SETS OF "POSITIVE" FUNCTIONS IN $H$-SYSTEMS

BY

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Introduction. In [1] Ambrose has defined $H$-systems to be Hilbert spaces in which multiplication is "partially defined." If $H$ is such a system and $a$ is in $H$, then $L_a$ and $R_a$ are the (not necessarily everywhere defined) operators of left and right multiplication by $a$ and the bounded algebra of $H$, written $A(H)$, is $[a | L_a$ and $R_a$ are everywhere defined$]$. We define the associated ring of operators of $H$, written $W(H)$, to be the weak closure of $[L_a | a$ is in $A(H)]$.

If $G$ is a separable, locally compact, unimodular group and $H(G)$ is the $L^2$ space of $G$ under Haar measure with multiplication "partially defined" by convolution as in [1], then $H(G)$ is an $H$-system. The left regular representation represents $G$ faithfully as a group of unitary operators on $H(G)$ each of which commutes with every element of $[R_f | f \in A(H(G))]$. However, it is known [6 or 7] that $W(H)$ is the commutant of $[R_f | f \in A(H)]$ so that $l(G) \subset W(H(G))$. If we define $P(G) = [f \in H(G) | f$ is almost everywhere positive on $G]$, then the elements of $l(G)$ have the further property that $l(x)P(G) \subset P(G)$. The main result of §1 is that these properties completely characterize $l(G)$, i.e., the only unitary operators in $W(H(G))$ which take $P(G)$ into itself are the elements of $l(G)$. Using this result we prove that groups whose $H$-systems are isomorphic in a manner preserving positivity are themselves isomorphic. Similar results for the $L_1$ algebra of a group have been obtained by Kawada [8] and Wendel [9].

The question now arises: given an $H$-system $H$ and a subset $P$ of $H$, when is $H$ the $H$-system of the group of unitary operators in $W(H)$ which take $P$ into itself? In §2 a set of necessary and sufficient conditions is found and by means of these it is shown that any homomorphism of $H(G_1)$ onto a left ideal in $H(G_2)$ which preserves positivity arises in a natural way from a homomorphism of $G_2$ onto $G_1$.

1. Characterization of $l(G)$. Throughout this section we assume that $G$ is a fixed separable, locally compact, unimodular group.

Lemma 1.1. If $G' = [U \in W(H(G)) | UP(G) \subset P(G)$ and $U$ is unitary], then $G'$ is a topological group in the strong operator topology. $G \subset G'$ and the topology of $G$ is that induced from the strong topology on $G'$.

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(*) The numbers in brackets refer to the bibliography at the end of the paper.

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(2) If $P$ is a property of some elements of a set $S$, then we write $[s | P(s)]$ for the subset consisting of these elements. In general, we use the notation of [2] for the elementary operations on sets. We write $c(A)$ for the characteristic function of the set $A$.

(3) The proof of this in [1] is incorrect; see [3].

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Proof. If $U$ and $V$ are in $G'$, then trivially $UV$ is, and for any $f$ and $g$ in $P(G)$, $(U^*f, g) = (f, Ug) \geq 0$, i.e., the inner product of $U^*f$ by any element of $P(G)$ is positive, so $U^*f$ is in $P(G)$, so $U^* \in G'$. $G'$ is strongly closed in the group of all unitaries in $W(G)$ which is known to be a topological group, so $G'$ is a topological group.

Since continuous functions of compact support are dense in $H(G)$, sets of the form $\{U \in G' \mid \|Uf - f\| < \alpha\}$ for such $f$ form a base for the strong topology on $G'$. If $M$ is the measure of the support of $f$, then there is a neighborhood $A$ of the identity in $G$ such that if $x \in A$, then $\|l(x)f(y) - f(y)\| < aM^{-1/2}$ for all $y$ so that $\|l(x)f(y) - f(y)\| < a$, i.e., $l(A) \subset \{U \in G' \mid \|Uf - f\| < \alpha\}$. Hence every strong open set of $G$ is open. Conversely, if $A$ is a neighborhood of the identity in $G$, we can find a neighborhood $B$ of the identity satisfying $BB^{-1} \subset A$, and if $xB$ and $B$ are not disjoint, then $xb_1 = b_2$ for some $b_1$ and $b_2$ in $B$ so that $x = b_2b_1^{-1} \in BB^{-1} \subset A$. Hence, if $x \in C(A)$, we have $\|c(xB) - c(B)\| = 2^{1/2}\|c(B)\|$ so that $G \cap \{U \mid \|Uc(B) - c(B)\| < 2^{1/2}\|c(B)\|\} \subset A$. This shows that open sets in $G$ are strongly open and completes the proof of the lemma.

