COVERINGS OF A TOPOLOGICAL SPACE

BY

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Introduction. One of the most interesting applications of the theory of lattices is point set topology. Beginning with the work of Stone [6], the close connection between the topological structure of a space S and the algebraic structure of the lattice of open (or closed) sets of S has been carefully investigated. A standard procedure in these investigations has been to represent the points of a space as ideals (or dual ideals) of a lattice. In the present paper, we shall use this technique to study the set of all coverings of a topological space.

The paper is in four parts. The first contains the topological preliminaries and definitions. In the second part, the concept of a covering ideal is introduced and the correspondence between covering ideals and topological spaces is proved. Part three shows how continuous mappings can be interpreted in terms of homomorphisms of covering ideals. Finally, in part four, two simple applications of the theory are presented.

Notation. Set operations will be denoted by rounded symbols. Thus $\cup$, $\cap$, $\subseteq$ will stand for set union, intersection, and inclusion respectively. The symbol $\subset$ is reserved for proper inclusion. If A and B are sets, $A-B=\{a\in A \mid a\notin B\}$. The complement (in some fixed set) of a set A is written A^c. Finally, the empty set is denoted $\varnothing$.

Sharp cornered symbols will be used to denote lattice operations or relations. Thus $\vee$, $\wedge$, $\leqq$ denote respectively the lattice join, meet, and order. The zero and unit of a lattice will be represented by o and u. In a Boolean algebra, the complement of an element a is written a'.

A mapping ϕ of elements of some set A to the elements of a set B induces a mapping (again denoted by ϕ) of 2^A to 2^B as follows: $\phi(A_1)=\{\phi a \mid a\in A_1\}$ for $A_1\subseteq A$.

If S is a topological space and A is any subset of S, the closure and interior of A are respectively denoted A^- and A^o. We observe that $(A\cap B)^{-o}=A^{-o}\cap B^{-o}$ if A and B are open sets. An open set A which satisfies $A=A^{-o}$ is called regular. The collection of all regular open sets of S forms a complete Boolean algebra which will be denoted by $B(S)$. Any space with a neighborhood basis composed of regular open sets is called semi-regular.

0. Algebraic introduction.

It is convenient to collect some algebraic definitions which can not be found in [1] or other standard references.

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DEFINITION 0.1. A subset I of a partially ordered set is called a *terminal subset* if $a \in I$ implies $b \in I$ for all $b \geqq a$.

DEFINITION 0.2. Let P be a set which is partially ordered by $\leqq$. Define a quasi-ordering of the set of all subsets of P by $A < B$ if, for any $a \in A$, there exists $b \in B$ satisfying $a \leqq b$.

Some elementary properties of $<$ which will be used often are:

(1) if $A \subseteq B$, then $A < B$,

(2) $A < \emptyset$ if and only if $A = \emptyset$,

(3) if C is a terminal subset of P and $A < B$, then $A \cap C < B \cap C$.

DEFINITION 0.3. Let P be a partially ordered set with a zero o. A subset P' of P is called *dense* in P, if for any $a \neq o$ in P, $b \neq o$ exists in P' with $b \leqq a$.

DEFINITION 0.4. A *semi-lattice* (abbreviated s.l. is a partially ordered set with a zero element in which every pair of elements has a g.l.b. A mapping π of one semi-lattice into another is a homomorphism if it satisfies $\pi(a \wedge b) = \pi a \wedge \pi b$ and $\pi o = o$.

REMARK 1. In the literature, the existence of a zero is not included in the definition of a s.l.

REMARK 2. It is possible to define a dual ideal of a s.l. just as a dual ideal of a lattice is defined. In speaking of these, the adjective "dual" will often be omitted. Since lattice ideals in the usual sense will never be considered, this can cause no confusion.

DEFINITION 0.5. Let P be a semi-lattice. Define the commutative and associative operation $\wedge$ on the set of all subsets of P by $A \wedge B = \{a \wedge b \mid a \in A, b \in B\}$.

(4) If A, B and C are subsets of P, then $A < B \wedge C$ if and only if $A < B$ and $A < C$.

DEFINITION 0.6. A semi-lattice P is *disjunctive* if, whenever $a \nleqq b$, $c \in P$ exists satisfying $a \wedge c \neq o$ and $b \wedge c = o$.

REMARK. If P' is a dense sub-s.l. of the disjunctive s.l. P, then P' is also disjunctive. In particular, any dense sub-s.l. of a Boolean algebra is disjunctive.

PROPOSITION 0.7 (STONE-GLIVENKO-BÜCHI). *Let P be a disjunctive semi-lattice. Then there exists a complete Boolean algebra Q and an isomorphism π of P onto a dense subset of Q. Moreover, π is unique in the sense that if Q_0 is a complete Boolean algebra and π_0 is an isomorphism of P onto a dense subset of Q_0, then $\pi_0 \pi^{-1}$ can be extended to an isomorphism of Q onto Q_0.*

Proof. See [2].

PART I. P-SPACES

In part one we shall define a class of topological spaces—the P-spaces. These are the spaces which will be studied in the remainder of the paper.

1. Fundamental definitions.

PROPOSITION 1.1. *Let $P=\{o, a, b, \cdots\}$ be a semi-lattice with a zero o. Let $S=\{X, Y, \cdots\}$ be a (possibly empty) set of proper (not empty and not all of P) dual ideals of P. Then S becomes a T_0 topological space (whose points are the distinct ideals $X, Y, \cdots$) if the sets of the form*

$$S(a) = \{X \in S \mid a \in X\} \tag{1}$$

are taken as an open neighborhood basis. Moreover, $S(a)\cap S(b)=S(a\wedge b)$ and $S(o)=\emptyset$. The mapping $a\rightarrow S(a)$ is one-to-one if and only if the following is satisfied whenever $a\nleq b$:

$(\mathrm{M}_{a,b})$ *there exists $X \in S$ with $a \in X$ and $b \notin X$.*

REMARK. In general $S(a\vee b)\neq S(a)\cup S(b)$. (See 2.2 below.)

Proof of 1.1. It will be sufficient to show that $S(a)\cap S(b)=S(a\wedge b)$. If $X\in S(a)\cap S(b)$, then $a\in X$ and $b\in X$. Since X is a dual ideal, $a\wedge b\in X$ and $X\in S(a\wedge b)$. Therefore $S(a)\cap S(b)\subseteq S(a\wedge b)$. On the other hand, if $a\wedge b\in X$, then $a\in X$ and $b\in X$, so $S(a\wedge b)\subseteq S(a)\cap S(b)$.

DEFINITION 1.2. Let P be a semi-lattice. A space S whose points are proper dual ideals of P will be called a *P-space* if the topology of S is obtained by taking all the sets of the form (1) as a neighborhood basis. If, in addition, S satisfies condition $\mathrm{M}_{a,b}$ whenever $a\nleq b$, then S is called a *representative P-space*.

A characterization of those T_0 spaces which are homeomorphic to P-spaces is given by 1.1 and its converse:

PROPOSITION 1.3. *Let P be a semi-lattice. Let S' be a T_0 space and P' a collection of open sets of S' such that* (i) *P' is a basis for the open sets of S',* (ii) *P' is closed under intersection and contains the empty set, and* (iii) *there is a homomorphism π of P onto P'. Let S be the P-space whose points are the proper dual ideals*

$$X_x = \{a \in P \mid x \in \pi a\}, \qquad \textit{where } x \in S'.$$

Then the mapping $X_x\rightarrow x$ is a homeomorphism of S onto S', mapping the open set $S(a)=\{X_x\in S \mid a\in X_x\}$ onto πa. Finally, if π is an isomorphism, S is a representative P-space.

The following simple results will be needed later.

LEMMA 1.4. *Let T be an arbitrary subset of the P-space S. Then the closure of T is $T^-=\{X\in S \mid X\subseteq \cup T\}$ (where $\cup T=\cup\{Y \mid Y\in T\}$).*

Proof. If $X\subseteq \cup T$, and if $a\in X$, then $a\in Y$ for some $Y\in T$. Thus every neighborhood of X contains a point of T so that $X\in T^-$. Conversely, if $X\in T^-$, every neighborhood of X contains a point of T. Hence if $a\in X$, then Y exists in T so that $a\in Y$; that is, $X\subseteq \cup T$.

LEMMA 1.5. *A P-space S is a T_1 space if and only if $X \subset Y$ never holds for X and Y in S.*

LEMMA 1.6. *A representative P-space S is a Hausdorff space if and only if for any X and Y of S with $X \neq Y$, there exists $a \in X$ and $b \in Y$ such that $a \wedge b = 0$.*

LEMMA 1.7. *Let S_1 and S_2 be P-spaces. Define $S_1 \leqq S_2$ if, for any $X_1 \in S_1$, there exists $X_2 \in S_2$ such that $X_2 \subseteq X_1$. Then $\leqq$ is a quasi-ordering on the set of all P-spaces and a partial ordering on the set of all T_1 P-spaces.*

In parts two and three of this paper, we shall be concerned mostly with P-spaces which are compact and T_1. It is convenient to introduce some notation for the set of all such spaces.

