THE SPACE OF POINT HOMOTOPIC MAPS INTO THE CIRCLE(1)

BY

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1. **Introduction.** The space $C(X)$ of real bounded continuous functions on a topological space has been studied extensively ([9], [7], [6](2), etc.). More recently some of this theory has been extended to the space of functions into certain Banach spaces [5].

In the present paper, we consider the space of point-homotopic continuous maps into the circle. The circle, R_{2q} (reals mod $2q$), can be made into an abelian group, complete under an invariant metric. Then $R_{2q}(X)$, the space of point-homotopic continuous functions from X into R_{2q}, is in a natural way an abelian group, complete under an invariant metric. We give a characterization of $R_{2q}(X)$, for X a compact connected space, as an abelian group, complete under an invariant metric (Theorem 6.4), and a proof that for compact X, the metric group properties of $R_{2q}(X)$ determine the topology on X (Theorem 7.1).

The characterization is obtained by imposing conditions which insure the existence of a pseudo-multiplication by scalars (Theorem 2.2), and the existence of sufficiently many "characters" of the group (Theorems 3.7, 3.10 and 3.11). The points of X are found among the "characters" of the group by investigating certain Banach spaces associated with the group (§4). Certain new linear functionals are defined and a Banach space characterization of $C(X)$, for compact X, is given (Theorem 5.4). That the metric group properties of $R_{2q}(X)$ determine the topology on a compact X follows quickly from the similar theorem for Banach spaces [10].

2. **Some metric group properties of $R_{2q}(X)$.** The circle R_{2q} is taken to be the factor group of the reals R by the subgroup $I_{2q}=\{n(2q)\}$ where n is any integer. Thus R_{2q} is an abelian group. We denote by j the natural homomorphism of R onto R_{2q}. (j_{2q} would be more precise. However, no confusion results from the omission of the subscript.) We define $j^{-1}: R_{2q}\to R$ by $j^{-1}(a)=\alpha$ such that $-q<\alpha\leqq q$ and $j(\alpha)=a$. It follows immediately that

$$j(j^{-1}(a)) = a$$

and that for $|\alpha|<q, j^{-1}(j(\alpha))=\alpha$.

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(2) Numbers in brackets refer to the bibliography at the end of this paper.

If we define $\rho(a)=|j^{-1}(a)|$, then the function $d(a, b)=\rho(a-b)$ is an invariant metric on R_{2q} under which R_{2q} is complete. The space of all continuous functions from a topological space X into R_{2q}, denoted by R_{2q}^X, is made into a metric abelian group by defining

$$(f_1+f_2)(x) = f_1(x)+f_2(x), \quad \rho(f) = \sup_{x\in X}\{\rho(f(x))\}, \quad \text{and} \quad d(f_1,f_2) = \rho(f_1-f_2).$$

The metric is invariant, convergence is uniform convergence, and R_{2q}^X is complete. Now the space of point-homotopic continuous functions from X into R_{2q}, $R_{2q}(X)$, is the component of the identity in R_{2q}^X [4]. Thus it is a closed subgroup of R_{2q}^X, and it too is an abelian group complete under an invariant metric. Moreover, for each $\epsilon>0$, $U_\epsilon=\{f\in R_{2q}(X)\mid \rho(f)<\epsilon\}$ generates $R_{2q}(X)$. In what follows, the word group will denote an abelian group, complete under an invariant metric, and generated by $U_\epsilon=\{a\mid\rho(a)<\epsilon\}$ for each $\epsilon>0$. In addition we assume, with no loss of generality, that $q\geqq 1$.

DEFINITION 2.1. If $\alpha\in R$ and $a\in U_1\subset R_{2q}$, then $\alpha a=j(\alpha j^{-1}(a))$.

This pseudo-multiplication by real scalars can be extended to $U_1\subset R_{2q}(X)$ by defining $(\alpha f)(x)=\alpha(f(x))$. Since each operation used in Definition 2.1 is continuous (j^{-1} is continuous on U_1), the function $\alpha f\in R_{2q}^X$. Moreover $\{tf\}$ for $0\leqq t\leqq\alpha$ is a homotopy from $\theta(x)\equiv\theta$(3) $=j(0)$ to αf and so $\alpha f\in R_{2q}(X)$.

Some of the properties of scalar multiplication in a Banach space are preserved by this pseudo-multiplication. Thus it can be readily verified that, for $\alpha, \beta\in R$ and $a, b\in U_1\subset R_{2q}(X)$, the following relations hold.

(P1) $|\alpha|\rho(a)<1\rightarrow\beta(\alpha a)=(\beta\alpha)(a)$.

(P2) $(\alpha+\beta)a=\alpha a+\beta a$.

(P3) $\rho(a)+\rho(b)<1\rightarrow\alpha(a+b)=\alpha a+\alpha b$.

(P4) $|\alpha|\rho(a)<1\rightarrow\rho(\alpha a)=|\alpha|\rho(a)$.

(P5) $1a=a$.

DEFINITION 2.2. A group G is a pseudo-Banach space if a multiplication by reals can be defined on $U_1\subset G$ which satisfies P1–P5.

Thus we have

THEOREM 2.1. *$R_{2q}(X)$ is a pseudo-Banach space.*

The next theorem shows that the property of being a pseudo-Banach space is a metric group property.

THEOREM 2.2. *A group G is a pseudo-Banach space if and only if, for each $a\in U_1\subset G$, there exists a unique isomorphic isometry, i_a: $[0, \rho(a)]\rightarrow G$ such that $i_a(\rho(a))=a$. $\{[0, \rho(a)]$ represents the closed interval in R with end points at 0 and $\rho(a)$. The isomorphism applies whenever α, β and $\alpha+\beta$ all belong to $[0, \rho(a)]$.$\}$*

Proof. (a) Suppose G is a pseudo-Banach space. For each $a\in U_1\subset G$, de-

(3) The symbol θ will denote the identity element in a group. The symbol 0 will be reserved for the zero of the reals.

fine $i_a(\alpha)=(\alpha/\rho(a))a$. Then by P5, $i_a(\rho(a))=a$; by P2, i_a is an isomorphism; and by P4, i_a is an isometry. Now suppose i_a' is another such map. It follows from P5 and P2 that for m any positive integer $ma=\sum_{i=1}^m a$. Thus for m and n positive integers such that $m\leqq n$ we have $i_a(m\rho(a)/n)=(m/n)a=(m/n)(i_a'(\rho(a)))=(m/n)(ni_a'(\rho(a)/n))$ and $i_a'(m\rho(a)/n)=mi_a'(\rho(a)/n)$. But by P1, $mi_a'(\rho(a)/n)=(m/n)(ni_a'(\rho(a)/n))$. Thus i_a and i_a' are equal on a dense set of $[0,\rho(a)]$ and since they are isometries they must be identical(4).

(b) Suppose i_a is a unique isomorphic isometry taking $\rho(a)$ into a. For $\alpha\in R$ and $\alpha>0$ define $\bar{\alpha}$ to be the smallest integer such that $\bar{\alpha}\geqq\alpha$. Define

$$\begin{aligned} \alpha a &= \bar{\alpha}\left[i_a\left(\frac{\alpha}{\bar{\alpha}}\rho(a)\right)\right] && \text{for } \alpha>0,\\ \alpha a &= \theta && \text{for } \alpha=0,\\ \alpha a &= -((-\alpha)a) && \text{for } \alpha<0. \end{aligned}$$

Since inverses and multiplication by integers are well defined in any group, the preceding definitions give a precise meaning to αa.

The proof that this multiplication satisfies P1–P5 involves much intricate detail and is not given here. It may be found in the author's dissertation.

LEMMA 2.1. *If G is a pseudo-Banach space, then for each $b\in G$ and each ϵ such that $0<\epsilon\leqq 1$, there exists $\alpha\in R$ and $a\in U_\epsilon$ such that $\alpha a=b$.*

Proof. Since U_ϵ generates G, there exist elements $a_1,\cdots,a_n$ in U_ϵ such that $b=\sum_{i=1}^n a_i$. Choose $\alpha\geqq n$ and let $a=\sum_{i=1}^n (1/\alpha)a_i$. Then

$$\rho(a)\leqq\sum_{i=1}^n\rho((1/\alpha)a_i).$$

By P4, $\rho((1/\alpha)a_i)=(1/\alpha)\rho(a_i)$ and so

$$\rho(a)\leqq\sum_{i=1}^n\rho\left(\frac{1}{\alpha}a_i\right)=\sum_{i=1}^n\frac{1}{\alpha}\rho(a_i)<\frac{n\epsilon}{\alpha}\leqq\epsilon.$$

Thus $a\in U_\epsilon$. Moreover $\sum_{i=1}^n\rho((1/\alpha)a_i)<\epsilon\leqq 1$ and so by repeated application of P3 $\alpha a=\sum_{i=1}^n\alpha((1/\alpha)a_i)$. But by P1, $\alpha((1/\alpha)a_i)=a_i$ and so $\alpha a=\sum_{i=1}^n a_i=b$.

LEMMA 2.2. *If G a is pseudo-Banach space, and if for $a\in U_1$ and $\alpha\in R$ and different from zero, $\alpha a=\theta$, then either $a=\theta$ or $\rho(a)\geqq 2/|\alpha|$.*

Proof. If $\theta=\alpha a=(\alpha/2+\alpha/2)a$ then, by P2, $(\alpha/2)a=-((\alpha/2)a)=(-\alpha/2)a$ and $(1/2)((\alpha/2)a)=(1/2)((-(\alpha/2))a)$. Now if $\rho(a)<2/|\alpha|$, then $|\alpha/2|\rho(a)<1$), and we have by P1 that $(1/2)((\alpha/2)a)=(\alpha/4)a$ and $(1/2)((-\alpha/2)a)$

(4) Since P3 was not used in establishing the existence and uniqueness of i_a, the proof of sufficiency will prove that P3 is a consequence of P1, P2, P4, and P5. This can easily be established directly.

$=(-\alpha/4)a$ and $(\alpha/4)a=(-\alpha/4)a=-(\alpha/4)a$. Thus $\theta=(\alpha/4)a+(\alpha/4)a=(\alpha/2)a$. But by P4, $\rho((\alpha/2)a)=|\alpha/2|\rho(a)$ and if this is zero, $\rho(a)=0$ and $a=\theta$.

LEMMA 2.3. *If G a pseudo-Banach space, the map of $R\times U_1\rightarrow G$ given by $(\alpha, a)\rightarrow\alpha a$ is continuous.*

Proof. We show that the neighborhood V of (α_0, a_0) defined by $V=\{(\alpha, a)\mid |\alpha-\alpha_0|<\min[\epsilon/2, 1/\rho(a)]$ and $\rho(a-a_0)<\min[\epsilon/2|\alpha_0|, 1/|\alpha_0|, 1-\rho(a_0)]\}$ maps into the ϵ neighborhood of $\alpha_0 a_0$.

For $(\alpha, a)\in V$

$$\begin{aligned}\alpha a-\alpha_0 a_0 &= \alpha(a-a_0)+\alpha a_0-\alpha_0 a_0 = \alpha(a-a_0)+(\alpha-\alpha_0)a_0\\ &= \alpha_0(a-a_0)+(\alpha-\alpha_0)(a-a_0)+(\alpha-\alpha_0)a_0\\ &= \alpha_0(a-a_0)+(\alpha-\alpha_0)a\end{aligned}$$

by P3, P2, P2, and P3 respectively. Then $\rho(\alpha a-\alpha_0 a_0)\leqq\rho(\alpha_0(a-a_0))+\rho((\alpha-\alpha_0)a)<\epsilon/2+\epsilon/2<\epsilon$ by P1 and P4.

In what follows we shall use properties P1–P5 without specific reference, taking care always that the hypotheses of the statements are satisfied.

DEFINITION 2.3. An element $h\in G$ is a root of unity if there exists an integer n such that $nh=\theta$. The set of roots of unity we denote by H and the closure of H in G by $\overline{H}$.

H and $\overline{H}$ are subgroups of G in the usual sense.

The elements of $\overline{H}\subset R_{2q}(X)$ have special metric properties as well. The following lemma makes this explicit for the case where X is a connected space.

LEMMA 2.4. *If X is connected, then $h\in\overline{H}\subset R_{2q}(X)$ is a constant function and if $\rho(h)<1$, then for $g\in R_{2q}(X)$ such that $\rho(h)+\rho(g)<1$ we have either $\rho(h+g)=\rho(h)+\rho(g)$ or $\rho(h-g)=\rho(h)+\rho(g)$.*

Proof. Suppose $h\in H\subset R_{2q}(X)$. Then there exists an integer n such that $n(h(x))\equiv\theta$ for all $x\in X$. Thus $h(X)\subset A=\{a\in R_{2q}\mid na=\theta\}$. But $h(X)$ is connected while the set A is discrete and so h is a constant function. The definition of the metric then implies that the elements of $\overline{H}$ are constant functions. From this fact plus the definition of the function ρ, the second part follows immediately.