Lemma 1.2. $G'$ as above. If $U \in G'$ and $S$ is any set in $G$ of positive finite measure, then for some positive number $a(S)$ and measurable set $V(S)$, $U(c(S)) = a(S)c(V(S))$.

Proof. For any $a > 0$ define $f_a$ and $g_a$ by $f_a(x) = U(c(S))(x)$ if this is greater than $a$, $f_a(x) = 0$ otherwise, and $g_a = U(c(S)) - f_a$. It will be sufficient to show that, for every $a$, either $f_a$ or $g_a$ is zero. Now, $c(S) = U^*f_a + U^*g_a$, but $U^*f_a$ and $U^*g_a$ are a.e. positive functions satisfying $(U^*f_a, U^*g_a) = (f_a, g_a) = 0$; hence, for some measurable sets $S_a$ and $S_a'$ whose union is $S$ and whose intersection is of zero measure, $U^*f_a = c(S_a)$ and $U^*g_a = c(S_a')$. If neither $f_a$ nor $g_a$ is zero, we can find an $x$ in $G$ for which the Haar measure of $S_a \times x^{-1} \cap S_a'$ is not zero. Define $T_a = S_a \cap S_a' \times x$ and $T_a' = S_a \times x^{-1} \cap S_a' = T_a \times x^{-1}$. Since $c(T_a)$ and $c(S_a') - c(T_a')$ are orthogonal functions in $P(G)$, so are $U(c(T_a'))$ and $U(c(S_a') - c(T_a'))$, $c(T_a')$ so must be restrictions of $U(c(S_a))$ to subsets of its support. Similarly, $U(c(T_a'))$ is a restriction of $g_a$ so that for a.a. $x$ in $G$, $U(c(T_a'))(x) \leq a$ and $U(c(T_a'))(x)$ is either 0 or $>a$. But, if $r$ is the right regular representation, $U(c(T_a')) = U(r(x)c(T_a')) = r(x)U(c(T_a'))$, which is impossible.

Lemma 1.3. In the above lemma, $a(S) = 1$.

Proof. If $S_1 \subset S_2$, then $U(c(S_1))$ is a restriction of $U(c(S_2))$ so $a(S_1) = a(S_2)$. If $S_1$ and $S_2$ are arbitrary, choose an $x$ in $G$ so that $S_1 \cap (S_2 x) = T$ has nonzero measure, then $a(S_1)c(U(Tx^{-1})) = U(r(x)c(T)) = r(x)a(S_1)c(U(T))$, so $a(S_1) = a(S_2) = a$.

By a basic sequence we shall mean a countable set $(S_n)$ of neighborhoods.
of the identity having the property that if $S$ is any neighborhood of the identity, then $S_n \subseteq S$ for large enough $n$. If $(S_n)$ is a basic sequence, then $\lim (1/\|c(S_n)\|_1) L_{U(c(S_n))}$ approaches the identity operator strongly.

Now let $(S_n)$ be a basic sequence so that

$$1 \leq \lim \inf \left( \frac{1}{\|c(S_n)\|_1} \right) L_{U(c(S_n))} \|c(S_n)\|_1 \geq \lim \inf \left( \frac{1}{\|c(S_n)\|_1} \right) c(U(S_n)) \|c(S_n)\|_1.$$

But $\|c(S_n)\|_1 = (c(S_n), c(S_n)) = a^2 \|c(U(S_n))\|_1$, and substituting this in the above gives $1 \leq 1/a$. Applying this to $U^*$ which multiplies characteristic functions by $1/a$ gives the opposite inequality and completes the proof.

For the basic sequence $(S_n)$ let $F_n = (1/\|c(S_n)\|_1) c(S_n)$.

**Lemma 1.4.** If $F_n$ and $S_n$ are as above and $m$ is Haar measure on $G$, then for every integer $n$ there is an $x$ in $G$ and an integer $k$ for which $m(U(S_k) \cap xS_n) \geq (n/(n+1)) m(S_n) k$.