DEFINITION 1.8. Let P be a semi-lattice. Denote by Σ_P the set of all compact T_1 P-spaces (including the empty space). Two spaces of Σ_P are equal if and only if they consist of the same dual ideals. The relation $\leqq$ of 1.7 partially orders Σ_P.

2. **Semi-regular P-spaces.** In this section, we shall examine those P-spaces for which P is a complete Boolean algebra.

THEOREM 2.1. *Let P be a complete Boolean algebra and let S be a representative P-space. Then the regular open sets of S are precisely those of the form $S(a)$.*

Proof. From the assumption that S is a representative P-space, it follows that $S(a)^- = \{X \in S \mid a \wedge b \neq 0 \text{ all } b \in X\}$. Thus $S(a)^{-c} = \{X \in S \mid a \wedge b = 0 \text{ some } b \in X\} = \{X \in S \mid a' \in X\} = S(a')$. Hence $S(a)^{-o} = S(a)^{-c-c} = S(a')^{-c} = S((a')')$ $= S(a)$, that is, $S(a)$ is a regular open set.

Thus the mapping $a \to S(a)$ is a meet isomorphism of the complete Boolean algebra P onto a dense subset of the complete Boolean algebra of regular open sets of S. By 0.7, this mapping must be onto. That is, every regular open set of S is of the form $S(a)$ for some $a \in P$.

COROLLARY 2.2. *Let P be a complete Boolean algebra. Then any representative P-space is a semi-regular T_0 topological space whose Boolean algebra of regular open sets is isomorphic to P. Conversely, if P is isomorphic to the Boolean algebra of regular open sets of the semi-regular T_0 space S, then S is homeomorphic to a representative P-space.*

Proof. By 2.1, 1.1, and 1.3.

REMARK. When P is a complete Boolean algebra, Definition 1.2 can be simplified. In fact, if P is a disjunctive semi-lattice, then a P-space S is representative if and only if condition $\mathrm{M}_{a,o}$ is satisfied for all $a \neq o$.

EXAMPLE 2.3. Let S be a (nonempty) semi-regular T_1 space which satisfies the second countability axiom and has no isolated points. Then it is proved in [1, p. 177] that $B(S)$ (the Boolean algebra of regular open sets of S) is iso-

morphic to the completion by cuts of the free Boolean algebra with countably many generators.

LEMMA 2.4. *Let S be a dense subspace of the topological space S_0. If A is a regular open set of S_0, then $A \cap S$ is regular open in S and the mapping $A \to A \cap S$ is an isomorphism of $B(S_0)$ onto $B(S)$.*

Proof. Obvious.

COROLLARY 2.5. *Let S be a space in which the isolated points are dense. Then $B(S)$ is isomorphic to the Boolean algebra of all subsets of the set of isolated points in S.*

By combining 2.3 and 2.5, it is possible to characterize $B(S)$ whenever S is a semi-regular T_1 space which satisfies the second countability axiom.

PART II. COVERING IDEALS

The chief object of part two is to formulate and prove the main theorem of the paper (Theorem 4.5). In the first section, we introduce the concept of a covering ideal and show how P-spaces give rise to covering ideals. The precise relation between P-spaces and covering ideals is established in the next section. Finally, the covering ideals which correspond to compact Hausdorff spaces are characterized.

Throughout part two, P will denote a fixed semi-lattice. In the notation, explicit indication of dependence on P will be omitted when there is no danger of confusion.

3. Covering ideals.

DEFINITION 3.1. A finite, nonempty subset α of P is called a *P-covering* (or simply a covering). The collection of all P-coverings will be denoted $\mathcal{L}$ $(=\mathcal{L}_P)$. For $\alpha, \beta \in \mathcal{L}$, define $\alpha < \beta$ and $\alpha \wedge \beta$ as in 0.2 and 0.5.

A nonempty subset $\mathfrak{a}$ of $\mathcal{L}$ is called a *P-covering ideal* (or just a covering ideal) if it has the properties:

(i) $\alpha_1, \alpha_2 \in \mathfrak{a}$ implies $\alpha_1 \wedge \alpha_2 \in \mathfrak{a}$,

(ii) $\alpha \in \mathfrak{a}$ and $\beta > \alpha$ implies $\beta \in \mathfrak{a}$.

The set of all P-covering ideals will be denoted Γ $(=\Gamma_P)$. Let Γ be partially ordered by inclusion.

DEFINITION 3.2. Let S be a P-space. If $\alpha \in \mathcal{L}$ satisfies $S = \bigcup \{S(a) \mid a \in \alpha\}$, or equivalently, $X \cap \alpha \neq \emptyset$ for all $X \in S$, then α is called a *covering of S*. Denote by $\mathfrak{a}(S)$ the set of all coverings of S.

Our first lemma is an obvious consequence of the definitions.

LEMMA 3.3. *For any P-space S, $\mathfrak{a}(S)$, if it is not empty, is a covering ideal. If $S_1 \leqq S_2$ (see 1.7), then $\mathfrak{a}(S_1) \supseteq \mathfrak{a}(S_2)$. If S is the empty space, $\mathfrak{a}(S) = \mathcal{L}$.*

DEFINITION 3.4. Let $\mathfrak{a}$ be a covering ideal and let (a, b) be an ordered pair

of elements of P. Then $\mathfrak{a}$ is called a *representative* covering ideal if the following condition is satisfied whenever $a \nleq b$:

($\mathrm{N}_{a,b}$) there exists $\alpha \in \mathfrak{a}$ such that $a, b \in \alpha$ and $\alpha - \{a\} \notin \mathfrak{a}$.

For convenience, $\mathrm{N}_{a,o}$ can be replaced by the equivalent

(N_a) there exists $\alpha \in \mathfrak{a}$ such that $\alpha - \{a\} \notin \mathfrak{a}$.

LEMMA 3.5. *Let $S \in \Sigma_P$ (see* 1.8), *and let (a, b) be an ordered pair of elements of P. Then $\mathfrak{a}(S)$ satisfies $\mathrm{N}_{a,b}$ if and only if S satisfies $\mathrm{M}_{a,b}$ of* 1.1. *In particular $\mathfrak{a}(S)$ satisfies N_a if and only if $a \in X$ for some $X \in S$.*

Proof. If S satisfies $\mathrm{M}_{a,b}$, $X \in S$ exists so that $a \in X$, $b \notin X$. Suppose Y is a point of S different from X. Since S is a T_1 space, there exists $c \in Y$ with $c \notin X$. By compactness, choose $c_1, \cdots, c_n$ so that $c_i \notin X$ for all i and $\alpha = \{a, b, c_1, \cdots, c_n\}$ is a covering of S. Since $c_i \notin X$ and $b \notin X$, $\alpha - \{a\} = \{b, c_1, \cdots, c_n\} \notin \mathfrak{a}(S)$. Hence $\mathfrak{a}(S)$ satisfies $\mathrm{N}_{a,b}$.

Conversely, if $\mathfrak{a}(S)$ satisfies $\mathrm{N}_{a,b}$, then $\alpha \in \mathfrak{a}(S)$ exists with $a, b \in \alpha$ and $\alpha - \{a\} \notin \mathfrak{a}(S)$. This means $(\alpha - \{a\}) \cap X = \emptyset$ for some $X \in S$. In particular $b \notin X$. Since $\alpha \in \mathfrak{a}(S)$, $a \in X$. Thus S satisfies $\mathrm{M}_{a,b}$.

COROLLARY 3.6. *If $S \in \Sigma$, $\mathfrak{a}(S)$ is a representative covering ideal if and only if S is a representative P-space (Definition* 1.2).

Now we shall determine the algebraic relation between $\mathfrak{a}(S)$ and the points (dual ideals) of the compact T_1 P-space S.

DEFINITION 3.7. Let $\mathfrak{a}$ be a covering ideal. A dual ideal I of P is said to be *under* $\mathfrak{a}$ if $\alpha \cap I \neq \emptyset$ for all $\alpha \in \mathfrak{a}$.

LEMMA 3.8. *Let $S \in \Sigma$ and assume S is not empty. Suppose I is a dual ideal of P which is under $\mathfrak{a}(S)$. Then there is a point X of S with $X \subseteq I$.*

Proof. Otherwise, for each $X \in S$, there exists $a \in X$ with $a \notin I$. By compactness, it would be possible to choose $\alpha = \{a_1, \cdots, a_n\}$ (nonempty) covering S with $a_i \notin I$ for all i. Then $\alpha \in \mathfrak{a}(S)$ and $\alpha \cap I = \emptyset$, contrary to the assumption that I is under $\mathfrak{a}(S)$.

LEMMA 3.9. *Let $S \in \Sigma$. Then every point of S is minimal in the set of dual ideals under $\mathfrak{a}(S)$.*

Proof. Clearly, every point of S is under $\mathfrak{a}(S)$. Let $X \in S$. Suppose I is under $\mathfrak{a}(S)$ and $I \subseteq X$. By 3.8, there is a point Y in S with $Y \subseteq I \subseteq X$. Since S is a T_1 space, this implies $X = Y = I$. Thus X is minimal under $\mathfrak{a}(S)$.

