THE SPACE OF POINT HOMOTOPIIC MAPS
INTO THE CIRCLE

BY

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1. Introduction. The space $C(X)$ of real bounded continuous functions on a topological space has been studied extensively ([9], [7], [6](2), etc.). More recently some of this theory has been extended to the space of functions into certain Banach spaces [5].

In the present paper, we consider the space of point-homotopic continuous maps into the circle. The circle, $\mathbb{R}^{2q}$ (reals mod $2q$), can be made into an abelian group, complete under an invariant metric. Then $\mathbb{R}^{2q}(X)$, the space of point-homotopic continuous functions from $X$ into $\mathbb{R}^{2q}$, is in a natural way an abelian group, complete under an invariant metric. We give a characterization of $\mathbb{R}^{2q}(X)$, for $X$ a compact connected space, as an abelian group, complete under an invariant metric (Theorem 6.4), and a proof that for compact $X$, the metric group properties of $\mathbb{R}^{2q}(X)$ determine the topology on $X$ (Theorem 7.1).

The characterization is obtained by imposing conditions which insure the existence of a pseudo-multiplication by scalars (Theorem 2.2), and the existence of sufficiently many "characters" of the group (Theorems 3.7, 3.10 and 3.11). The points of $X$ are found among the "characters" of the group by investigating certain Banach spaces associated with the group ($\S$4). Certain new linear functionals are defined and a Banach space characterization of $C(X)$, for compact $X$, is given (Theorem 5.4). That the metric group properties of $\mathbb{R}^{2q}(X)$ determine the topology on a compact $X$ follows quickly from the similar theorem for Banach spaces [10].

2. Some metric group properties of $\mathbb{R}^{2q}(X)$. The circle $\mathbb{R}^{2q}$ is taken to be the factor group of the reals $R$ by the subgroup $I_{2q} = \{n(2q)\}$ where $n$ is any integer. Thus $\mathbb{R}^{2q}$ is an abelian group. We denote by $j$ the natural homomorphism of $R$ onto $\mathbb{R}^{2q}$. ($j_{2q}$ would be more precise. However, no confusion results from the omission of the subscript.) We define $j^{-1}: \mathbb{R}^{2q} \to R$ by $j^{-1}(\alpha) = \alpha$ such that $-q < \alpha \leq q$ and $j(\alpha) = a$. It follows immediately that

$$j(j^{-1}(\alpha)) = a$$

and that for $|\alpha| < q$, $j^{-1}(j(\alpha)) = \alpha$.

Presented to the Society, September 5, 1953; received by the editors October 9, 1953.

(*) This paper is a revised version of the author's doctoral dissertation submitted to the University of Michigan. He wishes to thank Professor Sumner B. Myers for his help and encouragement. A portion of the work was done under Office of Naval Research contract N8-onr-71400.

(*) Numbers in brackets refer to the bibliography at the end of this paper.
If we define $p(a) = |j^{-1}(a)|$, then the function $d(a, b) = p(a - b)$ is an invariant metric on $R_{2q}$ under which $R_{2q}$ is complete. The space of all continuous functions from a topological space $X$ into $R_{2q}$, denoted by $R_{2q}^X$, is made into a metric abelian group by defining

$$(f_1 + f_2)(x) = f_1(x) + f_2(x), \quad \rho(f) = \sup_{x \in X} \{ \rho(f(x)) \}, \quad \text{and} \quad d(f_1, f_2) = \rho(f_1 - f_2).$$

The metric is invariant, convergence is uniform convergence, and $R_{2q}^X$ is complete. Now the space of point-homotopic continuous functions from $X$ into $R_{2q}$, $R_{2q}(X)$, is the component of the identity in $R_{2q}^X$ [4]. Thus it is a closed subgroup of $R_{2q}^X$, and it too is an abelian group complete under an invariant metric. Moreover, for each $\epsilon > 0$, $U_{\epsilon} = \{ f \in R_{2q}(X) \mid \rho(f) < \epsilon \}$ generates $R_{2q}(X)$. In what follows, the word group will denote an abelian group, complete under an invariant metric, and generated by $U_{\epsilon} = \{ a \mid \rho(a) < \epsilon \}$ for each $\epsilon > 0$. In addition we assume, with no loss of generality, that $q \geq 1$.

**Definition 2.1.** If $a \in R$ and $a \in U_1 \subset R_{2q}$, then $\alpha a = j(\alpha^{-1}(a))$.

This pseudo-multiplication by real scalars can be extended to $U_1 \subset R_{2q}(X)$ by defining $(\alpha f)(x) = \alpha(f(x))$. Since each operation used in Definition 2.1 is continuous ($j^{-1}$ is continuous on $U_1$), the function $\alpha f \in R_{2q}^X$. Moreover, $\{ tf \}$ for $0 \leq t \leq \alpha$ is a homotopy from $\theta(x) = \theta(t) = j(0)$ to $\alpha f$ and so $\alpha f \in R_{2q}(X)$.

Some of the properties of scalar multiplication in a Banach space are preserved by this pseudo-multiplication. Thus it can be readily verified that, for $\alpha, \beta \in R$ and $a, b \in U_1 \subset R_{2q}(X)$, the following relations hold.

(P1) $|\alpha|\rho(a) < 1 \Rightarrow \beta(\alpha a) = (\beta \alpha)(a)$.

(P2) $(\alpha + \beta)a = \alpha a + \beta a$.

(P3) $\rho(a) + \rho(b) < 1 \Rightarrow (a + b) = \alpha a + \beta b$.

(P4) $|\alpha|\rho(a) < 1 \Rightarrow \rho(\alpha a) = |\alpha|\rho(a)$.

(P5) $1a = a$.

**Definition 2.2.** A group $G$ is a pseudo-Banach space if a multiplication by reals can be defined on $U_1 \subset G$ which satisfies P1–P5.

Thus we have

**Theorem 2.1.** $R_{2q}(X)$ is a pseudo-Banach space.

The next theorem shows that the property of being a pseudo-Banach space is a metric group property.

**Theorem 2.2.** A group $G$ is a pseudo-Banach space if and only if, for each $a \in U_1 \subset G$, there exists a unique isomorphic isometry, $i_a: [0, \rho(a)] \rightarrow G$ such that $i_a(\rho(a)) = a$. \{ $[0, \rho(a)]$ represents the closed interval in $R$ with end points at $0$ and $\rho(a)$. The isomorphism applies whenever $\alpha, \beta$ and $\alpha + \beta$ all belong to $[0, \rho(a)]$.\}

**Proof.** (a) Suppose $G$ is a pseudo-Banach space. For each $a \in U_1 \subset G$, de-

(1) The symbol $\theta$ will denote the identity element in a group. The symbol $0$ will be reserved for the zero of the reals.
fine \( i_\alpha(\alpha) = (\alpha/\rho(\alpha))\alpha \). Then by P5, \( i_\alpha(\rho(\alpha)) = \alpha \); by P2, \( i_\alpha \) is an isomorphism; and by P4, \( i_\alpha \) is an isometry. Now suppose \( i_\alpha' \) is another such map. It follows from P5 and P2 that for \( m \) any positive integer \( ma = \sum_{n=1}^{m} a \). Thus for \( m \) and \( n \) positive integers such that \( m \leq n \) we have \( i_{\alpha}(mp(\alpha)/n) = (m/n)a \) and \( i_{\alpha'}(mp(\alpha)/n) = (m/n)mi_{\alpha'}(\rho(\alpha)/n) \) and \( i_{\alpha'}(mp(\alpha)/n) = mi_{\alpha'}(\rho(\alpha)/n) \). But by P1, \( mi_{\alpha}(\rho(\alpha)/n) = (m/n)i_{\alpha}(\rho(\alpha)/n) \). Thus \( i_\alpha \) and \( i_\alpha' \) are equal on a dense set of \([0, \rho(\alpha)]\) and since they are isometries they must be identical\(^4\).

(b) Suppose \( i_\alpha \) is a unique isomorphic isometry taking \( \rho(\alpha) \) into \( a \). For \( \alpha \in R \) and \( \alpha > 0 \) define \( \bar{\alpha} \) to be the smallest integer such that \( \bar{\alpha} \geq \alpha \). Define

\[
\alpha a = \begin{cases} 
\bar{\alpha} \left[ i_\alpha \left( \frac{\alpha}{\bar{\alpha}} \rho(\alpha) \right) \right] & \text{for } \alpha > 0, \\
\theta & \text{for } \alpha = 0, \\
-((-\alpha)a) & \text{for } \alpha < 0.
\end{cases}
\]

Since inverses and multiplication by integers are well defined in any group, the preceding definitions give a precise meaning to \( \alpha a \).

The proof that this multiplication satisfies P1–P5 involves much intricate detail and is not given here. It may be found in the author’s dissertation.

**Lemma 2.1.** If \( G \) is a pseudo-Banach space, then for each \( b \in G \) and each \( \epsilon \) such that \( 0 < \epsilon \leq 1 \), there exists \( \alpha \in R \) and \( a \in U_\epsilon \) such that \( \alpha a = b \).

**Proof.** Since \( U_\epsilon \) generates \( G \), there exist elements \( a_1, \ldots, a_n \) in \( U_\epsilon \) such that \( b = \sum_{i=1}^{n} a_i \). Choose \( \alpha \geq n \) and let \( a = \sum_{i=1}^{n} (1/\alpha)a_i \). Then

\[
\rho(a) \leq \sum_{i=1}^{n} \rho((1/\alpha)a_i).
\]

By P4, \( \rho((1/\alpha)a_i) = (1/\alpha)\rho(a_i) \) and so

\[
\rho(a) \leq \sum_{i=1}^{n} \rho\left(\frac{1}{\alpha} a_i\right) = \sum_{i=1}^{n} \frac{1}{\alpha} \rho(a_i) < \frac{ne}{\alpha} \leq \epsilon.
\]

Thus \( a \in U_\epsilon \). Moreover \( \sum_{i=1}^{n} \rho((1/\alpha)a_i) < \epsilon \leq 1 \) and so by repeated application of P3 \( \alpha a = \sum_{i=1}^{n} \alpha(1/\alpha)a_i \). But by P1, \( \alpha(1/\alpha)a_i = a_i \) and so \( \alpha a = \sum_{i=1}^{n} a_i = b \).

**Lemma 2.2.** If \( G \) is a pseudo-Banach space, and if for \( a \in U_1 \) and \( \alpha \in R \) and different from zero, \( \alpha a = \theta \), then either \( a = \theta \) or \( \rho(a) \geq 2/|\alpha| \).

**Proof.** If \( \theta = \alpha a = (\alpha/2 + \alpha/2)a \) then, by P2, \( (\alpha/2)a = \alpha/2(\alpha/2)a \) and \( (1/2)((\alpha/2)a) = (1/2)((\alpha/2)a) \). Now if \( \rho(a) < 2/|\alpha| \), then \( |\alpha/2| \rho(a) < 1 \), and we have by P1 that \((1/2)((\alpha/2)a) = (\alpha/4)a \) and \((1/2)((\alpha/2)a) \)

\(^4\) Since P3 was not used in establishing the existence and uniqueness of \( i_\alpha \), the proof of sufficiency will prove that P3 is a consequence of P1, P2, P4, and P5. This can easily be established directly.
\(-\alpha/4)a\) and \((\alpha/4)a = (-\alpha/4)a = -(\alpha/4)a\). Thus \(\theta = (\alpha/4)a + (\alpha/4)a = (\alpha/2)a\). But by P4, \(\rho((\alpha/2)a) = \alpha/2|\rho(a)|\) and if this is zero, \(\rho(a) = 0\) and \(a = \theta\).

**Lemma 2.3.** If \(G\) a pseudo-Banach space, the map of \(R \times U_1 \to G\) given by \((\alpha, a) \to \alpha a\) is continuous.

**Proof.** We show that the neighborhood \(V\) of \((\alpha_0, a_0)\) defined by

\[
V = \{(\alpha, a) \mid |\alpha - \alpha_0| < \min \left\{\frac{\varepsilon}{2}, 1/|\rho(a)|\right\} \text{ and } \rho(a - a_0) < \min \left\{\frac{\varepsilon}{2}|\alpha_0|, 1/|\alpha_0|, 1 - \rho(a_0)\right\}\}
\]

maps into the \(\varepsilon\) neighborhood of \(\alpha_0 a_0\).

For \((\alpha, a) \in V\)

\[
\alpha a - \alpha_0 a_0 = \alpha(a - a_0) + \alpha_0 a - \alpha_0 a_0 = \alpha(a - a_0) + (\alpha - \alpha_0)a_0
\]

\[
= \alpha_0(a - a_0) + (\alpha - \alpha_0)(a - a_0) + (\alpha - \alpha_0)a_0
\]

\[
= \alpha_0(a - a_0) + (\alpha - \alpha_0)a
\]

by P3, P2, P2, and P3 respectively. Then \(\rho(\alpha a - \alpha_0 a_0) \leq \rho(\alpha_0(a - a_0)) + \rho((\alpha - \alpha_0)a) < \varepsilon/2 + \varepsilon/2 < \varepsilon\) by P1 and P4.