**Proof.**

$$L_{U F_n} c(S_n)(x) = (1/m(S_n)) (c(U(S_n)) c(S_n))(x) = (1/m(S_n)) m(S_n \cap U(S_n)) = (1/m(S_n)) m(xS_n \cap U(S_n)),$$

so that if the lemma is false, $L_{U F_n} c(S_n)(x) \leq (n/(n+1))$ for all $x$ and $k$. However, $L_{U F_n} c(S_n)(x)$ approaches $U$ strongly, so this is impossible.

**Theorem 1.1.** $G = G'$.

**Proof.** If $U \subseteq G'$ we can choose, for some sequence $S_n$, integers $k(n)$ and elements $x_n$ in $G$ to satisfy Lemma 1.4. We wish to show that $l(x_n^{-1}) L_{U F_n(k(n))}$ approaches the identity strongly. $l(x_n^{-1}) \equiv f_n$ where $f_n = (1/m(S_k(n))) c(x_n^{-1} U(S_k(n)))$. If we define $T_n = x_n^{-1} U(S_k(n)) \cap S_n$ then, since the $T_n$ have nonzero measure and get arbitrarily small, the sequence $(1/m(T_n)) L_{c(T_n)}$ approaches the identity strongly. However, $\|L_{f_n} - (1/m(T_n)) L_{c(T_n)}\|_1 \leq \|l_n - (1/m(T_n)) c(T_n)|_1 = 2(1 - m(T_n)/m(S_k(n))) \to 0$ since $m(T_n) \geq (n/(n+1)) m(S_k(n))$. Hence, $l(x_n^{-1}) L_{U F_n(k(n))}$ approaches the identity strongly so $l(x_n)$ approaches $U$ strongly. The strong convergence of $l(x_n)$ implies that $(x_n)$ is a Cauchy sequence and $U = l(\lim x_n)$.

If $H$ is any $H$-system with elements $a$ and $b$, then we write $ab$ for their product when it is defined. Consistent with this notation, if $f$ and $g$ are functions in $L_2$ of $G$, we write $fg$ for their convolution and not their pointwise product.

**Lemma 1.5.** If $G_1$ and $G_2$ are separable, locally compact, unimodular groups and $w$ is a linear transformation of $H(G_1)$ into $H(G_2)$ satisfying:

1. $w(H(G_1))$ is a left ideal in $H(G_2)$,
2. $w(P(G_1)) \subseteq P(G_2)$,
3. for any $f$ and $g$ in $H(G_1)$, $(w(f), w(g)) = (f, g)$,

The referee has outlined a different proof of this theorem which does not require separability.
(4) if $f$ and $g$ are in $H(G_1)$ and $fg$ is defined then $w(f)w(g)$ is defined and $w(fg) = w(f)w(g)$,

then there is a homomorphism $\tilde{w}$ of $G_2$ into $G_1$ such that $l(x)w(f) = w(l(\tilde{w}(x)))f$
for any $x$ in $G_2$ and $f$ in $H(G_1)$.

Proof. If $f$ is in $H(G_1)$ and $x$ is in $G_2$, then $l(x)w(f)$ is in $w(H(G_1))$, so there
is a unique element $T(x)f$ in $H(G_1)$ satisfying $wT(x)(f) = l(x)w(f)$. Clearly
$T(x)$ is an isometric linear transformation. If $f$ and $g$ are in $P(G_1)$, then
$(T(x)f, g) = (wT(x)(f), w(g)) = (l(x)w(f), w(g)) \geq 0$, so $T(x)P(G_1) \subset P(G_1)$.
Also, $T(x)$ is in $W(G_1)$ since $W(G_1)$ is the commutant of $[R_f | f \in A(G_1)]$ and for any $f \in A(G_1), g \in H(G_1), wT(x)R_f(g) = wT(x)(gf) = l(x)(w(g)w(f)) = w(T(x)g)w(f) = wR_fT(x)(g)$, i.e., $T(x)R_f = R_fT(x)$.

The map $T: G_2 \to W(G_1)$ satisfies (i), $T(x)T(y) = T(xy)$, and (ii), $T(x)^* = T(x^{-1})$. These follow from $wT(x)T(y) = l(x)wT(y) = l(x)yw = l(xy)w = wT(xy)$ and $(T(x)f, g) = (wT(x)f, w(g)) = (w(f), l(x^{-1})w(g)) = (w(f), wT(x^{-1})g) = (f, T(x^{-1})g)$ respectively. Equation (ii), plus the fact that $T(e) = I$, implies that $T(x)$ is unitary; hence, $T(x) = l(w(x))$ for some $w(x)$ in $G_1$ and equation (i) implies that $\tilde{w}$ is a homomorphism.