COROLLARY 3.10. *The proper dual ideals of P which are minimal under $\mathfrak{a}(S)$ are precisely the points of S.*

Proof. By 3.9, every point of S is minimal under $\mathfrak{a}(S)$. Conversely, if $S \neq \emptyset$, and if I is minimal under $\mathfrak{a}(S)$, there is a point X of S with $X \subseteq I$. By minimality, this implies $X = I$. If S is empty, $\mathfrak{a}(S) = \mathcal{L}$ and no proper ideal is under $\mathfrak{a}(S)$.

4. The fundamental theorem.

DEFINITION 4.1. For $\mathfrak{a}\in\Gamma$, denote by $S(\mathfrak{a})$ the P-space of all proper dual ideals of P which are minimal under $\mathfrak{a}$.

Corollary 3.10 shows that for any $S_0\in\Sigma$, $S(\mathfrak{a}(S_0))=S_0$. The remainder of this section will be devoted to showing that for any $\mathfrak{a}_0\in\Gamma$, $\mathfrak{a}(S(\mathfrak{a}_0))=\mathfrak{a}_0$. First we will prove an existence theorem for dual ideals of P.

PROPOSITION 4.2. *Suppose $\mathfrak{a}\in\Gamma$ and $\mathfrak{a}\neq\mathcal{L}$. Let γ be a collection of coverings such that* (i) *if β_1, $\beta_2\in\gamma$, then $\beta_1\cup\beta_2\in\gamma$, and* (ii) $\mathfrak{a}\cap\gamma=\varnothing$. *Then a proper dual ideal X exists which is minimal under $\mathfrak{a}$ and satisfies $X\cap\beta=\varnothing$ for all $\beta\in\gamma$.*

Proof. By the maximal principle, extend γ to a collection of coverings γ_0 which is maximal satisfying (i) and (ii). Since $\{0\}\notin\mathfrak{a}$, $\gamma_0\neq\varnothing$. Define $J=\bigcup\{\beta\mid\beta\in\gamma_0\}$.

If $\alpha=\{a_1,\cdots,a_n\}\in\mathfrak{a}$, put $\alpha^*=\{a_i\in\alpha\mid a_i\nleq b,\text{ all } b\in J\}$. Then $\alpha^*\neq\varnothing$, since otherwise, for each $a_i\in\alpha$, b_i and β_i exist with $a_i\leqq b_i\in\beta_i\in\gamma_0$. This implies $\alpha<\beta_1\cup\cdots\cup\beta_n$, contrary to $\mathfrak{a}\cap\gamma_0=\varnothing$.

Put $X=J^c$. Then $X\cap\beta=\varnothing$ for all $\beta\in\gamma$ and $\alpha^*\subseteq X$ for all $\alpha\in\mathfrak{a}$. Consequently, $X\cap\alpha\neq\varnothing$ for all $\alpha\in\mathfrak{a}$. It must be shown that X is a dual ideal.

If $a\in X$, $\{a\}\notin\gamma_0$, so by the maximality of γ_0, $\alpha\in\mathfrak{a}$ and $\beta\in\gamma_0$ exist satisfying $\alpha<\beta\cup\{a\}$. This is possible only if $\alpha^*<\{a\}$. Conversely, $\alpha^*<\{a\}$ obviously implies $a\in X$.

If $b\geqq a$ and $a\in X$, then $\alpha^*<\{a\}<\{b\}$ for some $\alpha\in\mathfrak{a}$. Hence $b\in X$. If a_1, $a_2\in X$ and α_1, $\alpha_2\in\mathfrak{a}$ satisfy $\alpha_1^*<\{a_1\}$, $\alpha_2^*<\{a_2\}$, then $(\alpha_1\wedge\alpha_2)^*<\alpha_1^*\wedge\alpha_2^*<\{a_1\}\wedge\{a_2\}=\{a_1\wedge a_2\}$. Hence $a_1\wedge a_2\in X$ and X is a proper ideal.

Finally, to show that X is minimal under $\mathfrak{a}$, suppose I is a dual ideal under $\mathfrak{a}$ which is contained in X. Let $\gamma_1=\{\beta\in\mathcal{L}\mid\beta\cap I=\varnothing\}$. Then γ_1 satisfies (i) and (ii) and $\gamma_1\supseteq\gamma_0$. Since γ_0 is maximal, $\gamma_1=\gamma_0$. If $a\in X$, $\{a\}\notin\gamma_0=\gamma_1$, so $a\in I$. Thus $X\subseteq I$ and the proof is complete.

COROLLARY 4.3. *Suppose $\mathfrak{a}\in\Gamma$ and $\mathfrak{a}\neq\mathcal{L}$. Let I be a dual ideal which is under $\mathfrak{a}$. Then there exists a proper dual ideal $X\subseteq I$ which is minimal under $\mathfrak{a}$.*

Proof. Put $\gamma=\{\beta\in\mathcal{L}\mid\beta\cap I=\varnothing\}$. Then γ satisfies (i) and (ii) of 4.2. Hence a proper ideal X exists which is minimal under $\mathfrak{a}$ and satisfies $X\cap\beta=\varnothing$ for all $\beta\in\gamma$. If $a\in X$, then $\{a\}\notin\gamma$, so $a\in I$. Therefore $X\subseteq I$.

LEMMA 4.4. *If $\mathfrak{a}$ is a covering ideal, then $S(\mathfrak{a})\in\Sigma$.*

Proof. If $\mathfrak{a}=\mathcal{L}$, then $S(\mathfrak{a})$ is the empty space. Suppose then that $\mathfrak{a}\neq\mathcal{L}$. It must be shown that $S(\mathfrak{a})$ is a compact T_1 space. If X_1 and X_2 are minimal under $\mathfrak{a}$, then $X_1\subset X_2$ is impossible; thus, by 1.5, $S(\mathfrak{a})$ is a T_1 space.

To prove that $S(\mathfrak{a})$ is compact, it is sufficient to show that if Q is any subset of P with the property $S(\mathfrak{a})=\bigcup\{S(a)\mid a\in Q\}$ (where $S(a)=\{X\in S(\mathfrak{a})\mid a\in X\}$), there is a finite subset of Q which is in $\mathfrak{a}$. Suppose otherwise: $\mathfrak{a}\cap\gamma=\varnothing$, where γ denotes the collection of all nonempty finite subsets of Q. Then

$\gamma \subseteq \mathcal{L}$ and γ is closed under finite unions, so by 4.2 there is a proper ideal X which is minimal under $\mathfrak{a}$ and satisfies $X \cap \beta = \emptyset$ for all $\beta \in \gamma$. Thus $X \in S(\mathfrak{a})$, and $a \notin X$ for all $a \in Q$. This contradicts $S(\mathfrak{a}) = \bigcup \{S(a) \mid a \in Q\}$ and shows that some finite subset of Q covers $S(\mathfrak{a})$. Hence $S(\mathfrak{a})$ is compact.

THEOREM 4.5. *The mapping $S_0 \to \mathfrak{a}(S_0)$ and $\mathfrak{a}_0 \to S(\mathfrak{a}_0)$ defines a Galois connection (see* [1]*) between Σ and Γ. Moreover, if $S_0 \in \Sigma$ and $\mathfrak{a}_0 \in \Gamma$, then $S(\mathfrak{a}(S_0)) = S_0$ and $\mathfrak{a}(S(\mathfrak{a}_0)) = \mathfrak{a}_0$.*

Proof. The first assertion follows from 3.3, 4.4, and Definitions 3.2 and 4.1. By 3.10, $S(\mathfrak{a}(S_0)) = S_0$. The only thing left to prove is $\mathfrak{a}(S(\mathfrak{a}_0)) \subseteq \mathfrak{a}_0$. Assume $\mathfrak{a}_0 \neq \mathcal{L}$. If $\alpha \notin \mathfrak{a}_0$, then by 4.2 there exists X which is minimal under $\mathfrak{a}_0$ and satisfies $\alpha \cap X = \emptyset$. Hence $X \in S(\mathfrak{a}_0)$ and $\alpha \notin \mathfrak{a}(S(\mathfrak{a}_0))$. Thus $\mathfrak{a}(S(\mathfrak{a}_0)) \subseteq \mathfrak{a}_0$. If $\mathfrak{a}_0 = \mathcal{L}$, then $S(\mathfrak{a}_0) = \emptyset$ and $\mathfrak{a}(S(\mathfrak{a}_0)) = \mathcal{L} = \mathfrak{a}_0$.

EXAMPLE 4.6. Let P be a Boolean algebra and define $\mathfrak{a} = \{\alpha \in \mathcal{L}_P \mid \bigvee \alpha = u\}$ (where $\bigvee \alpha$ denotes $\bigvee \{a \mid a \in \alpha\}$). Suppose I is a proper dual ideal of P which is under $\mathfrak{a}$. Then if $a \in P$, $\{a, a'\} \in \mathfrak{a}$, so $\{a, a'\} \cap I \neq \emptyset$. That is, either $a \in I$ or $a' \in I$. Hence I is maximal. Conversely, if I is a maximal proper dual ideal of P, then I is prime. Thus if $\alpha \in \mathfrak{a}$, $\bigvee \alpha = u \in I$ and therefore $\alpha \cap I \neq \emptyset$. This proves that the ideals which are minimal under $\mathfrak{a}$ are precisely the maximal dual ideals of P. Consequently $S(\mathfrak{a})$ is the Boolean space associated with P.