We are led to the following definitions.

DEFINITION 2.4. If $\rho(a)+\rho(b)<1$ and if $\rho(a)+\rho(b)=\rho(a+b)$, then the pair $\{a, b\}$ is positive.

DEFINITION 2.5. If $a\in U_1\subset G$ and if for all $b\in U_{1-\rho(a)}\subset G$ either $\{a, b\}$ or $\{a, -b\}$ is positive, then a is a constant of G.

DEFINITION 2.6. A pseudo-Banach space G is a space with constants if $\overline{H}\neq\{\theta\}$ (5) and if all the elements of $\overline{H}\cap U_1$ are constants of G.

(5) We assume $\overline{H}\neq\{\theta\}$, as otherwise G is essentially a Banach space.

THEOREM 2.3. *If X is a connected space, then $R_{2q}(X)$ is a space with constants.*

Proof. Theorem 2.1 and Lemma 2.4.

3. **Subspaces and characters.** A basic theorem in the classification of Banach spaces is that every Banach space B is equivalent to a closed subspace of $C(X)$ for some compact topological space X [1]. The points of X are found among the continuous linear functionals on B. The existence of sufficiently many such functionals is assured by the Hahn-Banach theorem [3, p. 55]. In this section we prove that under modified definitions of equivalence and subspace, every space with constants is equivalent to a subspace of $R_{2q}(X)$ for some $q \geqq 1$ and for some compact connected space X.

DEFINITION 3.1. Two groups G and $\widehat{G}$ are equivalent if there is an isomorphism $I: G \rightarrow \widehat{G}$ such that I is an isometry on U_1 and such that $I(U_1) = \widehat{U}_1$. {For this definition we do not require that G and $\widehat{G}$ be complete.} It is clear that the relation of equivalence is symmetric, reflexive, and transitive.

DEFINITION 3.2. A subset G' of a pseudo-Banach space G is a subspace of G if G' is a subgroup (in the ordinary sense) and if, for $\alpha \neq 0$ and $a \in U_1 \subset G$, $\alpha a \in G'$ if and only if $a \in G'$.

DEFINITION 3.3. If G' is a subspace of a pseudo-Banach space G, then $L: G' \rightarrow R_{2q}$ is a character of G' if

(P′1) $L(a+b) = L(a) + L(b)$,

(P′2) $|j^{-1}(L(a))| \leqq \rho(a)$ whenever $\rho(a) < 1$,

(P′3) $L(\alpha a) = \alpha L(a)$ whenever $\rho(a) < 1$.

From the definitions it is clear that G' may be all of G.

THEOREM 3.1. *The characters of a subspace G' of a pseudo-Banach space G are continuous on G'.*

Proof. By P′1, L is a homomorphism. But for $0 < \epsilon < 1$ and $a \in U_\epsilon \cap G'$,

$$|j^{-1}(L(a))| < \epsilon$$

and so L is continuous at the identity and therefore continuous on G'.

THEOREM 3.2. *If G' is a subspace of a pseudo-Banach space G and if $L: G' \rightarrow R_{2q}$ satisfies* P′1 *and* P′2, *then L is a character of G'.*

Proof. For $a \in U_1 \cap G'$ and n any positive integer, $n((1/n)a) = a$. Then for m any integer P′1 gives $(m/n)L(a) = (m/n)L(n((1/n)a)) = (m/n)(nL((1/n)a))$. But R_{2q} is itself a pseudo-Banach space and by P′2 $n\rho(L((1/n)a)) = n|j^{-1}(L((1/n)a))| \leqq n\rho((1/n)a) = \rho(a) < 1$. Thus we have $(m/n)(nL((1/n)a)) = mL((1/n)a) = L((m/n)a)$ by P′1, and $(m/n)L(a) = L((m/n)a)$. By Theorem 3.1 and Lemma 2.3 both $\alpha L(a)$ and $L(\alpha a)$ are continuous in α. Since they are equal on a dense set of R, they are equal for all $\alpha \in R$ and L satisfies P′3 on G'.

The usual boundedness restriction for linear functionals on a Banach space would translate here to $|j^{-1}(L(a))| \leqq M\rho(a)$. However, this plus P′1 does not imply P′3. The proof uses strongly that $M=1$ and the theorem is false without it. For let $G' \subset R_2([0,1])$ be the set $\{f \in R_2([0,1]) \mid f(x)=j(\alpha+\beta x)\}$ and define $L(j(\alpha+\beta x))=j(\alpha-(3/2)\beta)$. It is easily verified that L satisfies P′1 and that $|j^{-1}L(f)| \leqq 4\rho(f)$. However,

$$(1/2)L(j(-1/2+x)) = (1/2)j(-1/2-3/2) = (1/2)j(-2) = (1/2)\theta = \theta,$$

while $L((1/2)j(-1/2+x))=L(j(1/2)j^{-1}j(-1/2+x)))=L(j(-1/4+(1/2)x))$ $=j(-1/4-3/4)=j(-1)\neq\theta$.

Since in the construction of characters we have no other way of insuring that P′3 be satisfied we must use the stronger form given by P′2.

THEOREM 3.3. *If G' is a subspace of a pseudo-Banach space G and L' is a character of G', then there exists a character L of G such that $L=L'$ on G'.*

Proof. The proof is a modification of the similar theorem for Banach spaces [3, p. 28]. If $G'=G$ we are through. If $G'\neq G$, there exists an element $a\in(G-G')\cap U_{1/2}$, since $U_{1/2}$ generates G. For b_1 and b_2 any elements of $G'\cap U_{1/2}$ and for β_1 and β_2 real numbers such that $0<\beta_i\leqq 1$ and $\beta=\min(\beta_1,\beta_2)$ we have, by P′1 and P′2, that

$$j^{-1}\left\{L'\left(\frac{\beta}{\beta_1}b_1\right)-L'\left(\frac{\beta}{\beta_2}b_2\right)\right\} = j^{-1}\left\{L'\left(\frac{\beta}{\beta_1}b_1-\frac{\beta}{\beta_2}b_2\right)\right\}$$
$$\leqq \rho\left(\frac{\beta}{\beta_1}b_1-\frac{\beta}{\beta_2}b_2\right),$$

since

$$\left|j^{-1}\left(L'\left(\frac{\beta}{\beta_1}b_1\right)\right)\right|+\left|j^{-1}\left(L'\left(\frac{\beta}{\beta_2}b_2\right)\right)\right|<1,$$

$$j^{-1}\left\{L'\left(\frac{\beta}{\beta_1}b_1\right)-L'\left(\frac{\beta}{\beta_2}b_2\right)\right\} = j^{-1}\left\{L'\left(\frac{\beta}{\beta_1}b_1\right)\right\}-j^{-1}\left\{L'\left(\frac{\beta}{\beta_2}b_2\right)\right\}$$

and so

$$j^{-1}\left\{L'\left(\frac{\beta}{\beta_1}b_1\right)\right\}-j^{-1}\left\{L'\left(\frac{\beta}{\beta_2}b_2\right)\right\} \leqq \rho\left(\frac{\beta}{\beta_1}b_1-\frac{\beta}{\beta_2}b_2\right)$$
$$\leqq \rho\left(\frac{\beta}{\beta_1}b_1+\beta a\right)+\rho\left(\frac{\beta}{\beta_2}b_2+\beta a\right)$$

and so

$$-\frac{\beta}{\beta_2}\rho(b_2+\beta_2 a)-\frac{\beta}{\beta_2}j^{-1}(L'(b_2)) \leqq \frac{\beta}{\beta_1}\rho(b_1+\beta_1 a)-\frac{\beta}{\beta_1}j^{-1}(L'(b_1)).$$

Dividing by β gives

$$(3.1)\quad -\frac{1}{\beta_2}\rho(b_2+\beta_2 a)-\frac{1}{\beta_2}j^{-1}(L'(b_2)) \leqq \frac{1}{\beta_1}\rho(b_1+\beta_1 a)-\frac{1}{\beta_1}j^{-1}(L'(b_1)).$$

Since (3.1) holds for all β_1, β_2, b_1, and b_2 we have

$$(3.2)\quad \begin{aligned} m &= \operatorname*{l.u.b.}_{b_2,\beta_2}\left\{-\frac{1}{\beta_2}\rho(b_2+\beta_2 a)-\frac{1}{\beta_2}j^{-1}(L'(b_2))\right\} \\ &\leqq \operatorname*{g.l.b.}_{b_1,\beta_1}\left\{\frac{1}{\beta_1}\rho(b_1+\beta_1 a)-\frac{1}{\beta_1}j^{-1}(L'(b_1))\right\} = M. \end{aligned}$$

Let $G''=\{c\in G \mid c=\gamma a+b$ for any $\gamma\in R$ and $b\in G'\}$. For a fixed $c\in G''$, γ and b are uniquely determined. If $\gamma a+b=\gamma' a+b'$, then $(\gamma-\gamma')a=b'-b\in G'$. But $a\notin G'$ and G' is a subspace, thus $\gamma=\gamma'$ and so $b=b'$.

Choose $\alpha\in R$ such that $m\leqq\alpha\leqq M$ and define $L'':G''\to R_{2q}$ by $L''(c)=j(\gamma\alpha)+L'(b)$.

We show that G'' is a subspace and that L'' is a character of G''. That G'' properly contains G' and that $L''=L'$ on G' is immediate.

G'' is clearly a subgroup (in the usual sense). Suppose $c\in G''\cap U_1$ and $0\neq\delta\in R$. Let $\eta=2\max(|\delta|, (1/2)|\delta\gamma|)$. Then $(\eta/\delta)((\delta/\eta)c-(\delta\gamma/\eta)a)=c-\gamma a=b\in G'$. Therefore $(\delta/\eta)c-(\delta\gamma/\eta)a\in G'$ and $\eta((\delta/\eta)c-(\delta\gamma/\eta)a)=\delta c-(\delta\gamma)a\in G'$, and so $\delta c\in G''$. Now suppose $\rho(c)<1$, $0\neq\delta\in R$, and that $\delta c\in G''$. Then $\delta c=\gamma' a+b'$. Let $\eta=\max(2, |\gamma'/\delta|)$. Then $\eta\delta((1/\eta)c-(\gamma'/\eta\delta)a)=b'\in G'$ and $(1/\eta)c-(\gamma'/\eta\delta)a\in G'$, so that

$$\eta((1/\eta)c-(\gamma'/\eta\delta)a)=c-(\gamma'/\delta)a\in G', \quad\text{and}\quad c\in G''.$$

Thus we have proved that G'' is a subspace.

Now $L''(c_1+c_2)=L''(\gamma_1 a+b_1+\gamma_2 a+b_2)=L''((\gamma_1+\gamma_2)a+(b_1+b_2))=j((\gamma_1+\gamma_2)\alpha)+L'(b_1+b_2)=j(\gamma_1\alpha)+L'(b_1)+j(\gamma_2\alpha)+L'(b_2)=L''(c_1)+L''(c_2)$ and P′1 is satisfied. Now suppose $c=\gamma a+b\in G''$ and $\rho(c)<1$. If $\gamma=0$, P′2 is immediate. If $\gamma\neq0$, let $\delta=\max(2, |4/\gamma|)$. Then $\delta\gamma\{(1/\delta\gamma)c-(1/\delta)a\}=c-\gamma a=b\in G'$ and so $(1/\delta\gamma)c-(1/\delta)a\in G'$. Moreover $\rho((1/\delta\gamma)c-(1/\delta)a)<1/4+1/4=1/2$. Thus in (3.1) we may put $b_1=b_2=(1/\delta\gamma)c-(1/\delta)a$ and $\beta_1=\beta_2=(1/\delta)$. We get

$$\begin{aligned} -\delta\rho\left(\frac{1}{\delta\gamma}c\right)-\delta\left\{j^{-1}\left(L'\left(\frac{1}{\delta\gamma}c-\frac{1}{\delta}a\right)\right)\right\} &\leqq m\leqq\alpha\leqq M \\ &\leqq \delta\rho\left(\frac{1}{\delta\gamma}c\right)-\delta\left\{j^{-1}\left(L'\left(\frac{1}{\delta\gamma}c-\frac{1}{\delta}a\right)\right)\right\} \end{aligned}$$

and so $|\alpha/\delta+j^{-1}(L'((1/\delta\gamma)c-(1/\delta)a))|\leqq\rho((1/\delta\gamma)c)=(1/|\delta\gamma|)\rho(c)$ and $|\gamma\alpha+\delta\gamma j^{-1}(L'((1/\delta\gamma)c-(1/\delta)a))|\leqq\rho(c)<1$. But $j^{-1}(j(\beta))=\beta$ for $|\beta|\leqq1$ and so

$$\rho(c) \geqq \left| j^{-1}\left\{ j(\gamma\alpha) + j\left(\delta\gamma j^{-1}\left(L'\left(\frac{1}{\delta\gamma}c - \frac{1}{\delta}a\right)\right)\right)\right\}\right|$$

$$= \left| j^{-1}\left\{ j(\gamma\alpha) + \delta\gamma\left(L'\left(\frac{1}{\delta\gamma}c - \frac{1}{\delta}a\right)\right)\right\}\right|$$

$$= \left| j^{-1}\left\{ j(\gamma\alpha) + L'\left(\delta\gamma\left(\frac{1}{\delta\gamma}c - \frac{1}{\delta}a\right)\right)\right\}\right|$$

as L' is a character on G' and satisfies P'3. Thus $\rho(c) \geqq |j^{-1}\{j(\gamma\alpha)+L'(b)\}| = |j^{-1}\{L''(c)\}|$ and L'' satisfies P'2 on G''.