In what follows we shall use properties P1–P5 without specific reference, taking care always that the hypotheses of the statements are satisfied.

**Definition 2.3.** An element \(h \in G\) is a root of unity if there exists an integer \(n\) such that \(nh = \theta\). The set of roots of unity we denote by \(\bar{H}\) and the closure of \(H\) in \(G\) by \(\bar{H}\).

\(\bar{H}\) and \(\bar{H}\) are subgroups of \(G\) in the usual sense.

The elements of \(\bar{H} \subset R_{2q}(X)\) have special metric properties as well. The following lemma makes this explicit for the case where \(X\) is a connected space.

**Lemma 2.4.** If \(X\) is connected, then \(h \in \bar{H} \subset R_{2q}(X)\) is a constant function and if \(\rho(h) < 1\), then for \(g \in R_{2q}(X)\) such that \(\rho(h) + \rho(g) < 1\) we have either \(\rho(h + g) = \rho(h) + \rho(g)\) or \(\rho(h - g) = \rho(h) + \rho(g)\).

**Proof.** Suppose \(h \in H \subset R_{2q}(X)\). Then there exists an integer \(n\) such that \(n(h(x)) = \theta\) for all \(x \in X\). Thus \(h(X) \subset A = \{a \in R_{2q} \mid na = \theta\}\). But \(h(X)\) is connected while the set \(A\) is discrete and so \(h\) is a constant function. The definition of the metric then implies that the elements of \(\bar{H}\) are constant functions. From this fact plus the definition of the function \(\rho\), the second part follows immediately.

We are led to the following definitions.

**Definition 2.4.** If \(\rho(a) + \rho(b) < 1\) and if \(\rho(a) + \rho(b) = \rho(a + b)\), then the pair \(\{a, b\}\) is positive.

**Definition 2.5.** If \(a \in U_1 \subset G\) and if for all \(b \in U_1 - \rho(a) \subset G\) either \(\{a, b\}\) or \(\{a, -b\}\) is positive, then \(a\) is a constant of \(G\).

**Definition 2.6.** A pseudo-Banach space \(G\) is a space with constants if \(\bar{H} \neq \{\theta\}\) and if all the elements of \(\bar{H} \cap U_1\) are constants of \(G\).

\(^{(5)}\) We assume \(\bar{H} \neq \{\theta\}\), as otherwise \(G\) is essentially a Banach space.
Theorem 2.3. If $X$ is a connected space, then $R_{2q}(X)$ is a space with constants.

Proof. Theorem 2.1 and Lemma 2.4.

3. Subspaces and characters. A basic theorem in the classification of Banach spaces is that every Banach space $B$ is equivalent to a closed subspace of $C(X)$ for some compact topological space $X$ [1]. The points of $X$ are found among the continuous linear functionals on $B$. The existence of sufficiently many such functionals is assured by the Hahn-Banach theorem [3, p. 55]. In this section we prove that under modified definitions of equivalence and subspace, every space with constants is equivalent to a subspace of $R_{2q}(X)$ for some $q \geq 1$ and for some compact connected space $X$.

Definition 3.1. Two groups $G$ and $\hat{G}$ are equivalent if there is an isomorphism $I: G \to \hat{G}$ such that $I$ is an isometry on $U_1$ and such that $I(U_1) = \hat{O}_1$. {For this definition we do not require that $G$ and $\hat{G}$ be complete.} It is clear that the relation of equivalence is symmetric, reflexive, and transitive.

Definition 3.2. A subset $G'$ of a pseudo-Banach space $G$ is a subspace of $G$ if $G'$ is a subgroup (in the ordinary sense) and if, for $\alpha \neq 0$ and $a \in U_1 \subseteq G$, $\alpha a \in G'$ if and only if $a \in G'$.

Definition 3.3. If $G'$ is a subspace of a pseudo-Banach space $G$, then $L: G' \to R_{2q}$ is a character of $G'$ if

$(P'1)$ $L(a+b) = L(a) + L(b),$  
$(P'2)$ $|j^{-1}(L(a))| \leq \rho(a)$ whenever $\rho(a) < 1,$  
$(P'3)$ $L(\alpha a) = \alpha L(a)$ whenever $\rho(a) < 1.$

From the definitions it is clear that $G'$ may be all of $G$.

Theorem 3.1. The characters of a subspace $G'$ of a pseudo-Banach space $G$ are continuous on $G'$.

Proof. By $P'1$, $L$ is a homomorphism. But for $0 < \epsilon < 1$ and $a \in U_1 \cap G'$,

$|j^{-1}(L(a))| < \epsilon$

and so $L$ is continuous at the identity and therefore continuous on $G'$.

Theorem 3.2. If $G'$ is a subspace of a pseudo-Banach space $G$ and if $L: G' \to R_{2q}$ satisfies $P'1$ and $P'2$, then $L$ is a character of $G'$.

Proof. For $a \in U_1 \cap G'$ and $n$ any positive integer, $n(1/n)a = a$. Then for $m$ any integer $P'1$ gives $(m/n)L(a) = (m/n)L(n((1/n)a)) = (m/n)(nL((1/n)a)).$ But $R_{2q}$ is itself a pseudo-Banach space and by $P'2$ $\rho(L((1/n)a)) = n|j^{-1}(L((1/n)a))| \leq \rho((1/n)a) = \rho(a) < 1.$ Thus we have $(m/n)(nL((1/n)a)) = mL((1/n)a) = L((m/n)a)$ by $P'1$, and $(m/n)L(a) = L((m/n)a)$. By Theorem 3.1 and Lemma 2.3 both $\alpha L(a)$ and $L(\alpha a)$ are continuous in $\alpha$. Since they are equal on a dense set of $R$, they are equal for all $\alpha \in R$ and $L$ satisfies $P'3$ on $G'$.

The usual boundedness restriction for linear functionals on a Banach
space would translate here to $|j^{-1}(L(a))| \leq M\rho(a)$. However, this plus $P'1$ does not imply $P'3$. The proof uses strongly that $M=1$ and the theorem is false without it. For let $G' \subset R_4([0,1])$ be the set \( \{f \in R_4([0,1]) \mid f(x) = j(\alpha + \beta x)\} \) and define $L(j(\alpha + \beta x)) = j(\alpha - (3/2)\beta)$. It is easily verified that $L$ satisfies $P'1$ and that $|j^{-1}L(f)| \leq 4\rho(f)$. However,

\[(1/2)L(j(-1/2 + x)) = (1/2)j(-1/2 - 3/2) = (1/2)j(-2) = (1/2)\theta = \theta,\]

while $L((1/2)j(-1/2 + x)) = L(j(1/2)j^{-1}j(-1/2 + x))) = L(j(-1/4 + (1/2)x)) = j(-1/4 - 3/4) = j(-1) \neq \theta$.

Since in the construction of characters we have no other way of insuring that $P'3$ be satisfied we must use the stronger form given by $P'2$.

**Theorem 3.3.** If $G'$ is a subspace of a pseudo-Banach space $G$ and $L'$ is a character of $G'$, then there exists a character $L$ of $G$ such that $L = L'$ on $G'$.

**Proof.** The proof is a modification of the similar theorem for Banach spaces [3, p. 28]. If $G' = G$ we are through. If $G' \neq G$, there exists an element $a \in (G - G') \cap U_{1/2}$, since $U_{1/2}$ generates $G$. For $b_1$ and $b_2$ any elements of $G' \cap U_{1/4}$ and for $\beta_1$ and $\beta_2$ real numbers such that $0 < \beta_i \leq 1$ and $\beta = \min(\beta_1, \beta_2)$ we have, by $P'1$ and $P'2$, that

\[j^{-1}\left\{L'(\frac{\beta}{\beta_1}b_1) - L'(\frac{\beta}{\beta_2}b_2)\right\} = j^{-1}\left\{L'(\frac{\beta}{\beta_1}b_1 - \frac{\beta}{\beta_2}b_2)\right\},\]

since

\[|j^{-1}\left(L'(\frac{\beta}{\beta_1}b_1)\right)| + |j^{-1}\left(L'(\frac{\beta}{\beta_2}b_2)\right)| < 1,\]

\[j^{-1}\left\{L'(\frac{\beta}{\beta_1}b_1) - L'(\frac{\beta}{\beta_2}b_2)\right\} = j^{-1}\left\{L'(\frac{\beta}{\beta_1}b_1)\right\} - j^{-1}\left\{L'(\frac{\beta}{\beta_2}b_2)\right\},\]

and so

\[j^{-1}\left\{L'(\frac{\beta}{\beta_1}b_1)\right\} - j^{-1}\left\{L'(\frac{\beta}{\beta_2}b_2)\right\} \leq \rho\left(\frac{\beta}{\beta_1}b_1 - \frac{\beta}{\beta_2}b_2\right),\]

and so

\[-\frac{\beta}{\beta_2}\rho(b_2 + \beta a) - \frac{\beta}{\beta_1}j^{-1}(L'(b_2)) \leq \frac{\beta}{\beta_1}\rho(b_1 + \beta a) - \frac{\beta}{\beta_1}j^{-1}(L'(b_1)).\]
Dividing by $\beta$ gives

\[
(3.1) \quad \frac{1}{\beta_2} \rho(b_2 + \beta_2 a) - \frac{1}{\beta_2} j^{-1}(L'(b_2)) \leq \frac{1}{\beta_1} \rho(b_1 + \beta_1 a) - \frac{1}{\beta_1} j^{-1}(L'(b_1)).
\]

Since (3.1) holds for all $\beta_1, \beta_2, a_1,$ and $a_2$ we have

\[
m = \text{l.u.b.} \left\{ - \frac{1}{\beta_2} \rho(b_2 + \beta_2 a) - \frac{1}{\beta_2} j^{-1}(L'(b_2)) \right\}
\]

\[
\leq \text{g.l.b.} \left\{ \frac{1}{\beta_1} \rho(b_1 + \beta_1 a) - \frac{1}{\beta_1} j^{-1}(L'(b_1)) \right\} = M.
\]

Let $G'' = \{ c \in G \mid c = \gamma a + b \text{ for any } \gamma \in R \text{ and } b \in G' \}$. For a fixed $c \in G''$, $\gamma$ and $b$ are uniquely determined. If $\gamma a + b = \gamma' a + b'$, then $(\gamma - \gamma')a = b' - b \in G'$. But $a \in G'$ and $G'$ is a subspace, thus $\gamma = \gamma'$ and so $b = b'$.

Choose $\alpha \in R$ such that $m \leq \alpha \leq M$ and define $L'': G'' \to R_{2q}$ by $L''(c) = j(\gamma a) + L'(b)$.

We show that $G''$ is a subspace and that $L''$ is a character of $G''$. That $G''$ properly contains $G'$ and that $L'' = L'$ on $G'$ is immediate.

$G''$ is clearly a subgroup (in the usual sense). Suppose $c \in G'' \cap U_1$ and $0 \neq \delta \in R$. Let $\eta = 2 \max \left( \frac{1}{\delta}, \frac{1}{2} \frac{1}{|\delta|} \right)$. Then $(\eta/\delta)((\delta/\eta)c - (\delta \gamma/\eta)a) = c - \gamma a = b \in G'$. Therefore $(\delta/\eta)c - (\delta \gamma/\eta)a \in G'$ and $\eta((\delta/\eta)c - (\delta \gamma/\eta)a) = \delta c - (\delta \gamma)a \in G'$, and so $\delta c \in G''$. Now suppose $\rho(c) < 1$, $0 \neq \delta \in R$, and that $\delta c \in G''$. Then $\delta c = \gamma a + b'$. Let $\eta = \max (2, \frac{1}{\gamma/\delta})$. Then $\eta((1/\eta)c - (\gamma/\eta)a) = b' \in G'$ and $(1/\eta)c - (\gamma/\eta)a \in G'$, so that

$$\eta((1/\eta)c - (\gamma/\eta)a) = c - (\gamma/\delta)a \in G', \quad c \in G''.$$ 

Thus we have proved that $G''$ is a subspace.

