THE INVERSION OF THE GENERALIZED FOURIER TRANSFORM BY ABELIAN SUMMABILITY

BY

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1. The fundamental ideas of the generalized Fourier transform have been presented by S. Bochner [1; 2], for functions which are either locally integrable or locally square integrable. A parallel line of thought has been followed by N. Wiener [7] and H. R. Pitt [4] in their theory of generalized harmonic analysis. While the parts of the Wiener theory which are connected with the idea of the generalized Fourier transform do not treat functions which are as large at infinity as those treated by Bochner, they do come closer to giving a theory for L_p integrability than does Bochner (see [4]). More recently L. Schwartz [5] has given a treatment of the generalized Fourier transform by quite different methods.

One of the weak points of the theory has been the lack of satisfactory inversion formulas, particularly for the generalized transform of order greater than one. The main part of this paper is devoted to showing that by the use of Abelian summability, the generalized Fourier transform may be inverted in the metric of the proper function space.

The last part of the paper attempts to unify some of the ideas of generalized harmonic analysis and the generalized Fourier transform and to present a generalization of a well known theorem of Titchmarsh [6, Theorem 74].

The function space L_p^k, $1 \leqq p < \infty$, consists of those functions for which the norm

$$\left(\int_{-1}^{1} |f(x)|^p dx + \int_{|x|>1} \left|\frac{f(x)}{x^k}\right|^p dx\right)^{1/p} = \|f\| \tag{1}$$

is finite. It is easily seen that this is a norm.

Let

$$K_k(x) = \begin{cases} \dfrac{1}{|x|^k}, & |x| > 1, \\ 1, & |x| \leqq 1. \end{cases} \tag{2}$$

Then L_∞^k consists of those functions for which the norm,

$$\operatorname*{ess\,sup}_{-\infty < x < \infty} |f(x)K_k(x)| = \|f\|_k, \tag{3}$$

is finite.

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We say f is uniformly continuous in L_∞^k if $f\in L_\infty^k$ and, given any $\epsilon>0$, there is a δ_1 such that

$$|f(x+\delta)-f(x)|\,K_k(x)<\epsilon \qquad \text{for all } |\delta|<\delta_1 \text{ and all } x. \tag{4}$$

It will be convenient to introduce a somewhat weaker topology in the space L_∞^k than that given by (3). We shall say $f_n(x)$ converges to $f(x)$ in the ω topology of L_∞^k as $n\to\infty$ if, given any $h(x)\in L_1^0$,

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}|f(x)-f_n(x)|\,K_k(x)h(x)dx=0. \tag{5}$$

If

$$\phi\in L_p^{k-1},\qquad 1<p<\infty,\qquad L_1^k,\quad\text{or}\quad L_\infty^{k-2},$$

then it has a generalized kth Fourier transform defined by

$$\begin{aligned} E(k,x)&=\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty}\frac{\phi(t)}{(it)^k}\left\{e^{ixt}-L_k(t,x)\right\}dt\quad\text{where}\\ L_k(t,x)&=\begin{cases}\displaystyle\sum_{n=0}^{k-1}\frac{(itx)^n}{n!}, & |t|\leqq 1,\\ 0, & |t|>1.\end{cases}\end{aligned} \tag{6}$$

(The special case $1<p\leqq 2$ where functions in L_p^{k-1} have a generalized $(k-1)$ Fourier transform will be treated in §3.)

2. The first problem is to invert (6.) The final inversion result is contained in Theorem 3 which shows the inversion exists in the sense of the metrics defined above.

Let

$$\begin{aligned}\phi_\epsilon(s)&=\frac{(-1)^k}{(2\pi)^{1/2}}(\epsilon-is)^k\int_{-\infty}^{0}E(k,x)e^{x(\epsilon-is)}dx\\ &\quad+\frac{(-1)^k}{(2\pi)^{1/2}}(-\epsilon-is)^k\int_{0}^{\infty}E(k,x)e^{x(-\epsilon-is)}dx.\end{aligned} \tag{7}$$

THEOREM 1.

$$\begin{aligned}\phi_\epsilon(s)&=\frac{\epsilon}{\pi}\int_{-1}^{1}\frac{\phi(t)dt}{(s-t)^2+\epsilon^2}\\ &\quad+\frac{1}{2\pi}\int_{|t|>1}\frac{\phi(t)}{(-it)^k}\left\{\frac{(\epsilon-is)^k}{\epsilon-is+it}-\frac{(-\epsilon-is)^k}{-\epsilon-is+it}\right\}dt.\end{aligned}$$

This can be proved by a direct if somewhat involved computation. One need only substitute (6) in (7), interchange order of integration, and evaluate.