By Theorem 3.2, L'' is a character of G''. Then by transfinite induction there exists a character L of G such that $L=L'$ on G'.

Theorem 3.3 does not prove the existence of characters on a pseudo-Banach space G. We must first exhibit a subspace G' of G and a character of G'. At first glance, the real multiples of an element in U_1 might seem to do for G'. But this is not necessarily a subspace of G (Corollary 1 to Theorem 3.5). We show, however, that $\overline{H}$ is a subspace of G and that if G is a space with constants, there exists a character taking $\overline{H}$ into R_{2q} for some $q \geqq 1$.

THEOREM 3.4. *If G is a pseudo-Banach space, $\overline{H}$ is a subspace of G.*

Proof. Suppose $h \in \overline{H} \cap U_1$ and $0 \neq \alpha \in R$. Then there exist $h_i \in H \cap U_1$ and integers p_i and q_i such that $h_i \to h$ and $p_i/q_i \to \alpha$. Since $h_i \in H$, there exist integers n_i such that $n_i h_i = \theta$. Then $n_i q_i((p_i/q_i)h_i) = p_i(n_i h_i) = \theta$ and so $(p_i/q_i)h_i \in H$. But by Lemma 2.3, $(p_i/q_i)h_i \to \alpha h$ and so $\alpha h \in \overline{H}$.

Now suppose $0 \neq \alpha \in R$, $h \in U_1$, and $\alpha h \in \overline{H}$. If $\alpha < 0$, then $\alpha h = -\{(-\alpha)h\}$ and $(-\alpha)h \in \overline{H}$. Thus we may assume $\alpha > 0$. There exist $h_i \in H$ such that $h_i \to \alpha h$. Thus there exists an I, such that $\rho(\alpha h - h_i) < 1$ whenever $i \geqq I$ and so $(1/\bar{\alpha})(\alpha h - h_i)$ is defined for $i \geqq I$. Moreover $\bar{\alpha}[(\alpha/\bar{\alpha})h - (1/\bar{\alpha})(\alpha h - h_i)] = \alpha h - \alpha h + h_i \in H$ and since $\bar{\alpha}$ is an integer, $a_i = (\alpha/\bar{\alpha})h - (1/\bar{\alpha})(\alpha h - h_i) \in H$. But $a_i \to (\alpha/\bar{\alpha})h$ and so $(\alpha/\bar{\alpha})h \in \overline{H}$. Since $(\alpha/\bar{\alpha})h \in U_1$, by the first part of the proof $(\bar{\alpha}/\alpha)((\alpha/\bar{\alpha})h) = h \in \overline{H}$.

LEMMA 3.1. *If $\{a, b\}$ is positive (Definition 2.4) and if $\alpha \geqq 0$, $\beta \geqq 0$, and $\alpha\rho(a) + \beta\rho(b) < 1$, then $\{\alpha a, \beta b\}$ is positive.*

Proof. For either α or β equal to zero, the result is immediate. We assume $\alpha \geqq \beta > 0$. Since $\alpha\rho(a) + \beta\rho(b) < 1$, $\alpha\rho(a) = \rho(\alpha a)$ and $\beta\rho(b) = \rho(\beta b)$. Thus $(1/\alpha)\rho(\alpha a + \beta b) \leqq (1/\alpha)(\rho(\alpha a) + \rho(\beta b)) = \rho(a) + (\beta/\alpha)\rho(b) \leqq \rho(a) + \rho(b) < 1$, and $(1/\alpha)\rho(\alpha a + \beta b) = \rho((1/\alpha)(\alpha a + \beta b))$. But $\rho(\alpha a) + \rho(\beta b) < 1$ and so $(1/\alpha)(\alpha a + \beta b) = (1/\alpha)(\alpha a) + (1/\alpha)(\beta b)$ and since $\alpha\rho(a) < 1$ and $\beta\rho(b) < 1$, $(1/\alpha)(\alpha a + \beta b) = a + (\beta/\alpha)b$. Now $\rho(a + (\beta/\alpha)b) = \rho(a + b - (1 - \beta/\alpha)b) \geqq \rho(a+b) - (1 - \beta/\alpha)\rho(b) = \rho(a) + \rho(b) - (1 - \beta/\alpha)\rho(b) = \rho(a) + (\beta/\alpha)\rho(b)$. But the opposite inequality is always true and so $\rho(a) + (\beta/\alpha)\rho(b) = \rho(a + (\beta/\alpha)b) = \rho((1/\alpha)(\alpha a + \beta b)) = (1/\alpha)\rho(\alpha a + \beta b)$. Thus $\rho(\alpha a + \beta b) = \alpha\rho(a) + \beta\rho(b) = \rho(\alpha a) + \rho(\beta b) < 1$ and so

$\{\alpha a, \beta b\}$ is positive.

LEMMA 3.2. *If G is a space with constants, $h_1 \in \overline{H}$, $h_2 \in \overline{H}$, and $\{h_1, h_2\}$ is positive, then $\rho(h_1 - h_2) = |\rho(h_1) - \rho(h_2)|$.*

Proof. $(1/2)h_1 + (1/2)h_2 \in \overline{H} \cap U_{1/2}$ and is therefore a constant of G. Moreover $\rho((1/2)h_1 + (1/2)h_2) + \rho((1/2)h_1 - (1/2)h_2) \leqq \rho(h_1) + \rho(h_2) < 1$, and so either $\{(1/2)h_1 + (1/2)h_2, (1/2)h_1 - (1/2)h_2\}$ or $\{(1/2)h_1 + (1/2)h_2, (1/2)h_2 - (1/2)h_1\}$ is positive. If the first pair is positive we have

$$\rho((1/2)h_1 + (1/2)h_2) + \rho((1/2)h_1 - (1/2)h_2) = \rho((1/2)h_1 + (1/2)h_2 + (1/2)h_1 - (1/2)h_2) = \rho(h_1)$$

Thus $\rho((1/2)h_1 - (1/2)h_2) = \rho(h_1) - \rho((1/2)h_1 + (1/2)h_2) = (1/2)\rho(h_1) - (1/2) \cdot \rho(h_2)$ by Lemma 3.1. If the second pair is positive we have $\rho((1/2)h_1 - (1/2)h_2) = (1/2)\rho(h_2) - (1/2)\rho(h_1)$. Since $\rho((1/2)h_1 - (1/2)h_2) \geqq 0$ we have in either case that $\rho((1/2)h_1 - (1/2)h_2) = (1/2)|\rho(h_1) - \rho(h_2)|$ and multiplication by 2 gives

$$\rho(h_1 - h_2) = |\rho(h_1) - \rho(h_2)|.$$

THEOREM 3.5. *If G is a space with constants, $\theta \neq h \in \overline{H} \cap U_1$, and $h_0 \in \overline{H}$, there exists $\alpha \in R$ such that $\alpha h = h_0$. In particular if $\rho(h_0) < 1$, then $h_0 = \pm(\rho(h_0)/\rho(h))h$.*

Proof. By Lemma 2.1, there exist $h_0' \in U_{1-\rho(h)}$ and $\beta \in R$ such that $\beta h_0' = h_0$. Now h is a constant of G and so either $\{h, h_0'\}$ or $\{h, -h_0'\}$ is positive. If $\{h, h_0'\}$ is positive, then by Lemma 3.1, $\{(\rho(h_0')/2\rho(h))h, (1/2)h_0'\}$ is positive. By Theorem 3.4, both these elements belong to $\overline{H}$ and so by Lemma 3.2,

$$\rho\left(\frac{\rho(h_0')}{2\rho(h)}\, h - \frac{1}{2}\, h_0'\right) = \left|\rho\left(\frac{\rho(h_0')}{2\rho(h)}\, h\right) - \rho\left(\frac{1}{2}\, h_0'\right)\right|$$
$$= \left|\frac{1}{2}\,\rho(h_0') - \frac{1}{2}\,\rho(h_0')\right| = 0$$

and so

$$\frac{1}{2}\, h_0' = \frac{\rho(h_0')}{2\rho(h)}\, h \quad \text{and} \quad h_0 = 2\beta\left(\frac{1}{2}\, h_0'\right) = \frac{\beta\rho(h_0')}{\rho(h)}\, h.$$

If $\{h, -h_0'\}$ is positive we get $h_0 = (-\beta\rho(h_0')/\rho(h))h$ and the first part of the theorem is proved.

Now if $\rho(h_0) < 1$, we may choose $h_0' = (1 - \rho(h))h_0$ and $\beta = (1/(1 - \rho(h))$. Then $h_0 = \pm(\rho(h_0)/\rho(h))h$.

COROLLARY 1. *If G' is a subspace of a space with constants, then $G' \supset \overline{H}$.*

Proof. Since $\theta \in G'$, $H \cap U_1 \subset G'$ as $h \in H \cap U_1$ implies there exists an n such that $nh = \theta$. Therefore $\alpha h \in G'$ for all $\alpha \in R$ and so $\overline{H} \in G'$.

COROLLARY 2. *If G' is a closed subspace of a space with constants, then G' is a space with constants.*

Proof. By Corollary 1, $G' \supset \overline{H} \neq \{\theta\}$ and since it is closed it is complete.

LEMMA 3.3. *If G is a space with constants, and $\theta \neq h \in \overline{H} \cap U_1$, there exists a real number $\alpha_h > 0$ such that $\alpha_h h = \theta$ and such that $0 < \alpha < \alpha_h$ implies $\alpha h \neq \theta$.*

Proof. Let $A = \{\alpha > 0 \mid \alpha h = \theta\}$. By Lemma 2.2, A is equal to $\{\alpha \geqq 2/\rho(h) \mid \alpha h = \theta\}$ and by Lemma 2.3, A is closed. Thus if A is not empty, $\alpha_h = \text{g.l.b.}_{\alpha \in A}\, \alpha$ has the required property. But A cannot be empty. For choose $h_0 \in H$ such that $h_0 \neq \theta$. Then there exist an integer n_0 such that $n_0 h_0 = \theta$ and, by Theorem 3.5, a real number $\alpha \neq 0$ such that $\alpha h = h_0$. Thus $\theta = n_0(\alpha h) = (n_0\alpha)h = -(n_0\alpha)h = (-n_0\alpha)h$. Now either $n_0\alpha$ or $-n_0\alpha$ is positive and so belongs to A.

COROLLARY. *$\alpha h = \theta$ if and only if $\alpha = n\alpha_h$ for some integer n.*

DEFINITION 3.4. Let $q_h = (1/2)\alpha_h \rho(h)$. By Lemma 2.2, $q_h \geqq 1$.

LEMMA 3.4. *If G is a space with constants, and $h \in \overline{H} \cap U_1$ and $h \neq \theta$, then $\overline{H}$ is equivalent to R_{2q_h}.*

Proof. For $h_0 \in \overline{H}$, there exists, by Theorem 3.5, $\alpha \in R$ such that $\alpha h = h_0$. Define $l_h: \overline{H} \to R_{2q_h}$ by $l_h(h_0) = j(\alpha\rho(h))$. We show that l_h is uniquely defined and gives an equivalence between $\overline{H}$ and R_{2q_h}.

(a) If $h_0 = \alpha h = \beta h$, then $(\beta - \alpha)h = \theta$ and $\beta - \alpha = n\alpha_h$ (corollary to Lemma 3.3). Thus $j(\alpha\rho(h)) - j(\beta\rho(h)) = j((\alpha-\beta)\rho(h)) = j(n\alpha_h\rho(h)) = j(n(2q_h)) = \theta$ and l_h is uniquely defined.

(b) $l_h(h_1 + h_2) = l_h(\alpha_1 h + \alpha_2 h) = j((\alpha_1 + \alpha_2)\rho(h)) = j(\alpha_1\rho(h)) + j(\alpha_2\rho(h)) = l_h(h_1) + l_h(h_2)$ and l_h is a homomorphism.