Now $L''((c_1 + c_2) = L''((\gamma_1 a + b_1 + \gamma_2 a + b_2) = L''((\gamma_1 + \gamma_2) a + (b_1 + b_2))$

\[= j((\gamma_1 + \gamma_2) a) + L'(b_1 + b_2) = j(\gamma a) + L'(b_1) + j(\gamma a) + L'(b_2) = L''((c_1) + L''((c_2)) \text{ and } P'1 \text{ is satisfied. }$

Now suppose $c = \gamma a + b \in G''$ and $\rho(c) < 1$. If $\gamma = 0, P'2$ is immediate. If $\gamma \neq 0$, let $\delta = \max (2, \frac{4}{|\gamma|})$. Then $\delta \gamma \left\{ (1/\delta \gamma)c - (1/\delta)c \right\}$$
= c - \gamma a = b \in G'$ and so $(1/\delta \gamma)c - (1/\delta)c \in G'$. Moreover $\rho((1/\delta \gamma)c - (1/\delta)c) < 1/4 + 1/4 = 1/2$. Thus in (3.1) we may put $b_1 = b_2 = (1/\delta \gamma)c - (1/\delta)c$ and $\beta_1 = \beta_2 = (1/\delta)$. We get

\[
- \delta \rho \left( \frac{1}{\delta \gamma} c \right) - \delta \{ j^{-1}(L'(\frac{1}{\delta \gamma} c - \frac{1}{\delta} a)) \} \leq m \leq \alpha \leq M
\]

\[
\leq \delta \rho \left( \frac{1}{\delta \gamma} c \right) - \delta \{ j^{-1}(L'(\frac{1}{\delta \gamma} c - \frac{1}{\delta} a)) \}
\]

and so $|\alpha/\delta + j^{-1}(L'(1/\delta \gamma)c - (1/\delta)c)| \leq \rho((1/\delta \gamma)c) = (1/|\delta \gamma|) \rho(c)$ and $|\gamma \alpha + \delta \gamma j^{-1}(L'(1/\delta \gamma)c - (1/\delta)c)| \leq \rho(c) < 1$. But $j^{-1}(j(\beta)) = \beta$ for $|\beta| \leq 1$ and so
\[
\rho(c) \geq \left| j^{-1}\left\{ j(\gamma \alpha) + j\left( \delta \gamma j^{-1}\left( L'\left( \frac{1}{\delta \gamma} c - \frac{1}{\delta} a \right) \right) \right) \right\} \right|
\]
\[
= \left| j^{-1}\left\{ j(\gamma \alpha) + \delta \gamma \left( L'\left( \frac{1}{\delta \gamma} c - \frac{1}{\delta} a \right) \right) \right\} \right|
\]
\[
= \left| j^{-1}\left\{ j(\gamma \alpha) + L'\left( \frac{1}{\delta \gamma} c - \frac{1}{\delta} a \right) \right\} \right|
\]
as \( L' \) is a character on \( G' \) and satisfies \( \mathcal{P}'3 \). Thus \( \rho(c) \geq \left| j^{-1}\{ j(\gamma \alpha) + L'(b) \} \right| = \left| j^{-1}\{ L''(c) \} \right| \) and \( L'' \) satisfies \( \mathcal{P}'2 \) on \( G'' \).

By Theorem 3.2, \( L'' \) is a character of \( G'' \). Then by transfinite induction there exists a character \( L \) of \( G \) such that \( L = L' \) on \( G' \).

Theorem 3.3 does not prove the existence of characters on a pseudo-Banach space \( G \). We must first exhibit a subspace \( G' \) of \( G \) and a character of \( G' \). At first glance, the real multiples of an element in \( U \) might seem to do for \( G' \). But this is not necessarily a subspace of \( G \) (Corollary 1 to Theorem 3.5). We show, however, that \( \overline{H} \) is a subspace of \( G \) and that if \( G \) is a space with constants, there exists a character taking \( \overline{H} \) into \( R_\mathbb{q} \) for some \( q \geq 1 \).

**Theorem 3.4.** If \( G \) is a pseudo-Banach space, \( \overline{H} \) is a subspace of \( G \).

**Proof.** Suppose \( h \in \overline{H} \cap U_1 \) and \( 0 \neq \alpha \in R \). Then there exist \( h_i \in H \cap U_1 \) and integers \( p_i \) and \( q_i \) such that \( h_i \to h \) and \( p_i/q_i \to \alpha \). Since \( h_i \in H \), there exist integers \( n_i \) such that \( n_i h_i = \theta \). Then \( n_i q_i ((p_i/q_i) h_i) = p_i (n_i h_i) = \theta \) and so \((p_i/q_i) h_i \in H \). But by Lemma 2.3, \((p_i/q_i) h_i \to ah \) and so \( ah \in \overline{H} \).

Now suppose \( 0 \neq \alpha \in R, h \in U_1, \) and \( ah \in \overline{H} \). If \( \alpha < 0 \), then \( ah = - \{ (-\alpha) h \} \) and \( (-\alpha) h \in \overline{H} \). Thus we may assume \( \alpha > 0 \). There exist \( h_i \in H \) such that \( h_i \to ah \). Thus there exists an \( I \), such that \( \rho (ah - h_i) < 1 \) whenever \( i \geq I \) and so \((1/\alpha)(ah - h_i)\) is defined for \( i \geq I \). Moreover \( \bar{h} [(\alpha/\bar{h}) h - (1/\alpha)(ah - h_i)] = ah - ah + h_i \in H \) and since \( \bar{h} \) is an integer, \( a_i = (\alpha/\bar{h}) h - (1/\alpha)(ah - h_i) \in H \). But \( a_i \to (\alpha/\bar{h}) h \) and so \((\alpha/\bar{h}) h \in \overline{H} \). Since \((\alpha/\bar{h}) h \in U_1 \), by the first part of the proof \((\alpha/\bar{h}) (\alpha/\bar{h}) h) = h \in \overline{H} \).

**Lemma 3.1.** If \( \{a, b\} \) is positive (Definition 2.4) and if \( \alpha \geq 0, \beta \geq 0, \) and \( \alpha a + \beta b < 1 \), then \( \{a, b\} \) is positive.

**Proof.** For either \( a \) or \( \beta \) equal to zero, the result is immediate. We assume \( \alpha \geq \beta > 0 \). Since \( \alpha a + \beta b < 1 \), \( \alpha a = \rho (\alpha a) \) and \( \beta b = \rho (\beta b) \). Thus \( (1/\alpha) \rho (\alpha a + \beta b) \leq (1/\alpha) (\rho (\alpha a) + \rho (\beta b)) = \rho (a) + (\beta/\alpha) \rho (b) \leq \rho (a) + \rho (b) < 1, \) and \( (1/\alpha) \rho (\alpha a + \beta b) = \rho ((1/\alpha) (\alpha a + \beta b)) \). But \( \rho (\alpha a) + \rho (\beta b) < 1 \) and so \( (1/\alpha) (\alpha a + \beta b) = a + (\beta/\alpha) b \). Now \( \rho (a + (\beta/\alpha) b) = \rho (a + \beta b - (1 - (\beta/\alpha)) b) \geq \rho (a + b) - (1 - (\beta/\alpha)) \rho (b) = \rho (a) + \rho (b) - (1 - (\beta/\alpha)) \rho (b) = \rho (a) + (1 - (\beta/\alpha)) \rho (b) < 1, (1/\alpha) (\alpha a + \beta b) = a + (\beta/\alpha) b \). But the opposite inequality is always true and so \( \rho (a + (\beta/\alpha) b) = \rho (a + (\beta/\alpha) b) = \rho (a) + (\beta/\alpha) \rho (b) = \rho (a) + (\beta/\alpha) \rho (b) < 1, (1/\alpha) (\alpha a + \beta b) = a + (\beta/\alpha) b \). Thus \( \rho (\alpha a + \beta b) = \rho (a) + \beta b = \rho (\alpha a) + \rho (b) < 1 \) and so
Lemma 3.2. If $G$ is a space with constants, $h_1 \in \mathcal{H}$, $h_2 \in \mathcal{H}$, and \{h$_1$, h$_2$\} is positive, then $\rho(h_1 - h_2) = |\rho(h_1) - \rho(h_2)|$.

Proof. $(1/2)h_1 + (1/2)h_2 \in \mathcal{H} \cap U_{1/2}$ and is therefore a constant of $G$. Moreover $\rho((1/2)h_1 + (1/2)h_2) + \rho((1/2)h_1 - (1/2)h_2) \leq \rho(h_1) + \rho(h_2) < 1$, and so either \{(1/2)h$_1$ + (1/2)h$_2$, (1/2)h$_1$ - (1/2)h$_2$\} or \{(1/2)h$_1$ + (1/2)h$_2$, (1/2)h$_2$ - (1/2)h$_1$\} is positive. If the first pair is positive we have

\[
\rho((1/2)h_1 + (1/2)h_2) + \rho((1/2)h_1 - (1/2)h_2) = \rho(h_1)
\]

Thus $\rho((1/2)h_1 - (1/2)h_2) = \rho(h_1) - \rho((1/2)h_1 + (1/2)h_2) = (1/2)\rho(h_1) - (1/2)\rho(h_2)$ by Lemma 3.1. If the second pair is positive we have $\rho((1/2)h_1 - (1/2)h_2) = (1/2)\rho(h_2) - (1/2)\rho(h_1)$. Since $\rho((1/2)h_1 - (1/2)h_2) \geq 0$ we have in either case that $\rho((1/2)h_1 - (1/2)h_2) = (1/2)|\rho(h_1) - \rho(h_2)|$ and multiplication by 2 gives

$\rho(h_1 - h_2) = |\rho(h_1) - \rho(h_2)|$.

Theorem 3.5. If $G$ is a space with constants, $\theta \neq h \in \mathcal{H} \cap U_1$, and $h_0 \in \mathcal{H}$, there exists $\alpha \in \mathbb{R}$ such that $\alpha h = h_0$. In particular if $\rho(h_0) < 1$, then $h_0 = \pm (\rho(h_0)/\rho(h))h$.

Proof. By Lemma 2.1, there exist $h_0 \in U_{1-\rho(h)}$ and $\beta \in \mathbb{R}$ such that $\beta h_0' = h_0$. Now $h$ is a constant of $G$ and so either \{h, h$_0'$\} or \{h, -h$_0'$\} is positive. If \{h, h$_0'$\} is positive, then by Lemma 3.1, \{(\rho(h$_0'$)/2\rho(h))h, (1/2)h$_0'$\} is positive. By Theorem 3.4, both these elements belong to $\mathcal{H}$ and so by Lemma 3.2,

$$\rho\left(\frac{\rho(h_0')}{2\rho(h)} h - \frac{1}{2} h_0'\right) = \left|\rho\left(\frac{\rho(h_0')}{2\rho(h)} h - \frac{1}{2} h_0'\right)\right|$$

$$= \left|\frac{1}{2} \rho(h_0') - \frac{1}{2} \rho(h_0')\right| = 0$$

and so

$$\frac{1}{2} h_0' = \frac{\rho(h_0')}{2\rho(h)} h \text{ and } h_0 = 2\beta\left(\frac{1}{2} h_0'\right) = \frac{\beta \rho(h_0')}{\rho(h)} h.$$ 

If \{h, -h$_0'$\} is positive we get $h_0 = (-\beta \rho(h_0')/\rho(h))h$ and the first part of the theorem is proved.

Now if $\rho(h_0) < 1$, we may choose $h_0' = (1 - \rho(h))h_0$ and $\beta = (1/(1 - \rho(h)))$. Then $h_0 = \pm (\rho(h_0)/\rho(h))h$.

Corollary 1. If $G'$ is a subspace of a space with constants, then $G' \subseteq \mathcal{H}$.

Proof. Since $\theta \in G'$, $H \cap U_1 \subseteq G'$ as $h \in H \cap U_1$ implies there exists an $n$ such that $nh = \theta$. Therefore $\alpha h \in G'$ for all $\alpha \in \mathbb{R}$ and so $\mathcal{H} \subseteq G'$. 
Corollary 2. If $G'$ is a closed subspace of a space with constants, then $G'$ is a space with constants.

Proof. By Corollary 1, $G' \subseteq \mathcal{H} \neq \{\theta\}$ and since it is closed it is complete.

Lemma 3.3. If $G$ is a space with constants, and $\theta \neq h \in \mathcal{H} \cap U_1$, there exists a real number $\alpha_h > 0$ such that $\alpha_h h = \theta$ and such that $0 < \alpha < \alpha_h$ implies $\alpha h \neq \theta$.

Proof. Let $A = \{\alpha > 0 \mid \alpha h = \theta\}$. By Lemma 2.2, $A$ is equal to $\{\alpha \geq 2/\rho(h) \mid \alpha h = \theta\}$ and by Lemma 2.3, $A$ is closed. Thus if $A$ is not empty, $\alpha_h = \text{g.l.b.}_{\alpha \in A} \alpha$ has the required property. But $A$ cannot be empty. Choose $h_0 \in \mathcal{H}$ such that $h_0 \neq \theta$. Then there exist an integer $n_0$ such that $n_0 h_0 = \theta$ and, by Theorem 3.5, a real number $\alpha \neq 0$ such that $\alpha h = h_0$. Thus $\theta = n_0 (\alpha h) = h_0 = (n_0 \alpha) h = (\alpha n_0 h) h$. Now either $n_0 \alpha$ or $-n_0 \alpha$ is positive and so belongs to $A$.

Corollary. $\alpha h = \theta$ if and only if $\alpha = n_0 \alpha_h$ for some integer $n$.