To show the continuity of $\tilde{w}$, let $f$ be an element of $H(G_1)$ and $S = [x \in G_1 | l(x)f - f\| < a]$; then $\tilde{w}^{-1}(S) = [y \in G_2 | l(\tilde{w}(y))f - f\| < a] = [y \in G_2 | l(y)w(f) - w(f)\| < a]$, which is open. Since sets of this form are a sub-basis for the topology of $G_1$, this completes the proof.

Theorem 1.2. If $G_1$ and $G_2$ are locally compact, separable, unimodular
groups, and $w$ is a linear map of $H(G_1)$ onto $H(G_2)$ satisfying the conditions
of Lemma 1.5, then $\tilde{w}$ is an isomorphism onto.

Proof. Trivially $w^{-1}$ satisfies conditions (1) and (3) of Lemma 1.5. If $f$ is
in $P(G_2)$ and $g$ is in $P(G_1)$, then $(w^{-1}(f), g) = (f, w(g)) \geq 0$, so $w^{-1}(f)$ is in
$P(G_1)$, i.e., condition (2) is satisfied. To prove (4) it will be sufficient [1] to
show that if $gf$ is defined in $H(G_2)$ and $h$ is in $A(G_1)$, then $(w^{-1}(g), zw^{-1}(f)^*) = (w^{-1}(gf), z)$. Trivially $w^{-1}(f)^* = w^{-1}(f^*)$ so $(w^{-1}(g), zw^{-1}(f)^*) = (g, w(z)f^*) = (gf, w(z)) = (w^{-1}(gf), z)$. Hence Lemma 1.5 gives a homomorphism $\tilde{w}^{-1}$ of
$G_1$ into $G_2$ and $l(\tilde{w}(\tilde{w}^{-1}(x))) = w(\tilde{w}^{-1}(x)) = w(w^{-1}l(x)) = l(x)$ so $\tilde{w}\tilde{w}^{-1}(x) = x$ and similarly $\tilde{w}^{-1}\tilde{w}(x) = x$, which completes the proof.

The assertion of Theorem 1.2 is not true if the assumption of positivity of $W$ is dropped. Ambrose proved [1, Theorem 10] that all Abelian $H$-systems are essentially the same algebraically except for dimension and it is an immediate corollary of this that any two finite Abelian groups of the same order
have isomorphic $H$-systems.

2. HP systems. We shall say that a subset $P$ of a Hilbert space $H$ is a
set of non-negative functions in $H$ if there is a representation $\phi$ of $H$ as the $L_2$
of some measure space such that $\phi(P)$ is the set of almost everywhere non-
negative functions in this $L_2(\mathbb{L})$. We write $x \leq y$ to mean that $y - x$ is in $P$, and $x \leq S$ to mean that $\{s - x | s \in S\} \subset P$. For any countable set $Q \subset P$ there is defined an element $\inf Q$ in $P$ and if, for some $y$, $x \leq y$ for all $x$ in $Q$, there is also defined an element $\sup Q \leq y$ in $P$ having all the usual properties. If $Q$ is a convex subset of $P$ we write $\inf Q$ for the unique element of minimal norm in the uniform closure of $Q$ and if, for some $y$, $Q \leq y$ we write $\sup Q$ for $\inf \{x | Q \leq x\}$. These definitions are consistent with one another.

If $H$ is a proper $H$-system let $C(H)$ be the dense subset consisting of all finite sums of products. We shall be concerned with the linear map $[\cdot]$ from $C(H)$ to the set of weakly continuous functions on $W(H)$ defined by $[\sum_{i} f_i](T) = \sum (f_i, T(g^*))$. (Note that this map is well defined for by [10, p. 76] we can find a set $(x_a)$ of approximate left identities in $H$ and since $H$ is separable we can choose a countable subset $(x_n)$ which is still a set of approximate left identities and then $[x](T) = \lim (x, Tg_i)$.)

**Definition.** A pair $(H, P)$ is an $HP$ system if $H$ is a proper $H$ system, $P$ is a set of non-negative functions in $H$, and the following conditions are satisfied; when $G$ is the group of unitaries in $W(H)$ which carry $P$ inside itself:

1. $C(H) \cap P$ is dense in $P$.
2. If $(f_i)$ is a countable subset of $C(H)$ whose sup exists and $\sup (\{f_i\}) \geq [f]$ for some $f$ in $C(H) \cap P$, then $\sup (f_i) \geq f$.
3. If $N$ is any strong neighborhood of $I$ in $G$ there is a nonzero $f$ in $C(H) \cap P$ with $[f]$ vanishing outside $N$.