(c) If $l_h(h_0) = \theta$, then $\alpha\rho(h) = 2nq_h$ and $\alpha = n\alpha_h$ and $h_0 = \alpha h = \theta$. Thus l_h is an isomorphism.

(d) If $\rho(h_0) < 1$, $h_0 = \pm(\rho(h_0)/\rho(h))h$ by Theorem 3.5. Thus $|j^{-1}(l_h(h_0))| = |j^{-1}(j((\rho(h_0)/\rho(h))\rho(h)))| = \rho(h_0)$ and l_h is an isometry on $\overline{H} \cap U_1$.

(e) Suppose $a \in U_1 \subset R_{2q_h}$. Let $\alpha = (1/\rho(h))\{j^{-1}(a)\}$ and $h_0 = \alpha h$. Then $h_0 \in U_1 \subset \overline{H}$ and $l_h(h_0) = j(j^{-1}(a)) = a$. Thus l_h maps $U_1 \cap \overline{H}$ onto $U_1 \cap R_{2q_h}$.

Thus (Definition 3.1) l_h gives an equivalence between $\overline{H}$ and R_{2q_h}.

COROLLARY. *If $h' \in \overline{H} \cap U_1$ and $h' \neq \theta$, then $q_{h'} = q_h$ and $l_{h'} = \pm l_h$.*

Proof. By the lemma, $\overline{H}$ is equivalent to both R_{2q_h} and $R_{2q_{h'}}$ and so R_{2q_h} is equivalent to $R_{2q_{h'}}$. This implies immediately that $q_h = q_{h'}$. Now the only continuous isomorphisms of R_{2q} onto itself are the identity and the reflection $(a \to -a)$. But $l_{h'}(l_h^{-1})$ is such a map and so $l_{h'} = \pm l_h$.

Thus we may drop the subscript h from q_h and define $q = (1/2)\alpha_h\rho(h)$ for any $h \in \overline{H} \cap U_1$ such that $h \neq \theta$. We choose one of the two equivalence mappings of $\overline{H}$ onto R_{2q} and denote it by l. The other is then $-l$.

We have already proved

THEOREM 3.6. *If G is a space with constants, then $l:\overline{H}\to R_{2q}$ is a character of $\overline{H}$.*

THEOREM 3.7. *If G is a space with constants, then for each $a\in U_1$, there exists a character L of G such that $L=l$ on $\overline{H}$ and $|j^{-1}(L(a))|=\rho(a)$.*

Proof. By Theorems 3.3, 3.4, and 3.6 there exist characters of G equal to l on $\overline{H}$. If $a\in\overline{H}$, $|j^{-1}(L(a))|=|j^{-1}(l(a))|=\rho(a)$ and we are through. Suppose $a\notin\overline{H}$. For each $h\in\overline{H}\cap U_1$, $l(h)$ or $l(-h)=j(\rho(h))$. Choose $h_0\in\overline{H}\cap U_{1/2}$ such that $h_0\neq\theta$ and $l(h_0)=j(\rho(h_0))$. Since h_0 is a constant of G, there exists b such that $b=\pm(1/2)a$ and $\{h_0, b\}$ is positive. From the proof of Theorem 3.3, there is a character L of G equal to l on $\overline{H}$ such that $L(b)=j(M)$ where $M=\text{g.l.b.}_{h_1\in\overline{H}\cap U_{1/2},\ 0<\beta_1\leq 1}\{(1/\beta_1)\rho(h_1+\beta_1 b)-(1/\beta_1)j^{-1}(l(h_1))\}$. By Theorem 3.5, $h_1=\pm(\rho(h_1)/\rho(h_0))h_0$.

(a) If $h_1=(\rho(h_1)/\rho(h_0))h_0$, then by Lemma 3.1 $\rho(h_1+\beta_1 b)=\rho(h_1)+\beta_1\rho(b)$, and thus $(1/\beta_1)\rho(h_1+\beta_1 b)-(1/\beta_1)j^{-1}(l(h_1))=(1/\beta_1)\rho(h_1)+\rho(b)-(1/\beta_1)\rho(h_1)=\rho(b)$.

(b) If $h_1=-(\rho(h_1)/\rho(h_0))h_0$, then

$$\frac{1}{\beta_1}\rho(h_1+\beta_1 b)-\frac{1}{\beta_1}j^{-1}(l(h_1))\geq\rho(b)-\frac{1}{\beta_1}\rho(h_1)+\frac{1}{\beta_1}\rho(h_1)=\rho(b).$$

Thus $M=\rho(b)$ and $L(b)=j(\rho(b))$. Then

$$|j^{-1}(L(a))|=|j^{-1}(L(\pm 2b))|=|j^{-1}(\pm 2L(b)|=2|j^{-1}j(\rho(b))|=2\rho(b)=\rho(a).$$

Let G be a space with constants. The set of characters of G which are extensions of l is a topological space under the point open topology. We denote this space by S.

THEOREM 3.8. *The space S is connected.*

Proof. Suppose L_0 and L_1 belong to S. For $0\leq\alpha\leq 1$ we define L_α: $G\to R_{2q}$ as follows. For $b\in G$, there exist $a\in U_1$ and $\gamma\in R$ such that $\gamma a=b$ (Lemma 2.1). We put $L_\alpha(b)=L_0(((1-\alpha)\gamma)a)+L_1((\alpha\gamma)a)$. Using strongly the fact that $L_0=L_1=l$ on $\overline{H}$, one may verify that $L_\alpha(b)$ is uniquely defined and that $L_\alpha\in S$. Since $|j^{-1}(L_\alpha(b)-L_{\alpha_0}(b))|\leq|j^{-1}L_0(((\alpha_0-\alpha)\gamma)a)|+|j^{-1}L_1(((\alpha-\alpha_0)\gamma)a)|\leq 2|\gamma|\,|\alpha-\alpha_0|$, the map $\alpha\to L_\alpha$ is a continuous curve connecting L_0 to L_1 in S. Thus S is connected.

THEOREM 3.9. *The space S is compact.*

Proof. See Theorem 6.1 which is independently proved. A direct proof, duplicating the proof that the unit sphere in a conjugate space is compact in the weak-star topology, can be given.

THEOREM 3.10. *If $\rho(a)\geq 1$, there exists $L\in S$, such that $|j^{-1}(L(a))|\geq 1$.*

Proof. Suppose $|j^{-1}(L(a))| < 1$ for all $L \in S$. There exist $b \in U_{1/3}$ and $\beta \in R$ such that $\beta b = a$ (Lemma 2.1). Now $\rho(a) \geqq 1$ and so $|\beta| \geqq 3$ which implies that $\beta[L(b) - (1/\beta)L(a)] = \beta L(b) - L(a) = \theta$ for all $L \in S$. The function $f: S \to R_{2q}$ defined by $f(L) = L(b) - (1/\beta)L(a)$ is continuous (Lemma 2.3 and the definition of the point open topology). Since S is connected (Theorem 3.8), $f(S)$ is connected. Now the set $C = \{c \in R_{2q} | \beta c = \theta\}$ is totally disconnected, and $f(S) < C$. Thus $f(S) \equiv c_0$. Moreover $|j^{-1}(c_0)| \leqq |j^{-1}(L(b))| + |j^{-1}((1/\beta)L(a))| < 2/3$.

Let $h = l^{-1}(c_0)$. Then $L(b-h) = L(b) - c_0 = (1/\beta)L(a)$, and $|j^{-1}(L(b-h))| < 1/|\beta|$ for all $L \in S$. But there exists $L_0 \in S$ such that $|j^{-1}(L_0(b-h))| = \rho(b-h)$ (Theorem 3.7) and so $\rho(b-h) < 1/|\beta|$. Thus $1 > |\beta|\rho(b-h) = \rho(\beta b - \beta h) = \rho(a - \beta h)$. But $l(\beta h) = \beta l(h) = \beta c_0 = \theta$ and $\beta h = \theta$. But then $\rho(a) < 1$, contradicting the hypothesis.

Theorems 3.7 and 3.10 combine to give

THEOREM 3.11. *If G is a space with constants and if $b \neq \theta$, then there exists $L \in S$ such that $L(b) \neq \theta$.*

THEOREM 3.12. *A group G is equivalent to a closed subspace G' of $R_{2q}(X)$ for some $q \geqq 1$ and for some compact, connected space X, if and only if G is a space with constants.*

Proof. (a) By Theorem 2.3 and Corollary 2 to Theorem 3.5, G' is a space with constants. But G is equivalent to G' and it follows that G is a space with constants.

(b) Suppose G is a space with constants. Then G uniquely determines an R_{2q} for $q \geqq 1$ (Lemma 3.4 and corollary), and the space S is a compact, connected space. We define $I(b)\{L\} = L(b)$ for all $L \in S$. The choice of the point open topology on S insures that $I(b)$ is a continuous function on S and so $I: G \to R_{2q}^S$.

By P′1 and Theorem 3.11, I is an isomorphism. Since U_1 generates G, $I(U_1)$ generates $I(G)$. But I is an isometry on U_1, by P′2 and Theorem 3.7, and so $I(U_1) \subset R_{2q}(S)$ and therefore $I(G) \subset R_{2q}(S)$. Moreover if $\rho(I(b)) < 1$, then $\rho(b) < 1$ (Theorem 3.10) and so $I(U_1) = \hat{U}_1 \subset I(G)$. Thus we have proved that $I: G \to I(G) \subset R_{2q}(S)$ and that G is equivalent to $I(G)$. It remains only to show that $I(G)$ is a closed subspace of $R_{2q}(S)$.

Suppose $f \in I(G) \cap \hat{U}_1$, then there exists $b \in U_1$ such that $I(b) = f$. Then $(\alpha f)(L) = \alpha(I(b)(L)) = \alpha L(b) = L(\alpha b) = I(\alpha b)(L) \in I(G)$.

Now suppose that $f \in R_{2q}(S)$, $\rho(f) < 1$ and for some $\alpha \neq 0$, $\alpha f \in I(G)$. Thus $\alpha f = I(b)$ for some $b \in G$. There exist $\beta \in R$ and $a \in U_1$ such that $\beta a = b$. Let $\gamma = \max\ [4, |4\beta/\alpha|]$. Then $\gamma\alpha[(\beta/\gamma\alpha)(I(a)(L)) - (1/\gamma)f(L)] = \beta L(a) - \alpha f(L) = L(b) - L(b) = \theta$. Then, as in the proof of Theorem 3.10, $(\beta/\gamma\alpha)(I(a)(L)) - (1/\gamma)f(L) \equiv c_0 \in R_{2q}$ and $\rho(c_0) < 1/2$. Let $h = l^{-1}(c_0)$. Then $(1/\gamma)f = (\beta/\gamma\alpha)I(a) - I(h) \in I(G) \cap \hat{U}_1$ and so $\gamma((1/\gamma)f) = f \in I(G)$. Thus $I(G)$ is a subspace of

$R_{2q}(S)$. Since an equivalence map is a local isometry in both directions, completeness is preserved and the completeness of G implies that $I(G)$ is complete and therefore closed.

4. **The associated Banach spaces.** If G is equivalent to $R_{2q}(X)$, the elements x of X give rise to characters of G. We wish to be able to identify these characters in terms of the metric group properties of G. We begin by examining certain Banach spaces associated with G. In this section, G is assumed to be a space with constants.

For $L_0 \in S$, we denote by G_0 the set $\{a \in G \mid L_0(a) = \theta\}$.

LEMMA 4.1. *If $a \in G_0$, there exists $b \in G_0 \cap U_1$ and $\beta \in R$ such that $\beta b = a$.*

Proof. By Lemma 2.1, there exists $c \in U_{1/2}$ and $\beta \in R$ such that $\beta c = a$. Let $h = l^{-1}(L_0(c))$. Then $L_0(c-h) = \theta$ and $\rho(c-h) \leqq \rho(c) + \rho(h) = \rho(c) + |j^{-1}(L_0(c))| \leqq 2\rho(c) < 1$. Thus $b = c - h \in G_0 \cap U_1$. Then $\beta b = \beta(c-h) = \beta c - \beta h = a - \beta h$. But $l(\beta h) = L_0(\beta h) = L_0(\beta h - a) = -L_0(\beta(c-h)) = -\beta L_0(c-h) = \theta$ and so $\beta h = \theta$ and $\beta b = a$.