Let

$$(8)\qquad H(s,\epsilon,t)=\begin{cases}-\dfrac{1}{2\epsilon(-i)^k}\{(-\epsilon-is+it)(\epsilon-is)^k \\ \qquad\qquad -(\epsilon-is+it)(-\epsilon-is)^k\} & \text{for } |t|>1 \\ 1 & \text{for } |t|\leqq 1.\end{cases}$$

Then

$$(9)\qquad \phi_\epsilon(s)=\frac{\epsilon}{\pi}\int_{-\infty}^{\infty}\frac{\phi(t)H(s,\epsilon,t)K_k(t)}{(s-t)^2+\epsilon^2}\,dt.$$

Several lemmas will be useful in the proof of the inversion theorem.

LEMMA 1. *If $\phi(t)\in L_p^k$, $1\leqq p<\infty$, then*

$$(10)\qquad \tau(u)=\int_{-\infty}^{\infty}\left|\,[\phi(t)-\phi(u+t)]K_k(t)\right|^p dt$$

is a continuous function of u and $\tau(0)=0$.

Proof. This is an easy generalization of the classical theorem $k=0$ (see [2, p. 98]).

LEMMA 2.

$$1=\frac{\epsilon}{\pi}\int_{-\infty}^{\infty}\frac{H(s,\epsilon,t)K_k(t)dt}{(s-t)^2+\epsilon^2}.$$

Proof. We need only show that if $\phi(t)\equiv 1$, then $\phi_\epsilon(s)\equiv 1$. If we take the 1st generalized transform $E(1, x)$ of $\phi(t)\equiv 1$ we find

$$E(1,x)=\left(\frac{\pi}{2}\right)^{1/2}\operatorname{sgn} x.$$

Computing $\phi_\epsilon(s)$ for this we find

$$\phi_\epsilon(s)=\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty}e^{-isx}e^{-\epsilon|x|}dE(1,x)=1.$$

Integrating by parts k times, equation (7) is obtained with $\phi_\epsilon(s)=1$. The use of Theorem 1 then proves the lemma.

LEMMA 3. *Suppose $|K(x)|<c_0/(1+x^2)$ and $g(t)=\int_{-\infty}^{\infty}K(x)f(t-x)dx$. Then*

$$\text{(a)}\qquad |f(x)|<\frac{c}{1+|x|^2}\quad \textit{implies}\quad |g(t)|<\frac{c_1}{1+|t|^2},$$

(b) $$|f(x)| < \frac{c}{1+|x|} \quad \textit{implies} \quad |g(t)| < \frac{c_1}{1+|t|},$$

(c) $$(1+|x|)f(x) \in L_p^0 \quad \textit{implies} \quad (1+|t|)g(t) \in L_p^0, \qquad 1 < p \leqq \infty.$$

Proof. In (a) and (b) assume $t>0$. The proof for $t<0$ is, of course, the same.

(a) $$\begin{aligned} |g(t)| &\leqq \int_{-\infty}^{t/2} |f(t-x)|\,|K(x)|\,dx + \int_{t/2}^{\infty} |f(t-x)|\,|K(x)|\,dx \\ &\leqq \int_{-\infty}^{t/2} \frac{c}{1+|t-x|^2}\,|K(x)|\,dx + \int_{t/2}^{\infty} |f(t-x)|\,\frac{c_0}{1+x^2}\,dx \\ &\leqq \frac{c}{|t/2|^2+1}\int_{-\infty}^{\infty}|K(x)|\,dx + \frac{c_0}{|t/2|^2+1}\int_{-\infty}^{\infty}|f(x)|\,dx \\ &\leqq \frac{c_1}{1+|t|^2}. \end{aligned}$$

