

SOME INTEGRALGEOMETRIC THEOREMS⁽¹⁾

BY

HERBERT FEDERER

1. Introduction. Assuming that A and B are analytic subsets of Euclidean n -space E_n , we consider for each isometry f of E_n the intersection $A \cap f(B)$. We prove (Theorem 5.7) that certain rectifiability properties of A in dimension k and of B in dimension l , where $k+l \geq n$, imply corresponding rectifiability properties of $A \cap f(B)$ in dimension $k+l-n$ for almost all f , and (Theorem 6.2) that in the presence of such rectifiability properties the integral over the group of isometries of the $k+l-n$ dimensional Hausdorff measure of $A \cap f(B)$ is proportional to the product of the k dimensional Hausdorff measure of A and the l dimensional Hausdorff measure of B .

2. Definitions. The notational conventions used in this paper are in general consistent with those of [F2], [F3], and [F4].

2.1 DEFINITION. If n is a positive integer, then E_n is the Euclidean space of dimension n , \mathcal{L}_n is the n dimensional Lebesgue measure over E_n , G_n is the group of all orthogonal transformations of E_n , and ϕ_n is the Haar measure over G_n such that $\phi_n(G_n) = 1$.

If $z \in E_n$, then T_z is the translation of E_n defined by the formula

$$T_z(x) = z + x \text{ for } x \in E_n.$$

If $a \in E_n$ and $r > 0$, then

$$K(a, r) = E_n \cap \{x \mid |x - a| < r\}$$

is the open sphere with center a and radius r .

2.2 DEFINITION. If μ is a measure over X and ν is a measure over Y , then the measure $\mu \otimes \nu$ over the cartesian product $X \times Y$ is defined as follows: For $P \subset X \times Y$, $(\mu \otimes \nu)(P)$ is the infimum of all numbers of the form

$$\sum_{i=1}^{\infty} \mu(A_i) \cdot \nu(B_i),$$

where A_1, A_2, A_3, \dots are μ measurable subsets of X ; B_1, B_2, B_3, \dots are ν measurable subsets of Y ; and

$$P \subset \bigcup_{i=1}^{\infty} A_i \times B_i.$$

[It is agreed here, as is usual in measure theory, that $0 \cdot \infty = \infty \cdot 0 = 0$.]

Presented to the Society, February 28, 1953; received by the editors June 5, 1953.

⁽¹⁾ This work was supported in part by a research grant from the National Science Foundation.

2.3 REMARK. There is a natural one-to-one correspondence between the group of isometries of E_n and the cartesian product space $E_n \times G_n$; in fact each isometry is uniquely representable in the form

$$T_z \circ R,$$

with $z \in E_n$ and $R \in G_n$. This correspondence associates with the measure $L_n \otimes \phi_n$ over $E_n \times G_n$ a certain measure over the group of isometries, which is readily seen to be a two-sided Haar measure. The integral of a function g with respect to this Haar measure equals

$$\int g(T_z \circ R) d(L_n \otimes \phi_n)(z, R).$$

2.4 DEFINITION. If $k < n$ are positive integers, then the functions

$$p_n^k: E_n \rightarrow E_k \quad \text{and} \quad \eta_n^k: E_k \rightarrow E_n$$

are defined by the formulae

$$\begin{aligned} p_n^k(x) &= (x_1, \dots, x_k) \in E_k && \text{for } x = (x_1, \dots, x_n) \in E_n, \\ \eta_n^k(y) &= (y_1, \dots, y_k, 0, \dots, 0) \in E_n && \text{for } y = (y_1, \dots, y_k) \in E_k. \end{aligned}$$

2.5 DEFINITION. If $m < n$ are positive integers, then Λ_n^m is the set of all m dimensional planes contained in E_n ,

$$Z_n^m = E_n \cap \{x \mid x_i = 0 \text{ for } i = 1, \dots, n - m\} \in \Lambda_n^m,$$

and the function

$$\lambda_n^m: G_n \times E_{n-m} \rightarrow \Lambda_n^m$$

is defined by the formula

$$\lambda_n^m(R, z) = (R \circ T_{\eta_n^{n-m}(z)})(Z_n^m) \text{ for } (R, z) \in G_n \times E_{n-m}.$$

2.6 REMARK. The function λ_n^m maps $G_n \times E_{n-m}$ onto Λ_n^m . Associating with each subset of Λ_n^m the $\phi_n \otimes \mathcal{L}_{n-m}$ measure of its counterimage under λ_n^m , we obtain a measure over Λ_n^m which is readily seen to be invariant under the transformations of Λ_n^m induced by the isometries of E_n . The integral of a function g with respect to this Haar measure equals

$$\int (g \circ \lambda_n^m) d(\phi_n \otimes \mathcal{L}_{n-m}).$$

2.7 REMARK. If $R \in G_n$ and $z \in E_{n-m}$, then

$$\lambda_n^m(R, z) = E_n \cap \{x \mid (p_n^{n-m} \circ \text{inv } R)(x) = z\}.$$

2.8 DEFINITION. In terms of the function Γ of classical analysis we define the functions α, β, γ by the formulae

$$\alpha(k) = \frac{2^k}{\Gamma(k+1)} \Gamma\left(\frac{1}{2}\right)^{k-1} \Gamma\left(\frac{k+1}{2}\right) \quad \text{for } k \geq 0,$$

$$\beta(n, k) = \frac{\alpha(k)\alpha(n-k)}{\alpha(n) \binom{n}{k}} = \frac{\Gamma\left(\frac{k+1}{2}\right) \Gamma\left(\frac{n-k+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{n+1}{2}\right)} \quad \text{for } n \geq k \geq 0,$$

$$\gamma(n, k, l) = \frac{\beta(n, k)\beta(n, l)}{\beta(n, k+l-n)\beta(2n-k-l, n-l)}$$

$$= \frac{\Gamma\left(\frac{k+1}{2}\right) \Gamma\left(\frac{l+1}{2}\right)}{\Gamma\left(\frac{k+l-n+1}{2}\right) \Gamma\left(\frac{n+1}{2}\right)}$$

for $n \geq k \geq 0, n \geq l \geq 0, k+l \geq n$.

2.9 DEFINITION. If X is a metric metrized by ρ , then

$$\text{diam}_\rho(S) = \sup_{x \in S, y \in S} \rho(x, y)$$

for any non-empty set $S \subset X$; if S is the empty set, then $\text{diam}_\rho(S) = 0$. Furthermore if $k \geq 0$ and $S \subset X$, then

$$\chi_\rho^k(S) = \alpha(k) 2^{-k} [\text{diam}_\rho(S)]^k.$$

If $A \subset X$, then the k dimensional Hausdorff measure

$$\mathfrak{H}_\rho^k(A)$$

is defined as follows: For each $r > 0$ consider the infimum of all numbers of the form

$$\sum_{S \in \mathcal{F}} \chi_\rho^k(S),$$

where \mathcal{F} is a countable covering of A with

$$\sup_{S \in \mathcal{F}} \text{diam}_\rho(S) \leq r.$$

As r decreases, this infimum does not decrease; its limit, as r approaches 0, equals $\mathfrak{H}_\rho^k(A)$.

We further agree that if $X = E_n$ and ρ is the usual metric of E_n , then \mathfrak{H}_ρ^k shall be designated by

$$\mathfrak{C}_n^k$$

2.10 REMARK. $\mathfrak{C}_\rho^0(A)$ equals the number (possibly ∞) of elements of A .

2.11 DEFINITION. A function f on E_k to E_n is *Lipschitzian* if and only if there exists an $M < \infty$ such that

$$|f(x) - f(x')| \leq M |x - x'| \text{ whenever } x \in E_k \text{ and } x' \in E_k.$$

2.12 DEFINITION. A subset A of E_n is *k rectifiable* if and only if there exists a Lipschitzian function on E_k to E_n which maps some bounded subset of E_k onto A .