LEMMA 4.2. *If $b_1, b_2 \in G_0 \cap U_1$, $\beta_1, \beta_2 \in R$ and $\beta_1 b_1 = \beta_2 b_2$, then $|\beta_1| \rho(b_1) = |\beta_2| \rho(b_2)$ and for any $\alpha \in R$, $(\alpha\beta_1)b_1 = (\alpha\beta_2)b_2 \in G_0$.*

Proof. From P′3, $(\alpha\beta_1)b_1$ and $(\alpha\beta_2)b_2 \in G_0$. Now if $\beta_1 = \beta_2 = 0$, the conclusions follow immediately. Thus we may assume $|\beta_1| \geqq |\beta_2|$ and $\beta_1 \neq 0$. Then $2\beta_1((1/2)b_1 - (\beta_2/2\beta_1)b_2) = \theta$ and so $(1/2)b_1 - (\beta_2/2\beta_1)b \in \overline{H}$ (Theorem 3.4). But $l((1/2)b_1 - (\beta_2/2\beta_1)b_2) = L_0((1/2)b_1 - (\beta_2/2\beta_1)b_2) = \theta$ and so $(1/2)b_1 = (\beta_2/2\beta_1)b_2$ (Lemma 3.4). Therefore $(1/2)\rho(b_1) = \rho((1/2)b_1) = \rho((\beta_2/2\beta)b_2) = (|\beta_2|/2|\beta_1|)\rho(b_2)$ and $|\beta_1|\rho(b_1) = |\beta_2|\rho(b_2)$. Moreover, multiplying our equality by $2\alpha\beta_1$ gives $(\alpha\beta_1)b_1 = (\alpha\beta_2)b_2$.

Using the β and b of Lemma 4.1, we define

(1) for each $a \in G_0$, $\rho'(a) = |\beta|\rho(b)$, and

(2) for each $a \in G_0$ and each $\alpha \in R$, $\alpha \times a = (\alpha\beta)b$.

The uniqueness of these definitions follows from Lemma 4.2.

Let G_0' be the space whose elements and underlying algebraic group structure are those of G_0, but with this new metric and multiplication by reals. That is, using a' to denote the element a in G_0', we have $\|a'\| = \rho'(a)$ and $\alpha a' = (\alpha \times a)'$.

One may readily verify

THEOREM 4.1. *G_0' is a Banach space, and G_0' is equivalent to G_0.*

Let G' be the vector direct sum of G_0' and the reals, $G' = G_0' \oplus Re$. For $a' + \alpha e \in G'$, we define $\|a' + \alpha e\| = \gamma\rho(((1/\gamma) \times a) + h)$ where $\gamma > \max\ [2\|a'\|, 2|\alpha|\ ]$ and $h = l^{-1}(j(\alpha/\gamma))$.

LEMMA 4.3. *$\|a' + \alpha e\|$ is uniquely defined.*

Proof. Suppose $\gamma_1, \gamma_2 > \max\ [2\|a'\|, 2|\alpha|\ ]$ and $\gamma_1 \geqq \gamma_2$. Then $l((\gamma_2/\gamma_1)h_2)$

$=(\gamma_2/\gamma_1)l(h_2)=(\gamma_2/\gamma_1)j(\alpha/\gamma_2)=j(\alpha/\gamma_1)=l(h_1)$ and $(\gamma_2/\gamma_1)h_2=h_1$. Thus $(\gamma_2/\gamma_1)\cdot((1/\gamma_2)\times a+h_2)=(\gamma_2/\gamma_1)((\beta/\gamma_2)b+h_2)=(\beta/\gamma_1)b+h_1=(1/\gamma_1)\times a+h_1$. Thus $\gamma_1\rho((1/\gamma_1)\times a+h_1)=\gamma_1\rho((\gamma_2/\gamma_1)((1/\gamma_2)\times a+h_2))=\gamma_2\rho((1/\gamma_2)\times a+h_2)$.

A direct verification then gives

THEOREM 4.2. *$G'=G_0'\oplus Re$ is a Banach space.*

DEFINITION 4.1. An element b of a Banach space B is a unit element if for every $a\in B$, either $\|a+b\|=\|a\|+1$ or $\|a-b\|=\|a\|+1$ [8].

LEMMA 4.4. *The element $e=\theta'+1e\in G'$ is a unit element.*

Proof. For any $a'+\alpha e\in G'$, choose $\gamma>\max\ [2\|a'\|,\ 2|\alpha|+2]$. Now $\bar{h}=l^{-1}(j(1/\gamma))$ is a constant of G. Assume $\{(1/\gamma)\times a+h,\ \bar{h}\}$ is positive. Then $\|a'+\alpha e+e\|=\gamma\rho((1/\gamma)\times a+l^{-1}(j((\alpha+1)/\gamma)))=\gamma\rho((1/\gamma)\times a+h+\bar{h})=\gamma\rho((1/\gamma)\times a+h)+\gamma\rho(\bar{h})=\|a'+\alpha e\|+1$. If $\{-(1/\gamma)\times a-h,\ \bar{h}\}$ is positive, the same argument gives $\|a'+\alpha e-e\|=\|a'+\alpha e\|+1$.

LEMMA 4.5. *λ_0: $G'\to R$, defined by $\lambda_0(a'+\alpha e)=\alpha$, is a linear functional of norm* 1.

Proof. λ_0 is clearly linear and clearly $\|\lambda_0\|\geqq 1$. But $|\lambda_0(a'+\alpha e)|=|\alpha|=|\gamma|\,|\alpha/\gamma|=|\gamma|\,|j^{-1}(L_0((1/\gamma)\times a+h)|\leqq|\gamma|\rho((1/\gamma)\times a+h)=\|a'+\alpha e\|$ and so $\|\lambda_0\|=1$.

For a fixed $q\geqq 1$, there is a natural mapping of $C(X)$, the Banach space of bounded, continuous, real-valued functions on X, into $R_{2q}(X)$ given by $(j(b))(x)=j(b(x))$. For $G'(X)\subset C(X)$, we assume for $j(G'(X))$ the metric, group properties induced on it as a subset of $R_{2q}(X)$.

THEOREM 4.3. *If X is compact, then $j(C(X))$ is equivalent to $R_{2q}(X)$.*

Proof. Theorem 1 of [4].

LEMMA 4.6. *If X is connected, and if $G'(X)$ is a linear subspace of $C(X)$ containing the function $e(x)\equiv 1$, then $j(G'(X))$ is a subspace of $R_{2q}(X)$.*

Proof. Since the map j is a homomorphism, $j(G'(X))$ is an algebraic subgroup of $R_{2q}(X)$.

(a) Suppose $a'(x)\in G'(X)$ and $j(a'(x))\in U_1\subset R_{2q}(X)$. Consider the function $j^{-1}(j(a'(x)))-a'(x)=f(x)$. Since j^{-1} is continuous on U_1, f is continuous and since X is connected, $f(X)$ is a connected set. But $j(f(x))\equiv\theta$ and so $f(X)\subset I_{2q}=\{n(2q)\}$. Thus $j^{-1}(j(a'(x)))=a'(x)+2n_0qe(x)$ for some fixed integer n_0. Thus $j^{-1}(j(a'(x)))$ and $\alpha j^{-1}(j(a'(x)))\in G'(X)$. Then $\alpha\{j(a'(x))\}=j(\alpha j^{-1}(j(a'(x))))\in j(G'(X))$.

(b) Now suppose $b(x)\in U_1\subset R_{2q}(X)$ and for some $\alpha\neq 0$, $\alpha b(x)=j(a'(x))$ for some $a'(x)\in G'(X)$. Choose $\gamma>\max\ ((4/|\alpha|)\|a'(x)\|,\ 4)$. Then $\gamma\alpha[(1/\gamma)b(x)-j(a'(x)/\gamma\alpha)]\equiv\theta$. Then by Theorem 3.4 and Lemma 2.4 $(1/\gamma)b(x)-j(a'(x)/\gamma\alpha)\equiv j(\beta e(x))\in j(G'(X))$. Thus $(1/\gamma)b(x)\in j(G'(X))\cap U_1$.

By (a), $\gamma((1/\gamma)b(x))=b(x)\in j(G'(X))$.

THEOREM 4.4. *If X is connected and $G'(X)$ is a linear subspace of $C(X)$, then G is equivalent to $j(G'(X))$ if and only if*
(1) *$G'(X)$ contains $e(x)\equiv 1$, and*
(2) *there exists an equivalence map of $G'=G_0'\oplus Re$ onto $G'(X)$ such that $e\to\pm e(x)$ under this equivalence.*

Proof. (a) Suppose G is equivalent to $j(G'(X))$. Let $i\colon G\to j(G'(X))$ be the equivalence map. Then for $\theta\neq h\in\overline{H}\subset G$, $i(h)\in\overline{H}\subset R_{2q}(X)$ and by Lemma 2.4, $i(h)=j(\beta e(x))$ for some $\beta\neq 2nq$ for any n. Thus, there exists $f(x)\in G'(X)$ such that $j(f(x))=j(\beta e(x))$. Since X is connected, $f(x)=(\beta+2n_0q)e(x)$ and since $G'(X)$ is a linear subspace and $\beta+2n_0q\neq 0$, $e(x)\in G'(X)$. Thus condition (1) is satisfied.

We proceed to prove condition (2). It is clear that the map $i(h)=j(\beta)$ is an equivalence map of $\overline{H}$ onto R_{2q}. Thus on $\overline{H}$, $i=\pm l$. Define $\bar{e}(x)=+e(x)$ or $\bar{e}(x)=-e(x)$ depending on whether $i=+l$ or $i=-l$. Then for $a'+\alpha e\in G'$, choose $\gamma>\|a'\|$ and define $I(a'+\alpha e)=\gamma\{j^{-1}[i((1/\gamma)\times a)]\}+\alpha\bar{e}(x)$.

(1) I is uniquely defined for if $\delta\geqq\gamma$, $(1/\delta)\times a=(\gamma/\delta)((1/\gamma)\times a)$ and

$$
\begin{aligned}
\delta\left\{j^{-1}\left[i\left(\frac{1}{\delta}\times a\right)\right]\right\} &= \delta\left\{j^{-1}\left[i\left(\frac{\gamma}{\delta}\left(\frac{1}{\gamma}\times a\right)\right)\right]\right\}\\
&= \delta\left\{j^{-1}\left[\frac{\gamma}{\delta}\,i\left(\frac{1}{\gamma}\times a\right)\right]\right\}\\
&= \delta\left\{j^{-1}j\left[\frac{\gamma}{\delta}\left(j^{-1}\left(i\left(\frac{1}{\gamma}\times a\right)\right)\right)\right]\right\}\\
&= \gamma\left\{j^{-1}\left[i\left(\frac{1}{\gamma}\times a\right)\right]\right\}.
\end{aligned}
$$

(2) $I(G')\subset G'(X)$. For if $a'+\alpha e\in G'$, $i((1/\gamma)\times a)=j(f(x))$ for some $f(x)$ in $G'(X)$. Then $\gamma\{j^{-1}[i((1/\gamma)\times a)]\}+\alpha\bar{e}(x)=\gamma(f(x)+2nqe(x))+\alpha\bar{e}(x)\in G'(X)$.

(3) $I(e)=\bar{e}(x)$.

(4) I is linear. It is clearly a homomorphism. Moreover if $b'=\beta a'=(\beta\times a)'$, $I(b'+\beta(\alpha e))=\gamma|\beta|\ \{j^{-1}[i((1/\gamma|\beta|)\times b)]\}+\beta\alpha\bar{e}(x)=\gamma|\beta|\ \{j^{-1}[i((1/\gamma|\beta|)\times(\beta\times a))]\}+\beta\alpha\bar{e}(x)=\gamma|\beta|\,(\beta/|\beta|)\{j^{-1}[i((1/\gamma)\times a)]\}+\beta\alpha\bar{e}(x)=\beta I(a'+\alpha e)$.

(5) I is norm-preserving. For if $\|a'+\alpha e\|<1/6$, $|\alpha|<1/6$ by Lemma 4.5 and so $\|a'\|<1/3$ by the triangle inequality. Then we may put $\gamma=1$ in the definition of $\|a'+\alpha e\|$, and we have

$$
\begin{aligned}
\|a'+\alpha e\| &= \rho(a+h) = \rho(i(a+h)) = \rho(i(a)+i(h))\\
&= \rho(i(a)+j\{(j^{-1}l(h))(\bar{e}(x))\}) = \|j^{-1}(i(a)+j\{(j^{-1}l(h))(\bar{e}(x))\})\|\\
&= \|j^{-1}(i(a)+j(\alpha\bar{e}(x)))\| = \|j^{-1}(i(a))+\alpha\bar{e}(x)\| = \|I(a'+\alpha e)\|
\end{aligned}
$$

(again taking $\gamma=1$). Thus I is norm-preserving on $U_{1/6}$. But I is linear and so I is norm-preserving on G'.

(6) I maps G' onto $G'(X)$. For suppose $b'(x)\in G'(X)$. There exists $a\in G_0$ and $h\in\overline{H}$ such that $i(a+h)=j(b'(x))$. But $j(I(a'))=i(a)=j(b'(x))-i(h)$ $=j(b'(x))-j\{(j^{-1}l(h))(\bar{e}(x))\}=j\{b'(x)-(j^{-1}l(h))(\bar{e}(x))\}$ and $I(a')=b'(x)$ $-(j^{-1}l(h)+2nq)(\bar{e}(x))$. Thus $I(a'+(j^{-1}l(h)+2nq)e)=b'(x)$.