5. **Normal covering ideals.** In this section, we shall formulate a sufficient condition on a covering ideal $\mathfrak{a}$ in order that $S(\mathfrak{a})$ be a Hausdorff space. The results contained in 5.2 and 5.4 below follow directly from Theorems VI 4.4 and V 8.9 in [7] and proofs will be omitted.

DEFINITION 5.1. Let $\alpha, \beta \in \mathfrak{a}$. Then α is a *star-refinement* of β ($\alpha^* < \beta$) if for any $a \in \alpha$, there exists $b \in \beta$ such that if $c \in \alpha$ and $a \wedge c \neq 0$, then $c \leqq b$.

A sequence of coverings $\alpha_0, \alpha_1, \alpha_2, \cdots$ is called *normal* if $\alpha_{j+1}{}^* < \alpha_j$ for all j. If $\mathfrak{a}$ is a covering ideal, then α is called normal in $\mathfrak{a}$ if α is the first term of a normal sequence of coverings in $\mathfrak{a}$. A covering ideal is called normal if every $\alpha \in \mathfrak{a}$ is normal in $\mathfrak{a}$.

PROPOSITION 5.2. *Let $\mathfrak{a}$ be a representative covering ideal. Then $S(\mathfrak{a})$ is Hausdorff if and only if $\mathfrak{a}$ is normal.*

PROPOSITION 5.3. *Let $\mathfrak{a}$ be a representative covering ideal. Then $S(\mathfrak{a})$ satisfies the second countability axiom if and only if there is a countable subset A of P such that if $\beta \in \mathfrak{a}$, there exists $\alpha \in \mathfrak{a}$ with $\alpha < \beta$ and $\alpha \subseteq A$.*

Proof. The sufficiency of this condition is obvious. The necessity follows from 1.1 and 4.5 by an easy compactness argument.

PROPOSITION 5.4. *Let $\mathfrak{a}$ be a representative covering ideal. Then $S(\mathfrak{a})$ is a compactum if and only if there is a normal sequence of coverings which is coinitial in $\mathfrak{a}$.*

Part III. Mappings

By Theorem 4.5, the coverings rather than the points can be considered as the primitive elements of a compact T_1 space. This shift of viewpoint can be useful, as the examples of part four will show. It is natural then to try to relate the topological properties of a space directly to the algebraic properties of its covering ideal. With this aim, we have already considered the concepts of Hausdorff separation and perfect separability. In part three the comparison between topology and algebra will be continued with studies of subspaces, homeomorphisms, and continuous mappings of P-spaces.

6. Closed subspaces of a P-space.

Definition 6.1. Let $S\in\Sigma$. Denote

$$I(S) = \bigcup\{X \mid X \in S\} = \{a \in P \mid \mathfrak{a}(S) \text{ satisfies } N_a\}. \tag{1}$$

(See 3.4 and 3.5.)

It is evident that $I(S)$ is a terminal subset of P (see 0.1). If P is a disjunctive semi-lattice, $I(S)=P-\{o\}$ if and only if S is a representative P-space.

Lemma 6.2. *Let $S_0\in\Sigma$. Suppose S is a closed subspace of S_0. Put $\mathfrak{a}=\mathfrak{a}(S_0)$. Then $S\in\Sigma$ and*

(2) *if $a\in I(S)$, there exists $\alpha\in\mathfrak{a}$ satisfying $\alpha-\{a\}\not>\beta\cap I(S)$ for all $\beta\in\mathfrak{a}$,*

(3) $\mathfrak{a}(S)=\{\alpha\in\mathcal{L} \mid \alpha>\beta\cap I(S),$ *some* $\beta\in\mathfrak{a}\}$.

Proof. Obviously S is a compact T_1 P-space, i.e., $S\in\Sigma$. Suppose $a\in I(S)$, Then $a\in X$ for some $X\in S$. By compactness, $\alpha\in\mathfrak{a}$ exists satisfying $X\cap(\alpha-\{a\})=\emptyset$. But X is under $\mathfrak{a}$, so if $\beta\in\mathfrak{a}$, $\alpha-\{a\}\not>\beta\cap X$. Since $X\subseteq I(S)$, this implies (2).

To prove (3), note that if $\alpha>\beta\cap I(S)$ for some $\beta\in\mathfrak{a}$, and if $X\in S$, then $\alpha\cap X>\beta\cap I(S)\cap X=\beta\cap X\neq\emptyset$. Hence $\alpha\in\mathfrak{a}(S)$. Conversely, suppose $\alpha\in\mathfrak{a}(S)$. Let $Y\in S_0-S$. Since S is closed, $a\in P$ exists with $Y\in S_0(a)\subseteq S_0-S$. Thus $a\notin I(S)$. By compactness, $a_1, \cdots, a_n$ exist with $a_j\notin I(S)$ all j and $\beta=\{a_1, \cdots, a_n\}\cup\alpha\in\mathfrak{a}$. Then $\alpha=\beta\cap I(S)$.

Lemma 6.3. *Let $S_0\in\Sigma$. Suppose S_1 and S_2 are two closed subspaces of S_0. Then $I(S_1)\subseteq I(S_2)$ if and only if $S_1\subseteq S_2$.*

Proof. Suppose $S_1\not\subseteq S_2$. Let $X\in S_1$, $X\notin S_2$. Since S_2 is closed, $a\in P$ exists satisfying $X\in S_0(a)\subseteq S_0-S_2$. Then $a\in I(S_1)$, $a\notin I(S_2)$. Thus $I(S_1)\not\subseteq I(S_2)$. The converse is obvious.

Definition 6.4. Let $\mathfrak{a}\in\Gamma$ and suppose I is any terminal subset of P. Then $a\in P$ is called *essential in I with respect to* $\mathfrak{a}$ if $\alpha\in\mathfrak{a}$ exists such that $\alpha-\{a\}\not>\beta\cap I$ for all $\beta\in\mathfrak{a}$. If every $a\in I$ is essential in I with respect to $\mathfrak{a}$, then I is called an $\mathfrak{a}$-*closed subset* of P.

If S is a closed subspace of $S(\mathfrak{a})$, then $I(S)$ is $\mathfrak{a}$-closed by (2). The converse is a corollary of the next lemma.

LEMMA 6.5. *Let $\mathfrak{a}\in\Gamma$ and suppose I is a terminal subset of P. Define*

$$S = S(\mathfrak{a}, I) = \{X \in S(\mathfrak{a}) \mid X \subseteq I\}. \tag{4}$$

Then S is a closed subspace of $S(\mathfrak{a})$ and $I(S)=\{a\in P \mid a$ is essential in I with respect to $\mathfrak{a}\}$.

Proof. By 1.4, S is closed in $S(\mathfrak{a})$. If $a\in I(S)$, then by (2), a is essential in $I(S)$ with respect to $\mathfrak{a}$. But $I(S)\subseteq I$ by (1) and (4), so a is essential in I with respect to $\mathfrak{a}$.

Suppose $a\notin I(S)$, that is, $a\notin X$ for all $X\in S$. Let $\alpha\in\mathfrak{a}$. Then $(\alpha-\{a\})\cap X=\alpha\cap X\neq\varnothing$ for all $X\in S$. If $X\notin S$, $X\not\subseteq I$ by (4), so $b\in X$ exists with $b\notin I$. By compactness, $\beta=\{b_1, \cdots, b_n\}\cup(\alpha-\{a\})$ exists in $\mathfrak{a}(S(\mathfrak{a}))=\mathfrak{a}$ with $b_j\notin I$ for all j. Then $\alpha-\{a\}>\beta\cap I$. This proves that a is not essential in I with respect to $\mathfrak{a}$.

COROLLARY 6.6. *If S is a closed subspace of $S(\mathfrak{a})$, then $S(\mathfrak{a}, I(S))=S$. If I is an $\mathfrak{a}$-closed subset of P, $I(S(\mathfrak{a}, I))=I$.*

7. Homeomorphisms.

LEMMA 7.1. *Let π be an isomorphism of the semi-lattice P_1 into the semi-lattice P_2. Suppose $\mathfrak{a}$ is a P_1-covering ideal. Denote by $[\pi\mathfrak{a}]$ the P_2-covering ideal $\{\alpha\in\mathcal{L}_{P_2} \mid \alpha>\pi\alpha_1,\ \alpha_1\in\mathfrak{a}\}$. Then π^*: $X\to\pi^{-1}X$ maps $S_2=S([\pi\mathfrak{a}])$ homeomorphically onto $S_1=S(\mathfrak{a})$.*

Proof. A dual ideal I of P_2 is under $[\pi\mathfrak{a}]$ if and only if $\pi^{-1}I$ is under $\mathfrak{a}$. In particular, if $X\in S_2$, then $\pi^{-1}X$ is a point of S_1. Thus π^* maps S_2 into S_1. If $Y_1\in S_1$, then $[\pi Y_1]=\{a_2\in P_2 \mid a_2\geqq\pi a_1,\ a_1\in Y_1\}$ is a dual ideal of P_2 which is minimal under $[\pi\mathfrak{a}]$. Since $\pi^*[\pi Y_1]=\pi^{-1}[\pi Y_1]=Y_1$, π^* is onto. Also, if $X\in S_2$,

$$[\pi(\pi^*X)] = X \tag{1}$$

(because X is minimal under $[\pi\mathfrak{a}]$). Thus π^* is one-to-one. Finally, if $a\in P_1$,

$$\pi^*S_2(\pi a) = \{\pi^{-1}X \mid X \in S_2(\pi a)\} = \{Y_1 \in S_1 \mid a \in Y_1\} = S_1(a).$$

By (1), the sets $S_2(\pi a)$ form a basis for S_2. Hence π^* is a homeomorphism.