2.13 DEFINITION. A subset A of E_n is *Hausdorff k rectifiable* if and only if

$$\mathfrak{C}_n^k(A) < \infty$$

and there exist k rectifiable subsets B_1, B_2, B_3, \dots of E_n such that

$$\mathfrak{C}_n^k\left(A - \bigcup_{i=1}^{\infty} B_i\right) = 0.$$

2.14 REMARK. A set is Hausdorff k rectifiable if and only if it is (\mathfrak{C}_n^k, k) rectifiable, as defined in [F3].

When discussing k rectifiable or Hausdorff k rectifiable subsets of E_n in this paper, we shall always make the tacit assumption that $k < n$.

2.15 DEFINITION. For each non-empty set $U \subset E_n$ we let d_U be the function on E_n such that

$$d_U(x) = \inf_{y \in U} |y - x|.$$

2.16 DEFINITION. Suppose $A \subset E_n, a \in E_n,$ and $U \in \Lambda_n^k$. Then we say that

A is restricted at a by U

if and only if $a \in U$ and for each $\epsilon > 0$ there is a $\delta > 0$ such that

$$A \cap K(a, \delta) \subset \{x \mid d_U(x) \leq \epsilon \mid x - a \mid \}.$$

2.17 REMARK. The preceding definition is closely related to Definition 2.28 of [F3]. Using the notation of [F3], we note that if $R \in G_n, a \in G_n$ and U is the k dimensional plane through a perpendicular to $\square_n^{n-k}(R, a)$, then

$$|P_R^{n-k}(x - a)| = d_U(x) \text{ for } x \in E_n,$$

and consequently

$$\diamond_n^{n-k}(R, \eta, a) = \{x \mid d_U(x) > (1 + \eta^2)^{-1/2} \mid x - a \mid \}$$

whenever $0 < \eta < \infty$. Taking account of this connection and applying Theorems 3.7, 4.7, 5.1, 5.2, 5.7 of [F3] together with the proposition that if f is a Lipschitzian function on E_k to E_n then \mathcal{L}_k almost all of the set of those points

of E_k at which f has a nonsingular differential can be represented as a countable union of compact sets on each of which f is univalent, one readily obtains the following two results:

(1) If $\mathfrak{H}_n^k(A) < \infty$ and if for \mathfrak{H}_n^k almost all a in A it is true that A is restricted at a by some k dimensional plane, then A is Hausdorff k rectifiable.

(2) If A is a Hausdorff k rectifiable subset of E_n , then there are compact k rectifiable subsets $A_1, A_2, A_3 \dots$ of E_n such that

$$\mathfrak{H}_n^k\left(A - \bigcup_{i=1}^{\infty} A_i\right) = 0$$

and such that for each positive integer i and each point $a \in A_i$ it is true that A_i is restricted at a by some k dimensional plane.

2.18 DEFINITION. If μ is a measure over X and g is a function on X to the extended real number system, then

$$\int^* g d\mu$$

is the upper Lebesgue integral of g with respect to μ ; it equals the infimum of

$$\int h d\mu,$$

where h is a μ integrable function such that

$$g(x) \leq h(x) \leq \infty \text{ for } x \in X.$$

3. Some inequalities involving Hausdorff measure. The central result of this section is Theorem 3.2. Since this proposition appears to be of basic interest for the theory of Hausdorff measure, it is formulated and proved in somewhat greater generality than the applications in the present paper require. For the immediate purpose it would be sufficient to consider only the special case in which $Z = E_u$; then Lemma 3.1 could be replaced by the isodiametric inequality.

If the space Y in Theorem 3.2 consists of a single point and $\psi(Y) = 1$, then $X \times Y$, $\mathfrak{H}_p^{u+v} \otimes \psi$, and $\mathfrak{H}_r^u \otimes \psi$ may obviously be replaced by X , \mathfrak{H}_p^{u+v} , and \mathfrak{H}_r^u respectively; thus f becomes a function on X to Z with Lipschitz constant M , P becomes a subset of X , and there results the inequality

$$\int^* \mathfrak{H}_p^v(\{x \mid x \in P \text{ and } f(x) = z\}) d\mathfrak{H}_r^u z \leq M^u \frac{\alpha(u)\alpha(v)}{\alpha(u+v)} \mathfrak{H}_p^{u+v}(P).$$

Two special cases of this inequality have been established previously:

Eilenberg treated in [E] the case in which $u = 1$, $Z = E_1$ and, for all $x \in X$, $f(x)$ equals the distance between x and a fixed subset of X .

Besicovitch and Moran considered in Theorem 1 of [BM] the case in

which $X = E_2, Z = E_1, f = p_1^2$ and the subset P of E_2 is the cartesian product of two subsets A and B of E_1 [when convenient we identify $E_m \times E_n$ and E_{m+n}] with $\mathfrak{C}_1^u(A) < \infty$ and $\mathfrak{C}_1^v(B) < \infty$; here the inequality reduces to

$$\mathfrak{C}_1^u(A) \mathfrak{C}_1^v(B) \leq \frac{\alpha(u)\alpha(v)}{\alpha(u+v)} \mathfrak{C}_2^{u+v}(A \times B).$$

3.1 LEMMA. Suppose τ metrizes $Z, A \subset Z, u \geq 0$, and

$$\mathfrak{C}_\tau^u(A) < \infty.$$

For $s > 1$ and $r > 0$, let $B(s, r)$ be the set of all those points z of Z such that

$$\mathfrak{C}_\tau^u(A \cap W) \leq s\chi_\tau^u(W)$$

whenever $z \in W \subset Z$ and $\text{diam}_\tau(W) < r$. Then:

- (1) For $s > 1$ and $r > 0, B(s, r)$ is a closed subset of Z .
- (2) $\mathfrak{C}_\tau^u[A - \bigcap_{s>1} \bigcup_{r>0} B(s, r)] = 0$.

The first of these two statements is subject to routine verification, while the second can be proved very much like Theorem 3.5 of [F3]. We omit the details.

3.2 THEOREM. If

- (i) X, Y, Z are separable and complete metric spaces, metrized by ρ, σ, τ respectively,
- (ii) f is a Baire function on $X \times Y$ to $Z, 0 < M < \infty$, and

$$\tau[f(x, y), f(x', y)] \leq M\rho(x, x')$$

whenever, $x \in X, x' \in X, y \in Y$,

(iii) ψ is such a measure over Y that all Borel subsets of Y are ψ measurable and every ψ measurable set is contained in a Borel set of equal ψ measure,

(iv) $u \geq 0, v \geq 0, P \subset X \times Y$ and $f(P)$ is the union of countably many sets of finite \mathfrak{C}_τ^u measure.

Then

$$\begin{aligned} & \int_{Z \times Y}^* \mathfrak{C}_\rho^v(\{x \mid (x, y) \in P \text{ and } f(x, y) = z\}) d(\mathfrak{C}_\tau^u \otimes \psi)(z, y) \\ & \leq M^u \frac{\alpha(u)\alpha(v)}{\alpha(u+v)} (\mathfrak{C}_\rho^{u+v} \otimes \psi)(P). \end{aligned}$$

Proof. We let

$$g = M^u \frac{\alpha(u)\alpha(v)}{\alpha(u+v)}$$

and suppose that $s > 1$.