Thus I is a Banach space equivalence and clearly an equivalence in our sense.

(b) Now suppose $I:G'\rightarrow G'(X)$ is the hypothesized equivalence. Since $l^{-1}L_0:G\rightarrow\overline{H}$ is a continuous projection, $G=G_0\oplus\overline{H}$ is a direct sum. Thus we may define $J:G\rightarrow j(G'(X))$ by $J(a+h)=j\{I(a')+(j^{-1}(l(h)))I(e)\}$.

(1) J is clearly a homomorphism.

(2) If $J(a+h)=\theta$, $j\{I(a')+(j^{-1}(l(h)))I(e)\}\equiv\theta$ and $I(a')\equiv[2nq$ $+j^{-1}(l(h))]I(e)$ (since $I(e)=\bar{e}(x)$). But G' and therefore $G'(X)$ is a direct sum and so $I(a')\equiv 0$ and $j^{-1}(l(h))=-2nq=0$ as $-q<j^{-1}(l(h))\leqq q$. Thus $a=h=\theta$ and J is an isomorphism.

(3) If $f(x)\in j(G'(X))$, there exist $a\in G_0$ and $\alpha\in R$ such that $f(x)=j(I(a')$ $+\alpha I(e))=j\{I(a')+(j^{-1}(j(\alpha)))I(e)\}$. Let $h=l^{-1}(j(\alpha))$; then $J(a+h)=f(x)$ and J maps G onto $j(G'(X))$.

(4) Suppose $\rho(a+h)<1$. Now $\rho(J(a+h))=\|j^{-1}\{j(I(a')+(j^{-1}(l(h)))I(e))\}\|$. Since $j^{-1}j(\alpha)=\alpha$ for $|\alpha|<1$, we prove $\rho(J(a+h))=\rho(a+h)$ by showing that $\|I(a')+(j^{-1}(l(h)))I(e)\|=\rho(a+h)<1$. Now G' and $G'(X)$ are Banach spaces and so an equivalence between them in our sense is a Banach space equivalence. Thus $\|I(a')+(j^{-1}(l(h)))I(e)\|=\|a'+j^{-1}(l(h))e\|$. Now $\rho(h)=|j^{-1}(l(h))|$ $=|j^{-1}(L_0(h))|=|j^{-1}(L_0(a+h))|\leqq\rho(a+h)<1$. Thus choosing $\gamma=\max$ [4, $4\|a'\|$], we have $\|a'+j^{-1}(l(h))e\|=\gamma\rho((1/\gamma)\times a+(1/\gamma)h)$ since $(1/\gamma)h$ $=l^{-1}\{j[(1/\gamma)j^{-1}(l(h))]\}$. But $\gamma\{(1/\gamma)\times a+(1/\gamma)h-(1/\gamma)(a+h)\}=\theta$ and so $b=(1/\gamma)\times a+(1/\gamma)h-(1/\gamma)(a+h)\in\overline{H}$. But $l(b)=L_0(b)=\theta$ and so $b=\theta$. Therefore $\gamma\rho((1/\gamma)\times a+(1/\gamma)h)=\gamma\rho((1/\gamma)(a+h))=\rho(a+h)$. Thus J is an isometry on $U_1\subset G$.

(5) Suppose $\rho(J(a+h))<1$. Then for some fixed integer n_0, $2n_0q-1$ $<I(a')+j^{-1}(l(h))I(e)<2n_0q+1$ for all $x\in X$. Thus $\|I(a')+\{j^{-1}(l(h))-2n_0q\}$ $\cdot I(e)\|=\|a'+\{j^{-1}(l(h))-2n_0q\}e\|<1$. By Lemma 4.5, $|j^{-1}(l(h))-2n_0q|<1$ and since $-q<j^{-1}(l(h))\leqq q$ we have $n_0=0$ and $\rho(h)=|j^{-1}(l(h))|<1$. Thus again $\gamma\rho((1/\gamma)\times a+(1/\gamma)h)=\rho(J(a+h))<1$. But since $\gamma\rho((1/\gamma)\times a+(1/\gamma)h)$ <1, $\gamma\rho((1/\gamma)\times a+(1/\gamma)h)=\rho(a+h)$ and J maps $U_1\subset G$ onto $U_1\subset j(G'(X))$.

Thus G is equivalent to $j(G'(X))$.

5. **Some theorems on Banach spaces.** We have shown that a group G is equivalent to $R_{2q}(X)$ for some compact, connected space X if and only if (1) G is a space with constants and (2) G' is equivalent to $C(X)$ (Theorems 4.3 and 4.4). In the usual characterizations of a Banach space G' as $C(X)$, the points of X are found in G'^* (the set of linear functionals of G'). We wish to give a characterization in terms of the group G. In §6, we show that the characters

of G correspond naturally to a subset of the linear functionals of G_0'. However, the F_T's of G_0' [8], or the extreme points of the unit sphere of $G_0'^*$ [2], do not in general give the required space X.

Specifically, for $G'=G_0'\oplus Re$, we look for a space $E\subset G_0'^*$ such that the natural correspondence $a'+\alpha e\to\xi(a')+\alpha$ $(\xi\in E)$ is an equivalence, and such that G' is equivalent to $C(X)$ for some X if and only if this mapping takes G' onto $C(E)$.

Let B' be a Banach space with a unit element e.

DEFINITION 5.1. *If* λ_0 is a linear functional on B' of norm 1 whose value at e is 1, then $B=\{b\in B'\mid \lambda_0(b)=0\}$ is a positive hyperplane of B'. B clearly is a Banach space and $B'=B\oplus Re$ is a direct sum.

DEFINITION 5.2. A functional $\lambda\in B^*$ is essentially positive (relative to B') if for all $b\in B$, and $\alpha\in R$, $|\lambda(b)+\alpha|\leqq\|b+\alpha e\|$.

In what follows the topology in B^* is the weak-star (point open) topology.

LEMMA 5.1. *The set* $\mathfrak{S}$ *of essentially positive linear functionals is closed in* B^*.

Proof. Suppose $\lambda'\in B^*$ and $\lambda'\notin\mathfrak{S}$. Then there exists $b\in B$, $\alpha\in R$ such that $|\lambda'(b)+\alpha|>\|b+\alpha e\|$. The set $V=\{\lambda\in B^*\mid |\lambda(b)-\lambda'(b)|<(|\lambda'(b)+\alpha|-\|b+\alpha e\|)/2\}$ is open and contains λ'. For $\lambda\in V$, $|\lambda(b)+\alpha|\geqq|\lambda'(b)+\alpha|-|\lambda(b)-\lambda'(b)|>(|\lambda'(b)+\alpha|+\|b+\alpha e\|)/2>\|b+\alpha e\|$. Thus $V\cap\mathfrak{S}=\phi$ and $\mathfrak{S}$ is closed.

LEMMA 5.2. $\mathfrak{S}$ *is compact.*

Proof. For $\alpha=0$, $\lambda\in\mathfrak{S}$, $|\lambda(b)|\leqq\|b\|$ for all $b\in B$. Thus $\mathfrak{S}$ is contained in Σ' the unit sphere in B^*. But Σ is compact in the weak-star topology [1], $\mathfrak{S}$ is closed by Lemma 5.1, and $\mathfrak{S}$ is compact.

DEFINITION 5.3. For $\lambda\in\mathfrak{S}$, $M(\lambda)=\{b\in B\mid\lambda(b)\geqq\lambda'(b)$ for all $\lambda'\in\mathfrak{S}\}$.

We may order the sets $M(\lambda)$ by inclusion.

DEFINITION 5.4. A functional $\xi\in\mathfrak{S}$ is a maximal functional of $\mathfrak{S}$ if $M(\xi)$ is a maximal set in the ordering of the sets $M(\lambda)$.

It can be shown that in the natural imbedding of $\mathfrak{S}$ into Σ' (the unit sphere in B'^*), the maximal functionals do not in general map into either F_T's or extreme points of Σ'.

THEOREM 5.1. *If* $\lambda_0\in\mathfrak{S}$, *there exists a maximal functional* ξ, *such that* $M(\xi)\supset M(\lambda_0)$.

Proof. By Zorn's lemma, $M(\lambda_0)$ is contained in a maximal linearly ordered chain $\{M(\lambda_\mu)\}$. Define $E(\mu)=\{\lambda\in\mathfrak{S}\mid\lambda(b)=\lambda_\mu(b)$ for all $b\in M(\lambda_\mu)\}$.

(1) $E(\mu)$ is not empty as $\lambda_\mu\in E(\mu)$.

(2) If $M(\lambda_{\mu_1})\subset M(\lambda_{\mu_2})$ then $E(\mu_1)\supset E(\mu_2)$. For if $\lambda\in E(\mu_2)$, $\lambda(b)=\lambda_{\mu_2}(b)\geqq\lambda_{\mu_1}(b)$ for all $b\in M(\lambda_{\mu_2})$. Thus $\lambda(b)\geqq\lambda_{\mu_1}(b)$ for all $b\in M(\lambda_{\mu_1})$. But the opposite inequality always holds and so $\lambda(b)=\lambda_{\mu_1}(b)$ for all $b\in M(\lambda_{\mu_1})$ and so $\lambda\in E(\mu_1)$.

(3) $E(\mu)$ is closed for $E(\mu)=\bigcap_{b\in M(\lambda_\mu)}\{\lambda\in\mathfrak{S}\mid\lambda(b)=\lambda_\mu(b)\}$ and is the intersection of closed sets.

Thus $\{E(\mu)\}$ is a family of closed, non-empty sets of $\mathfrak{S}$, linearly ordered by inclusion. Since $\mathfrak{S}$ is compact, there exists $\xi\in\bigcap_\mu\{E(\mu)\}$. For any λ_μ in our chain, we now have $\xi(b)=\lambda_\mu(b)\geqq\lambda(b)$ for all $b\in M(\lambda_\mu)$ and all $\lambda\in\mathfrak{S}$. Thus $M(\xi)\supset M(\lambda_\mu)$. If $M(\lambda_{\mu'})\supset M(\xi)$, then $M(\lambda_{\mu'})$ belongs to the chain (the chain is maximal) and so $M(\xi)\supset M(\lambda_{\mu'})$. Thus ξ is a maximal functional and since $M(\lambda_0)\in\{M(\lambda_\mu)\}$, $M(\xi)\supset M(\lambda_0)$.

THEOREM 5.2. *If B is a positive hyperplane of B', a Banach space with a unit element e, then for each $b'\in B'$, $b'=b+\beta e$, there exists a maximal functional ξ of $\mathfrak{S}$ such that $|\xi(b)+\beta|=\|b'\|$.*

Proof. By the Hahn-Banach extension theorem [3, p. 28], there exists a $\lambda_0'\in B'^*$ such that $\|\lambda_0'\|=1$, $\lambda_0'(e)=1$, and $\lambda_0'(b')=\inf_{\alpha\in R}\|b'+\alpha e\|-\alpha$. Now e is a unit element. We assume first that $\|b'+e\|=\|b'\|+1$. Then for $\alpha\geqq0$, Lemma 3.1 implies that $\|b'+\alpha e\|-\alpha=\|b'\|+\alpha-\alpha=\|b'\|$. (The condition $\|b'\|+\alpha\|e\|<1$ is not needed in a Banach space.) For $\alpha<0$, $\|b'+\alpha e\|-\alpha\geqq\|b'\|-|\alpha|-\alpha=\|b'\|$. Thus $\inf_{\alpha\in R}\|b'+\alpha e\|-\alpha=\|b'\|$ and $\lambda_0'(b')=\|b'\|$.

Let λ_0 be the functional λ_0' cut down to B. Since $\|\lambda_0'\|=1$ and $\lambda_0'(e)=1$, λ_0 is an element of $\mathfrak{S}$. Moreover for all $\lambda\in\mathfrak{S}$, $\lambda(b)+\beta\leqq\|b+\beta e\|$ and so, for the b and β defined by b', $\lambda(b)\leqq\|b+\beta e\|-\beta=\lambda_0(b)$. Thus $b\in M(\lambda_0)$. But there exists a maximal functional ξ such that $M(\xi)\supset M(\lambda_0)$, Theorem 5.1. Moreover on $M(\lambda_0)$, $\xi=\lambda_0$ and so $\xi(b)+\beta=\lambda_0(b)+\beta=\lambda_0'(b+\beta e)=\lambda_0'(b')=\|b'\|$.

Now if $\|b'-e\|=\|-b'+e\|=\|b'\|+1$, the same argument proves the existence of a maximal functional ξ, such that $\xi(-b)-\beta=\|b'\|$. Since one of these two conditions must apply we have shown the existence of a maximal functional ξ, such that $|\xi(b)+\beta|=\|b'\|$.