If $G$ is a separable, locally compact, unimodular group, $H$ its $H$-system, and $P$ the almost everywhere non-negative functions in $H$, then, by Theorem 1.1, $(H, P)$ is an $HP$ system. The main result of this section is that the converse is also true.

We assume until further notice that $(H, P)$ is a fixed $HP$ system, and write $C$ for $C(H) \cap P$.

**Lemma 2.1.** $C = \{f | f \in C(H) \text{ and } [f] \geq 0\}$, $P = P^*$, and if $p$ and $q$ are in $P$ and $pq$ is defined, then $pq$ is in $P$.

**Proof.** If $f$ is in $C(H)$ and $[f] \geq 0$, then $f$ is in $C$ by condition 2. If $f$ is in $C$ and $[f] \leq -e < 0$ on some open set $N$, choose $h$ in $C$ with $|\{h\}(U)| \leq e$ and $[h]$ vanishing outside $N$, then $\sup (\{h\}, [f]) \geq [f + 2h]$ so by condition 2, $f + h \geq \sup (f, h) \geq f + 2h$ which is impossible.

If $f$ is in $C$ then $[f^*]$ is the complex conjugate of $[f]$, hence $f^*$ is in $C$ and by condition 1 this implies $P = P^*$.

Finally $[pq](U) = (p, Uq^*) \geq 0$ so $pq$ is in $C$.

(*) Nagy, in [4], proves that $P$ is a set of non-negative functions in $H$ if and only if the following conditions are satisfied: $(u, v) \geq 0$ for every $u$ and $v$ in $P$, if $(u, v) \geq 0$ for every $v$ in $P$ then $u$ is in $P$, and if $u_1, u_2, v_1$, and $v_2$ are in $P$ and $u_1 + u_2 = v_1 + v_2$, then there are elements $w_{11}, w_{12}, w_{21}, w_{22}$ in $P$ such that $u_1 = w_{11} + w_{12}$ and $v_1 = w_{11} + w_{21}$ for $i = 1, 2$. 

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If \( f \) is in \( C \) and \( A \) is a subset of \( G \), we say that \( f \) covers \( A \) if \( \{f\}(U) \geq 1 \) for all \( U \) in \( A \), and we say that \( A \) is bounded if there is an element of \( C \) which covers it. If \( A \) is bounded, \( \Gamma(A) \) is to be the (nonempty) set \( \left\{ \sup F \mid F \subset C, F \leq f \right\} \) for some \( f \), and there exists an enumerable set of sets \( X_i \subset A \) and elements \( f_i \) in \( F \) such that \( f_i \) covers \( X_i \) and \( \sum X_i = A \). \( \Gamma(A) \) is convex since if \( F_1 \) and \( F_2 \) are subsets of \( C \) satisfying the above conditions, then so does the set \( F = \{(1/2)(f_i + f_j) \mid f_i \text{ is in } F_i \} \) and \( \sup F = (1/2)(\sup F_1 + \sup F_2) \). We define, for bounded \( A \), \( d(A) = \inf \Gamma(A) \).

**Lemma 2.2.** If the sets \( A, B, \) and \( A_i \) are bounded, \( A \subset B \), and \( U \) an element of \( G \), then

(i) \( d(A) \leq d(B) \),
(ii) \( d(A_i) \leq \inf (d(A_i)) \),
(iii) \( A^{-1} \) is bounded and \( d(A^{-1}) = d(A)^* \),
(iv) \( UA \) is bounded and \( d(UA) = Ud(A) \),
(v) if \( A = \sum A_i \), then \( d(A) = \sup (d(A_i)) \).

**Proof.** The first four assertions are trivial and in the fifth it is clear that \( d(A) = \inf (d(A_i)) \). Choose subsets \( F_i \) of \( C \) so that \( \|\sup F_i - d(A_i)\|^2 \leq \epsilon 2^{-i} \), then \( \sup (\sup F_i) = \sup (\sum F_i) \geq d(A) \) and \( \|d(A) - \sup (d(A_i))\|^2 \leq \|\sup (F_i) - d(A_i)\|^2 \leq \sum \|\sup (F_i) - d(A_i)\|^2 \leq \epsilon \).