Definition 3.4. Let $q_h = (1/2) \alpha_h \rho(h)$. By Lemma 2.2, $q_h \geq 1$.

Lemma 3.4. If $G$ is a space with constants, and $h \in \mathcal{H} \cap U_1$ and $h \neq \theta$, then $\mathcal{H}$ is equivalent to $R_{2q_h}$.

Proof. For $h_0 \in \mathcal{H}$, there exists, by Theorem 3.5, $\alpha \in R$ such that $\alpha h = h_0$. Define $l_h : \mathcal{H} \rightarrow R_{2q_h}$ by $l_h(h_0) = j(\alpha h)$. We show that $l_h$ is uniquely defined and gives an equivalence between $\mathcal{H}$ and $R_{2q_h}$.

(a) If $h_0 = \alpha h = \beta h$, then $(\beta - \alpha) h = \theta$ and $\beta - \alpha = n_0 \alpha_h$ (corollary to Lemma 3.3). Thus $j(\alpha h) - j(\beta h) = j((\alpha - \beta) h) = j(n_0 \alpha h \rho(h)) = j((2q_h) h) = \theta$ and $l_h$ is uniquely defined.

(b) $l_h(h_1 + h_2) = l_h(\alpha_1 h_1 + \alpha_2 h_2) = j((\alpha_1 + \alpha_2) h) = j(\alpha_1 h) + j(\alpha_2 h) = l_h(h_1) + l_h(h_2)$ and $l_h$ is a homomorphism.

(c) If $l_h(h_0) = \theta$, then $\alpha h = 2nq_h$ and $\alpha = n_0 \alpha_h$ and $h_0 = \alpha h = \theta$. Thus $l_h$ is an isomorphism.

(d) If $\rho(h_0) < 1$, $h_0 = \pm (\rho(h_0)/\rho(h)) h$ by Theorem 3.5. Thus $|j^{-1}(l_h(h_0))| = |j^{-1}(j((\rho(h_0)/\rho(h)) h))| = \rho(h_0)$ and $l_h$ is an isometry on $\mathcal{H} \cap U_1$.

(e) Suppose $a \in U_1 \subseteq R_{2q_h}$. Let $\alpha = (1/\rho(h)) \{j^{-1}(a)\}$ and $h_0 = \alpha h$. Then $h_0 \in \mathcal{H} \cap U_1$ and $l_h(h_0) = j(j^{-1}(a)) = a$. Thus $l_h$ maps $U_1 \cap \mathcal{H}$ onto $U_1 \cap R_{2q_h}$.

Thus (Definition 3.1) $l_h$ gives an equivalence between $\mathcal{H}$ and $R_{2q_h}$.

Corollary. If $h' \in \mathcal{H} \cap U_1$ and $h' \neq \theta$, then $q_{h'} = q_h$ and $l_{h'} = \pm l_h$.

Proof. By the lemma, $\mathcal{H}$ is equivalent to both $R_{2q_h}$ and $R_{2q_h'}$ and so $R_{2q_h}$ is equivalent to $R_{2q_h'}$. This implies immediately that $q_h = q_{h'}$. Now the only continuous isomorphisms of $R_{2q}$ onto itself are the identity and the reflection $(a \rightarrow -a)$. But $l_{h'}(l_h^{-1})$ is such a map and so $l_{h'} = \pm l_h$.

Thus we may drop the subscript $h$ from $q_h$ and define $q = (1/2) \alpha_h \rho(h)$ for any $h \in \mathcal{H} \cap U_1$ such that $h \neq \theta$. We choose one of the two equivalence mappings of $\mathcal{H}$ onto $R_{2q}$ and denote it by $l$. The other is then $-l$.

We have already proved
Theorem 3.6. If $G$ is a space with constants, then $l: \mathcal{H} \to \mathbb{R}^{2q}$ is a character of $\mathcal{H}$.

Theorem 3.7. If $G$ is a space with constants, then for each $a \in U_1$, there exists a character $L$ of $G$ such that $L = l$ on $\mathcal{H}$ and $|j^{-1}(L(a))| = \rho(a)$.

**Proof.** By Theorems 3.3, 3.4, and 3.6 there exist characters of $G$ equal to $l$ on $\mathcal{H}$. If $a \in \mathcal{H}$, $|j^{-1}(L(a))| = |j^{-1}(l(a))| = \rho(a)$ and we are through. Suppose $a \notin \mathcal{H}$. For each $h \in \mathcal{H} \cap U_1$, $l(h)$ or $l(-h) = j(\rho(h))$. Choose $h_0 \in \mathcal{H} \cap U_{1/2}$ such that $h_0 \neq 0$ and $l(h_0) = j(\rho(h_0))$. Since $h_0$ is a constant of $G$, there exists $b$ such that $b = \pm (1/2)a$ and $\{h_0, b\}$ is positive. From the proof of Theorem 3.3, there is a character $L$ of $G$ equal to $l$ on $\mathcal{H}$ such that $L(b) = j(M)$ where $M = \text{g.l.b.} h_0 \in \mathcal{H} \cap U_{1/2}, c \leq b \leq 1 \{ (1/\beta_1) \rho(h_1 + \beta_1 b) - (1/\beta_1) j^{-1}(l(h_1)) \}$. By Theorem 3.5, $h_1 = \pm (\rho(h_1)/\rho(h_0)) h_0$.

(a) If $h_1 = (\rho(h_1)/\rho(h_0)) h_0$, then by Lemma 3.1 $\rho(h_1 + \beta_1 b) = \rho(h_1) + \beta_1 \rho(b)$, and thus $\rho(h_1 + \beta_1 b) = (1/\beta_1) \rho(h_1) + \rho(b) - (1/\beta_1) \rho(h_1) = \rho(b)$.

(b) If $h_1 = - (\rho(h_1)/\rho(h_0)) h_0$, then

\[
\frac{1 - \rho(h_1 + \beta_1 b)}{\beta_1} - \frac{1}{\beta_1} j^{-1}(l(h_1)) \geq \rho(b) - \frac{1}{\beta_1} \rho(h_1) + \frac{1}{\beta_1} \rho(h_1) = \rho(b).
\]

Thus $M = \rho(b)$ and $L(b) = j(\rho(b))$. Then

\[
|j^{-1}(L(a))| = |j^{-1}(L(\pm 2b))| = |j^{-1}(\pm 2L(b))| = 2 |j^{-1}j(\rho(b))| = 2 \rho(b) = \rho(a).
\]

Let $G$ be a space with constants. The set of characters of $G$ which are extensions of $l$ is a topological space under the point open topology. We denote this space by $S$.

Theorem 3.8. The space $S$ is connected.

**Proof.** Suppose $L_0$ and $L_1$ belong to $S$. For $0 \leq \alpha \leq 1$ we define $L_\alpha: G \to \mathbb{R}^{2q}$ as follows. For $b \in G$, there exist $a \in U_1$ and $\gamma \in \mathbb{R}$ such that $\gamma a = b$ (Lemma 2.1). We put $L_\alpha(b) = L_0(((1 - \alpha)\gamma)a + L_1((\alpha \gamma)a)$. Using strongly the fact that $L_0 = L_1 = l$ on $\mathcal{H}$, one may verify that $L_\alpha(b)$ is uniquely defined and that $L_\alpha \in S$. Since $|j^{-1}(L_\alpha(b) - L_\alpha(b))| \leq j^{-1}L_0(((\alpha_0 - \alpha)\gamma)a) + j^{-1}L_1(((\alpha - \alpha_0)\gamma)a) \leq 2 |\gamma| |\alpha - \alpha_0|$, the map $\alpha \to L_\alpha$ is a continuous curve connecting $L_0$ to $L_1$ in $S$. Thus $S$ is connected.

Theorem 3.9. The space $S$ is compact.

**Proof.** See Theorem 6.1 which is independently proved. A direct proof, duplicating the proof that the unit sphere in a conjugate space is compact in the weak-star topology, can be given.

Theorem 3.10. If $\rho(a) \geq 1$, there exists $L \in S$, such that $|j^{-1}(L(a))| \geq 1$. 
Proof. Suppose $|j^{-1}(L(a))| < 1$ for all $L \in S$. There exist $b \in U_{1/3}$ and $\beta \in R$ such that $\beta b = a$ (Lemma 2.1). Now $\rho(a) \geq 1$ and so $|\beta| \geq 3$ which implies that $\beta [L(b) - (1/\beta)L(a)] = \beta L(b) - L(a) = \theta$ for all $L \in S$. The function $f: S \rightarrow R_{2q}$ defined by $f(L) = L(b) - (1/\beta)L(a)$ is continuous (Lemma 2.3 and the definition of the point open topology). Since $S$ is connected (Theorem 3.8), $f(S)$ is connected. Now the set $C = \{c \in R_{2q} | \beta c = \theta \}$ is totally disconnected, and $f(S) < C$. Thus $f(S) = c_0$. Moreover $|j^{-1}(c_0)| \leq |j^{-1}(L(b))| + |j^{-1}((1/\beta)L(a))| < 2/3$.

Let $h = l^{-1}(c_0)$. Then $L(b - h) = L(b) - c_0 = (1/\beta)L(a)$, and $|j^{-1}(L(b - h))| < 1/|\beta|$ for all $L \in S$. But there exists $L_0 \in S$ such that $|j^{-1}(L_0(b - h))| = \rho(b - h)$ (Theorem 3.7) and so $\rho(b - h) < 1/|\beta|$. Thus $1 > |\beta| \rho(b - h) = \rho(\beta b - \beta h) = \rho(a - \beta h)$. But $l(\beta h) = \beta l(h) = \beta c_0 = \theta$ and $\beta h = \theta$. But then $\rho(a) < 1$, contradicting the hypothesis.

Theorems 3.7 and 3.10 combine to give

Theorem 3.11. If $G$ is a space with constants and if $b \neq \theta$, then there exists $L \in S$ such that $L(b) \neq \theta$.

Theorem 3.12. A group $G$ is equivalent to a closed subspace $G'$ of $R_{2q}(X)$ for some $q \geq 1$ and for some compact, connected space $X$, if and only if $G$ is a space with constants.

Proof. (a) By Theorem 2.3 and Corollary 2 to Theorem 3.5, $G'$ is a space with constants. But $G$ is equivalent to $G'$ and it follows that $G$ is a space with constants.

(b) Suppose $G$ is a space with constants. Then $G$ uniquely determines an $R_{2q}$ for $q \geq 1$ (Lemma 3.4 and corollary), and the space $S$ is a compact, connected space. We define $I(b) \{L\} = L(b)$ for all $L \in S$. The choice of the point open topology on $S$ insures that $I(b)$ is a continuous function on $S$ and so $I: G \rightarrow R_{2q}$.

By $P'1$ and Theorem 3.11, $I$ is an isomorphism. Since $U_1$ generates $G$, $I(U_1)$ generates $I(G)$. But $I$ is an isometry on $U_1$, by $P'2$ and Theorem 3.7, and so $I(U_1) \subset R_{2q}(S)$ and therefore $I(G) \subset R_{2q}(S)$. Moreover if $\rho(I(b)) < 1$, then $\rho(b) < 1$ (Theorem 3.10) and so $I(U_1) = \overline{U}_1 \subset I(G)$. Thus we have proved that $I: G \rightarrow I(G) \subset R_{2q}(S)$ and that $G$ is equivalent to $I(G)$. It remains only to show that $I(G)$ is a closed subspace of $R_{2q}(S)$.

Suppose $f \in I(G) \cap \overline{U}_1$, then there exists $b \in U_1$ such that $I(b) = f$. Then $(\alpha f)(L) = \alpha(I(b)(L)) = \alpha L(b) = L(ab) = I(ab)(L) \in I(G)$.

Now suppose that $f \in R_{2q}(S)$, $\rho(f) < 1$ and for some $\alpha \neq 0$, $\alpha f \in I(G)$. Thus $\alpha f = I(b)$ for some $b \in G$. There exist $\beta \in R$ and $a \in U_1$ such that $\beta a = b$. Let $\gamma = \max \{4, |4\beta/\alpha| \}$. Then $\gamma \alpha[(\beta/\gamma\alpha)(I(a)(L)) - (1/\gamma)f(L)] = \beta L(a) - \alpha f(L) = L(b) - L(b) = \theta$. Then, as in the proof of Theorem 3.10, $(\beta/\gamma\alpha)(I(a)(L)) - (1/\gamma)f(L) = c_0 \in R_{2q}$ and $\rho(c_0) < 1/2$. Let $h = l^{-1}(c_0)$. Then $(1/\gamma)f = (\beta/\gamma\alpha)I(a) - I(h) \in I(G) \cap \overline{U}_1$ and so $\gamma((1/\gamma)f) = f \in I(G)$. Thus $I(G)$ is a subspace of
Since an equivalence map is a local isometry in both directions, completeness is preserved and the completeness of $G$ implies that $I(G)$ is complete and therefore closed.