THEOREM 5.3. *If B is a positive hyperplane of B', a Banach space with a unit element e, and E is the space of maximal functionals of $\mathfrak{S}$, then B' is equivalent to a closed, linear subspace of $C(E)$.*

Proof. We map $b'=b+\beta e\to f(\xi)=\xi(b)+\beta$. The weak star topology on $E\subset B^*$ insures the continuity of f. Since $|\xi(b)+\beta|\leqq\|b+\beta e\|$, $f(\xi)$ is bounded. This map of $B'\to C(E)$ is clearly linear, and by Theorem 5.2 it is norm-preserving. Thus B' is equivalent to its image in $C(E)$ and since B' is a complete, linear space, its image is a closed, linear subspace of $C(E)$.

THEOREM 5.4. *A Banach space B' is equivalent to $C(X)$ for some compact space X if and only if*

(1) *B' has a unit element and*

(2) *there exists a positive hyperplane B of B', such that for any $b\in B$ and $\beta\in R$, there exists $\bar{b}\in B$ and $\bar{\beta}\in R$, such that $\xi(\bar{b})+\bar{\beta}=|\xi(b)+\beta|$ for all maximal functionals $\xi\in S$.*

Proof. (a) Suppose B' is equivalent to $C(X)$ for some compact X. Then $e(x)\equiv 1$ is a unit element. Let B be any positive hyperplane of B' (one exists by the Hahn-Banach theorem). To show that condition (2) is necessary we need only show that every maximal functional corresponds to a point of X ($\xi(b)=b(x_0)$ for some $x_0\in X$), for $b'(x)\in C(X)$ implies $|b'(x)|\in C(X)$.

Let $X_b=\{x\in X \mid b(x)=\sup_{x\in X} b(x)\}$. Since X is compact X_b is not empty. Now the functional λ_0: $b\to b(x_0)$ is an element of $\mathfrak{S}$. Thus for any $\lambda\in\mathfrak{S}$ and $b\in M(\lambda)$, $\lambda(b)\geqq\lambda_0(b)=b(x_0)$ for all $x_0\in X$. Thus for $b\in M(\lambda)$, $\lambda(b)\geqq\sup_{x\in X} b(x)$. Now choose $\alpha=\|b\|$. Then $\lambda(b)\leqq\|b+\alpha e\|-\alpha=\sup_{x\in X}(b(x)+\alpha e(x))-\alpha=\sup_{x\in X} b(x)$. Thus $\lambda(b)=\sup_{x\in X} b(x)$ for $b\in M(\lambda)$. Now for b_i any finite set of elements of $M(\lambda)$ we have $\sum_{i=1}^n b_i\in M(\lambda)$ and so

$$\sup_{x\in X}\left[\sum_{i=1}^n b_i(x)\right]=\lambda\left(\sum_{i=1}^n b_i\right)=\sum_{i=1}^n(\lambda(b_i))=\sum_{i=1}^n\sup_{x\in X}(b_i(x)).$$

But this implies that $\bigcap_{i=1}^n X_{b_i}$ is not empty. Since X is compact and X_b is closed we have that there exists an $x_1\in X$ such that $x_1\in\bigcap_{b\in M(\lambda)} X_b$. Then λ_1: $b\to b(x_1)$ is equal to λ on $M(\lambda)$ and so we have $M(\lambda_1)\supset M(\lambda)$. Now suppose $\lambda=\xi$ is a maximal functional. Thus $M(\xi)=M(\lambda_1)$ and $\xi(b)=b(x_1)$ for all $b\in M(\lambda_1)$. But $b\in B$ such that there exists an $\alpha\in R$ such that $b(x_1)+\alpha=\|b+\alpha e\|$ certainly belong to $M(\lambda_1)$. Thus $\xi=\lambda_1$ on these elements and by Lemma 2.3 of [8], $\xi=\lambda_1$ on B and so all maximal functionals correspond to points of X. {The preceding also proves that all points of X give rise to maximal functionals.}

(b) Now suppose (1) and (2) are satisfied. Let $\overline{E}$ be the closure of E in B^*. Since $E\subset\mathfrak{S}$, and $\mathfrak{S}$ is compact, $\overline{E}$ is compact. Moreover, the map $b'=b+\alpha e\to f(\xi)=\xi(b)+\alpha$ for $\xi\in\overline{E}$ is an equivalence map (Theorem 5.3, the addition of elements of $\mathfrak{S}$ to E to form $\overline{E}$ does not change this property). Thus B is equivalent to Γ, a closed, linear subspace of $C(\overline{E})$. Then by the theorem of Kakutani [6], $\Gamma=C(\overline{E})$ if

(1) whenever $\xi_1, \xi_2\in\overline{E}$ and $\xi_1\neq\xi_2$ there exists $f\in\Gamma$ such that $f(\xi_1)\neq f(\xi_2)$,

(2) Γ contains a nontrivial constant function, and

(3) Γ is lattice closed.

If $f(\xi_1)=f(\xi_2)$ for all f in Γ, then $\xi_1(b)+\alpha=\xi_2(b)+\alpha$ for all $b\in B$, and so $\xi_1=\xi_2$.

Moreover $0+e\in B'$ maps into the function $f(\xi)\equiv 1$ and Γ contains a nontrivial constant function.

Finally

$$\begin{aligned}&\max_{\min}\{\xi(b_1)+\alpha_1, \xi(b_2)+\alpha_2\}\\ &\qquad=\frac{1}{2}\{\xi(b_1)+\xi(b_2)+\alpha_1+\alpha_2\pm|\xi(b_2)-\xi(b_1)+\alpha_2-\alpha_1|\}\end{aligned}$$

and by condition (2) both these functions are in Γ, and Γ is lattice closed.

Thus B' is equivalent to $C(\overline{E})$. By the remark in the proof of the converse all the elements of $\overline{E}$ are maximal and so $\overline{E}=E$.

LEMMA 5.3. *If X is compact, then X is connected if and only if $e(x)\equiv 1$ and $e(x)\equiv -1$ are the only unit elements of $C(X)$.*

PROOF. (a) If V is a nontrivial open and closed set in X, then $e(x)\equiv 1$ on V and $e(x)\equiv -1$ on the complement of V is a unit element of $C(X)$.

(b) Suppose $f\in C(X)$ is a unit element. Then $\|f\|=\|0+f\|=\|0\|+1=1$, and so $|f(x)|\leqq 1$ for all x. Now suppose that for some $x_0\in X$, $|f(x_0)|<1$. There exists $b\in C(X)$ such that $\|b\|=1$, $b(x_0)=1$, and $b(x)\equiv 0$ wherever $f(x)=1$. Then $\|b+f\|<2$ which contradicts the hypothesis that f is a unit element. Thus if f is a unit element, $|f(x)|\equiv 1$ for all $x\in X$. But X is connected and so either $f(x)\equiv 1$ or $f(x)\equiv -1$.

Suppose B' is a Banach space with a unit element e, and B_1 and B_2 are positive hyperplanes of B'. For $b_2\in B_2$, there is a unique $b_1\in B_1$ and $\alpha\in R$ such that $b_2=b_1+\alpha e$. For $\lambda_1\in \mathfrak{S}_1$ we define $[i(\lambda_1)](b_2)=\lambda_1(b_1)+\alpha$. It is clear that i is a 1-1 map of $\mathfrak{S}_1$ onto $\mathfrak{S}_2$.

LEMMA 5.4. *If $\xi_1\in\mathfrak{S}_1$ is a maximal functional of $\mathfrak{S}_1$, then $i(\xi_1)\in\mathfrak{S}_2$ is a maximal functional of $\mathfrak{S}_2$.*

Proof. Let $M_2=(M(\xi_1)+Re)\cap B_2$. For $b_2\in M_2$, $[i(\xi_1)](b_2)=\xi_1(b_1)+\alpha \geqq\lambda_1(b_1)+\alpha=[i(\lambda_1)](b_2)$ for all $\lambda_1\in\mathfrak{S}_1$. Thus $M\{i(\xi_1)\}\supset M_2$. Suppose $M(\lambda_2) \supset M(i(\xi_1))$, and $M(\lambda_2)\neq M(i(\xi_1))$. Then $M_1=(M(\lambda_2)+Re)\cap B_1$ contains $M(\xi_1)$ properly and moreover $M(i^{-1}(\lambda_2))\supset M_1$. But this contradicts the maximality of ξ_1 and so $M(\lambda_2)=M(i(\xi_1))$ and $i(\xi_1)$ is a maximal functional of $\mathfrak{S}_2$.

6. **A characterization of $R_{2q}(X)$.** Let G be a space with constants. For $L\in S$, we define the functional $I_0(L):G_0'\rightarrow R$ by $[I_0(L)](a')=\alpha j^{-1}(L((1/\alpha)\times a))$ for $|\alpha|>\|a'\|$. The notation is that of §§3 and 4.

LEMMA 6.1. *$I_0(L)$ is uniquely defined.*

Proof. The ambiguity of definition arises in the choice of α. However, for $|\gamma|\geqq|\alpha|$, $(\alpha/\gamma)((1/\alpha)\times a)=(\alpha/\gamma)\times((1/\alpha)\times a)=(1/\gamma)\times a$. Thus $\gamma j^{-1}(L((1/\gamma)\times a))=\gamma j^{-1}(L((\alpha/\gamma)((1/\alpha)\times a)))=\gamma j^{-1}((\alpha/\gamma)L((1/\alpha)\times a))=\gamma j^{-1} (j((\alpha/\gamma)j^{-1}L((1/\alpha)\times a)))=\gamma((\alpha/\gamma)j^{-1}(L((1/\alpha)\times a)))=\alpha j^{-1}(L((1/\alpha)\times a))$.

LEMMA 6.2. *The functional $I_0(L)$ is an element of $\mathfrak{S}_0$, the set of positive linear functionals of G_0' (with respect to G').*

Proof. (1) $[I_0(L)](a_1'+a_2')=\alpha j^{-1}(L((1/\alpha)\times(a_1+a_2)))=\alpha j^{-1}(L((1/\alpha)\times a_1) +L((1/\alpha)\times a_2))$. But we may choose $\alpha>\|a_1'\|+\|a_2'\|$ which makes j^{-1} a homomorphism and so $I_0(L)$ is a homomorphism.

(2) $[I_0(L)]((\beta\times a)')=\alpha j^{-1}L((1/\alpha)\times(\beta\times a))$ where we may choose $\alpha >\max\ [|\beta|\|a'\|, \|a'\|]$. Then $\alpha j^{-1}L((1/\alpha)\times(\beta\times a))=\alpha j^{-1}L(\beta\times((1/\alpha)\times a)) =\alpha j^{-1}L(\beta((1/\alpha)\times a))=\alpha j^{-1}\beta L((1/\alpha)\times a)=\alpha j^{-1}(j(\beta j^{-1}(L((1/\alpha)\times a))))=\beta(\alpha j^{-1} L((1/\alpha)\times a))=\beta[I_0(L)](a')$ and so $I_0(L)$ is linear.

(3) For $a'+\beta e\in G'$, choose $\alpha>\|a'\|+|\beta|$ and put $h=l^{-1}(j(\beta/\alpha))$. Then $|[I_0(L)](a')+\beta| = |\alpha j^{-1}(L((1/\alpha)\times a))+\beta| = \alpha|j^{-1}(L((1/\alpha)\times a))+\beta/\alpha| = \alpha|j^{-1}(L((1/\alpha)\times a)+j(\beta/\alpha))| = \alpha|j^{-1}(L((1/\alpha)\times a+h))| \leqq \alpha\rho((1/\alpha)\times a+h) = \|a'+\beta e\|$ and so $I_0(L)\in\mathfrak{S}_0$.

LEMMA 6.3. *The map I_0: $S\to\mathfrak{S}_0$ is a homeomorphism onto.*

Proof. (1) If $I_0(L_1)=I_0(L_2)$, then $L_1=L_2$ on $G_0\cap U_1$ and since $G_0\cap U_1$ generates G_0, $L_1=L_2$ on G_0. But $L_1=L_2=l$ on $\overline{H}$ and so $L_1=L_2$ on G. Thus $L_1=L_2$ and I_0 is 1-1.