THEOREM 7.2. *Let P_1 and P_2 be complete Boolean algebras and suppose $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are representative P_1- and P_2-covering ideals respectively. If π is an isomorphism of P_1 onto P_2 such that $\pi\mathfrak{a}_1=\mathfrak{a}_2$, then π induces a homeomorphism π^* of $S_2=S(\mathfrak{a}_2)$ onto $S_1=S(\mathfrak{a}_1)$; π^* is uniquely determined by the condition $\pi(\pi^*X)=X$. If ϕ is a homeomorphism of S_2 onto S_1, then ϕ induces an isomorphism ϕ^+ of P_1 onto P_2 with $\phi^+\mathfrak{a}_1=\mathfrak{a}_2$; ϕ^+ is uniquely determined by the condition $\phi^+(\phi X)=X$. Finally $(\pi^*)^+=\pi$ and $(\phi^+)^*=\phi$.*

Proof. The theorem follows easily from 7.1.

COROLLARY 7.3. *Let P be a complete Boolean algebra and let $\mathfrak{a}_1$ and $\mathfrak{a}_2$ be representative P-covering ideals. Then $S(\mathfrak{a}_1)$ is homeomorphic to $S(\mathfrak{a}_2)$ if and only if there is an automorphism π of P such that $\pi\mathfrak{a}_1=\mathfrak{a}_2$.*

COROLLARY 7.4. *Let P be a complete Boolean algebra and let $\mathfrak{a}$ be a representative covering ideal. If ϕ is a homeomorphism of $S(\mathfrak{a})$ onto itself, then $\phi^+\mathfrak{a}=\mathfrak{a}$. Conversely, if π is an automorphism of P satisfying $\pi\mathfrak{a}=\mathfrak{a}$, then π^* is a homeomorphism of $S(\mathfrak{a})$ onto itself.*

Proof. If $\pi\mathfrak{a}=\mathfrak{a}$ and X is minimal under $\mathfrak{a}$, then $\pi^*X=\pi^{-1}X$ is minimal under $\pi^{-1}\mathfrak{a}=\mathfrak{a}$. Hence π^* maps $S(\mathfrak{a})$ onto itself.

8. Continuous mappings.

THEOREM 8.1. *Let P_1 and P_2 be semi-lattices. Suppose $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are representative P_1- and P_2-covering ideals respectively. Denote $S_1=S(\mathfrak{a}_1)$ and $S_2=S(\mathfrak{a}_2)$. Assume that S_1 is a Hausdorff space. Finally, let π be a homomorphism of P_1 into P_2 which satisfies*

$$\pi\mathfrak{a}_1 \subseteq \mathfrak{a}_2. \tag{1}$$

Then there is a unique continuous mapping π^* of S_2 into S_1 such that

$$\pi(\pi^*X) \subseteq X, \qquad X \in S_2. \tag{2}$$

Moreover,

$$(\pi^*)^{-1}S_1(a) \subseteq S_2(\pi a) \subseteq (\pi^*)^{-1}(S_1(a)^-) \tag{3}$$

for all $a\in P_1$. Finally, $\pi^*S_2=S(\mathfrak{a}_1, I)$ (see 6.5), where $I=\{a\in P_1 \mid \pi a\neq 0\}$.

Proof. Let $X\in S_2$. Then $\pi^{-1}X$ is a proper dual ideal of P_1 which is under $\mathfrak{a}_1$. For if $\alpha\in\mathfrak{a}_1$, $\pi\alpha\in\mathfrak{a}_2$, so $\pi\alpha\cap X\neq\varnothing$. Thus $\alpha\cap\pi^{-1}X\neq\varnothing$. By 4.3, $Y\in S_1$ exists satisfying $Y\subseteq\pi^{-1}X$. This Y is unique. Indeed, suppose Y_1, $Y_2\subseteq\pi^{-1}X$, where Y_1 and Y_2 are distinct points of S_1. By 1.6, $o=a\wedge b$ for some $a\in Y_1$ and $b\in Y_2$. Hence $o\in\pi^{-1}X$—a contradiction since $\pi o=o\notin X$. Thus (2) defines a unique mapping π^*: $S_2\to S_1$.

To prove (3), notice that $(\pi^*)^{-1}S_1(a)\subseteq S_2(\pi a)$ follows directly from (2). If $\pi a\in X$, then $a\in\pi^{-1}X$ so that $a\wedge b\neq 0$ for all $b\in\pi^*X$. Consequently, $\pi^*X\in S_1(a)^-$, and therefore $S_2(\pi a)\subseteq(\pi^*)^{-1}(S_1(a)^-)$.

Now suppose N is any open set of S_1. By (3) and the normality of S_1, $(\pi^*)^{-1}N=\bigcup\{(\pi^*)^{-1}S_1(a) \mid S_1(a)^-\subseteq N\}\subseteq\bigcup\{S_2(\pi a) \mid S_1(a)^-\subseteq N\}\subseteq\bigcup\{(\pi^*)^{-1}\cdot(S_1(a)^-) \mid S_1(a)^-\subseteq N\}=(\pi^*)^{-1}N$. Thus $(\pi^*)^{-1}N$ is open and π^* is continuous.

It remains to show that $\pi^*S_2=S(\mathfrak{a}_1, I)=\{Y_1\in S_1 \mid Y_1\subseteq I\}$. If $X\in S_2$, $\pi^*X\subseteq\pi^{-1}X=\{a\in P_1 \mid \pi a\in X\}\subseteq\{a\in P_1 \mid \pi a\neq o\}=I$. Thus $\pi^*S_2\subseteq S(\mathfrak{a}_1, I)$. If $Y_1\notin\pi^*S_2$, then since π^*S_2 is closed and S_1 is normal, $a\in Y_1$ exists satisfying $S_1(a)^-\cap\pi^*S_2=\varnothing$. By (3), $S_2(\pi a)\subseteq(\pi^*)^{-1}(S_1(a)^-)=(\pi^*)^{-1}[S_1(a)^-\cap\pi^*S_2]$

$=\varnothing$, so $\pi a=o$. Consequently $Y_1 \not\subseteq I$, that is, $Y_1 \notin S(\mathfrak{a}_1, I)$. Since Y_1 was arbitrary, $S(\mathfrak{a}_1, I) \subseteq \pi^* S_1$.

REMARK. If P_2 is a complete Boolean algebra, (3) can be sharpened to

$$[(\pi^*)^{-1} S_1(a)]^{-o} \subseteq S_2(\pi a). \tag{4}$$

In order to prove a converse of 8.1, it is necessary to assume that P_2 is a complete Boolean algebra. In compensation, it need not be assumed that S_1 is a Hausdorff space.

THEOREM 8.2. *Let P_1 be a semi-lattice and P_2 a complete Boolean algebra. Let S_1 and S_2 be representative P_1- and P_2-spaces respectively. Denote $\mathfrak{a}_1=\mathfrak{a}(S_1)$ and $\mathfrak{a}_2=\mathfrak{a}(S_2)$. Suppose ϕ is a continuous mapping of S_2 into S_1. Then the relation*

$$S_2(\phi^+ a) = (\phi^{-1} S_1(a))^{-o}, \qquad a \in P_1, \tag{5}$$

defines a unique (semi-lattice) homomorphism ϕ^+ of P_1 into P_2 such that

$$\phi^+ \mathfrak{a}_1 \subseteq \mathfrak{a}_2$$

and

$$\phi^+(\phi X) \subseteq X, \qquad X \in S_2.$$

The kernel of ϕ^+ is the complement of $I(\phi S_2)$ (see 6.1).

The proof of 8.2 is straightforward and will be omitted.

Theorems 8.1 and 8.2 are converses when the following conditions are fulfilled: P_1 is a disjunctive semi-lattice, P_2 is a complete Boolean algebra, $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are representative covering ideals, and $S_1=S(\mathfrak{a}_1)$ is a Hausdorff space. If these are satisfied, 8.1 and 8.2 imply $(\phi^+)^*=\phi$ for any continuous mapping ϕ of S_2 into S_1. However, in general, not all homomorphisms π of $\mathfrak{a}_1$ into $\mathfrak{a}_2$ satisfy $(\pi^*)^+=\pi$ (see 9.4).