Since $f(P)$ is the union of countably many sets of finite \mathfrak{C}_r^u measure, Lemma 3.1 enables us to construct a sequence of disjoint Borel sets Z_1, Z_2, Z_3, \dots of finite \mathfrak{C}_r^u measure and a sequence of positive numbers r_1, r_2, r_3, \dots such that

$$\mathfrak{C}_r^u \left[f(P) - \bigcup_{i=1}^{\infty} Z_i \right] = 0$$

and such that for each positive integer i the inequality

$$\mathfrak{C}_r^u(W) \leq s\chi_r^u(W)$$

holds whenever $W \subset Z_i$ and $\text{diam}_r(W) < r_i$.

Associating with each set $Q \subset X \times Y$ the function g_Q on $Z \times Y$ by the formula

$$g_Q(z, y) = \mathfrak{C}_\rho^v(\{x \mid (x, y) \in Q \text{ and } f(x, y) = z\})$$

for $(z, y) \in Z \times Y$, we divide the remainder of the argument into four parts.

Part 1. If i is a positive integer, F is a countable family of closed subsets of X such that

$$\sup_{C \in F} \text{diam}_\rho(C) < r_i/M,$$

h is the function on $Z \times Y$ such that

$$h(z, y) = \sum_{C \in F, z \in f(C \times \{y\})} \chi_\rho^v(C)$$

for $(z, y) \in Z \times Y$, and B is a Borel subset of Y , then

$$\int_{Z \times B} h d(\mathfrak{C}_r^u \otimes \psi) \leq sq \sum_{C \in F} \chi_\rho^{u+v}(C) \psi(B).$$

Proof. For each $C \in F$ we define

$$H_C = \{(z, y) \mid z \in f(C \times \{y\})\}$$

and let h_C be the characteristic function of H_C . Noting that the natural projection of $X \times Y \times Z$ onto $Z \times Y$ maps the Borel set

$$\{(x, y, z) \mid x \in C \text{ and } f(x, y) = z\}$$

onto H_C , we find that H_C is an analytic set and hence that h_C is an $\mathfrak{C}_r^u \otimes \psi$ measurable function.

Next we observe that

$$h = \sum_{C \in F} \chi_\rho^v(C) h_C.$$

Thus h is a non-negative $\mathfrak{C}_r^u \otimes \psi$ measurable function, and it will be sufficient to show that

$$\chi_\rho^v(C) \int_{Z_i \times B} h_C d(\mathfrak{C}_r^u \otimes \psi) \leq sq \chi_\rho^{u+v}(C) \psi(B)$$

whenever $C \in F$.

For this purpose we henceforth fix $C \in F$ and assume that $\psi(B) < \infty$. Inasmuch as $\mathfrak{C}_r^u(Z_i) < \infty$, the Fubini Theorem implies that

$$\begin{aligned} \int_{Z_i \times B} h_C d(\mathfrak{C}_r^u \otimes \psi) &= \int_B \int_{Z_i} h_C(z, y) d\mathfrak{C}_r^u z d\psi y \\ &= \int_B \mathfrak{C}_r^u [Z_i \cap f(C \times \{y\})] d\psi y. \end{aligned}$$

Furthermore if $y \in B$ then

$$\text{diam}_r [f(C \times \{y\})] \leq M \text{diam}_\rho(C) < r_i,$$

and consequently

$$\mathfrak{C}_r^u [Z_i \cap f(C \times \{y\})] \leq s \chi_r^u [f(C \times \{y\})] \leq s M^u \chi_\rho^u(C).$$

We conclude that

$$\begin{aligned} \chi_\rho^v(C) \int_{Z_i \times B} h_C d(\mathfrak{C}_r^u \otimes \psi) &\leq \chi_\rho^v(C) s M^u \chi_\rho^u(C) \psi(B) \\ &= sq \chi_\rho^{u+v}(C) \psi(B). \end{aligned}$$

Part 2. If i is a positive integer, A is a Borel subset of X , and B is a Borel subset of Y , then

$$\int_{Z_i \times Y}^* g_{A \times B} d(\mathfrak{C}_r^u \otimes \psi) \leq sq \mathfrak{C}_\rho^{u+v}(A) \psi(B).$$

Proof. We consider a sequence of countable closed coverings F_1, F_2, F_3, \dots of A such that

$$\begin{aligned} \sup_{C \in F_j} \text{diam}_\rho(C) &< r_i/M \text{ for } j = 1, 2, 3, \dots, \\ \lim_{j \rightarrow \infty} \sup_{C \in F_j} \text{diam}_\rho(C) &= 0, \\ \lim_{j \rightarrow \infty} \sum_{C \in F_j} \chi_\rho^{u+v}(C) &= \mathfrak{C}_\rho^{u+v}(A). \end{aligned}$$

Associating with each covering F_j the function h_j on $Z \times Y$ such that

$$h_j(z, y) = \sum_{C \in F_j, z \in f(C \times \{y\})} \chi_\rho^v(C)$$

for $(z, y) \in Z \times Y$, we infer from Fatou's Lemma and Part 1 that

$$\int_{Z_i \times B} \liminf_{j \rightarrow \infty} h_j(z, y) d(\mathcal{H}_\tau^u \otimes \psi)(z, y) \leq \liminf_{j \rightarrow \infty} \int_{Z_i \times B} h_j d(\mathcal{H}_\tau^u \otimes \psi) \leq sq \mathcal{H}_\rho^{u+v}(A) \psi(B).$$

Finally we note that if $(z, y) \in Z_i \times Y$ and

$$g_{A \times B}(z, y) \neq 0,$$

then $y \in B$ and

$$\{x \mid (x, y) \in A \times B \text{ and } f(x, y) = z\} = A \cap \{x \mid f(x, y) = z\} \\ \subset \bigcup_{C \in \mathcal{F}_j} C \cap \{x \mid f(x, y) = z\} = \bigcup_{C \in \mathcal{F}_j, z \in f(C \times \{y\})} C$$

for every positive integer j , hence

$$g_{A \times B}(z, y) \leq \liminf_{j \rightarrow \infty} h_j(z, y).$$

Part 3. If i is a positive integer and $Q \subset X \times Y$, then

$$\int_{Z_i \times Y}^* g_Q d(\mathcal{H}_\tau^u \otimes \psi) \leq sq (\mathcal{H}_\rho^{u+v} \otimes \psi)(Q).$$

Proof. Let $\epsilon > 0$. We choose Borel subsets A_1, A_2, A_3, \dots of X and Borel subsets B_1, B_2, B_3, \dots of Y such that

$$Q \subset \bigcup_{j=1}^{\infty} A_j \times B_j$$

and

$$\sum_{j=1}^{\infty} \mathcal{H}_\rho^{u+v}(A_j) \psi(B_j) \leq (\mathcal{H}_\rho^{u+v} \otimes \psi)(Q) + \epsilon.$$

Noting that

$$g_Q(z, y) \leq \sum_{j=1}^{\infty} g_{A_j \times B_j}(z, y) \text{ for } (z, y) \in (Z \times Y),$$

we use Part 2 to infer that

$$\int_{Z_i \times Y}^* g_Q d(\mathcal{H}_\tau^u \otimes \psi) \leq \sum_{j=1}^{\infty} \int_{Z_i \times Y}^* g_{A_j \times B_j} d(\mathcal{H}_\tau^u \otimes \psi) \\ \leq \sum_{j=1}^{\infty} sq \mathcal{H}_\rho^{u+v}(A_j) \psi(B_j) \leq sq [(\mathcal{H}_\rho^{u+v} \otimes \psi)(Q) + \epsilon].$$

Part 4. $\int^* g_P(\mathcal{H}_r^u \otimes \psi) \leq sq(\mathcal{H}_p^{u+v} \otimes \psi)(P)$.