4. The associated Banach spaces. If $G$ is equivalent to $R_{2\pi}(X)$, the elements $x$ of $X$ give rise to characters of $G$. We wish to be able to identify these characters in terms of the metric group properties of $G$. We begin by examining certain Banach spaces associated with $G$. In this section, $G$ is assumed to be a space with constants.

For $L_0 \subseteq S$, we denote by $G_0$ the set $\{a \in G | L_0(a) = \theta\}$.

**Lemma 4.1.** If $a \in G_0$, there exists $b \in G_0 \cap U_1$ and $\beta \in R$ such that $\beta b = a$.

**Proof.** By Lemma 2.1, there exists $c \in U_{1/2}$ and $\beta \in R$ such that $\beta c = a$. Let $h = l^{-1}(L_0(c))$. Then $L_0(c - h) = \theta$ and $\rho(c - h) \leq \rho(h) = \rho(c) + |f^{-1}(L_0(c))| \leq 2\rho(c) < 1$. Thus $b = c - h \in G_0 \cap U_1$. Then $\beta b = \beta(c - h) = \beta c - \beta h = a - \beta h$. But $l(\beta h) = L_0(\beta h) = L_0(\beta h - a) = -L_0(\beta(c - h)) = -\beta L_0(c - h) = \theta$ and so $\beta h = \theta$ and $\beta b = a$.

**Lemma 4.2.** If $b_1, b_2 \in G_0 \cap U_1$, $\beta_1, \beta_2 \in R$ and $\beta_1 b_1 = \beta_2 b_2$, then $|\beta_1| \rho(b_1) = |\beta_2| \rho(b_2)$ and for any $\alpha \in R$, $(\alpha \beta_1) b_1 = (\alpha \beta_2) b_2 \in G_0$.

**Proof.** From P'3, $(\alpha \beta_1) b_1$ and $(\alpha \beta_2) b_2 \in G_0$. Now if $\beta_1 = \beta_2 = 0$, the conclusions follow immediately. Thus we may assume $|\beta_1| \geq |\beta_2|$ and $\beta_1 \neq 0$. Then $2\beta_1((1/2)b_1 - (\beta_2/2\beta_1)b_2) = \theta$ and so $(1/2)b_1 - (\beta_2/2\beta_1)b_2 \in H$ (Theorem 3.4). But $l((1/2)b_1 - (\beta_2/2\beta_1)b_2) = L_0((1/2)b_1 - (\beta_2/2\beta_1)b_2) = \theta$ and so $(1/2)b_1 = (\beta_2/2\beta_1)b_2$ (Lemma 3.4). Therefore $(1/2)\rho(b_1) = \rho((1/2)b_1) = \rho((\beta_2/2\beta_1)b_2) = (|\beta_2|/2|\beta_1|)\rho(b_2)$ and $|\beta_1| \rho(b_1) = |\beta_2| \rho(b_2)$. Moreover, multiplying our equality by $2\alpha \beta_1$ gives $(\alpha \beta_1)b_1 = (\alpha \beta_2)b_2$.

Using the $\beta$ and $b$ of Lemma 4.1, we define

1. for each $a \in G_0$, $\rho'(a) = |\beta| \rho(b)$, and
2. for each $a \in G_0$ and each $\alpha \in R$, $\alpha \times a = (\alpha \beta)b$.

The uniqueness of these definitions follows from Lemma 4.2.

Let $G'$ be the space whose elements and underlying algebraic group structure are those of $G_0$, but with this new metric and multiplication by reals. That is, using $a'$ to denote the element $a$ in $G_0$, we have $\|a'\| = \rho'(a)$ and $\alpha a' = (\alpha \times a)'$.

One may readily verify

**Theorem 4.1.** $G'$ is a Banach space, and $G'$ is equivalent to $G_0$.

Let $G'$ be the vector direct sum of $G_0$ and the reals, $G' = G'_0 \oplus R\epsilon$. For $a' + \alpha \epsilon \in G'$, we define $\|a' + \alpha \epsilon\| = \gamma \rho((1/\gamma) \times a' + \epsilon)$ where $\gamma > \max \{2\|a'\|, 2\|\alpha\|\}$ and $h = l^{-1}(j(\alpha'/\gamma))$.

**Lemma 4.3.** $\|a' + \alpha \epsilon\|$ is uniquely defined.

**Proof.** Suppose $\gamma_1, \gamma_2 \geq \gamma \max \{2\|a'\|, 2\|\alpha\|\}$ and $\gamma_1 \geq \gamma_2$. Then $l((\gamma_2/\gamma_1)h_2)$
\[(\gamma_2/\gamma_1)l(h_2) = (\gamma_2/\gamma_1)j(\alpha/\gamma_2) = j(\alpha/\gamma_1) = l(h_1) \text{ and } (\gamma_2/\gamma_1)h_2 = h_1. \] Thus \((\gamma_2/\gamma_1) \cdot ((1/\gamma_2) \times a + h_2) = (\gamma_2/\gamma_1)((\beta/\gamma_2)b + h_2) = (\beta/\gamma_1)b + h_1 = (1/\gamma_1) \times a + h_1.\] Thus \(\gamma_1\rho((1/\gamma_1) \times a + h_1) = \gamma_2\rho((\gamma_2/\gamma_1)((1/\gamma_2) \times a + h_2)) = \gamma_2\rho(1/\gamma_2) \times a + h_2).\]

Thus a direct verification then gives

**Theorem 4.2.** \(G' = G' \oplus \text{Re} \) is a Banach space.

**Definition 4.1.** An element \(b\) of a Banach space \(B\) is a unit element if for every \(a \in B\), either \(\|a + b\| = \|a\| + 1\) or \(\|a - b\| = \|a\| + 1\) [8].

**Lemma 4.4.** The element \(e = \theta' + 1e \in G'\) is a unit element.

**Proof.** For any \(a' + ae \in G'\), choose \(\gamma > \max \{2\|a'\|, 2|\alpha| + 2\}\). Now \(h = l^{-1}(j(1/\gamma))\) is a constant of \(G\). Assume \(\{(l/\gamma)Xa + A\}\) is positive. Then \(\|a' + ae + e\| = \gamma\rho\{(l/\gamma)Xa + l^{-1}(j((\alpha + 1)/\gamma)))\} = \gamma\rho(1/\gamma)Xa + h + h\) \(= \gamma\rho(1/\gamma)Xa + h + h\). If \(- (1/\gamma)Xa - h, h\) is positive, the same argument gives \(\|a' + ae - e\| = \|a' + ae\| + 1\).

**Lemma 4.5.** \(\lambda_0: G' \to \mathbb{R},\) defined by \(\lambda_0(a' + ae) = \alpha,\) is a linear functional of norm 1.

**Proof.** \(\lambda_0\) is clearly linear and clearly \(\|\lambda_0\| \geq 1\). But \(\|\lambda_0(a' + ae)\| = |\alpha| = |\gamma| |\alpha/\gamma| = |\gamma| |j^{-1}(L_0((1/\gamma)Xa + h))a| \leq |\gamma| \rho((1/\gamma)Xa + h) = \|a' + ae\|\) and so \(\|\lambda_0\| = 1\).

For a fixed \(q \geq 1\), there is a natural mapping of \((C(X),\) the Banach space of bounded, continuous, real-valued functions on \(X,\) into \(R_{2q}(X)\) given by \((j(b)) (x) = j(b(x)).\) For \(G'(X) \subseteq C(X),\) we assume for \(j(G'(X))\) the metric, group properties induced on it as a subset of \(R_{2q}(X).\)

**Theorem 4.3.** If \(X\) is compact, then \(j(C(X))\) is equivalent to \(R_{2q}(X).\)

**Proof.** Theorem 1 of [4].

**Lemma 4.6.** If \(X\) is connected, and if \(G'(X)\) is a linear subspace of \((C(X),\) containing the function \(e(x) = 1,\) then \(j(G'(X))\) is a subspace of \(R_{2q}(X).\)

**Proof.** Since the map \(j\) is a homomorphism, \(j(G'(X))\) is an algebraic subgroup of \(R_{2q}(X).\)

(a) Suppose \(a'(x) \in G'(X)\) and \(j(a'(x)) \in U_1 \subseteq R_{2q}(X).\) Consider the function \(j^{-1}(j(a'(x))) - a'(x) = f(x).\) Since \(j^{-1}\) is continuous on \(U_1, f\) is continuous and since \(X\) is connected, \(f(X)\) is a connected set. But \(j(f(x)) \equiv \theta\) and so \(f(X) \subseteq \{n(2q)\}.\) Thus \(j^{-1}(j(a'(x))) = a'(x) + 2n_0q_e(x)\) for some integer \(n_0.\) Thus \(j^{-1}(j(a'(x))) = a_0^{-1}(j(a'(x))) \subseteq G'(X).\) Then \(\alpha\{j(a'(x))\} = j(\alpha^{-1}(j(a'(x)))) \subseteq j(G'(X)).\)

(b) Now suppose \(b(x) \in U_1 \subseteq R_{2q}(X)\) and for some \(\alpha \neq 0, ab(x) = j(a'(x))\) for some \(a'(x) \in G'(X).\) Choose \(\gamma > \max \{(4/|\alpha|)\|a'(x)\|, 4\}.\) Then \(\gamma \alpha \{((1/\gamma)b(x) - j(a'(x)/\gamma \alpha)) = j(\beta e(x)) \subseteq j(G'(X)).\) Thus \((1/\gamma)b(x) \subseteq j(G'(X)) \cap U_1.\)
By (a), \( \gamma((1/\gamma)b(x)) = b(x) \in j(G'(X)) \).

**Theorem 4.4.** If \( X \) is connected and \( G'(X) \) is a linear subspace of \( C(X) \), then \( G \) is equivalent to \( j(G'(X)) \) if and only if

1. \( G'(X) \) contains \( e(x) = 1 \), and
2. there exists an equivalence map of \( G' = G \oplus \mathbb{R} \) onto \( G'(X) \) such that \( e \to \pm e(x) \) under this equivalence.

**Proof.** (a) Suppose \( G \) is equivalent to \( j(G'(X)) \). Let \( i: G \to j(G'(X)) \) be the equivalence map. Then for \( \theta \neq \phi \in G \), \( i(\phi) \in \mathbb{H} \subset R_{\mathbb{R}}(X) \) and by Lemma 2.4, \( i(\phi) = j(\phi e(x)) \) for some \( \beta \neq 2nq \) for any \( n \). Thus, there exists \( f(x) \in G'(X) \) such that \( j(f(x)) = j(\phi e(x)) \). Since \( X \) is connected, \( f(x) \in G'(X) \) and since \( G'(X) \) is a linear subspace and \( \beta + 2nq \neq 0, e(x) \in G'(X) \). Thus condition (1) is satisfied.

We proceed to prove condition (2). It is clear that the map \( i(\phi) = j(\beta) \) is an equivalence map of \( H \) onto \( R_{\mathbb{R}} \). Thus on \( H \), \( i = \pm 1 \). Define \( \delta(x) = +e(x) \) or \( \delta(x) = -e(x) \) depending on whether \( i = +1 \) or \( i = -1 \). Then for \( a' + \alpha e \in G' \), choose \( \gamma > ||a'|| \) and define \( I(a' + \alpha e) = \gamma \{ j^{-1}[(1/(\gamma x))a] \} + \alpha \delta(x) \).

1. \( I \) is uniquely defined for \( \delta \geq \gamma, (1/\delta) x a = (\gamma/\delta)((1/\gamma) x a) \) and \( \delta \{ \gamma j^{-1} \left[ i \left( \frac{1}{\delta} \times a \right) \right] \} = \delta \{ \gamma j^{-1} \left[ i \left( \frac{1}{\gamma} \times a \right) \right] \} = \gamma \{ j^{-1} \left[ i \left( \frac{1}{\gamma} \times a \right) \right] \} \).

2. \( I(G') \subset G'(X) \). For if \( a' + \alpha e \in G' \), \( i((1/\gamma) x a) = j(f(x)) \) for some \( f(x) \) in \( G'(X) \). Then \( \gamma \{ j^{-1} i((1/\gamma) x a) \} + \alpha \delta(x) = \gamma j(f(x)) + 2\alpha e(x) \in G'(X) \).

3. \( I(e) = \delta(x) \).

4. \( I \) is linear. It is clearly a homomorphism. Moreover if \( \beta a' = (\beta x a)' \), \( I(b' + \beta (\alpha e)) = \gamma \beta \{ j^{-1} i((1/\gamma) x b) \} + \beta \alpha e(x) \) \( = \gamma \beta \{ j^{-1} i((1/\gamma) x a) \} + \beta \alpha e(x) \) \( = \beta I(a' + \alpha e) \).

