ , provided Γ is nonvoid and:

- (1) f_0 is in E^3 ;
- (2) f_0 is positive semi-regular, i.e., $\psi(J) = f_0(x_0, J)$ is convex for each x_0 in A ; and
- (3) f_0 is positive definite, i.e., $f_0(x, J) > 0$ for x in A and $J \neq 0$.

The purpose of this section is to show that condition (3) may be weakened, as follows:

(13.2) EXISTENCE THEOREM. *The minimum problem (13.1) has a solution in Γ if C is rectifiable and contained in A , and if (1), (2), and any one of the following hold:*

(3a) $(L, f_0) > 0$ for every basic closed L situated in A ;

(3b) *There exists an open set Ω containing A and $\epsilon_1 > 0$ such that $(L, f_0) \geq \epsilon_1 \alpha(L)$ for every closed polyhedron L situated in Ω ; or*

(3c) (i) $f_0(x, J) \geq 0$, whenever x is in A ;

(ii) $f_0(x, J) + f_0(x, -J) > 0$, for x in A and $J \neq 0$;

(iii) *The zeros⁽⁶⁾ of f_0 are contained in a closed subset X of A satisfying the hypotheses of (12.1)⁽⁷⁾.*

Proof. Since C is rectifiable and contained in A , Γ is nonvoid. Suppose (3a) or (3b) holds. By Theorem (1.4), an exact integrand ϕ and an $\epsilon > 0$ exist such that $f_0(x, J) - \phi(x, J) \geq \epsilon |J|$ for all x in A and all J . The integrand $f_0 - \phi$ is thus positive definite. Since ϕ is linear in J by (4.4) and f_0 is semi-regular, $f_0 - \phi$ is semi-regular. By the existence theorem cited above for positive definite, semi-regular integrands, an L_0 in Γ exists such that $(L_0, f_0 - \phi) \leq (L, f_0 - \phi)$ for all L in Γ . By (6.3), $\lambda(L_0) = \lambda(L)$ for all L in Γ . Then, for all L in Γ ,

$$\begin{aligned} (L_0, f_0) &= (L_0, f_0 - \phi) + (L_0, \phi) = (L_0, f_0 - \phi) + (\lambda(L_0), \phi) \\ &\leq (L, f_0 - \phi) + (\lambda(L), \phi) = (L, f_0 - \phi) + (L, \phi) = (L, f_0); \end{aligned}$$

i.e., L_0 is a solution for the minimum problem (13.1).

Suppose (3c) holds. Let L_1 be basic closed and situated in A . If L_1 is not situated in X , then $(L_1, f_0) > 0$ since f_0 is non-negative and strictly positive for x in $A - X$ and $J \neq 0$. If, on the other hand, L_1 is situated in X then, by (12.2), L_1 is concentrated at a point x_0 of X . Let $\psi(J) = f_0(x_0, J)$. Then $\psi(kJ) = k\psi(J)$ for $k \geq 0$, and ψ is convex since f_0 is semi-regular. Since the domain S^3 of ψ is a linear space (every skew symmetric 3×3 matrix J being of rank 0 or 2), it follows from these properties and (3c) (ii) that a linear function $\phi(J)$ exists such that $\phi(J) < \psi(J)$ for $J \neq 0$. As noted in the remarks following (4.5), ϕ is an exact integrand. Then since L_1 is closed, non-negative, and not 0:

$$0 = (L_1, \phi) < (L_1, \psi) = (L_1, f_0).$$

Thus (3c) implies (3a). This completes the proof of (13.2).

14. **Generalized minimum problems.** Minimum problems of the type of (13.1) may cease to have solutions identifiable as surfaces in the ordinary

⁽⁶⁾ A point x_0 in R^3 is called a zero of f_0 if $f_0(x_0, J) = 0$ for at least one $J \neq 0$.