Proof. Letting

$$T = f(P) - \bigcup_{i=1}^{\infty} Z_i,$$

we have $\mathcal{H}_r^u(T) = 0$ and consequently

$$(\mathcal{H}_r^u \otimes \psi)(T \times Y) = 0.$$

Inasmuch as

$$\{(z, y) \mid g_P(z, y) \neq 0\} \subset [f(P) \times Y] \subset (T \times Y) \cup \bigcup_{i=1}^{\infty} (Z_i \times Y),$$

we infer that

$$\int_{Z \times Y}^* g_P d(\mathcal{H}_r^u \otimes \psi) \leq \sum_{i=1}^{\infty} \int_{Z_i \times Y}^* g_P d(\mathcal{H}_r^u \otimes \psi).$$

Considering next, for each positive integer i , the set

$$R_i = \{(x, y) \mid f(x, y) \in Z_i\},$$

we see that

$$g_P(z, y) = g_{P \cap R_i}(z, y) \text{ whenever } (z, y) \in Z_i \times Y,$$

and use Part 3 to obtain

$$\int_{Z_i \times Y}^* g_P d(\mathcal{H}_r^u \otimes \psi) = \int_{Z_i \times Y}^* g_{P \cap R_i} d(\mathcal{H}_r^u \otimes \psi) \leq sq(\mathcal{H}_p^{u+v} \otimes \psi)(P \cap R_i).$$

The statement of Part 4 now follows from the inequality

$$\sum_{i=1}^n (\mathcal{H}_p^{u+v} \otimes \psi)(P \cap R_i) \leq (\mathcal{H}_p^{u+v} \otimes \psi)(P),$$

which is true because R_1, R_2, R_3, \dots are disjoint Borel subsets of $X \times Y$.

Since s is an arbitrary number greater than 1, the theorem is an immediate consequence of Part 4.

3.3 THEOREM. If $u < n$ are positive integers, $k \geq u$, and $P \subset E_n \times G_n$, then

$$\begin{aligned} \int_{E_u \times G_n}^* \mathcal{H}_n^{k-u} [\{a \mid (a, R) \in P\} \cap \lambda_n^{n-u}(R, z)] d(\mathcal{L}_u \otimes \phi_n)(z, R) \\ \leq \frac{\alpha(u)\alpha(k-u)}{\alpha(k)} (\mathcal{H}_n^k \otimes \phi_n)(P). \end{aligned}$$

Proof. Letting f be the function on $E_n \times G_n$ to E_u such that

$$f(a, R) = (p_n^u \circ \text{inv } R)(a)$$

for $(a, R) \in E_n \times G_n$, we see that f is a continuous function and that

$$|f(a, R) - f(a', R)| \leq |a - a'|$$

whenever $a \in E_n, a' \in E_n$, and $R \in G_n$. Accordingly the preceding theorem (applied with $v = k - u$) yields the inequality

$$\begin{aligned} \int^* \mathcal{H}_n^{k-u}(\{a \mid (a, R) \in P \text{ and } (p_n^u \circ \text{inv } R)(a) = z\}) d(\mathcal{L}_u \otimes \phi_n)(z, R) \\ \leq \frac{\alpha(u)\alpha(k-u)}{\alpha(k)} (\mathcal{H}_n^k \otimes \phi_n)(P), \end{aligned}$$

and reference to 2.7 completes the proof.

3.4 COROLLARY. *If $u < n$ are positive integers, $k \geq u$, and $A \subset E_n$, then*

$$\int_{E_u \times G_n}^* \mathcal{H}_n^{k-u}[A \cap \lambda_n^{n-u}(R, z)] d(\mathcal{L}_u \otimes \phi_n)(z, R) \leq \frac{\alpha(u)\alpha(k-u)}{\alpha(k)} \mathcal{H}_n^k(A).$$

Proof. We apply the preceding theorem with

$$P = A \times G_n,$$

and note that

$$(\mathcal{H}_n^k \otimes \phi_n)(P) = \mathcal{H}_n^k(A)\phi_n(G_n) = \mathcal{H}_n^k(A).$$

3.5 THEOREM. *If $v \geq 0$ and*

$$P \subset E_n \times E_n \times G_n = E_{2n} \times G_n,$$

then

$$\begin{aligned} \int_{E_n \times G_n}^* \mathcal{H}_n^v(\{a \mid (a, (\text{inv } R)(a - z), R) \in P\}) d(\mathcal{L}_n \otimes \phi_n)(z, R) \\ \leq 2^{n/2} \frac{\alpha(n)\alpha(v)}{\alpha(n+v)} (\mathcal{H}_{2n}^{n+v} \otimes \phi_n)(P). \end{aligned}$$

Proof. Let f be the function on $E_n \times E_n \times G_n$ to E_n such that

$$f(a, b, R) = a - R(b) \text{ for } (a, b, R) \in E_n \times E_n \times G_n.$$

Clearly f is a continuous function and

$$\begin{aligned} |f(a, b, R) - f(a', b', R)| &= |a - a' - R(b - b')| \\ &\leq |a - a'| + |b - b'| \leq 2^{1/2} |(a, b) - (a', b')| \end{aligned}$$

whenever a, b, a', b' are points of E_n and $R \in G_n$.

Applying Theorem 3.2 with $X = E_{2n}, Y = G_n, Z = E_n, M = 2^{1/2}, u = n$, and

$\psi = \phi_n$ we obtain the inequality

$$\int^* \mathfrak{H}_{2n}^v(\{(a, b) \mid (a, b, R) \in P \text{ and } a - R(b) = z\}) d(\mathcal{L}_n \otimes \phi_n)(z, R) \leq 2^{n/2} \frac{\alpha(u)\alpha(v)}{\alpha(u+v)} (\mathfrak{H}_{2n}^{n+v} \otimes \phi_n)(P).$$

Finally we consider a fixed point $(z, R) \in E_n \times G_n$ and note that

$$\begin{aligned} \{(a, b) \mid (a, b, R) \in P \text{ and } a - R(b) = z\} &= \{(a, b) \mid (a, b, R) \in P \text{ and } b = (\text{inv } R)(a - z)\} \\ &= \{(a, (\text{inv } R)(a - z)) \mid (a, (\text{inv } R)(a - z), R) \in P\}; \end{aligned}$$

hence this subset of E_{2n} is mapped by the projection p_n^{2n} onto the subset

$$\{a \mid (a, (\text{inv } R)(a - z), R) \in P\}$$

of E_n and consequently

$$\begin{aligned} \mathfrak{H}_n^v(\{a \mid (a, (\text{inv } R)(a - z), R) \in P\}) &\leq \mathfrak{H}_{2n}^v(\{(a, b) \mid (a, b, R) \in P \text{ and } a - R(b) = z\}). \end{aligned}$$

3.6 COROLLARY. *If $A \subset E_n, B \subset E_n, k \geq 0, l \geq 0$, and $k+l \geq n$, then*

$$\begin{aligned} \int_{E_n \times G_n}^* \mathfrak{H}_n^{k+l-n}[A \cap (T_z \circ R)(B)] d(\mathcal{L}_n \otimes \phi_n)(z, R) &\leq 2^{n/2} \frac{\alpha(n)\alpha(k+l-n)}{\alpha(k+l)} \mathfrak{H}_{2n}^{k+l}(A \times B). \end{aligned}$$

Proof. We apply Theorem 4.3 with $v = k+l-n$ and

$$P = A \times B \times G_n.$$

Since $\phi_n(G_n) = 1$ we have

$$(\mathfrak{H}_{2n}^{k+l} \otimes \phi_n)(P) = \mathfrak{H}_{2n}^{k+l}(A \times B).$$

Furthermore if $(z, R) \in E_n \times G_n$, then

$$\begin{aligned} \{a \mid (a, (\text{inv } R)(a - z), R) \in P\} &= A \cap \{a \mid (\text{inv } R)(a - z) \in B\} \\ &= A \cap \{a \mid a - z \in R(B)\} \\ &= A \cap (T_z \circ R)(B). \end{aligned}$$

4. On certain cartesian products. It is known (see [BM]) that the $k+l$ dimensional Hausdorff measure of a cartesian product $A \times B$ can be infinite even though both the k dimensional measure of A and the l dimensional measure of B are finite. Here it will be shown that this pathological

situation never occurs in case B is l rectifiable. It will also be proved that if A has k dimensional measure zero and B is l rectifiable, then $A \times B$ has $k+l$ dimensional measure zero.