PROPOSITION 8.3. *Let P_1, P_2, and P_3 be semi-lattices. Suppose $\mathfrak{a}_1$, $\mathfrak{a}_2$, and $\mathfrak{a}_3$ are representative P_1-, P_2-, and P_3-covering ideals respectively. Assume $S(\mathfrak{a}_1)$ and $S(\mathfrak{a}_2)$ are Hausdorff spaces. Finally, suppose π_1: $P_1 \to P_2$ and π_2: $P_2 \to P_3$ are homomorphisms which satisfy $\pi_1 \mathfrak{a}_1 \subseteq \mathfrak{a}_2$ and $\pi_2 \mathfrak{a}_2 \subseteq \mathfrak{a}_3$. Then $(\pi_2 \pi_1)^* = \pi_1^* \pi_2^*$.*

This follows directly from 8.1. The analogous equation $(\phi_2 \phi_1)^+ = \phi_1^+ \phi_2^+$ is usually false. (Compare (4) and (5).)

9. Homeomorphisms into a compact Hausdorff space.

LEMMA 9.1. *Let P_1 and P_2 be semi-lattices. Let $S_1=S(\mathfrak{a}_1)$, $S_2=S(\mathfrak{a}_2)$ be representative compact Hausdorff P_1- and P_2-spaces respectively. Suppose π is a homomorphism of P_1 into P_2 with $\pi \mathfrak{a}_1 \subseteq \mathfrak{a}_2$, such that π^* is a one-to-one mapping of S_2 into S_1. Then $\pi \mathfrak{a}_1$ is coinitial in $\mathfrak{a}_2$.*

Proof. Let $c \in P_2$. Suppose $c \in X \in S_2$. Then $\pi^* X \in \pi^* S_2(c)$ and since

π^* is a homeomorphism, $\pi^*S_2(c)$ is an open set of π^*S_2 in its relative topology. Hence, because S_1 is normal, $a\in P_1$ exists satisfying $\pi^*X\in S_1(a)\cap\pi^*S_2 \subseteq S_1(a)^-\cap\pi^*S_2\subseteq\pi^*S_2(c)$. By 8.1 (3), $S_2(\pi a)\subseteq(\pi^*)^{-1}(S_1(a)^-)=(\pi^*)^{-1}[S_1(a)^- \cap\pi^*S_2]\subseteq(\pi^*)^{-1}[\pi^*S_2(c)]=S_2(c)$. Putting $c=o$ in this last equation also shows that if $S_1(a)^-\cap\pi^*S_2=\emptyset$, then $\pi a=o$.

Now suppose $\beta\in\mathfrak{a}_2$. By the result of the above paragraph and the compactness of π^*S_2, it is possible to choose $a_1, \cdots, a_n$ in P_1 so that $\pi^*S_2 \subseteq S_1(a_1)\cup\cdots\cup S_1(a_n)$ and $\{\pi a_1, \cdots, \pi a_n\}<\beta$. Since S_1 is compact Hausdorff, $b_1, \cdots, b_m$ exist so that $S_1(b_j)^-\cap\pi^*S_2=\emptyset$ for all j and $\alpha=\{a_1, \cdots, a_n, b_1, \cdots, b_m\}\in\mathfrak{a}_1$. Then by the last statement of the first paragraph, $\pi\alpha <\{\pi a_1, \cdots, \pi a_n\}<\beta$.

COROLLARY 9.2. *Let P_1 be a semi-lattice and P_2 a complete Boolean algebra. If ϕ is a one-to-one continuous mapping of the representative P_2-space $S(\mathfrak{a}_2)$ into the compact Hausdorff representative P_1-space $S(\mathfrak{a}_1)$, then $\phi^+\mathfrak{a}_1$ is coinitial in $\mathfrak{a}_2$.*

Combining 9.1 with the following theorem, we get a precise characterization of those homomorphisms π (with $\pi\mathfrak{a}_1\subseteq\mathfrak{a}_2$) for which π^* is one-to-one.

THEOREM 9.3. *Let P_1 and P_2 be semi-lattices. Let $S_1=S(\mathfrak{a}_1)$, $S_2=S(\mathfrak{a}_2)$ be representative P_1- and P_2-spaces respectively. Assume S_1 is compact Hausdorff. Suppose π is a homomorphism of P_1 into P_2 such that $\pi\mathfrak{a}_1$ is a coinitial subset of $\mathfrak{a}_2$. Then*

(i) *π maps P_1 onto a dense subset of P_2,*

(ii) *π^* is a one-to-one mapping of S_2 into S_1 and*

(iii) *if P_2 is a complete Boolean algebra, $(\pi^*)^+=\pi$ if and only if $I=\{a\in P_1 \mid \pi a\neq 0\}$ is an $\mathfrak{a}_1$-closed subset of P_1 (see 6.4).*

Proof. The assumption that $\pi\mathfrak{a}_1$ is coinitial in the representative P_2-covering ideal $\mathfrak{a}_2$ obviously implies (i).

Let $X\in S_2$. Denote by $[X]$ the dual ideal $\{b_2\in P_2 \mid b_2\geqq\pi a,\ a\in\pi^*X\}$. By 8.1 (2), $[X]\subseteq X$. It will now be shown that $[X]$ is under $\mathfrak{a}_2$ and therefore (since X is minimal under $\mathfrak{a}_2$) $X=[X]$. Let $\beta\in\mathfrak{a}_2$. Then by hypothesis, $\alpha\in\mathfrak{a}_1$ exists so that $\pi\alpha<\beta$. Since π^*X is under $\mathfrak{a}_1$, $\pi^*X\cap\alpha\neq\emptyset$. Hence $\emptyset\neq[X]\cap\pi\alpha <[X]\cap\beta$. Because β was arbitrary, $[X]$ is under $\mathfrak{a}_2$ and $[X]=X$. Now if X and Y in S_2 are such that $\pi^*X=\pi^*Y$, then $[X]=[Y]$. Consequently $X=Y$. This proves that π^* is one-to-one. By 8.1, $\pi^*S_2=S(\mathfrak{a}_1, I)$ (see 6.5), where $I=\{a \mid \pi a\neq 0\}$.

Suppose I is $\mathfrak{a}_1$-closed. By 6.6, $I=I(S(\mathfrak{a}_1, I))$, so $\pi b\neq o$ implies $b\in\pi^*X$ for some $X\in S_2$. Let $a\in P_1$. Suppose $b\in P_1$ satisfies $o\neq\pi b\leqq\pi a$. Then for some $X\in S_2$, $b\in\pi^*X$ and $\pi a\in[X]=X$. Therefore $(\pi^*)^{-1}S_1(b)\cap S_2(\pi a)\neq\emptyset$. Since πP_1 is dense in the complete Boolean algebra P_2, it follows that $((\pi^*)^{-1}S_1(a))^- \supseteq S_2(\pi a)$, so by 8.1 (4), $S_2(\pi a)=((\pi^*)^{-1}S_1(a))^{-o}=S_2((\pi^*)^+a)$. Thus $(\pi^*)^+=\pi$.

Conversely, if $(\pi^*)^+=\pi$, $S_2(\pi a)=((\pi^*)^{-1}S_1(a))^{-o}$ for every $a\in P_1$. Thus

$\pi a \neq o$ implies $S_2(\pi a) \cap (\pi^*)^{-1} S_1(a) \neq \emptyset$, so $X \in S_2(\pi a)$ exists with $a \in \pi^* X$. Therefore, $I \subseteq \bigcup \{\pi^* X \mid X \in S_2\} = I(S(\mathfrak{a}_1, I))$. By 6.5, it follows that I is $\mathfrak{a}_1$-closed. The proof of the theorem is complete.

EXAMPLE 9.4. We shall use 9.3 to construct a homomorphism π which does not satisfy $(\pi^*)^+ = \pi$.

Let S_1 be the real closed interval $[-1, 1]$. Let P_1 be the Boolean algebra of regular open sets of this interval. According to 1.3 and 4.5, S_1 is homeomorphic to the compact Hausdorff representative P_1-space $S(\mathfrak{a}_1)$, where $\mathfrak{a}_1$ is the set of P_1-coverings of S_1. Define $X_0 = \{a \in P_1 \mid 0 \in S_1(a)\}$ and let I be a (proper) maximal dual ideal of P_1 with $X_0 \subset I$. Since $X = X_0$ is the only point of $S(\mathfrak{a}_1)$ satisfying $X \subseteq I$, it follows from 6.6 that I is not $\mathfrak{a}_1$-closed.

Let P_2 be the two-element Boolean algebra $\{o, u\}$. Define π to be the unique homomorphism of P_1 on P_2 with the kernel I^c. This π maps $\mathfrak{a}_1$ onto the representative P_2-covering ideal $\mathfrak{a}_2 = \{\{u\}, \{o, u\}\}$. Thus the hypotheses of 9.3 are satisfied. By 9.3 (iii), π cannot satisfy $(\pi^*)^+ = \pi$.

PART IV. APPLICATIONS

Two applications of the theory developed in the preceding parts will now be presented. First, we shall study compactification, the process of imbedding a topological space as a dense subset in a compact space. The second application is a method of obtaining a topological space from a ring or a lattice. No special effort will be made to obtain new results. Our only aim is to exhibit typical applications of Theorem 4.5.