**Lemma 2.3.** If \( A \) and \( B \) are closed and bounded, then \( d(AC \cap B) = \inf (d(A), d(B)) \). If further \( A \subset B \), then \( d(B - A) = d(B) - d(A) \).

**Proof.** Suppose \( A \) and \( B \) are disjoint. For any \( V \) in \( B \) there is some neighborhood \( N \) of the identity for which \( VN \) does not intersect \( A \). Choose \( f_0 \) according to assumption 3 for this \( N \) and let \( f = 2f_0/\max |f_0| \) so that \( d(A) = \inf (|Uf|, |Vf|) = 0 \) if \( U \) is not in \( A \). In this case \( |Uf + Vf| = \sup (|Uf|, |Vf|) \) so that \( Uf + Vf \leq \inf (Uf, Vf) \) by assumption 2 and this implies that \( \inf (Uf, Vf) = 0 \) so we must have \( (Uf, Vf) = 0 \). For each \( U, Uf \) covers some neighborhood of \( V \) and we can choose a countable subcovering \( (Uf) \) of \( A \). Then \( d(A), Vf) \leq \sup (Uf, Vf) \). Again we can choose a countable subcovering of \( B \) from among all such \( Vf \)'s so \( d(A), d(B)) = 0 \), and hence \( \inf (d(A), d(B)) = 0 \).

We can now prove the second assertion. If \( (N_i) \) is a basic sequence, then by the previous lemma \( d(B - A) + d(A) = \lim (d(B - AN_i) + d(A)) = \lim d(B - AN_i + A) \leq d(B) \). The opposite inequality is trivially true for any bounded sets \( B - A \) and \( A \).

The first assertion now follows from \( \inf (d(A), d(B)) = \inf (d(A - A \cap B), d(B - A \cap B)) + d(A \cap B) = \lim \inf (d(A - (A \cap B)N_i), d(B - (A \cap B)N_i) + d(A \cap B) = d(A \cap B) \).

The set \( R_0 = \left\{ \sum^n(B_i - A_i) \mid A_i \text{ and } B_i \text{ are closed and bounded, } B_i \subset A_i \right\} \text{ and the summands are mutually disjoint} \) is a ring.

**Lemma 2.4.** If \( X_1 \) and \( X_2 \) are in \( R_0 \) and are disjoint, then \( \inf (d(X_1), d(X_2)) \)
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... 0, and 

$$d(X_1 \cup X_2) = d(X_1) + d(X_2).$$

If \( X_i \) are mutually disjoint and \( \sum_i X_i = X \) is in \( R_0 \), then 

$$d(X) = \sum_i d(X_i).$$

**Proof.** If \( X = \sum (A_i - B_i) \) and \( (N_i) \) is a basic sequence, then \( X \) is the limit (on \( k \)) of the closed sets \( \sum (A_i - B_i N_k) \) and this by the previous lemma implies the first assertion. The other two are immediate consequences of this one.

The above lemma says that the measure \( m \) on \( R_0 \) defined by: 

$$m(X) = \|d(X)\|^2$$

is countably additive, hence can be extended to the \( \sigma \)-ring \( R \) generated by \( R_0 \).

**Lemma 2.5.** The measure \( m \) is both left and right invariant and \( (G, R, m) \) is a measurable group [2, p. 257].

**Proof.** Since \( d \) is left invariant on \( R_0 \) so is \( m \), and if \( X \) is in \( R_0 \),

$$m(XU) = \|d(XU)\|^2 = \|d(XU)\|^2 = \|d(U^{-1}X^{-1})\|^2 = \|d(X^{-1})\|^2$$

This extends trivially to \( R \). To complete the proof we must show that the shearing transformation \( T: (U, V) \rightarrow (U, UV) \) of \( G \times G \) onto itself preserves measurability. Since \( R \) is generated by the open bounded sets which it contains, it will be sufficient to show that \( T(A \times B) \) is measurable if \( A \) and \( B \) are open and bounded. But if \( (U, V) \) is in \( T(A \times B) \), that is, \( U \) is in \( A \) and \( V \) is in \( UB \), and \( N \) is a bounded neighborhood of the identity with \( NU \subset A \) and \( N^{-1}N \subset UB^{-1} \), then \( NU \times NV \subset T(A \times B) \), and if \( (N_i, U_i \times N_i V_i) \) is a countable subcovering, \( T(A \times B) = \sum (N_i U_i \times N_i V_i) \).