4.1 LEMMA. *If $k > 0$, $A \subset E_n$, and $B \subset E_l$, then*

$$\mathfrak{C}_{n+l}^{k+l}(A \times B) \leq (l + 1)^{(k+l)/2} 2^{-l} \frac{\alpha(k + l)}{\alpha(k)} \mathfrak{C}_n^k(A) \mathfrak{C}_l^1(B).$$

Proof. It is easy to reduce the general statement to the special case in which B is an l dimensional cube. We henceforth assume that B is a cube with side length b . Letting

$$c = (l + 1)^{(k+l)/2} 2^{-l} \frac{\alpha(k + l)}{\alpha(k)},$$

we divide the argument into two parts.

Part 1. If $S \subset E_n$, then $S \times B$ has a finite covering H such that

$$\text{diam}(T) = (l + 1)^{1/2} \text{diam}(S) \text{ for } T \in H$$

and

$$\sum_{T \in H} \chi_{n+l}^{k+l}(T) \leq c \chi_n^k(S) [b + \text{diam}(S)]^l.$$

Proof. Let $r = \text{diam}(S)$ and let m be the least integer such that $rm \geq b$. Then

$$r(m - 1) < b, \quad rm < b + r,$$

and B can be covered by m^l cubes with side length r . Forming the cartesian product of S with each of these cubes, we obtain a covering H of $S \times B$ such that H has m^l elements and the diameter of each element of H is equal to

$$(l + 1)^{1/2} r.$$

It follows that

$$\begin{aligned} \sum_{T \in H} \chi_{n+l}^{k+l}(T) &= m^l \alpha(k + l) 2^{-(k+l)} (l + 1)^{k+l/2} r^{k+l} \\ &= c \alpha(k) 2^{-k} r^k (rm)^l \leq c \chi_n^k(S) (b + r)^l. \end{aligned}$$

Part 2. If $\epsilon > 0$ and if F is a countable covering of A such that

$$\sup_{S \in F} \text{diam}(S) \leq \epsilon,$$

then $A \times B$ has a countable covering G such that

$$\sup_{T \in G} \text{diam}(T) \leq (l + 1)^{1/2} \epsilon$$

and

$$\sum_{T \in G} \chi_{n+l}^{k+l}(T) \leq c \sum_{S \in F} \chi_n^k(S)(b + \epsilon)^l.$$

Proof. Using Part 1 we associate with each $S \in F$ a covering H_S of $S \times B$ such that

$$\text{diam}(T) \leq (l + 1)^{1/2}\epsilon \text{ for } T \in H_S$$

and

$$\sum_{T \in H_S} \chi_{n+l}^{k+l}(T) \leq c \chi_n^k(S)(b + \epsilon)^l.$$

Then the family

$$G = \bigcup_{S \in F} H_S$$

is a suitable covering of $A \times B$.

Noting that $\mathcal{L}_l(B) = b^l$, we see that the inequality

$$\mathfrak{H}_{n+l}^{k+l}(A \times B) \leq c \mathfrak{H}_n^k(A) \mathcal{L}_l(B)$$

is an immediate consequence of Part 2.

4.2 THEOREM. *If $A \subset E_n$, $k > 0$, and B is an l rectifiable subset of E_p , then*

- (1) $\mathfrak{H}_n^k(A) < \infty$ implies that $\mathfrak{H}_{n+p}^{k+l}(A \times B) < \infty$,
- (2) $\mathfrak{H}_n^k(A) = 0$ implies that $\mathfrak{H}_{n+p}^{k+l}(A \times B) = 0$.

Proof. Choosing a bounded set $C \subset E_l$ and a Lipschitzian function f which maps C onto B , we consider the function g on $A \times C$ defined by the formula

$$g(x, y) = (x, f(y)) \text{ for } (x, y) \in A \times C.$$

Obviously the range of g is $A \times B$. Furthermore if M is a Lipschitz constant for f , then M is also a Lipschitz constant for g and consequently

$$\begin{aligned} \mathfrak{H}_{n+p}^{k+l}(A \times B) &\leq M^{k+l} \mathfrak{H}_{n+l}^{k+l}(A \times C) \\ &\leq M^{k+l} (l + 1)^{(k+l)/2} 2^{-l} \frac{\alpha(k + l)}{\alpha(k)} \mathfrak{H}_n^k(A) \mathcal{L}_l(C). \end{aligned}$$

From this inequality the statements (1) and (2) follow immediately.

5. The rectifiability of intersections. In this section the connection between rectifiability and restrictedness is used to prove that a Hausdorff k rectifiable subset of E_n has Hausdorff $k - u$ rectifiable intersections with almost all $n - u$ dimensional planes, and also that a Hausdorff k rectifiable

subset of E_n has Hausdorff $k+l-n$ rectifiable intersections with almost all isometric images of an l rectifiable set.

5.1 LEMMA. *If $X \subset E_n$, $Y \subset E_n$, θ is the origin of E_n , U and V are vector subspaces of E_n , X is restricted at θ by U , and Y is restricted at θ by V , then $X \cap Y$ is restricted at θ by $U \cap V$.*

Proof. Let $\epsilon > 0$.

The non-negative continuous function $d_U + d_V$ vanishes only on $U \cap V$, and consequently has a positive lower bound λ on the compact set

$$C = E_n \cap \{x \mid |x| = 1 \text{ and } d_{U \cap V}(x) \geq \epsilon\}.$$

Let $0 < \mu < \lambda/2$. If $\theta \neq x \in E_n$, then the condition

$$d_U(x) \leq \mu |x| \quad \text{and} \quad d_V(x) \leq \mu |x|$$

implies that

$$\begin{aligned} d_U\left(\frac{x}{|x|}\right) + d_V\left(\frac{x}{|x|}\right) &< \lambda, \\ \frac{x}{|x|} &\notin C, \quad d_{U \cap V}\left(\frac{x}{|x|}\right) < \epsilon, \\ d_{U \cap V}(x) &< \epsilon |x|. \end{aligned}$$

Choosing a positive number δ for which

$$X \cap K(\theta, \delta) \subset \{x \mid d_U(x) \leq \mu |x|\}$$

and

$$Y \cap K(\theta, \delta) \subset \{x \mid d_V(x) \leq \mu |x|\},$$

we conclude that

$$\begin{aligned} X \cap Y \cap K(\theta, \delta) &\subset \{x \mid d_U(x) \leq \mu |x| \text{ and } d_V(x) \leq \mu |x|\} \\ &\subset \{x \mid d_{U \cap V}(x) \leq \epsilon |x|\}. \end{aligned}$$

5.2 LEMMA. *If U and V are vector subspaces of E_n and*

$$\dim(U) + \dim(V) \geq n,$$

then

$$\dim[U \cap R(V)] = \dim(U) + \dim(V) - n$$

for ϕ_n almost all R in G_n .