5. \( I \) is norm-preserving. For if \( ||a' + \alpha e|| < 1/6, ||x|| < 1/6 \) by Lemma 4.5 and so \( ||a'|| < 1/3 \) by the triangle inequality. Then we may put \( \gamma = 1 \) in the definition of \( ||a' + \alpha e|| \), and we have

\[
||a' + \alpha e|| = \rho(a + b) = \rho(i(a + b)) = \rho(i(a) + i(b)) = \rho(i(a) + j \{ j^{-1} i(b) \} e(x)) = \| j^{-1} i(a) + j \{ j^{-1} i(b) \} e(x) \| = \| j^{-1} i(a) + \alpha e(x) \| = \| I(a' + \alpha e) \|
\]
(again taking $\gamma = 1$). Thus $I$ is norm-preserving on $U_{1/q}$. But $I$ is linear and so $I$ is norm-preserving on $G'$.

(6) $I$ maps $G'$ onto $G'(X)$. For suppose $b'(x) \in G'(X)$. There exists $a \in G_0$ and $h \in \overline{H}$ such that $i(a + h) = j(b'(x))$. But $j(I(a')) = i(a) = j(b'(x)) - i(h) = j(b'(x)) - j^{-1}(l(h))(\tilde{e}(x)) = I(a') = b'(x) - (j^{-1}l(h) + 2\pi q)(\tilde{e}(x))$. Thus $I(a' + (j^{-1}l(h) + 2\pi q)e) = b'(x)$.

Thus $I$ is a Banach space equivalence and clearly an equivalence in our sense.

(b) Now suppose $I: G' \to G'(X)$ is the hypothesized equivalence. Since $l^{-1}L_0: G \to \overline{H}$ is a continuous projection, $G = G_0 \oplus \overline{H}$ is a direct sum. Thus we may define $J: G \to j(G'(X))$ by $J(a + h) = j\left\{I(a') + (j^{-1}(l(h)))\tilde{e}\right\}$.

(1) $J$ clearly a homomorphism.

(2) If $J(a + h) = \theta$, $j\left\{I(a') + (j^{-1}(l(h)))\tilde{e}\right\} = \theta$ and $I(a') = \left\{2\pi q + j^{-1}(l(h))\tilde{e}\right\}$ (since $I(\tilde{e}) = \tilde{e}(x))$. But $G'$ and therefore $G'(X)$ is a direct sum and so $I(a') = 0$ and $j^{-1}(l(h)) = -2\pi q = 0$ as $-q < j^{-1}(l(h)) \leq q$. Thus $a = h = \theta$ and $J$ is an isomorphism.

(3) If $f(x) \in j(G'(X))$, there exist $a \in G_0$ and $\alpha \in R$ such that $f(x) = j(I(a') + \alpha I(\tilde{e})) = j\left\{I(a') + (j^{-1}(\alpha))\tilde{e}\right\}$. Let $h = l^{-1}(\alpha)$; then $J(a + h) = f(x)$ and $J$ maps $G$ onto $j(G'(X))$.

(4) Suppose $\rho(a + h) < 1$. Now $\rho(J(a + h)) = \left\|J(I(a') + (j^{-1}(l(h)))\tilde{e}\right\|$. Since $j^{-1}(\alpha) = \alpha$ for $|\alpha| < 1$, we prove $\rho(J(a + h)) = \rho(a + h)$ by showing that $\left\|I(a') + (j^{-1}(l(h)))\tilde{e}\right\| = \rho(a + h) < 1$. Now $G'$ and $G'(X)$ are Banach spaces and so an equivalence between them in our sense is a Banach space equivalence. Thus $\left\|I(a') + (j^{-1}(l(h)))\tilde{e}\right\| = \left\|a' + (j^{-1}(l(h)))\tilde{e}\right\|$. Now $\rho(h) = \left\|j^{-1}(l(h))\right\| = \left\|j^{-1}(L_0(h))\right\| = \left\|j^{-1}(L_0(a + h))\right\| \leq \rho(a + h) < 1$. Thus choosing $\gamma = \max \left\{\frac{4}{\pi}a', \frac{4}{\pi}h\right\}$, we have $\left\|a' + (j^{-1}(l(h)))\tilde{e}\right\| = \gamma \rho((1/\gamma)Xa + (1/\gamma)h)$ since $(1/\gamma)h = l^{-1}(\gamma(1/\gamma)l^{-1}(l(h)))$. But $\gamma \left\{(1/\gamma)Xa + (1/\gamma)h - (1/\gamma)(a + h)\right\} = \theta$ and so $b = (1/\gamma)Xa + (1/\gamma)h - (1/\gamma)(a + h) \in \overline{H}$. But $l(b) = L_0(b) = \theta$ and so $b = \theta$. Therefore $\gamma \rho((1/\gamma)Xa + (1/\gamma)h) = \gamma \rho((1/\gamma)(a + h)) = \rho(a + h)$. Thus $J$ is an isometry on $U_1 \subset G$.

(5) Suppose $\rho(J(a + h)) < 1$. Then for some fixed integer $n_0$, $2nq - 1 < I(a') + j^{-1}(l(h))\tilde{e} < 2nq + 1$ for all $x \in X$. Thus $\left\|I(a') + j^{-1}(l(h))\tilde{e}\right\| < 2nq + 1$ and $\left\|I(a') + j^{-1}(l(h))\tilde{e}\right\|$. By Lemma 4.5, $\left\|j^{-1}(l(h))\tilde{e}\right\| < 1$. Thus again $\gamma \rho((1/\gamma)Xa + (1/\gamma)h) = \rho(J(a + h)) < 1$. But since $\gamma \rho((1/\gamma)Xa + (1/\gamma)h) < 1$, $\gamma \rho((1/\gamma)Xa + (1/\gamma)h) = \rho(a + h)$ and $J$ maps $U_1 \subset G$ onto $U_1 \subset j(G'(X))$.

Thus $G$ is equivalent to $j(G'(X))$.

5. Some theorems on Banach spaces. We have shown that a group $G$ is equivalent to $R_2(X)$ for some compact, connected space $X$ if and only if (1) $G$ is a space with constants and (2) $G'$ is equivalent to $C(X)$ (Theorems 4.3 and 4.4). In the usual characterizations of a Banach space $G'$ as $C(X)$, the points of $X$ are found in $G'^*$ (the set of linear functionals of $G'$). We wish to give a characterization in terms of the group $G$. In §6, we show that the characters
of $G$ correspond naturally to a subset of the linear functionals of $G'$. However, the $F_r$'s of $G' [8]$, or the extreme points of the unit sphere of $G_0^* [2]$, do not in general give the required space $X$.

Specifically, for $G' = G' \oplus \mathbb{R}e$, we look for a space $E \subset G_0^*$ such that the natural correspondence $a' + \alpha e \rightarrow \xi(a') + \alpha (\xi \in E)$ is an equivalence, and such that $G'$ is equivalent to $C(X)$ for some $X$ if and only if this mapping takes $G'$ onto $C(E)$.

Let $B'$ be a Banach space with a unit element $e$.

**Definition 5.1.** If $\lambda_0$ is a linear functional on $B'$ of norm 1 whose value at $e$ is 1, then $B = \{ b \in B' | \lambda_0(b) = 0 \}$ is a positive hyperplane of $B'$. $B$ clearly is a Banach space and $B' = B \oplus \mathbb{R}e$ is a direct sum.

**Definition 5.2.** A functional $\lambda \in B^*$ is essentially positive (relative to $B'$) if for all $b \in B$, and $\alpha \in \mathbb{R}$, $| \lambda(b) + \alpha | \geq \| b + \alpha e \|$.

In what follows the topology in $B^*$ is the weak-star (point open) topology.

**Lemma 5.1.** The set $S$ of essentially positive linear functionals is closed in $B^*$.

**Proof.** Suppose $\lambda' \in B^*$ and $\lambda' \notin S$. Then there exists $b \in B$, $\alpha \in \mathbb{R}$ such that $| \lambda'(b) + \alpha | > \| b + \alpha e \|$. The set $V = \{ \lambda \in B^* | | \lambda(b) - \lambda'(b) | < (| \lambda'(b) + \alpha | - | b + \alpha e | )/2 \}$ is open and contains $\lambda'$. For $\lambda \in V$, $| \lambda(b) + \alpha | \geq | \lambda'(b) + \alpha | - | \lambda(b) - \lambda'(b) | > (| \lambda'(b) + \alpha | + \| b + \alpha e \|)/2 \geq \| b + \alpha e \|$. Thus $V \cap S = \emptyset$ and $S$ is closed.

**Lemma 5.2.** $S$ is compact.

**Proof.** For $\alpha = 0$, $\lambda \in S$, $| \lambda(b) | \leq \| b \|$ for all $b \in B$. Thus $S$ is contained in $\Sigma'$ the unit sphere in $B^*$. But $\Sigma$ is compact in the weak-star topology [1], $S$ is closed by Lemma 5.1, and $S$ is compact.

**Definition 5.3.** For $\lambda \in S$, $M(\lambda) = \{ b \in B | \lambda(b) \geq \lambda'(b) \text{ for all } \lambda' \in S \}$.

We may order the sets $M(\lambda)$ by inclusion.

**Definition 5.4.** A functional $\xi \in S$ is a maximal functional of $S$ if $M(\xi)$ is a maximal set in the ordering of the sets $M(\lambda)$.

It can be shown that in the natural imbedding of $S$ into $\Sigma'$ (the unit sphere in $B^*$), the maximal functionals do not in general map into either $F_r$'s or extreme points of $\Sigma'$.

**Theorem 5.1.** If $\lambda_0 \in S$, there exists a maximal functional $\xi$, such that $M(\xi) \supset M(\lambda_0)$.

**Proof.** By Zorn's lemma, $M(\lambda_0)$ is contained in a maximal linearly ordered chain $\{ M(\lambda_0) \}$. Define $E(\mu) = \{ \lambda \in S | \lambda(b) = \lambda_0(b) \text{ for all } b \in M(\mu) \}$.

(1) $E(\mu)$ is not empty as $\lambda_0 \in E(\mu)$.

(2) If $M(\lambda_1) \subset M(\lambda_2)$ then $E(\mu_1) \supset E(\mu_2)$. For if $\lambda \in E(\mu_2)$, $\lambda(b) = \lambda_\mu_2(b) \geq \lambda_\mu_1(b) \text{ for all } b \in M(\lambda_\mu_2)$. Thus $\lambda(b) \geq \lambda_\mu_1(b)$ for all $b \in M(\lambda_\mu_1)$. But the opposite inequality always holds and so $\lambda(b) = \lambda_\mu_1(b) \text{ for all } b \in M(\lambda_\mu_1)$ and so $\lambda \in E(\mu_1)$. 


(3) \( E(\mu) \) is closed for \( E(\mu) = \bigcap_{b \in M(\lambda_0)} \{ \lambda \in S | \lambda(b) = \lambda_0(b) \} \) and is the intersection of closed sets.

Thus \( \{ E(\mu) \} \) is a family of closed, non-empty sets of \( S \), linearly ordered by inclusion. Since \( S \) is compact, there exists \( \xi \in \bigcap_{\mu} \{ E(\mu) \} \). For any \( \lambda_\mu \) in our chain, we now have \( \xi(b) = \lambda_\mu(b) \geq \lambda(b) \) for all \( b \in \bigcap_{\mu} M(\lambda_\mu) \) and all \( \lambda \in S \). Thus \( M(\xi) \supseteq M(\lambda_\mu) \). If \( M(\lambda_\mu') \supseteq M(\xi) \), then \( M(\lambda_\mu') \) belongs to the chain (the chain is maximal) and so \( \xi(b) \) is closed by inclusion. Since \( M(\lambda_0) \subseteq \{ M(\lambda_\mu) \}, M(\xi) \supseteq M(\lambda_0) \).

**Theorem 5.2.** If \( B \) is a positive hyperplane of \( B' \), a Banach space with a unit element \( e \), then for each \( b' \in B' \), \( b' = b + \beta e \), there exists a maximal functional \( \xi \) of \( S \) such that \( |\xi(b) + \beta| = |b'| \).

**Proof.** By the Hahn-Banach extension theorem [3, p. 28], there exists a \( \lambda_\xi \in B'^* \) such that \( \|\lambda_\xi\| = 1 \), \( \lambda_\xi(e) = 1 \), and \( \lambda_\xi(b') = \inf_{a \in B} \| b' + ae \| - \alpha \). Now \( e \) is a unit element. We assume first that \( \| b' + ae \| - \alpha = \| b' \| + \alpha - \alpha = \| b' \| \). Then for \( \alpha \geq 0 \), Lemma 3.1 implies that \( \| b' + ae \| - \alpha = \| b' \| + \alpha - \alpha = \| b' \| \). (The condition \( \| b' \| + \alpha \| e \| < 1 \) is not needed in a Banach space.) For \( \alpha < 0 \), \( \| b' + ae \| - \alpha \geq \| b' \| - \alpha = \| b' \| \). Therefore \( \inf_{a \in B} \| b' + ae \| - \alpha = \| b' \| \). Thus \( \lambda_\xi(b') = \| b' \| \).