**Lemma 2.6.** The Weil topology with respect to the measure \( m \) coincides with the strong topology.

**Proof.** A base for the Weil topology is given by sets of the form 

$$[U \mid m(\rho(S, US)) < \varepsilon]$$

(for \( S \) in \( R \) and \( \varepsilon > 0 \) where \( \rho \) is the symmetric difference). If \( S = \sum S_i \) where the \( S_i \) are mutually disjoint elements of \( R_0 \) and \( V \) is in the strongly open set \( \prod \{ U \mid U d(S_i) - d(S_i) \|^2 < \varepsilon 2^{-i} \} \), then 

$$m(\rho(S, VS)) \leq \sum \rho(S_i, VS) = \sum \| Vd(S_i) - d(S_i) \|^2 + \sum \rho(S_i, VS) < \varepsilon$$

if \( n \) is chosen large enough. Hence every Weil open set is strongly open. Conversely if \( N \) is a strong neighborhood of \( I \), choose a neighborhood \( S \) satisfying \( SS^{-1} \subset N \). Then if \( U \) is not in \( N \), \( S \cap US = 0 \) so inf \( (d(S), Ud(S)) = 0 \) so \( d(S), Ud(S) = 0 \) and hence \( [U \mid (d(S), Ud(S)) > 0] \subset N \). It only remains to show that \( d(S) \neq 0 \), but this is a trivial consequence of assumptions 2 and 3.

The above lemma implies that \( G \) is complete in the Weil topology, hence by Weil's theorem [2, p. 275] \( G \) is a locally compact group in this topology and \( m \) is its Haar measure.

Let \( S \) be the linear transformation of \( H(G) \) into \( H \) which takes \( c(X) \) into \( d(X) \) for \( X \) in \( R \). \( S \) takes positive elements into positive elements and \( (Sx, Sy) \)
If we define $T x = [x]$ for $x$ in $C(H)$, then for $x$ and $y$ in $P$, $(T x, T y) = \sup (a, b)$, $a$ and $b$ take on only a finite number of values, all non-negative, $a \leq [x]$ and $b \leq [y] \leq (x, y)$ since $(a, b) = (S a, S b)$ and $S a \leq x$, $S b \leq y$. Hence $T$ can be extended to a transformation of $H$ onto $H(G)$ which preserves positivity.

**Theorem 2.1.** $ST$ and $TS$ are the identity operators, $S$ and $T$ preserve positivity and take adjoints into adjoints. For every $U$ in $G$ we have $T U = l(U) T$ and $U S = S l(U)$. If $ab$ is defined in $H$, then $T a T b$ is defined in $H(G)$ and $T a T b = T(ab)$; if $xy$ is defined in $H(G)$, then $S x S y$ is defined in $H$ and $S x S y = S(xy)$.

**Proof.** To show that $TS$ is the identity it will be sufficient to show that $T S c(X) = c(X)$. Choose $(f^n) \subset C$ so that, for fixed $n$, $(f^n)$ gives a covering of $X$, $d(X) = \lim_n \sup \| f^n \|$ and $c(X) = \lim_n \sup \| f^n \|$ Then

$$T(d(X)) = \lim \sup \| f^n \| \sup \| f^n \| = c(X),$$

but since $\| T(d(X)) \| \leq \| c(x) \|$ this proves the assertion.

If $E = ST$, then $E(H) = S(H(G))$, $E$ preserves positivity, $E^* = E$, and $(E^* E x, y) = (E x, E y) \leq (x, y)$ for all $x$ and $y$ in $P$, which implies that $E^* E x \leq x$ for all $x$ in $P$. If $x$ is in $P$, then so is $p = E x - E^* E x = E x - E^* E x$ and, for any $y$, $(p, E y) = 0$. Hence if $z$ is in $A(H) \cap P \cap S(H(G))$, for example if $z = d(X)$ for small enough $X$, then $[p z(U)] = (p, U z^*) = 0$ since $U S(H(G)) = S(H(G)) = S(H(G))^*$ and hence $p z = 0$. But we can choose a $q$ in $P \cap A(H)$ with $0 < q \leq p$ and by assumption 3 we can find $\lambda > 0$ in $C(H) \cap P$ with $[\lambda] \leq \inf (\| z \|^2, \| q \|^2) / 2$ and support contained in

$$|U| \| U z - z \| < \| z \| / 2, |U p - p| < \| q \| / 2$$

so that $\lambda < z z^*$ and $\lambda < q q^*$, which implies $0 < \|z z^*\| < \|zz^*\| q q^* = (q z) = 0$, so $p = 0$. Thus $E = E^* E = E$ and, if $x$ is in $P$, then $x - E x = x - E^* E x \geq 0$ and, for any $y$, $(x - E x, E y) = 0$ so as before $x - E x = 0$, that is, $E = I$.