Proof. We note that the set

$$G_n \cap \{R \mid \dim[U \cap R(V)] \neq \dim(U) + \dim(V) - n\}$$

contains neither component of G_n , and is equal to the set of all common zeros of certain determinants which are analytic functions on the Lie group G_n . (See [F4, p. 540].)

5.3 LEMMA. *If A is a compact subset of E_n , B is a closed subset of E_n , m is a positive integer less than n , and C is the set of all those triples*

$$(a, z, R) \in E_n \times E_n \times G_n$$

for which the set

$$A \cap (T_z \circ R)(B)$$

is restricted to a by some m dimensional plane, then C is an analytic set.

Proof. For any positive integers i, j, k we let

$$P_{i,j,k}$$

be the set of all those quadruples

$$(a, z, R, U) \in E_n \times E_n \times G_n \times \Lambda_n^m$$

for which

$$A \cap (T_z \circ R)(B) \cap \{x \mid k^{-1} \leq |x - a| \leq j^{-1}\} \subset \{x \mid d_U(x) < i^{-1} |x - a|\}.$$

We also define

$$S = (E_n \times E_n \times G_n \times \Lambda_n^m) \cap \{(a, z, R, U) \mid a \in U\}$$

and observe that S is a closed set. Inasmuch as

$$S \cap \bigcap_{i=1}^{\infty} \bigcup_{j=1}^{\infty} \bigcap_{k=1}^{\infty} P_{i,j,k}$$

is equal to the set of all quadruples (a, z, R, U) such that

$$A \cap (T_z \circ R)(B) \text{ is restricted to } a \text{ by } U,$$

it will be sufficient to show that each set $P_{i,j,k}$ is open.

For this purpose we note that the elements of the set

$$Q_{i,j,k} = (E_n \times E_n \times G_n \times \Lambda_n^m) - P_{i,j,k}$$

are those quadruples (a, z, R, U) for which there exists a point x satisfying the four conditions

$$x \in A, \quad x \in (T_z \circ R)(B), \quad k^{-1} \leq |x - a| \leq j^{-1}, \quad d_U(x) \geq i^{-1} |x - a|.$$

Thus $Q_{i,j,k}$ is the projection image of a closed subset of the cartesian product space

$$E_n \times E_n \times G_n \times \Lambda_n^m \times A.$$

Since A is compact it follows that $Q_{i,j,k}$ is closed and consequently that $P_{i,j,k}$ is open.

5.4 LEMMA. *If $A \subset E_n$, $a \in E_n$, A is restricted at a by some k dimensional plane, and u is a positive integer less than k , then for ϕ_n almost all R in G_n the set*

$$A \cap \lambda_n^{n-u} [R, (\phi_n^u \circ \text{inv } R)(a)]$$

is restricted at a by some $k-u$ dimensional plane.

Proof. Let $X = T_{-a}(A)$. Since restrictedness is invariant under translations, the set X is restricted at the origin θ of E_n by a k dimensional vector subspace U of E_n . For each $R \in G_n$ we infer from Lemma 5.1, applied with $Y = V = R(Z_n^{n-u})$, that

$$X \cap R(Z_n^{n-u}) \text{ is restricted at } \theta \text{ by } U \cap R(Z_n^{n-u}).$$

Furthermore we know from Lemma 5.2 that for ϕ_n almost all R the equations

$$\dim [U \cap R(Z_n^{n-u})] = k + (n - u) - n = k - u$$

is valid, so that $X \cap R(Z_n^{n-u})$ is restricted at θ by some $k-u$ dimensional plane. Finally we observe (see [F3, p. 539]) that if $R \in G_n$ then

$$T_a[X \cap R(Z_n^{n-u})] = A \cap \lambda_n^{n-u} [R, (\phi_n^u \circ \text{inv } R)(a)].$$

5.5 THEOREM. *If A is a Hausdorff k rectifiable subset of E_n and u is a positive integer less than k , then*

$$A \cap \lambda_n^{n-u}(R, z) \text{ is Hausdorff } k - u \text{ rectifiable}$$

for $\mathcal{L}_u \otimes \phi_n$ almost all (z, R) in $E_u \times G_n$.

Proof. First we use Corollary 3.4 to infer that

$$\mathfrak{H}_n^{k-u} [A \cap \lambda_n^{n-u}(R, z)] < \infty$$

for $\mathcal{L}_u \otimes \phi_n$ almost all (z, R) .

Next we choose A_1, A_2, A_3, \dots according to the proposition (2) of Remark 2.17 and let

$$A_0 = A - \bigcup_{i=1}^{\infty} A_i.$$

Applying Corollary 3.4 once more, we find that

$$\mathfrak{H}_n^{k-u} [A_0 \cap \lambda_n^{n-u}(R, z)] = 0$$

for $\mathcal{L}_u \otimes \phi_n$ almost all (z, R) . To complete the proof we need therefore only show that if i is a positive integer then

$A \cap A_i \cap \lambda_n^{n-u}(R, z)$ is Hausdorff $k - u$ rectifiable

for $\mathcal{L}_u \otimes \phi_n$ almost all (z, R) .

We henceforth consider a fixed positive integer i and let Q be the set of all those ordered pairs

$$(a, R) \in A_i \times G_n$$

for which the set

$$A_i \cap \lambda_n^{n-u}[R, (p_n^u \circ \text{inv } R)(a)] = A_i \cap (T_a \circ R)(Z_n^{n-u})$$

is restricted at a by some $k - u$ dimensional plane.

Since Lemma 5.3 implies that Q is an analytic subset of $A_i \times G_n$, we can use the Fubini Theorem in conjunction with Lemma 5.4 to obtain

$$(\mathcal{H}_n^k \otimes \phi_n)[(A_i \times G_n) - Q] = 0,$$

whence it follows by Theorem 3.3 that

$$\mathcal{H}_n^{k-u}[\{a \mid (a, R) \in (A_i \times G_n) - Q\} \cap \lambda_n^{n-u}(R, z)] = 0$$

for $\mathcal{L}_u \otimes \phi_n$ almost all (z, R) .

Now suppose (z, R) is a point of $E_u \times G_n$ for which both

$$\mathcal{H}_n^{k-u}[A \cap \lambda_n^{n-u}(R, z)] < \infty$$

and

$$\mathcal{H}_n^{k-u}[\{a \mid (a, R) \in (A_i \times G_n) - Q\} \cap \lambda_n^{n-u}(R, z)] = 0,$$

and let

$$S = A \cap A_i \cap \lambda_n^{n-u}(R, z).$$

Then it is true for \mathcal{H}_n^{k-u} almost all a in S that $(a, R) \in Q$; inasmuch as

$$(p_n^u \circ \text{inv } R)(a) = z,$$

it follows from the definition of Q that S is restricted at a by some $k - u$ dimensional plane. Applying the proposition (1) of Remark 2.17 with A and k replaced by S and $k - u$ respectively, we conclude that the set S is Hausdorff $k - u$ rectifiable.

5.6 LEMMA. *If $A \subset E_n, B \subset E_n, a \in E_n, b \in E_n, A$ is restricted at a by some k dimensional plane, B is restricted at b by some l dimensional plane, and $k + l > n$, then for ϕ_n almost all R in G_n the set*

$$A \cap [T_{a-R(b)} \circ R](B)$$

is restricted at a by some $k + l - n$ dimensional plane.