(b) $$\begin{aligned} |g(t)| &\leqq \int_{-\infty}^{t/2} |f(t-x)|\,|K(x)|\,dx \\ &\quad + \int_{t/2}^{\infty} |f(t-x)| \left|\frac{c_0}{1+x^2}\right|^{1/2} \left|\frac{c_0}{1+x^2}\right|^{1/2} dx \\ &\leqq \int_{-\infty}^{t/2} \frac{c}{|x-t|+1}\,|K(x)|\,dx \\ &\quad + \left(\int_{t/2}^{\infty} \frac{|f(t-x)|^2 c_0}{1+x^2}\,dx\right)^{1/2} \left(\int_{t/2}^{\infty} \frac{c_0}{1+x^2}\,dx\right)^{1/2} \\ &\leqq \frac{c}{|t/2|+1}\int_{-\infty}^{\infty}|K(x)|\,dx \\ &\quad + \left(\frac{c_0}{1+|t/2|^2}\right)^{1/2} \left(\int_{-\infty}^{\infty}|f(x)|^2 dx\right)^{1/2} \left(\int_{-\infty}^{\infty}\frac{c_0}{1+x^2}\,dx\right)^{1/2} \\ &\leqq \frac{c_1}{1+|t|}. \end{aligned}$$

(c) We use the well known fact that the convolution of a function in L_1^0 and a function in L_p^0 is also in L_p^0. Therefore

$$\int_{-\infty}^{\infty} K(x)(t-x)f(t-x)dx \text{ is in } L_p^0.$$

Since we wish to show

$$t\int_{-\infty}^{\infty} K(x)f(t-x)dx \text{ is in } L_p^0,$$

it is only necessary to show

$$\int_{-\infty}^{\infty} K(x)xf(t-x)dx \text{ is in } L_p^0.$$

This follows immediately from the fact that $xK(x) \in L_p^0$, $1 < p \leqq \infty$, and that $f(x) \in L_1^0$ if $p < \infty$. If $p = \infty$ the result follows from (b).

THEOREM 2. (a) *If* $\phi \in L_1^k$, *then* $\phi_\epsilon \in L_1^k$.
(b) *If* $\phi \in L_p^{k-1}$, $1 < p < \infty$, *then* $\phi_\epsilon \in L_p^{k-1}$.
(c) *If* $\phi \in L_\infty^{k-2}$, *then* $\phi_\epsilon \in L_\infty^{k-2}$.

Proof. Note that under the various hypotheses $E(k, x)$ exists so that it is permissible to use the form of $\phi_\epsilon(s)$ given in Theorem 1.

(a) An inspection of $H(s, \epsilon, t)$ shows that it satisfies

$$\begin{aligned} |H(s, \epsilon, t)| &\leqq R(\epsilon)(|s|^k + |P_1(s, \epsilon)| + |t||s|^{k-1} + |t||P_2(s, \epsilon)|) \\ &\leqq R_1(\epsilon)(|s|^k + |t||s|^{k-1} + 1) \end{aligned} \tag{11}$$

where R and R_1 are continuous functions of ϵ, $P_i(s, \epsilon)$ are polynomials in ϵ and s with the highest power of s in P_1 being $k-1$ and in P_2 being $k-2$. From (9) we obtain

$$\begin{aligned} |K_k(s)\phi_\epsilon(s)| &= \left|\frac{\epsilon}{\pi}\int_{-\infty}^{\infty} \frac{\phi(t)H(s, \epsilon, t)K_k(t)K_k(s)dt}{(s-t)^2+\epsilon^2}\right| \\ &\leqq \frac{\epsilon}{\pi} R_1(\epsilon)\left\{\int_{-\infty}^{\infty} \frac{|\phi(t)||s|^k K_k(t)K_k(s)dt}{(s-t)^2+\epsilon^2}\right. \\ &\quad + \int_{-\infty}^{\infty} \frac{|\phi(t)||t||s|^{k-1}K_k(t)K_k(s)dt}{(s-t)^2+\epsilon^2} \\ &\quad \left. + \int_{-\infty}^{\infty} \frac{|\phi(t)|K_k(t)K_k(s)}{(s-t)^2+\epsilon^2}dt\right\}. \end{aligned} \tag{12}$$

The first of these integrals $\in L_1^0$ since $|s|^k K_k(s) \leqq 1$ and $\phi(t)K_k(t) \in L_1^0$. The second integral $\in L_1^0$ since if we integrate and interchange order of integration we get

$$\int_{-\infty}^{\infty} |\phi(t)||t| K_k(t)dt \int_{-\infty}^{\infty} \frac{K_k(s)|s|^{k-1}}{(s-t)^2+\epsilon^2} ds$$

and by part (b) of Lemma 3, the inner integral is $\leqq c/(1+|t|)$. It is trivial to show the third integral $\in L_1^0$.