Let \( \lambda_0 \) be the functional \( \lambda_\xi \) cut down to \( B \). Since \( \| \lambda_\xi \| = 1 \) and \( \lambda_\xi(e) = 1 \), \( \lambda_0 \) is an element of \( S \). Moreover for all \( \lambda \in S \), \( \lambda(b) + \beta \leq \| b + \beta e \| \) and so, for the \( b \) and \( \beta \) defined by \( b' \), \( \lambda(b) \leq \| b + \beta e \| - \beta = \lambda_0(b) \). Thus \( b \in \bigcap_{\mu} M(\lambda_\mu) \). But there exists a maximal functional \( \xi \) such that \( M(\xi) \supseteq M(\lambda_0) \), Theorem 5.1. Moreover on \( M(\lambda_0), \xi = \lambda_0 \) and so \( \xi(b) + \beta = \lambda_0(b) + \beta = \lambda_\xi(b + \beta e) = \lambda_\xi(b') = \| b' \| \).

Now if \( \| b' - e \| = \| -b' + e \| = \| b' \| + 1 \), the same argument proves the existence of a maximal functional \( \xi \), such that \( \xi(-b) - \beta = \| b' \| \). Since one of these two conditions must apply we have shown the existence of a maximal functional \( \xi \), such that \( |\xi(b) + \beta| = |b'| \).

**Theorem 5.3.** If \( B \) is a positive hyperplane of \( B' \), a Banach space with a unit element \( e \), and \( E \) is the space of maximal functionals of \( S \), then \( B' \) is equivalent to a closed, linear subspace of \( C(E) \).

**Proof.** We map \( b' = b + \beta e \mapsto f(\xi) = \xi(b) + \beta \). The weak star topology on \( E \subseteq B^* \) insures the continuity of \( f \). Since \( \| \xi(b) + \beta \| \leq \| b + \beta e \|, f(\xi) \) is bounded. This map of \( B' \to C(E) \) is clearly linear, and by Theorem 5.2 it is norm-preserving. Thus \( B' \) is equivalent to its image in \( C(E) \) and since \( B' \) is a complete, linear space, its image is a closed, linear subspace of \( C(E) \).

**Theorem 5.4.** A Banach space \( B' \) is equivalent to \( C(X) \) for some compact space \( X \) if and only if

1. \( B' \) has a unit element and
2. there exists a positive hyperplane \( B \) of \( B' \), such that for any \( b \in B \) and \( \beta \in R \), there exists \( b \in B \) and \( \beta \in R \), such that \( \xi(b) + \beta = |\xi(b) + \beta| \) for all maximal functionals \( \xi \in S \).
Proof. (a) Suppose $B'$ is equivalent to $C(X)$ for some compact $X$. Then $e(x) = 1$ is a unit element. Let $B$ be any positive hyperplane of $B'$ (one exists by the Hahn-Banach theorem). To show that condition (2) is necessary we need only show that every maximal functional corresponds to a point of $X$ ($f(x) = b(x_0)$ for some $x_0 \in X$), for $b'(x) \in C(X)$ implies $|b'(x)| \subseteq C(X)$.

Let $X_0 = \{ x \in X \mid b(x) = \sup_x b(x) \}$. Since $X$ is compact $X_0$ is not empty. Now the functional $\lambda_0 : b \to b(x_0)$ is an element of $S$. Thus for any $\lambda \in S$ and $b \in M(\lambda)$, $\lambda(b) = \sup_x b(x)$ for all $x_0 \in X$. Thus for $b \in M(\lambda)$, $\lambda(b) \geq \sup_x b(x)$. Now choose $\alpha = \|b\|$. Then $\lambda(b) \leq \|b + \alpha e\| - \alpha = \sup_x (b(x) + \alpha e(x)) - \alpha = \sup_x b(x)$. Thus $\lambda(b) = \sup_x b(x)$ for $b \in M(\lambda)$. Now for $b_i$ any finite set of elements of $M(\lambda)$ we have $\sum_{i=1}^n b_i \in M(\lambda)$ and so

$$\sup_{x \in X} \left( \sum_{i=1}^n b_i(x) \right) = \lambda \left( \sum_{i=1}^n \lambda(b_i) \right) = \sum_{i=1}^n \sup_x (b_i(x)).$$

But this implies that $\bigcap_{i=1}^n X_{b_i}$ is not empty. Since $X$ is compact and $X_0$ is closed we have that there exists an $x_1 \in X$ such that $x_1 \in \bigcap_{b \in M(\lambda)} X_b$. Then $\lambda_1 : b \to b(x_1)$ is equal to $\lambda$ on $M(\lambda)$ and so we have $M(\lambda_1) \subseteq M(\lambda)$. Now suppose $\lambda = \xi$ is a maximal functional. Thus $M(\xi) = M(\lambda_1)$ and $\xi(b) = b(x_1)$ for all $b \in M(\lambda)$. But $b \in B$ such that there exists an $\alpha \in R$ such that $b(x_1) + \alpha = \|b + \alpha e\|$ certainly belong to $M(\lambda_1)$. Thus $\xi = \lambda_1$ on these elements and by Lemma 2.3 of [8], $\xi = \lambda_1$ on $B$ and so all maximal functionals correspond to points of $X$. The preceding also proves that all points of $X$ give rise to maximal functionals.}

(b) Now suppose (1) and (2) are satisfied. Let $\overline{E}$ be the closure of $E$ in $B^*$. Since $E \subseteq S$, and $S$ is compact, $\overline{E}$ is compact. Moreover, the map $b' = b + \alpha e = f(\xi) = \xi(b) + \alpha$ for $\xi \in \overline{E}$ is an equivalence map (Theorem 5.3, the addition of elements of $S$ to $E$ to form $\overline{E}$ does not change this property). Thus $B$ is equivalent to $\Gamma$, a closed, linear subspace of $C(\overline{E})$. Then by the theorem of Kakutani [6], $\Gamma = C(\overline{E})$ if

1. whenever $\xi_1, \xi_2 \in \overline{E}$ and $\xi_1 \neq \xi_2$ there exists $f \in \Gamma$ such that $f(\xi_1) \neq f(\xi_2)$,
2. $\Gamma$ contains a nontrivial constant function, and
3. $\Gamma$ is lattice closed.

If $f(\xi_1) = f(\xi_2)$ for all $f$ in $\Gamma$, then $\xi_1(b) + \alpha = \xi_2(b) + \alpha$ for all $b \in B$, and so $\xi_1 = \xi_2$.

Moreover $0 + e \in B'$ maps into the function $f(\xi) = 1$ and $\Gamma$ contains a nontrivial constant function.

Finally

$$\max \{ \xi(b_1) + \alpha_1, \xi(b_2) + \alpha_2 \}$$

$$= \frac{1}{2} \{ \xi(b_1) + \xi(b_2) + \alpha_1 + \alpha_2 \pm |\xi(b_2) - \xi(b_1) + \alpha_2 - \alpha_1| \}$$

and by condition (2) both these functions are in $\Gamma$, and $\Gamma$ is lattice closed.
Thus $B'$ is equivalent to $C(\mathcal{E})$. By the remark in the proof of the converse all the elements of $\mathcal{E}$ are maximal and so $\mathcal{E} = E$.

**Lemma 5.3.** If $X$ is compact, then $X$ is connected if and only if $e(x) \equiv 1$ and $e(x) \equiv -1$ are the only unit elements of $C(X)$.

**Proof.** (a) If $V$ is a nontrivial open and closed set in $X$, then $e(x) \equiv 1$ on $V$ and $e(x) \equiv -1$ on the complement of $V$ is a unit element of $C(X)$.

(b) Suppose $f \in C(X)$ is a unit element. Then $\|f\| = \|0 + f\| = \|0\| + 1 = 1$, and so $|f(x)| \leq 1$ for all $x$. Now suppose that for some $x_0 \in X$, $|f(x_0)| < 1$. There exists $b \in C(X)$ such that $\|b\| = 1$, $b(x_0) = 1$, and $b(x) \equiv 0$ wherever $f(x) = 1$. Then $\|b + f\| < 2$ which contradicts the hypothesis that $f$ is a unit element. Thus if $f$ is a unit element, $|f(x)| \equiv 1$ for all $x \in X$. But $X$ is connected and so either $f(x) \equiv 1$ or $f(x) \equiv -1$.

Suppose $B'$ is a Banach space with a unit element $e$, and $B_1$ and $B_2$ are positive hyperplanes of $B'$. For $b \in B_2$, there is a unique $b_1 \in B_1$ and $a \in R$ such that $b_2 = b_1 + ae$. For $\lambda_1 \in S_1$ we define $[i(\lambda_1)](b_2) = \lambda_1(b_1) + a$. It is clear that $i$ is a 1-1 map of $S_1$ onto $S_2$.

**Lemma 5.4.** If $\xi_1 \in S_1$ is a maximal functional of $S_1$, then $i(\xi_1) \in S_2$ is a maximal functional of $S_2$.

**Proof.** Let $M_2 = (M(\xi_1) + Re) \cap B_2$. For $b_2 \in M_2$, $[i(\xi_1)](b_2) = \xi_1(b_1) + a \geq \lambda_1(b_1) + a = [i(\lambda_1)](b_2)$ for all $\lambda_1 \in S_1$. Thus $M \{ i(\xi_1) \} \supset M_2$. Suppose $M(\lambda_2) \supset M(i(\xi_1))$, and $M(\lambda_2) \neq M(i(\xi_1))$. Then $M_1 = (M(\lambda_2) + Re) \cap B_1$ contains $M(\xi_1)$ properly and moreover $M(i^{-1}(\lambda_2)) \supset M_1$. But this contradicts the maximality of $\xi_1$ and so $M(\lambda_2) = M(i(\xi_1))$ and $i(\xi_1)$ is a maximal functional of $S_2$.

6. A characterization of $R_2(X)$. Let $G$ be a space with constants. For $L \in S$, we define the functional $I_0(L) : G_0' \rightarrow R$ by $[I_0(L)](a') = \alpha^{-1}(L((1/\alpha) \times a))$ for $|\alpha| > ||a'||$. The notation is that of §§3 and 4.

**Lemma 6.1.** $I_0(L)$ is uniquely defined.

**Proof.** The ambiguity of definition arises in the choice of $\alpha$. However, for $|\gamma| \geq |\alpha|$, $(\alpha/\gamma)((1/\alpha) \times a) = (\alpha/\gamma) \times ((1/\alpha) \times a) = (1/\gamma) \times a$. Thus $\gamma^{-1}(L((1/\gamma) \times a)) = \gamma^{-1}(L((1/\alpha) \times a)) = \gamma^{-1}(\alpha/\gamma)(L((1/\alpha) \times a)) = \gamma^{-1}((\alpha/\gamma)L((1/\alpha) \times a)) = \gamma^{-1}(L((1/\alpha) \times a)) = \gamma^{-1}(L((1/\alpha) \times a))$.

**Lemma 6.2.** The functional $I_0(L)$ is an element of $S_0$, the set of positive linear functionals of $G_0'$ (with respect to $G'$).

**Proof.** (1) $[I_0(L)](a'_1 + a'_2) = \alpha^{-1}(L(((1/\alpha) \times (a_1 + a_2))) = \alpha^{-1}(L((1/\alpha) \times a_1) + L(1/\alpha \times a_2))$. But we may choose $\alpha > ||a'_1|| > ||a'_2||$ which makes $j^{-1}$ a homomorphism and so $I_0(L)$ is a homomorphism.

(2) $[I_0(L)](\beta \times a') = \alpha^{-1}(L((1/\alpha) \times (\beta \times a)))$ where we may choose $\alpha > \max \{ ||\beta|| ||a'||, ||a'|| \}$. Then $\alpha^{-1}L((1/\alpha) \times (\beta \times (a_1 \times a_2))) = (\alpha^{-1}((1/\alpha) \times a_1) \times a_2 = (\alpha^{-1}L((1/\alpha) \times a)) = \beta(\alpha^{-1}L((1/\alpha) \times a))) = \beta(\alpha^{-1}L((1/\alpha) \times a)) = \beta[I_0(L)](a')$ and so $I_0(L)$ is linear.
(3) For $a' + \beta e \in G'$, choose $\alpha > \|a'\| + |\beta|$ and put $h = l^{-1}(j(\beta/\alpha))$. Then 
$$
|\left[ I_0(L) \right](a') + \beta| = \alpha j^{-1}(L((1/\alpha) \times a)) + \beta = \alpha j^{-1}(L((1/\alpha) \times a)) + \beta/\alpha |
$$
$$
= \alpha j^{-1}(L((1/\alpha) \times a) + j(\beta/\alpha)) = \alpha j^{-1}(L((1/\alpha) \times a + h)) \leq \alpha \rho((1/\alpha) \times a + h) = \|a' + \beta e\| \text{ and so } I_0(L) \in S_0.
$$

**Lemma 6.3.** The map $I_0: S \to S_0$ is a homeomorphism onto.

**Proof.** (1) If $I_0(L_1) = I_0(L_2)$, then $L_1 = L_2$ on $G_0 \cap U_1$ and since $G_0 \cap U_1$ generates $G_0$, $L_1 = L_2$ on $G_0$. But $L_1 = L_2 = l$ on $\overline{H}$ and so $L_1 = L_2$ on $G$. Thus $L_1 = L_2$ and $I_0$ is 1-1.