Proof. Letting

$$X = T_{-a}(A), \quad Y = T_{-b}(B),$$

we see that X is restricted at the origin θ of E_n by a k dimensional vector subspace U of E_n and that Y is restricted at θ by an l dimensional vector subspace V of E_n .

If $R \in G_n$, then $R(Y)$ is restricted at θ by $R(V)$ and, according to Lemma 5.1, $X \cap R(Y)$ is restricted at θ by $U \cap R(V)$; inasmuch as

$$T_{a-R(b)} \circ R = T_a \circ R \circ T_{-b},$$

the set

$$A \cap [T_{a-R(b)} \circ R](B) = T_a[X \cap R(Y)]$$

is restricted at a by $T_a[U \cap R(V)]$.

Reference to Lemma 5.2 completes the proof.

5.7 THEOREM. *If A is a Hausdorff k rectifiable subset of E_n , B is an l rectifiable subset of E_n , and $k+l > n$, then*

$$A \cap (T_z \circ R)(B) \text{ is Hausdorff } k + l - n \text{ rectifiable}$$

for $\mathcal{L}_n \otimes \phi_n$ almost all (z, R) in $E_n \times G_n$.

Proof. First we use the statement (1) of Theorem 4.2 to obtain the inequality

$$\mathfrak{H}_{2n}^{k+l}(A \times B) < \infty,$$

and infer from Corollary 3.6 that

$$\mathfrak{H}_n^{k+l-n}[A \cap (T_z \circ R)(B)] < \infty$$

for $\mathcal{L}_n \otimes \phi_n$ almost all (z, R) .

Next we choose A_1, A_2, A_3, \dots according to the proposition (2) of Remark 2.17, and similarly select compact l rectifiable subsets B_1, B_2, B_3, \dots of E_n such that

$$\mathfrak{H}_n^l\left(B - \bigcup_{j=1}^{\infty} B_j\right) = 0$$

and such that for each positive integer j and each point $b \in B_j$ it is true that B_j is restricted at b by some l dimensional plane. Letting

$$A_0 = A - \bigcup_{i=1}^{\infty} A_i, \quad B_0 = B - \bigcup_{j=1}^{\infty} B_j,$$

we apply the statement (2) of Theorem 4.2 to infer that

$$\mathfrak{C}_{2n}^{k+l}(A_0 \times B) = 0$$

and

$$\mathfrak{C}_{2n}^{k+l}(A_i \times B_0) = \mathfrak{C}_{2n}^{l+k}(B_0 \times A_i) = 0 \text{ for } i = 1, 2, 3, \dots$$

Accordingly Corollary 3.6 implies that the conditions

$$\mathfrak{C}_n^{k+l-n}[A_0 \cap (T_z \circ R)(B)] = 0$$

and

$$\mathfrak{C}_n^{k+l-n}[A_i \cap (T_z \circ R)(B_0)] = 0 \text{ for } i = 1, 2, 3, \dots$$

hold for $\mathcal{L}_n \otimes \phi_n$ almost all (z, R) . Inasmuch as

$$\begin{aligned} [A \cap (T_z \circ R)(B)] &\subset [A_0 \cap (T_z \circ R)(B)] \cup \bigcup_{i=1}^{\infty} [A_i \cap (T_z \circ R)(B_0)] \\ &\cup \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} [A \cap A_i \cap (T_z \circ R)(B_j)] \end{aligned}$$

whenever $(z, R) \in E_n \times G_n$, the proof will be complete as soon as we have shown that if i and j are positive integers then

$$A \cap A_i \cap (T_z \circ R)(B_j) \text{ is Hausdorff } k + l - n \text{ rectifiable}$$

for $\mathcal{L}_n \otimes \phi_n$ almost all (z, R) .

We henceforth consider fixed positive integers i and j and let Q be the set of all those triples

$$(a, b, R) \in A_i \times B_j \times G_n$$

for which the set

$$A_i \cap [T_{a-R(b)} \circ R](B_j)$$

is restricted at a by some $k+l-n$ dimensional plane.

Since Lemma 5.3 implies that Q is an analytic subset of the cartesian product space

$$A_i \times B_j \times G_n = (A_i \times B_j) \times G_n,$$

and since

$$\mathfrak{C}_{2n}^{k+l}(A_i \times B_j) < \infty,$$

according to the statement (1) of Theorem 4.2, we can use the Fubini Theorem in conjunction with Lemma 5.6 to obtain

$$(\mathfrak{C}_{2n}^{k+l} \otimes \phi_n)[(A_i \times B_j \times G_n) - Q] = 0.$$

Applying Theorem 3.5 with $v = k + l - n$, we find that

$$\mathfrak{H}_n^{k+l-n} [\{a \mid (a, (\text{inv } R)(a - z), R) \in (A_i \times B_j \times G_n) - Q\}] = 0$$

for $\mathcal{L}_n \otimes \phi_n$ almost all (z, R) .

Now suppose (z, R) is a point of $E_n \times G_n$ for which both

$$\mathfrak{H}_n^{k+l-n} [A \cap (T_z \circ R)(B)] < \infty$$

and

$$\mathfrak{H}_n^{k+l-n} [\{a \mid (a, (\text{inv } R)(a - z), R) \in (A_i \times B_j \times G_n) - Q\}] = 0,$$

and let

$$S = A \cap A_i \cap (T_z \circ R)(B_j).$$

We observe that if $a \in S$ then

$$a \in (T_z \circ R)(B_j), \quad a - z \in R(B_j), \quad (\text{inv } R)(a - z) \in B_j, \\ (a, (\text{inv } R)(a - z), R) \in A_i \times B_j \times G_n.$$

Therefore it is true for \mathfrak{H}_n^{k+l-n} almost all a in S that

$$(a, (\text{inv } R)(a - z), R) \in Q;$$

inasmuch as

$$a - R[(\text{inv } R)(a - z)] = z,$$

it follows from the definition of Q that S is restricted at a by some $k + l - n$ dimensional plane. Applying the proposition (1) of Remark 2.17 with A and k replaced by S and $k + l - n$ respectively, we conclude that the set S is Hausdorff $k + l - n$ rectifiable.

6. The principal integralgeometric formula. Considering again the intersections of a Hausdorff k rectifiable subset of E_n with the isometric images of an l rectifiable set, we compute here the integral over the group of isometries of the $k + l - n$ dimensional Hausdorff measures of these intersections.

6.1 LEMMA. *If A and B are analytic subsets of E_n , A is Hausdorff k rectifiable, and B is $n - k$ rectifiable, then*

$$\int_{E_n \times G_n} \mathfrak{H}_n^0 [A \cap (T_z \circ R)(B)] d(\mathcal{L}_n \otimes \phi_n)(z, R) = \beta(n, k) \mathfrak{H}_n^k(A) \mathfrak{H}_n^{n-k}(B).$$

Proof. First we note, as a consequence of Theorem 4.1 of [F1], that the integrand in the above formula is an $\mathcal{L}_n \otimes \phi_n$ measurable function. In fact if f is the function on $A \otimes B \otimes G_n$ to $E_n \otimes G_n$ such that

$$f(a, b, S) = (a - S(b), S) \text{ for } (a, b, S) \in A \times B \times G_n,$$

and if $(z, R) \in E_n \times G_n$, then the multiplicity with which f assumes the value (z, R) is equal to the number of elements of the set

$$A \cap (T_z \circ R)(B).$$

Inasmuch as \mathcal{H}_n^k almost all of A is the union of countably many disjoint analytic k rectifiable sets, it will obviously be sufficient to show that the formula is valid in case either $\mathcal{H}_n^k(A) = 0$ or A is k rectifiable.