(b) Using (11) we obtain

$$
(13)\quad
\begin{aligned}
| K_{k-1}(s)\phi_\epsilon(s) | \leqq \frac{\epsilon}{\pi} R_1(\epsilon) \Bigg\{ & \int_{-\infty}^{\infty} \frac{| \phi(t) | | s |^k K_k(t) K_{k-1}(s) dt}{(s-t)^2+\epsilon^2} \\
& + \int_{-\infty}^{\infty} \frac{| \phi(t) | | t | | s |^{k-1} K_k(t) K_{k-1}(s) dt}{(s-t)^2+\epsilon^2} \\
& + \int_{-\infty}^{\infty} \frac{| \phi(t) | K_k(t) K_k(s) dt}{(s-t)^2+\epsilon^2} \Bigg\}.
\end{aligned}
$$

The first integral $\in L_p^0$ by (c) of Lemma 3 since $|t|\phi(t)K_k(t)\in L_p^0$. The second integral $\in L_p^0$ since $|s|^{k-1}K_{k-1}(s)\leqq 1$ and $|\phi(t)||t|K_k(t)\in L_p^0$. The third integral is trivial.

(c) We have

$$
(14)\quad
\begin{aligned}
| K_{k-2}(s)\phi_\epsilon(s) | \leqq \frac{\epsilon}{\pi} R_1(t) \Bigg\{ & \int_{-\infty}^{\infty} \frac{| \phi(t) | | s |^k K_k(t) K_{k-2}(s) dt}{(s-t)^2+\epsilon^2} \\
& + \int_{-\infty}^{\infty} \frac{| \phi(t) | | t | | s |^{k-1} K_k(t) K_{k-2}(s) dt}{(s-t)^2+\epsilon^2} \\
& + \int_{-\infty}^{\infty} \frac{| \phi(t) | K_k(t) K_{k-2}(s) dt}{(s-t)^2+\epsilon^2} \Bigg\}.
\end{aligned}
$$

The first integral is bounded by (a) of Lemma 3 since $|\phi(t)|K_k(t)<c/(1+t^2)$. The second integral is bounded by (b) of Lemma 3, and the third integral is obviously also bounded. Q.E.D.

THEOREM 3. (a) *If* $\phi\in L_1^k$, *then*

$$\lim_{\epsilon\to 0} \|\phi_\epsilon(s) - \phi(s)\|_{k,1} = 0.$$

(b) *If* $\phi\in L_p^{k-1}$, $1<p<\infty$, *then*

$$\lim_{\epsilon\to 0} \|\phi_\epsilon(s) - \phi(s)\|_{k-1,p} = 0.$$

(c) *If* $\phi\in L_\infty^{k-2}$ *then* $\phi_\epsilon(s)$ *converges to* $\phi(s)$ *in the* ω *topology of* L_∞^{k-2}.

(d) *If* ϕ *is uniformly continuous in* L_∞^{k-2}, *then*

$$\lim_{\epsilon\to 0} \|\phi_\epsilon(s) - \phi(s)\|_{k-2,\infty} = 0.$$

Proof. By Lemma 2

$$
\begin{aligned}
| \phi(s) - \phi_\epsilon(s) | &\leqq \frac{\epsilon}{\pi} \int_{-\infty}^{\infty} \frac{| \phi(s) - \phi(t) | | H(s,\epsilon,t) | K_k(t) dt}{(s-t)^2+\epsilon^2} \\
&= \frac{\epsilon}{\pi} \left\{ \int_{|s-t|<\delta} + \int_{|s-t|>\delta} \right\} = J_1 + J_2.
\end{aligned}
$$