⁽⁷⁾ This theorem remains true if C merely occupies 2-dimensional Hausdorff measure 0 and Γ is nonvoid (see footnote 5).

sense if the condition of semi-regularity of the integrand f_0 is dropped. To establish a general principle of minimum applicable to nonregular as well as regular problems, we therefore define to be a *generalized solution* (or, *generalized surface solution*) the limit of a w^* convergent minimizing sequence of Dirichlet surfaces. The existence of a generalized solution is insured as soon as a minimizing sequence with bounded norms (= Lebesgue areas) and situated in a fixed bounded subset of R^m exists [18, p. 64, Principle of Minimum]. We note further that by (A1) and (A4) below, the existence of a generalized solution to the minimum problem $\mu = \inf (L, f_0)$ over a set Γ of Dirichlet surfaces situated in a bounded subset of R^m is equivalent to the existence of a solution in the w^* closed convex span $c(\Gamma)$ of Γ . This is in turn equivalent to the existence of a hyperplane of support to $c(\Gamma)$ of the form $(L, f_0) = \mu$.

APPENDIX

In this appendix we prove two results ((A1) and (A4)) on extreme points, which we have referred to above. (A4) is a minor extension of the well known Krein-Milman theorem [8]. For completeness we give a direct proof of (A1) rather than attempt to deduce it from the bounded case, which is known (cf. Milman [11]), but apparently much less well known than the Krein-Milman theorem. The proof of (A1) given applies with only trivial modifications in establishing Milman's result, namely, that the extreme points of the w^* closed convex span of an arbitrary bounded subset Γ of the conjugate of a Banach space lie in the w^* closure of Γ .

(A1) *Let $A \subset R^m$ be compact. Then every extreme point of the w^* closed convex span $c(\Gamma)$ of a set Γ of non-negative linear functionals L situated in A is the w^* limit of a sequence in Γ .*

Proof. Suppose (A1) false. Then an extreme point L_0 of $c(\Gamma)$, and a w^* neighborhood

$$N = N(L_0; \epsilon; f_1, \dots, f_\nu) = [L \mid |(L, f_j) - (L_0, f_j)| < \epsilon, j = 1, \dots, \nu]$$

exist such that $N \cap \Gamma$ is void. By adjoining f_α to the set f_1, \dots, f_ν , if necessary, one can suppose that $f_1 = f_\alpha$. Let $N_1 = N(L_0; \epsilon/2; f_1, \dots, f_\nu)$. Since L_0 is in $c(\Gamma)$, L_0 is the w^* limit of a directed sequence of convex combinations

$$L_\alpha = \sum_{i=1}^{i_\alpha} k_{i\alpha} L_{i\alpha}$$

of elements $L_{i\alpha}$ of Γ . Since $L_0 = w^* \lim_\alpha L_\alpha$, all L_α are in N_1 for $\alpha > \alpha_0$. Suppose that

$$|(L_{i\alpha}, f_j) - (L_\alpha, f_j)| < \epsilon/2$$

for all $j=1, \dots, \nu$. Then a straightforward verification shows that $L_{i\alpha}$ would be in N for $\alpha > \alpha_0$, which is not possible since all $L_{i\alpha}$ are in Γ and $\Gamma \cap N$

is void. Hence each $i = 1, 2, \dots, i_\alpha$ for $\alpha > \alpha_0$ is in at least one of the sets

$$Q^+(j, \alpha) = [i \mid (L_{i\alpha}, f_j) - (L_\alpha, f_j) \geq \epsilon/2];$$

$$Q^-(j, \alpha) = [i \mid (L_{i\alpha}, f_j) - (L_\alpha, f_j) \leq -\epsilon/2].$$

Let

$$k^\pm(j, \alpha) = \sum_{i \in Q_\pm(j, \alpha)} k_{i\alpha}.$$

Then for $\alpha > \alpha_0$,

$$\sum_{j=1}^{\nu} [k^+(j, \alpha) + k^-(j, \alpha)] \geq \sum_{i=1}^{i_\alpha} k_{i\alpha} = 1.$$