$T$ and $S$ trivially preserve positivity and adjoints on the sets $C(H)$ and $d(X) | X$ in $R$ respectively, hence everywhere. If $f$ is in $C(H)$, then $[U f](V) = [f](U^{-1} V) = l(U)[f](V)$ so, by continuity, $T U = l(U) T$, and then $U S = S l(U) T S = S l(U)$.

If $T f$ is continuous and has compact support and $f g$ is defined, then $(T f)(T g)(U) = (T f, l(U)(T g)^*) = (T f, T U g^*) = (f, U g^*) = [f g](U)$. If $g h$ is defined in $H$ and $f$ is as before, then $(T g, T f T h^*) = (T g, T(f h^*)) = (g, f h^*) = (g h, f) = (T g h, T f)$ so $[1, p. 29]$ $T g T h$ is defined and equal to $T(g h)$. If $S a$ is in $A(H)$, then $S a S b = S(T(S a S b)) = S(ab)$ and by the same argument as before this implies the general case.

**Theorem 2.2.** The homomorphism $\tilde{\omega}$ whose existence is proved in Lemma 1.5 carries $G_2$ onto $G_1$. 


Proof. If \( f \) is in \( A(H(G_1)) \), \( \omega(g) \) is in \( H(G_2) \), and \( \omega(h) \) is the projection of \( z \) into \( \omega(H(G_1)) \), then:

\[
\left< (\omega(f) - \omega(f^*)^* \omega(g), \ z) = \left< (\omega(f) - \omega(f^*)^* \omega(g), \ \omega(h) - \omega(h) \right> - \left< \omega(g), \omega(f^* h) \right> = 0.
\]

Hence \( (\omega(f) - \omega(f^*)^*) \omega(g) = 0 \) and if \( (e_n) \) are a set of approximate identities in \( H(G_2) \), then:

\[
\left< (\omega(f^*)^* - \omega(f^*)) \ y, \ e_n \right> = 0 \text{ so } \omega(f^*)^* - \omega(f^*) \text{ is orthogonal to everything in } \omega(H(G_1)) \cap A(H(G_1)) \text{ which is dense in } \omega(H(G_1)) \text{ [1., p. 41] so}
\]

\[
\| \omega(f^*)^* \|^2 = < \omega(f^*), \omega(f^*) > , \text{ that is, } \omega(f^*)^* = \omega(f^*).
\]

Suppose \( \rho = \sum_{i} f_i g_i \) is in \( C(H(G_1)) \) and \( \rho \rho \geq 0 \) on \( l(\omega(G_2)) \), then if \( x \) is in \( G_2 \),

\[
\left< \omega(f^*)^* l(x), \omega(g^*) \right> = \sum_i \left< \omega(f_i), \omega(g_i^*) \right> = \sum_i \left< \omega(f_i), \omega_l(\omega(x))(g_i^*) \right> = \sum_i \left< f_i, \omega(\omega_l(x))(g_i^*) \right> = \left< \rho, l(\omega_l(x)) \right> \geq 0.
\]

Thus \( \omega(\rho) \) is in \( P(G_2) \) and if \( q \) is in \( P(G_1) \),

\[
\left< \omega(\rho), \omega(q) \right> = \left< \omega(\rho), \omega(q) \right> \geq 0 \text{ so } \rho \text{ is in } P(G_1). \text{ Now all the requirements of the definition of an HP system are satisfied for } H(G_1), P(G_1) \text{ with the group } G \text{ replaced by } l(\omega(G_2)) \text{ and the proof of Theorem 2.1 goes through as before, } \omega(G_2) \text{ being complete, to give } H(\omega(G_2)) \text{ isomorphic to } H(G_1) \text{ under a positivity preserving map so that, by Theorem 1.2, } G_1 \text{ is isomorphic to } \omega(G_2).}
\]

**Bibliography**