(2) Suppose $\lambda\in\mathfrak{S}_0$. Define $\overline{L}$: $G\to R_{2q}$ by $\overline{L}(a+h)=j(\lambda(a'))+l(h)$. $\overline{L}$ is certainly a homomorphism and so satisfies P′1. Moreover, if $\rho(a+h)<1$, $\rho(h)<1$ and $\rho(a+h)=\|a'+(j^{-1}(l(h)))e\|$. (See proof of Theorem 4.4.) Then $1>\rho(a+h)\geqq|\lambda(a')+j^{-1}(l(h))| = |j^{-1}(j(\lambda(a')+j^{-1}(l(h))))| = |j^{-1}(\overline{L}(a+h))|$ and so $\overline{L}$ satisfies P′2. Thus by Theorem 3.2, $\overline{L}$ is a character of G. Since $\overline{L}=l$ on $\overline{H}$, by definition, we have $\overline{L}\in S$. But $[I_0(\overline{L})](a')=\alpha j^{-1}\overline{L}((1/\alpha)\times a) = \alpha j^{-1}(j\lambda(((1/\alpha)\times a)'))=\alpha\lambda(((1/\alpha)\times a)')=\lambda(a')$ and so $I_0(\overline{L})=\lambda$ and I_0 maps S onto $\mathfrak{S}_0$.

(3) Since $\mathfrak{S}_0$ is compact (Lemma 5.2), and S is clearly Hausdorff, to show I_0 is a homeomorphism we need only show that I_0^{-1} is continuous. Now for $\bar\lambda\in\mathfrak{S}_0, a_i\in G_0, i=1,\cdots,n,$ and $1\geqq\epsilon>0$, $V=\{L\in S\,|\,|j^{-1}(L(a_i)-[I_0^{-1}(\bar\lambda)](a_i))|<\epsilon\}$ is a basic neighborhood of $I_0^{-1}(\bar\lambda)$ in S. We need choose the a_i's only from G_0 as for all $L\in S$, $L=l$ on $\overline{H}$ and $G=G_0\oplus\overline{H}$. Let $V'=\{\lambda\in\mathfrak{S}_0\,|\,|\lambda(a_i')-\bar\lambda(a_i')|<\epsilon\}$. Thus V' is a neighborhood of $\bar\lambda$ in $\mathfrak{S}_0$. If $L\in I_0^{-1}(V')$ we have that

$$\begin{aligned}|j^{-1}(L(a_i)-[I_0^{-1}(\bar\lambda)](a_i))| &= |j^{-1}(L(\alpha_i((1/\alpha_i)\times a_i))-[I_0^{-1}(\bar\lambda)](\alpha_i((1/\alpha_i)\times a_i)))|\\ &= |j^{-1}(\alpha_i(L((1/\alpha_i)\times a_i)-\alpha_i[I_0^{-1}(\bar\lambda)]((1/\alpha_i)\times a_i)))|\\ &= |j^{-1}j(\alpha_i j^{-1}L((1/\alpha_i)\times a_i)\\ &\qquad -\alpha_i[I_0^{-1}(\bar\lambda)]((1/\alpha_i)\times a_i))|\\ &= |j^{-1}j([I_0(L)](a_i')-\bar\lambda(a_i'))|\\ &= |[I_0(L)](a_i')-\bar\lambda(a_i')|<\epsilon,\end{aligned}$$

since $j^{-1}j(\beta)=\beta$ if $|\beta|<1$. Thus $I_0^{-1}(V')\subset V$, I_0^{-1} is continuous and I_0 is a homeomorphism.

We have immediately

THEOREM 6.1. *If G is a space with constants, S is compact.*

DEFINITION 6.1. For $\overline{L}\in S$ and $L_0\in S$, let $N_0(\overline{L})=\{a\in G_0\cap U_1\,|\,j^{-1}(\overline{L}(a))\geqq j^{-1}(L(a))$ for all $L\in S\}$.

As in §5 we order the sets $N_0(\overline{L})$ by inclusion.

DEFINITION 6.2. $F\in S$ is a maximal G_0 character if $N_0(F)$ is a maximal set in the ordering.

The correspondence between $N_0(\bar{L})$ and $M(I_0(\bar{L}))$ $\{$Definition 5.3$\}$ is quite direct.

LEMMA 6.4. $M(I_0(\bar{L}))=\{a'\in G_0' \mid$ *for* $\alpha>\|a'\|$, $(1/\alpha)\times a\in N_0(\bar{L})\}$.

Proof. $(I_0(\bar{L}))(a')=\alpha j^{-1}(\bar{L}((1/\alpha)\times a))$. Since $\alpha>0$, $(I_0(\bar{L}))(a')\geqq(I_0(L))(a')\rightarrow j^{-1}(\bar{L}((1/\alpha)\times a))\geqq j^{-1}(L((1/\alpha)\times a))$ and the lemma follows as I_0 maps S onto $\mathfrak{S}_0$ (Lemma 6.3).

A corollary of Lemma 6.4, obtained by putting $\alpha=1$, is

LEMMA 6.5. $N_0(\bar{L})=\{a\in G_0 \mid a'\in M(I_0(\bar{L}))$ *and* $\|a'\|<1\}$.

THEOREM 6.2. *F is a maximal G_0 character if and only if $I_0(F)$ is a maximal functional of $\mathfrak{S}_0$.*

Proof. (a) Suppose $M(\lambda)\supset M(I_0(F))$; then, by Lemma 6.5, $N_0(I_0^{-1}(\lambda))\supset N_0(F)$. But then if F is maximal, $N_0(I_0^{-1}(\lambda))\subset N_0(F)$ and, by Lemma 6.4, $M(\lambda)\subset M(I_0(F))$. Thus $M(I_0(F))$ is maximal and $I_0(F)$ is a maximal functional of $\mathfrak{S}_0$.

(b) Suppose $N_0(L)\supset N_0(F)$. Then Lemma 6.4 implies $M(I_0(L))\supset M(I_0(F))$. But if $I_0(F)$ is maximal, $M(I_0(L))\subset M(I_0(F))$ and by Lemma 6.5, $N_0(L)\subset N_0(F)$. Thus $N_0(F)$ is maximal and F is a maximal G_0 character.

The maximality of $F\in S$ does not depend on the choice of L_0.

DEFINITION 6.3. $F\in S$ is a maximal character if it is a maximal G_0 character for all $L_0\in S$.

THEOREM 6.3. *F is a maximal character if it is a maximal G_0 character for some $L_0\in S$.*

Proof. The theorem is an immediate consequence of Theorem 6.2 and Lemma 5.4.

The final characterization may now be given.

THEOREM 6.4. *A group G is equivalent to $R_{2q}(X)$ for some $q\geqq 1$ and for some compact, connected space X if and only if*

(1) *there exists a unique, isomorphic, isometry, i_a: $[0, \rho(a)]\rightarrow G$ such that $i_a(\rho(a))=a$,*

(2) *the elements of $\bar{H}\cap U_1$ are constants of G,*

(3) *the elements of $\bar{H}\cap U_1$ are the only constants of G,*

(4) *for $b\in G$, there exists $\bar{b}\in G$ such that $j^{-1}\{F(\bar{b})\}=|j^{-1}\{F(b)\}|$ for all maximal characters F of S.*

Proof. Suppose G is equivalent to $R_{2q}(X)$ for $q\geqq 1$ and for X a compact, connected space. Conditions (1) and (2) follow from Theorems 2.1, 2.2, and 2.3. Condition (3) follows quickly since a constant of $R_{2q}(X)$ must be of the form $f(x)\equiv a\in U_1\subset R_{2q}$ (see proof of Lemma 5.3) and so $f\in\bar{H}\cap U_1$. Now by

Theorems 4.3 and 4.4, G' is equivalent to $C(X)$. Then by Theorem 5.4, there exists a positive hyperplane $\tilde{G}$ of G' such that $G'=\tilde{G}\oplus Re$ and such that for each $a'\in\tilde{G}$ and each $\alpha\in R$, there exists $\bar{a}'\in\tilde{G}$ and $\bar{\alpha}\in R$ such that for all maximal functionals ξ of $\mathfrak{S}$, $\xi(\bar{a}')+\bar{\alpha}=|\xi(a')+\alpha|$. Let λ_0 be the functional which defined $\tilde{G}$. We redefine G_0 using the character $L_0=I_0^{-1}(\lambda_0)$. Then $G=G_0\oplus\overline{H}$ and $G_0'=\tilde{G}$. Now suppose $b\in U_1$. We have $b=a+h$ where $a\in G_0$ and $h\in\overline{H}$. Then there exist $\bar{a}'\in G_0'$ and $\bar{\alpha}\in R$ such that $\xi(\bar{a}')+\bar{\alpha}=|\xi(a')+j^{-1}(l(h))|$, for all maximal functionals ξ. Let $\bar{b}=\bar{a}+l^{-1}(j(\bar{\alpha}))$. Now by Theorems 6.2 and 6.3, the maximal characters F are exactly the elements $I_0^{-1}(\xi)$ where ξ are the maximal functionals. Thus $j^{-1}\{F(\bar{b})\}=j^{-1}\{(I_0^{-1}(\xi))(\bar{b})\}=j^{-1}\{j(\xi(\bar{a}'))+j(\bar{\alpha})\}=j^{-1}j\{\xi(\bar{a}')+\bar{\alpha}\}=j^{-1}j|\xi(a')+j^{-1}(l(h))|=|j^{-1}j(\xi(a')+j^{-1}(l(h)))|=|j^{-1}(F(b))|$ and condition (4) is satisfied.

(b) If (1) and (2) are satisfied, G is a space with constants. Choosing any $L_0\in S$, we have, as before, $G=G_0\oplus\overline{H}$, $G'=G_0'\oplus Re$, e is a unit element of G', and G_0' is a positive hyperplane of G'. For each $a'\in G_0'$, and each $\alpha\in R$, choose $\gamma>4\max(\|a'\|,|\alpha|)$ and let $h=l^{-1}j(\alpha/\gamma)$ and $b=(1/\gamma)\times a+h$. By condition (4), there exists $\bar{b}=\bar{c}+\bar{h}\in G$, such that $j^{-1}(F(\bar{b}))=|j^{-1}(F(b))|$ for all maximal characters F. Let $\bar{a}'=(\gamma\times\bar{c})'$ and $\bar{\alpha}=\gamma j^{-1}(l(\bar{h}))$. Then for ξ a maximal functional of $\mathfrak{S}_0$, $\xi(\bar{a}')+\bar{\alpha}=(I_0(F))(\bar{a}')+\bar{\alpha}=\gamma j^{-1}(F((1/\gamma)\times(\gamma\times\bar{c})))+\gamma j^{-1}(l(\bar{h}))=\gamma j^{-1}F(\bar{b})=|\gamma j^{-1}(F(b))|=|\gamma j^{-1}F((1/\gamma)\times a)+\gamma j^{-1}l(h)|=|\xi(a')+\alpha|$. Thus by Theorem 5.4, G' is equivalent to $C(X)$ for some compact X. Moreover if $a'+\alpha e$ is a unit of G', then $(1/\gamma)\times a+h$ is a constant of G. By condition (3), $(1/\gamma)\times a+h\in\overline{H}$ and so $a'=\theta$. But then $|\alpha|$ must equal 1 and so $\pm e$ are the only unit elements of G'. But $e(x)\equiv 1$ and $e(x)\equiv -1$ are unit elements of $C(X)$ and so e must map into either $e(x)\equiv 1$ or $e(x)\equiv -1$ under the equivalence and these are the only unit elements of $C(X)$ and so X is connected (Lemma 5.3). Finally by Theorems 4.4 and 4.3, G is equivalent to $R_{2q}(X)$.

7. The homeomorphism theorem.

THEOREM 7.1. *If X and Y are compact, and if $R_{2q_1}(X)$ is equivalent to $R_{2q_2}(Y)$, then $q_1=q_2$ and X is homeomorphic to Y.*

Proof. Let $T:R_{2q_1}(X)\to R_{2q_2}(Y)$ be the equivalence. Choose a positive integer n such that $2q_1/n<1$. Then $f_1\equiv j(2q_1/n)\in R_{2q_1}(X)$, $nf_1=\theta$ and $\rho(f_1)=2q_1/n<1$. Thus $n(Tf_1)=\theta$, and $\rho(Tf_1)=2q_1/n$. Since Y is compact, there exists $y_0\in Y$ such that $j^{-1}((Tf_1)(y_0))=\pm 2q_1/n$. But $n((Tf_1)(y_0))=\theta$ so that $nj^{-1}((Tf_1)(y_0))=2mq_2$, where m is an integer. Thus $q_1=\pm mq_2$ and q_1 is an integer multiple of q_2. The exact same proof, using T^{-1}, gives that q_2 is an integer multiple of q_1. Since q_1 and q_2 are both positive we have that $q_1=q_2$.

Now define the mapping $T^*:C(X)\to C(Y)$ by

$$(T^*\sigma)(y) = \gamma[j^{-1}\{T(j((1/\gamma)\sigma))(y)\}]$$

for $\gamma>\|\sigma\|$. It is easy to verify that T^* is a uniquely defined, linear, norm-preserving map of $C(X)$ onto $C(Y)$. By the Banach-Stone theorem [10], X is homeomorphic to Y.

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