Hence at least one $k^\pm(j, \alpha) = k_\alpha \geq 1/2\nu$. Let Q_α denote the corresponding $Q^\pm(j, \alpha)$, and

$$L'_\alpha = k_\alpha^{-1} \sum_{i \in Q_\alpha} k_{i\alpha} L_{i\alpha}.$$

Since $f_i = f_\alpha$ and all L in question are non-negative,

$$(A2) \quad a(L_\alpha) = (L_\alpha, f_1) < (L_0, f_1) + \epsilon/2 = a(L_0) + \epsilon/2$$

for $\alpha > \alpha_0$. It follows that $a(L'_\alpha) \leq k_\alpha^{-1} a(L_\alpha) \leq 2\nu a(L_\alpha) < 2\nu a(L_0) + \nu\epsilon$. Since the set of all L situated in A with norms bounded by $2\nu a(L_0) + \nu\epsilon$ is w^* compact and $1/2\nu \leq k_\alpha \leq 1$ for $\alpha > \alpha_0$, subdirected sequences $(k_\beta), (L'_\beta)$ of $(k_\alpha), (L'_\alpha)$ and k, L' exist such that $k = \lim_\beta k_\beta$ and $L' = w^* \lim_\beta L'_\beta$.

Now for each $\alpha > \alpha_0$,

$$k_\alpha \mid (L'_\alpha, f_j) - (L_\alpha, f_j) \mid = \left| \sum_{i \in Q_\alpha} k_{i\alpha} [(L_{i\alpha}, f_j) - (L_\alpha, f_j)] \right| \geq k_\alpha \frac{\epsilon}{2} \geq \frac{\epsilon}{4\nu}$$

for at least one j by the choice of Q_α . Since there are only a finite number of f_j , passage to the limit gives

$$(A3) \quad k \mid (L', f_j) - (L_0, f_j) \mid \geq \epsilon/4\nu$$

for at least one j .

If $k = 1$, then $L' = L_0$, contrary to (A3). Hence $k < 1$. Now all L'_β are in $c(\Gamma)$ by their definition; hence L' is in $c(\Gamma)$ since $c(\Gamma)$ is w^* closed. Since $k_{\beta \rightarrow k} < 1$,

$$L''_\beta = (1 - k_\beta)^{-1} (L_\beta - k_\beta L'_\beta)$$

converges w^* to $L'' = (1 - k)^{-1} (L_0 - kL')$ and $L_0 = kL' + (1 - k)L''$. All L''_β are in $c(\Gamma)$; hence, L'' is in $c(\Gamma)$. Since L_0 is an extreme point of $c(\Gamma)$, $L_0 = L' = L''$, contradicting (A3). Thus every extreme point L_0 of $c(\Gamma)$ is a w^* limit point of Γ . Since the w^* topology is metrizable by the McShane distance in the set

of non-negative linear functionals situated in A , each such extreme point is the limit of a sequence in Γ .

(A4) Let $A \subset R^m$ be compact and f_0 in $E^{m,A}$. Then the minimum $\mu = \inf (L, f_0)$ over a w^* closed convex set Γ of non-negative linear functionals L situated in A is attained at an extreme point of Γ if it is attained at all in Γ .

Proof. The set Γ_1 of L in Γ for which $(L, f_0) = \mu$ is w^* closed, convex, and by hypothesis nonempty. Since $a(L) = (L, f_a)$ for L non-negative, an L_1 in Γ_1 of smallest norm exists. The set Γ_2 of L in Γ such that $a(L) = a(L_1)$ and $(L, f_0) = \mu$ is w^* compact and convex. By the Krein-Milman theorem [8], an extreme point L_0 of Γ_2 exists. It follows easily from the linearity of $a(L)$ and (L, f_0) that L_0 is also an extreme point of Γ , which proves (A4).

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