10. Compactification.

The basic theorem on which most of the discussion of this section depends is

THEOREM 10.1. *Let S be a T_1 P-space (not necessarily compact). Let $\mathfrak{a}$ be a P-covering ideal. Then S is a subspace of $S(\mathfrak{a})$ if and only if* (i) $\mathfrak{a} \subseteq \mathfrak{a}(S)$ *and* (ii) *for any $X \in S$ and any $a \in X$, there exists $\alpha \in \mathfrak{a}$ such that $(\alpha - \{a\}) \cap X = \emptyset$. If S is a representative and* (i) *and* (ii) *are satisfied, then S is dense in $S(\mathfrak{a})$.*

Proof. Suppose (i) and (ii) are satisfied. Let $X \in S$. Then $X \cap \alpha \neq \emptyset$ for all $\alpha \in \mathfrak{a}(S)$, so by (i), X is under $\mathfrak{a}$. If D is a dual ideal of P with $D \subset X$, there exists $a \in X$ with $a \notin D$. By (ii), $\alpha \in \mathfrak{a}$ exists satisfying $D \cap (\alpha - \{a\}) \subseteq X \cap (\alpha - \{a\}) = \emptyset$. But $a \notin D$, so $\alpha \cap D = \emptyset$. It follows that X is minimal under $\mathfrak{a}$ and therefore $X \in S(\mathfrak{a})$. Since X was arbitrary, $S \subseteq S(\mathfrak{a})$.

Conversely, suppose S is a subset of $S(\mathfrak{a})$. Then a covering of $S(\mathfrak{a})$ is also a covering of S, so (i) is satisfied. By 4.5, any point of $S(\mathfrak{a})$ (and hence of S) has the property (ii). The last statement of the theorem is obvious.

REMARK. The P-space topology of a subset S of a P-space S_0 is precisely the relative topology of S induced by S_0.

EXAMPLE 10.2. Let P be the s.l. of open sets of the T_1 space S. Let $\mathfrak{a}$ be the set of all open coverings of S. The conditions of 10.1 are satisfied and it can be

shown (see Example 11.7) that $S(\mathfrak{a})$ is homeomorphic to the Wallman compactification of S.

EXAMPLE 10.3. Suppose again that P is the s.l. of open sets of the T_1 space S, but now let $\mathfrak{a}$ be the set of all (finite) normal coverings of S. Then (see [7]) $\mathfrak{a}$ satisfies 10.1 (ii) if and only if S is completely regular. In this case, $S(\mathfrak{a})$ is a Hausdorff compactification and, in fact, it is not hard to prove that $S(\mathfrak{a})$ is homeomorphic to the Tychonoff compactification of S.

EXAMPLE 10.4. By taking $P=B(S)$ and $\mathfrak{a}=$all P-coverings of S, Theorem 10.1 gives the following result: in order that a space S be homeomorphic to a dense subset of a semi-regular compact T_1 space, it is necessary and sufficient that S be a semi-regular T_1 space which satisfies:

(1) if $x\in A\subseteq S$, where A is a regular open set, then regular open sets $B_1, \cdots, B_n$ exist such that $x\notin B_j$ for all j, and $S=A\cup B_1\cup \cdots \cup B_n$.

It is obvious that (1) is satisfied if S is a regular space. However, examples of semi-regular Hausdorff spaces which do not satisfy (1) can be given.

EXAMPLE 10.5. Suppose S is a locally compact Hausdorff space. Let P be the s.l. of all open sets A such that either A^- or A^c is compact. If $\mathfrak{a}$ is the set of all P-coverings of S, the conditions of 10.1 are satisfied, so $S(\mathfrak{a})$ is a compactification of S. It is evident that $S(\mathfrak{a})$ has the same P-covering ideal as $\dot{S}$, the one-point compactification of S. Thus, by 4.5, $S(\mathfrak{a})$ is homeomorphic to $\dot{S}$.

EXAMPLE 10.6. Theorem 10.1 can be used to prove the following theorem of Hurewicz [3]:

A separable metric space of dimension $\leqq n$ can be imbedded as a dense subset in a compactum of dimension $\leqq n$.

Proof. Let P be the collection of all open sets of S. By 1.3, it can be assumed that S is a P-space. Choose a countable neighborhood basis $A_1, A_2, \cdots$. For each A_j, let B_{ji} be a sequence of open sets such that $\bigcap_i B_{ji} = (A_j)^c$. Let $\alpha_1, \alpha_2, \cdots$ be a simple enumeration of the coverings $\{A_j, B_{ji}\}$. Put $\beta_1=\alpha_1$ and by induction choose β_k $(k>1)$ so that (i) β_k is a P-covering of S, (ii) β_k is a star refinement of $\beta_{k-1}\wedge\alpha_k$, and (iii) β_k has order $\leqq n$. Let $\mathfrak{a}$ be the covering ideal generated by the normal sequence $\{\beta_k\}$. Then $\mathfrak{a}$ fulfills (i) and (ii) of 10.1 by the choice of the α's. Hence $S(\mathfrak{a})$ is a compactum (by 5.4) of dimension $\leqq n$ and S is homeomorphic to a dense subset of $S(\mathfrak{a})$.

PROPOSITION 10.7. *Let S be a representative T_1 P-space. Let $\mathfrak{a}$ and $\mathfrak{b}$ be P-covering ideals satisfying $\mathfrak{a}\subseteq\mathfrak{b}\subseteq\mathfrak{a}(S)$. If $\mathfrak{a}$ satisfies* (ii) *of* 10.1, *so does $\mathfrak{b}$. Moreover, if $\mathfrak{a}$ is a normal covering ideal and π is the identity isomorphism of P, then π^* (see* 8.1) *is a continuous mapping of $S(\mathfrak{b})$ onto $S(\mathfrak{a})$ which satisfies $\pi^*X=X$ for $X\in S$ and $\pi^*(S(\mathfrak{b})-S)=S(\mathfrak{a})-S$.*

Proof. By 4.5 and 8.1.

11. Rings and lattices. In this section, we shall show how a topological space can be obtained from a ring or distributive lattice. Instead of consider-

ing rings and lattices separately however, it is convenient to treat them both as special cases of semi-rings. A semi-ring $\mathcal{R}$ is an algebra with two binary operations, addition (written $+$) which is associative and commutative, and multiplication (indicated by juxtaposition) which is associative and satisfies the distributive laws $f(g+h)=(fg)+(fh)$ and $(g+h)f=(gf)+(hf)$. An element $o\in\mathcal{R}$ is called a zero if it satisfies $f+o=f$ and $fo=of=o$ for all $f\in\mathcal{R}$. An element $1\in\mathcal{R}$ is called a unit if it satisfies $f1=1f=f$ for all $f\in\mathcal{R}$.

LEMMA 11.1. *Let $\mathcal{R}$ be a semi-ring and suppose π is a product homomorphism of $\mathcal{R}$ into a semi-lattice P. Assume that P has a unit u. Let $\mathcal{M}$ be a multiplicative system in $\mathcal{R}$, that is, a nonempty subset of $\mathcal{R}$ which is closed under multiplication. Define*

$$\begin{aligned}\mathfrak{a}(\mathcal{M}) &= a(\mathcal{M};\mathcal{R},\pi,P)\\ &= \{\alpha\in\mathcal{L}_P \mid \alpha > \{\pi f_1,\cdots,\pi f_n\}, f_1+\cdots+f_n\in\mathcal{M}\}.\end{aligned}\tag{1}$$

Then $\mathfrak{a}(\mathcal{M})$ is a P-covering ideal.

Proof. *If $f_1+\cdots+f_n\in\mathcal{M}$ and $g_1+\cdots+g_m\in\mathcal{M}$,* then $f_1g_1+\cdots+f_ng_m=(f_1+\cdots+f_n)(g_1+\cdots+g_m)\in\mathcal{M}$. Hence $\{\pi f_1,\cdots,\pi f_n\}\wedge\{\pi g_1,\cdots,\pi g_m\}=\{\pi(f_1g_1),\cdots,\pi(f_ng_m)\}\in\mathfrak{a}(\mathcal{M})$. Consequently, $\mathfrak{a}(\mathcal{M})$ is a covering ideal which always contains $\{u\}$. (In particular, $\mathfrak{a}(\mathcal{M})$ is nonempty.)

LEMMA 11.2. *If $\mathcal{M}_1$ and $\mathcal{M}_2$ are multiplicative systems in $\mathcal{R}$, and if $\mathcal{M}_1\subseteq\mathcal{M}_2$, then $\mathfrak{a}(\mathcal{M}_1)\subseteq\mathfrak{a}(\mathcal{M}_2)$.*

Proof. Obvious.