If $\mathcal{H}_n^k(A) = 0$, then the formula is a consequence of Theorem 4.2 and Corollary 3.6.

If A is k rectifiable, then A and B can be represented as univalent (see [MF]) Lipschitzian images of an \mathcal{L}_k measurable subset of E_k and an \mathcal{L}_{n-k} measurable subset of E_{n-k} respectively, and the formula follows from Theorem 4.2 of [F2] and Theorem 5.9 of [F3].

6.2 THEOREM. *If A and B are analytic subsets of E_n , A is Hausdorff k rectifiable, B is l rectifiable, and $k+l \geq n$, then*

$$\int_{E_n \times G_n} \mathcal{H}_n^{k+l-n} [A \cap (T_z \circ R)(B)] d(\mathcal{L}_n \otimes \phi_n)(z, R) = \gamma(n, k, l) \mathcal{H}_n^k(A) \mathcal{H}_n^l(B).$$

Proof. In view of the preceding lemma we may assume that $k+l > n$. Letting h be the function on $E_{k+l-n} \times G_n \times E_n \times G_n$ such that

$$h(y, S, z, R) = \mathcal{H}_n^0 [A \cap \lambda_n^{2n-k-l}(S, y) \cap (T_z \circ R)(B)]$$

whenever $y \in E_{k+l-n}$, $S \in G_n$, $z \in E_n$, and $R \in G_n$, we divide our argument into three parts; the theorem is an immediate consequence of Part 2 and Part 3.

Part 1. h is an $\mathcal{L}_{k+l-n} \otimes \phi_n \otimes \mathcal{L}_n \otimes \phi_n$ measurable function.

Proof. Considering the function

$$f: A \times B \times G_n \times G_n \rightarrow E_{k+l-n} \times G_n \times E_n \times G_n$$

such that

$$f(a, b, U, V) = ((p_n^{k+l-n} \circ \text{inv } U)(a), U, a - V(b), V)$$

whenever $a \in A$, $b \in B$, $U \in G_n$ and $V \in G_n$, we note that if

$$(y, S, z, R) \in E_{k+l-n} \times G_n \times E_n \times G_n,$$

then the multiplicity with which f assumes the value (y, S, z, R) is equal to $h(y, S, z, R)$. Hence Part 1 is a consequence of Theorem 4.1 of [F1].

Part 2.

$$\begin{aligned} \int \mathcal{H}_n^{k+l-n} [A \cap (T_z \circ R)(B)] d(\mathcal{L}_n \otimes \phi_n)(z, R) \\ = \frac{1}{\beta(n, k+l-n)} \int hd(\mathcal{L}_{k+l-n} \otimes \phi_n \otimes \mathcal{L}_n \otimes \phi_n). \end{aligned}$$

Proof. According to Part 1 and the Fubini Theorem we have

$$\int hd(\mathcal{L}_{k+l-n} \otimes \phi_n \otimes \mathcal{L}_n \otimes \phi_n) = \int \int h(y, S, z, R)d(\mathcal{L}_{k+l-n} \otimes \phi_n)(y, S)d(\mathcal{L}_n \otimes \phi_n)(z, R).$$

Furthermore Theorem 5.7 implies that for $\mathcal{L}_n \otimes \phi_n$ almost all (z, R) the set

$$A \cap (T_z \circ R)(B)$$

is $k+l-n$ rectifiable, whence it follows by Theorem 9.7 of [F3] and by the formula for the integralgeometric measure given in Section 5 of [F4] that

$$\begin{aligned} \mathcal{H}_n^{k+l-n}[A \cap (T_z \circ R)(B)] &= \mathcal{F}_n^{k+l-n}[A \cap (T_z \circ R)(B)] \\ &= \frac{1}{\beta(n, k+l-n)} \int h(y, S, z, R)d(\mathcal{L}_{k+l-n} \otimes \phi_n)(y, S). \end{aligned}$$

Part 3.

$$\int hd(\mathcal{L}_{k+l-n} \otimes \phi_n \otimes \mathcal{L}_n \otimes \phi_n) = \frac{\beta(n, k)\beta(n, l)}{\beta(2n - k - l, n - l)} \mathcal{H}_n^k(A) \mathcal{H}_n^l(B).$$

Proof. According to Part 1 and the Fubini Theorem we have

$$\int hd(\mathcal{L}_{k+l-n} \otimes \phi_n \otimes \mathcal{L}_n \otimes \phi_n) = \int \int h(y, S, z, R)d(\mathcal{L}_n \otimes \phi_n)(z, R)d(\mathcal{L}_{k+l-n} \otimes \phi_n)(y, S).$$

Furthermore Theorem 5.5 implies that for $\mathcal{L}_{k+l-n} \otimes \phi_n$ almost all (y, S) the set

$$A \cap \lambda_n^{2n-k-l}(S, y)$$

is $n-l$ rectifiable, whence it follows by Lemma 6.1 that

$$\begin{aligned} \int h(y, S, z, R)d(\mathcal{L}_n \otimes \phi_n)(z, R) &= \beta(n, l) \mathcal{H}_n^{n-l}[A \cap \lambda_n^{2n-k-l}(S, y)] \mathcal{H}_n^l(B) \\ &= \beta(n, l) \mathcal{F}_n^{n-l}[A \cap \lambda_n^{2n-k-l}(S, y)] \mathcal{H}_n^l(B). \end{aligned}$$

Finally we use the theorem in Section 8 of [F4] with $m = n - l$ to compute

$$\begin{aligned} \int \mathcal{F}_n^{n-l}[A \cap \lambda_n^{2n-k-l}(S, y)]d(\mathcal{L}_{k+l-n} \otimes \phi_n)(y, S) \\ = \frac{\beta(n, k)}{\beta(2n - k - l, n - l)} \mathcal{F}_n^k(A) = \frac{\beta(n, k)}{\beta(2n - k - l, n - l)} \mathcal{H}_n^k(A). \end{aligned}$$

6.3 REMARK. The condition “ A and B are analytic subsets of E_n ” in Theorem 6.2 can be replaced by the weaker condition “ A is \mathfrak{H}_n^k measurable and B is \mathfrak{H}_n^l measurable.” This extension presents no difficulty because every Hausdorff measurable set of finite measure contains an F_σ of equal measure, and is contained in a G_δ of equal measure.

The problem whether or not the condition “ B is l rectifiable” can be replaced by the weaker condition “ B is Hausdorff l rectifiable” is still unsolved, even for the simplest case in which $k=1$, $l=1$ and $n=2$.

REFERENCES

- A. S. BESICOVITCH and P. A. P. MORAN
 BM. *The measure of product and cylinder sets*, Proc. London Math. Soc. vol. 20 (1945) pp. 110–120.
- S. EILENBERG
 E. *On ϕ measures*, Annales de la Société Polonaise de Mathématique vol. 17 (1938) pp. 252–253.
- H. FEDERER
 F1. *Surface area. I*, Trans. Amer. Math. Soc. vol. 55 (1944) pp. 420–437.
 F2. *Coincidence functions and their integrals*, Trans. Amer. Math. Soc. vol. 59 (1946) pp. 441–466.
 F3. *The (ϕ, k) rectifiable subsets of n space*, Trans. Amer. Math. Soc. vol. 62 (1947) pp. 114–192.
 F4. *Dimension and measure*, Trans. Amer. Math. Soc. vol. 62 (1947) pp. 536–547.
- A.P. MORSE and H. FEDERER
 MF. *Some properties of measurable functions*, Bull. Amer. Math. Soc. vol. 49 (1943) pp. 270–277.
- BROWN UNIVERSITY,
 PROVIDENCE, R. I.