(2) Suppose $X \in S_0$. Define $L: G \to R_{2q}$ by $L(a + h) = j(\lambda(a')) + l(h)$. $L$ is certainly a homomorphism and so satisfies $P'1$. Moreover, if $\rho(a + h) < 1$, $\rho(a + h) = \|a' + (j^{-1}(l(h)))e\|$. (See proof of Theorem 4.4.) Then 
$$
1 > \rho(a + h) = \|\lambda(a') + j^{-1}(l(h))\| = |j^{-1}(j(\lambda(a')) + j^{-1}(l(h)))| = j^{-1}(L(a + h)) |
$$
and so $L$ satisfies $P'2$. Thus by Theorem 3.2, $L$ is a character of $G$. Since $L = l$ on $\overline{H}$, by definition, we have $L \in S$. But $[I_0(L)](a') = \alpha j^{-1}(L((1/\alpha) \times a)) = \alpha j^{-1}(\lambda((1/\alpha) \times a')) = \lambda(a') \text{ and so } I_0(L) = \lambda$ and $I_0$ maps $S$ onto $S_0$.

(3) Since $S_0$ is compact (Lemma 5.2), and $S$ is clearly Hausdorff, to show $I_0$ is a homeomorphism we need only show that $I_0^{-1}$ is continuous. Now for $\lambda \in S_0, a_i \in G_0, i = 1, \ldots, n,$ and $1 \geq \epsilon > 0, V = \{L \in S | \|j^{-1}(L(a_i)) = [I_0^{-1}(\lambda)](a_i)\| < \epsilon\}$ is a basic neighborhood of $I_0^{-1}(\lambda)$ in $S$. We need choose the $a_i$ only from $G_0$, as for all $L \in S, L = l$ on $\overline{H}$ and $G = G_0 \oplus \overline{H}$. Let $V' = \{\lambda \in S_0 | \|\lambda(a_i) - \lambda(a_i')\| < \epsilon\}$. Thus $V'$ is a neighborhood of $\lambda$ in $S_0$. If $L \in I_0^{-1}(V')$ we have that 
$$
j^{-1}(L(a_i)) = \left[ I_0^{-1}(\lambda) \right](a_i)) = |j^{-1}(L(\alpha_i((1/\alpha_i) \times a_i)) = j^{-1}(L(\alpha_i((1/\alpha_i) \times a_i)) - j^{-1}(\lambda(1/\alpha_i))((1/\alpha_i) \times a_i))|
$$
$$
= |j^{-1}(\lambda((1/\alpha_i) \times a_i)) - j^{-1}(\lambda((1/\alpha) \times a_i))| < \epsilon,
$$
since $j^{-1}(j(\beta)) = \beta$ if $|\beta| < 1$. Thus $I_0^{-1}(V') \subset V, I_0^{-1}$ is continuous and $I_0$ is a homeomorphism.

We have immediately

**Theorem 6.1.** If $G$ is a space with constants, $S$ is compact.

**Definition 6.1.** For $L \in S$ and $L_0 \in S$, let $N_0(L) = \{a \in G_0 \cap U_1 | j^{-1}(L(a)) \geq f^{-1}(L(a))\}$ for all $L \in S$.

As in §5 we order the sets $N_0(L)$ by inclusion.

**Definition 6.2.** $F \in S$ is a maximal $G_0$ character if $N_0(F)$ is a maximal set in the ordering.
The correspondence between $N_0(L)$ and $M(I_0(L))$ \{Definition 5.3\} is quite direct.

**Lemma 6.4.** $M(I_0(L)) = \{a' \in G_0 \mid \alpha > ||a'||, (1/\alpha) \times a \in N_0(L)\}.$

**Proof.** $(I_0(L))(a') = x^{-1}((1/\alpha) \times a)).$ Since $\alpha > 0,$ $(I_0(L))(a') \supseteq (I_0(L))(a') \rightarrow j^{-1}(L((1/\alpha) \times a)) \supseteq j^{-1}(L((1/\alpha) \times a))$ and the lemma follows as $I_0$ maps $S$ onto $S_0$ (Lemma 6.3).

A corollary of Lemma 6.4, obtained by putting $\alpha = 1,$ is

**Lemma 6.5.** $N_0(L) = \{a \in G_0 \mid a' \in M(I_0(L)) \text{ and } ||a'|| < 1\}.$

**Theorem 6.2.** $F$ is a maximal $G_0$ character if and only if $I_0(F)$ is a maximal functional of $S_0.$

**Proof.** (a) Suppose $M(\lambda) \supset M(I_0(F));$ then, by Lemma 6.5, $N_0(I_0^{-1}(\lambda)) \supset N_0(F).$ But then if $F$ is maximal, $N_0(I_0^{-1}(\lambda)) \subset N_0(F)$ and, by Lemma 6.4, $M(\lambda) \subset M(I_0(F)).$ Thus $M(I_0(F))$ is maximal and $I_0(F)$ is a maximal functional of $S_0.$

(b) Suppose $N_0(L) \supset N_0(F).$ Then Lemma 6.4 implies $M(I_0(L)) \supset M(I_0(F)).$ But if $I_0(F)$ is maximal, $M(I_0(L)) \subset M(I_0(F))$ and by Lemma 6.5, $N_0(L) \subset N_0(F).$ Thus $N_0(F)$ is maximal and $F$ is a maximal $G_0$ character.

The maximality of $F \in S$ does not depend on the choice of $L_0.$

**Definition 6.3.** $F \in S$ is a maximal character if it is a maximal $G_0$ character for all $L_0 \in S.$

**Theorem 6.3.** $F \in S$ is a maximal character if it is a maximal $G_0$ character for some $L_0 \in S.$

**Proof.** The theorem is an immediate consequence of Theorem 6.2 and Lemma 5.4.

The final characterization may now be given.

**Theorem 6.4.** A group $G$ is equivalent to $R_{2q}(X)$ for some $q \geq 1$ and for some compact, connected space $X$ if and only if

1. there exists a unique, isomorphic, isometry, $i_\alpha: [0, \rho(a)] \rightarrow G$ such that $i_\alpha(\rho(a)) = a,$
2. the elements of $H \cap U_1$ are constants of $G,$
3. the elements of $H \cap U_1$ are the only constants of $G,$
4. for $b \in G,$ there exists $b' \in G$ such that $j^{-1}\{F(b')\} = j^{-1}\{F(b)\}$ for all maximal characters $F$ of $S.$

**Proof.** Suppose $G$ is equivalent to $R_{2q}(X)$ for $q \geq 1$ and for $X$ a compact, connected space. Conditions (1) and (2) follow from Theorems 2.1, 2.2, and 2.3. Condition (3) follows quickly since a constant of $R_{2q}(X)$ must be of the form $f(x) = a \in U_1 \subset R_{2q}$ (see proof of Lemma 5.3) and so $f \in H \cap U_1.$ Now by
Theorems 4.3 and 4.4, $G'$ is equivalent to $C(X)$. Then by Theorem 5.4, there exists a positive hyperplane $G$ of $G'$ such that $G' = \overline{G} \oplus \mathbb{R} e$ and such that for each $a' \in \overline{G}$ and each $\alpha \in \mathbb{R}$, there exists $\hat{a}' \in \overline{G}$ and $\alpha \in \mathbb{R}$ such that for all maximal functionals $\xi$ of $S$, $\langle \hat{a}' + \alpha \rangle = |\xi(a') + \alpha \rangle$. Let $\lambda_0$ be the functional which defined $\overline{G}$. We redefine $G_0$ using the character $\rho_0 = 1_{\lambda_0^{-1}}(\xi)$. Then $G = G_0 \oplus \mathbb{H}$ and $G_0' = \overline{G}$. Now suppose $b \in U_1$. We have $b = a + h$ where $a \in G_0$ and $h \in \mathbb{H}$. Then there exist $a' \in \overline{G}$ and $\alpha \in \mathbb{R}$ such that $\langle \hat{a}' + \alpha \rangle = |\xi(a') + \alpha \rangle$ for all maximal functionals $\xi$. Let $\hat{a} = a + l^{-1}(\rho(\hat{a}))$. Now by Theorems 6.2 and 6.3, the maximal characters $F$ are exactly the elements $1_{\lambda_0^{-1}}(\xi)$ where $\xi$ are the maximal functionals. Thus $j^{-1}(F(b)) = j^{-1}(1_{\lambda_0^{-1}}(\xi)(b)) = j^{-1}(j(\xi(a') + \alpha \xi)) = j^{-1}(j(\xi(a') + \alpha \xi) + j^{-1}(l(h))) = j^{-1}(j(\xi(a') + \alpha \xi) + j^{-1}(l(h))) = j^{-1}(F(b))$ and condition (4) is satisfied.

(b) If (1) and (2) are satisfied, $G$ is a space with constants. Choosing any $L_0 \in S$, we have, as before, $G = G_0 \oplus \mathbb{H}$, $G' = G_0' \oplus \mathbb{R} e$, $e$ is a unit element of $G'$, and $G_0'$ is a positive hyperplane of $G'$. For each $a' \in G_0'$ and each $\alpha \in \mathbb{R}$, choose $\gamma > 4$ max (\|a'\|, |\alpha \| and let $h = l^{-1}(\gamma \alpha \gamma)$. Then $j^{-1}(F(b)) = j^{-1}(F(b))$ for all maximal characters $F$. Let $\hat{a} = (\gamma \alpha \gamma)'$ and $\alpha = \gamma^{-1}(l(h))$. Then for $\xi$ a maximal functional of $S$, $\xi(a') + \alpha = (j_0(F))(a') + \alpha = \gamma^{-1}(F((1/\gamma) \times \xi(a'))) + \gamma^{-1}(l(h)) = \gamma^{-1}(F(b)) = \gamma^{-1}(F(b)) + |\gamma^{-1}(l(h))) = |\gamma^{-1}(l(h))) = |\gamma^{-1}(l(h)))$. Thus by Theorem 5.4, $G'$ is equivalent to $C(X)$ for some compact $X$. Moreover if $a' + \alpha$ is a unit of $G'$, then $(1/\gamma) \times a + h$ is a constant of $G$. By condition (3), $(1/\gamma) \times a + h \in \mathbb{H}$ and so $a' = \theta$. But then $|\alpha \|$ must equal 1 and so $\pm e$ are the only unit elements of $G'$. But $e(x) = 1$ and $e(x) = -1$ are unit elements of $C(X)$ and so $e$ must map into either $e(x) = 1$ or $e(x) = -1$ under the equivalence and these are the only unit elements of $C(X)$ and so $X$ is connected (Lemma 5.3). Finally by Theorems 4.4 and 4.3, $G$ is equivalent to $R_2\sigma(X)$.

7. The homeomorphism theorem.

THEOREM 7.1. If $X$ and $Y$ are compact, and if $R_2\sigma_1(X)$ is equivalent to $R_2\sigma_2(Y)$, then $\sigma_1 = \sigma_2$ and $X$ is homeomorphic to $Y$.

Proof. Let $T: R_2\sigma_1(X) \rightarrow R_2\sigma_2(Y)$ be the equivalence. Choose a positive integer $n$ such that $2q_1/n < 1$. Then $f_1 = j(2q_1/n) \in R_2\sigma_1(X)$, $n f_1 = \theta$ and $\rho(f_1) = 2q_1/n < 1$. Thus $n(Tf_1) = \theta$, and $\rho(Tf_1) = 2q_1/n$. Since $Y$ is compact, there exists $y_0 \in Y$ such that $j^{-1}((Tf_1)(y_0)) = \pm 2q_1/n$. But $n((Tf_1)(y_0)) = \theta$ so that $n j^{-1}((Tf_1)(y_0)) = 2mq_2$, where $m$ is an integer. Thus $q_1 = \pm m q_2$ and $q_1$ is an integer multiple of $q_2$. The exact same proof, using $T^{-1}$, gives that $q_2$ is an integer multiple of $q_1$. Since $q_1$ and $q_2$ are both positive we have that $q_1 = q_2$.

Now define the mapping $T^*: C(X) \rightarrow C(Y)$ by

$$
(T^*\sigma)(y) = \gamma [j^{-1}((1/\gamma) (\sigma))(y)]
$$

for $\gamma > |\sigma|$. It is easy to verify that $T^*$ is a uniquely defined, linear, norm-
preserving map of $C(X)$ onto $C(Y)$. By the Banach-Stone theorem [10], $X$ is homeomorphic to $Y$.

**Bibliography**


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