LEMMA 11.3. *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be multiplicative systems in $\mathcal{R}$ such that for any $f\in\mathcal{M}_1$, there exists $g\in\mathcal{R}$ satisfying $fg\in\mathcal{M}_2$ or $gf\in\mathcal{M}_2$. Then $\mathfrak{a}(\mathcal{M}_1)\subseteq\mathfrak{a}(\mathcal{M}_2)$.*

Proof. If $\alpha\in\mathfrak{a}(\mathcal{M}_1)$, then $\alpha>\{\pi f_1,\cdots,\pi f_n\}$ where $f_1+\cdots+f_n\in\mathcal{M}_1$. By hypothesis, $g\in\mathcal{R}$ exists so that (for instance) $f_1g+\cdots+f_ng=(f_1+\cdots+f_n)g\in\mathcal{M}_2$. But $\alpha>\{\pi f_1\wedge\pi g,\cdots,\pi f_n\wedge\pi g\}=\{\pi(f_1g),\cdots,\pi(f_ng)\}$, and therefore $\alpha\in\mathfrak{a}(\mathcal{M}_2)$.

COROLLARY 11.4. *If $\mathcal{R}$ contains a unit 1, then $\mathfrak{a}(\{1\})\subseteq\mathfrak{a}(\mathcal{M})$ for any multiplicative system $\mathcal{M}$.*

DEFINITION 11.5. Let $\mathcal{R}$ be a semi-ring and suppose π is a product homomorphism of $\mathcal{R}$ into a semi-lattice P. Assume that P has a unit. Put

$$\mathfrak{a}(\mathcal{R},\pi,P)=\bigcap\{\mathfrak{a}(\mathcal{M};\mathcal{R},\pi,P)\mid \mathcal{M}\text{ is a multiplicative system in }\mathcal{R}\}.\tag{2}$$

The space $S(\mathfrak{a}(\mathcal{R},\pi,P))$ will be called the (π,P)-space of $\mathcal{R}$.

LEMMA 11.6. *If $\mathcal{R}$ is a semi-ring with a unit 1 and π is a product homomorphism of $\mathcal{R}$ onto a sub-s.l. of P, then $\mathfrak{a}(\mathcal{R},\pi,P)=\mathfrak{a}(\{1\};\mathcal{R},\pi,P)$. Moreover, the (π,P)-space of $\mathcal{R}$ is homeomorphic to the $(\pi,\pi\mathcal{R})$-space of $\mathcal{R}$.*

Proof. By 11.4 and 7.1.

EXAMPLE 11.7. Let P be a distributive lattice with a zero and a unit. Then P is also a semi-ring with the operations $\vee$ and $\wedge$ as sum and product respectively. If π_0 is the identity mapping of P onto itself, $\mathfrak{a}=\mathfrak{a}(P, \pi_0, P)=\mathfrak{a}(\{u\}; P, \pi_0, P)=\{\alpha\in\mathcal{L}_P \mid \vee\alpha=u\}$. Suppose X is minimal under $\mathfrak{a}$. If $a\vee b\in X$, there exists $\alpha\in\mathfrak{a}$ such that $X\cap(\alpha-\{a\vee b\})=\emptyset$. But $(\alpha-\{a\vee b\})\cup\{a, b\}\in\mathfrak{a}$, so $a\in X$ or $b\in X$. Thus X is prime. Conversely, if X is prime, $\vee\alpha=u\in X$ implies $X\cap\alpha\neq\emptyset$. Consequently, $S(\mathfrak{a})$ is the P-space of all minimal prime ideals of P. This space is homeomorphic to the Wallman space associated with the dual of P (see [8]).

EXAMPLE 11.8. Let $\mathfrak{R}$ be the lattice of all real-valued continuous functions on a compact Hausdorff space S. Then $\mathfrak{R}$ is a semi-ring with the lattice operations $(f\vee g)(x)=\max\{f(x), g(x)\}$ and $(f\wedge g)(x)=\min\{f(x), g(x)\}$ as sum and product respectively. Let π be the product homomorphism of $\mathfrak{R}$ into $B(S)$ defined by $f\to\{x\in S \mid f(x)>0\}^{-\circ}$. We shall show that the $(\pi, B(S))$-space of $\mathfrak{R}$ is homeomorphic to S.

By 11.2 and 11.3,

$$\mathfrak{a}(\mathfrak{R}, \pi, B(S)) = \bigcap_{n=1}^{\infty} \mathfrak{a}(\langle n\rangle), \tag{3}$$

where $\langle n\rangle$ denotes the multiplicative system whose only element is the function on S which is equal everywhere to n. If $f_1\vee\cdots\vee f_m\in\langle n\rangle$, then $S\subseteq\bigcup_{k=1}^{m}\{x\in S \mid f_k(x)>0\}$, so $\{\pi f_1, \cdots, \pi f_m\}$ is a covering of S. Conversely, by the well known "partition theorem," if $\{A_1, \cdots, A_m\}$ is any covering of S by regular open sets, real-valued continuous functions $f_1, \cdots, f_m$ exist so that $\{x\in S \mid f_k(x)\neq 0\}\subseteq A_k$ and $f_1\vee\cdots\vee f_m\in\langle n\rangle$. Then $\{\pi f_1, \cdots, \pi f_m\}<\{A_1, \cdots, A_m\}$. Thus $\mathfrak{a}(\langle n\rangle)$ is the set of all $B(S)$-coverings of S. By (3), $\mathfrak{a}(\mathfrak{R}, \pi, B(S))$ is also the set of all $B(S)$-coverings of S. It follows from 1.3 and 4.5 that the $(\pi, B(S))$-space of $\mathfrak{R}$ is homeomorphic to S.

REMARK. The homomorphism π of 11.8 is uniquely determined by its kernel $\{f\in\mathfrak{R} \mid f\leqq 0\}$ (see [5]). Thus π depends on the zero function. However, it is clear that the result of 11.8 is the same if π is replaced by any homomorphism of the form $f\to\{x\in S \mid f(x)>f_0(x)\}^{-\circ}$, where $f_0\in\mathfrak{R}$ is fixed. Consequently, 11.8 gives a new proof of the theorem of Kaplansky [4] that a compact Hausdorff space is determined to within homeomorphism by its lattice of real-valued, continuous functions.

EXAMPLE 11.9. Let $\mathfrak{R}$ be a commutative ring which contains a unit. Let S be the space of maximal proper ideals of $\mathfrak{R}$ with the "hull-kernel" topology. Then S is a compact T_1 space and the collection P of all sets $A_f=\{X\in S \mid f\notin X\}$ is a basis for the open sets of S. The mapping $\pi: f\to A_f$ is obviously a product homomorphism of $\mathfrak{R}$ onto the semi-lattice P. We shall show that the (π, P)-space of $\mathfrak{R}$ is homeomorphic to S.

If $f_1+\cdots+f_n=1$, and if $X\in S$, then $f_i\notin X$ for at least one i. For other-

wise, $1=f_1+\cdots+f_n\in X$, contrary to the assumption that X is proper. Thus $A_{f_1}\cup\cdots\cup A_{f_n}=S$. Conversely, if $A_{f_1}\cup\cdots\cup A_{f_n}=S$, then the ideal generated by $\{f_1,\cdots,f_n\}$ is contained in no maximal proper ideal of $\mathfrak{R}$. Hence $\{f_1,\cdots,f_n\}$ generates the unit ideal and $g_1,\cdots,g_n$ exist such that $g_1f_1+\cdots+g_nf_n=1$. This shows that $\mathfrak{a}(\mathfrak{R},\pi,P)=\mathfrak{a}(\{1\};\mathfrak{R},\pi,P)$ is the set of all P-coverings of S. Thus, by 4.5, the (π,P)-space of $\mathfrak{R}$ is homeomorphic to S.

EXAMPLE 11.10. Let $\mathfrak{R}$ be a ring of functions on a locally compact Hausdorff space S. Assume that the functions of $\mathfrak{R}$ take their values in an integral domain with a unit. Finally suppose:

(i) for any $f\in\mathfrak{R}$, the set $\{x\in S|f(x)\neq 0\}$ is open,

(ii) if A is open in S and $x\in A$, then $f\in\mathfrak{R}$ exists so that $f(y)=0$ if $y\in A^c$ and $f(y)=1$ for all y in some neighborhood of x.

Let π be the mapping of $\mathfrak{R}$ into $B(S)$ defined by $f\to\{x\in S|f(x)\neq 0\}^{-o}$. Because of (i) and (ii), π is a product homomorphism which can be defined intrinsically in terms of the ring structure of $\mathfrak{R}$ (see [5]). Suppose P is a sub-algebra of $B(S)$ which contains $\pi\mathfrak{R}$. Then it is possible to show that

$$\mathfrak{a}_1\subseteq\mathfrak{a}(\mathfrak{R},\pi,P)\subseteq\mathfrak{a}_0, \tag{4}$$

where $\mathfrak{a}_0$ is the set of all P-coverings of S and $\mathfrak{a}_1$ is the collection of P-coverings which contain at least one regular open set whose complement is compact. Consequently, the (π,P)-space of $\mathfrak{R}$ is a compactification of S. If S is compact, then by 7.1, the (π,P)-space of $\mathfrak{R}$ is homeomorphic to S. Finally, if every element of $\mathfrak{R}$ has a compact carrier, and if P is the Boolean algebra of those regular open sets A such that either A^- or A^c is compact, then the (π,P)-space of $\mathfrak{R}$ is homeomorphic to $\dot{S}$.

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