Suppose, for the moment, that δ has been selected. Then as in Theorem 2,

$$\begin{aligned} |J_2| \leqq R_1(\epsilon)\frac{\epsilon}{\pi}\Big\{\int_{|s-t|>\delta}&\frac{|\phi(s)-\phi(t)|\,|s|^k K_k(t)}{(s-t)^2}\,dt \\ &+\int_{|s-t|>\delta}\frac{|\phi(s)-\phi(t)|\,|s|^{k-1}|t|\,K_k(t)dt}{(s-t)^2} \\ &+\int_{|s-t|>\delta}\frac{|\phi(s)-\phi(t)|\,K_k(t)dt}{(s-t)^2}\Big\}. \end{aligned}$$

Note that the expression in the bracket is independent of ϵ.

$$|J_2| \leqq R_1(\epsilon)\frac{\epsilon}{\pi}S(s). \tag{15}$$

If $\phi\in L_1^k$, then by the same proof as appears in Theorem 2, $K_k(s)S(s)\in L_1^k$. If $\phi\in L_p^{k-1}$, $1<p<\infty$, then so does $K_{k-1}(s)S(s)$, and if $\phi\in L_\infty^{k-2}$, then $K_{k-2}(s)S(s)$ does also. Therefore by choosing ϵ sufficiently small we can make $|J_2|$ arbitrarily small in the proper metric.

We need only show that we may select δ, independently of ϵ, so that $|J_1|$ is arbitrarily small in the proper metric.

(a) Suppose $\phi\in L_1^k$.

$$J_1K_k(s) = \frac{\epsilon}{\pi}\int_{-\delta}^{\delta}\frac{|\phi(s)-\phi(s-u)|\,|H(s,\epsilon,s-u)|\,K_k(s-u)K_k(s)du}{u^2+\epsilon^2}.$$

Then

$$\max_{-\delta<u<\delta}|H(s,\epsilon,s-u)|\,K_k(s-u)\leqq c,$$

so that

$$\begin{aligned} J_1K_k(s)ds &\leqq \frac{\epsilon c}{\pi}\int_{-\delta}^{\delta}\frac{du}{u^2+\epsilon^2}\int_{-\infty}^{\infty}|\phi(s)-\phi(s-u)|\,K_k(s)ds \\ &\leqq \frac{\epsilon c}{\pi}\max_{-\delta<u<\delta}\tau(u)\int_{-\infty}^{\infty}\frac{du}{u^2+\epsilon^2} = c\max_{-\delta<u<\delta}\tau(u). \end{aligned}$$

By Lemma 1 this goes to zero as $\delta\to 0$.

(b) Suppose $\phi\in L_p^{k-1}$.

$$\begin{aligned} J_1K_{k-1}(s) &= \frac{\epsilon}{\pi}\int_{-\delta}^{\delta}\frac{|\phi(s)-\phi(s-u)|\,|H(s,\epsilon,s-u)|\,K_k(s-u)K_{k-1}(s)du}{u^2+\epsilon^2} \\ &\leqq \frac{\epsilon c}{\pi}\int_{-\delta}^{\delta}|\phi(s)-\phi(s-u)|\,K_{k-1}(s)\left|\frac{1}{u^2+\epsilon^2}\right|^{1/p}\left|\frac{1}{u^2+\epsilon^2}\right|^{1/p'}du. \end{aligned}$$

$$|J_1K_{k-1}(s)|^p \leqq \frac{\epsilon^p c^p}{\pi^p}\int_{-\delta}^{\delta} |(\phi(s)-\phi(s-u))K_{k-1}(s)|^p \frac{1}{u^2+\epsilon^2}du$$
$$\cdot\left(\int_{-\delta}^{\delta}\frac{du}{u^2+\epsilon^2}\right)^{p/p'}.$$

$$\int_{-\infty}^{\infty}|J_1K_{k-1}(s)|^p ds \leqq \frac{\epsilon^p c^p}{\pi^p}\left(\int_{-\infty}^{\infty}\frac{du}{u^2+\epsilon^2}\right)^p \max_{-\delta<u<\delta}\tau(u) = c^p \max_{-\delta<u<\delta}\tau(u).$$

By Lemma 1 this goes to zero as $\delta\to 0$.

(c) Suppose $\phi\in L_\infty^{k-2}$. Let $f(s)\in L_1^0$. As in part (a)

$$\int_{-\infty}^{\infty}J_1K_{k-2}(s)f(s)ds \leqq \frac{\epsilon c}{\pi}\int_{-\delta}^{\delta}\frac{du}{u^2+\epsilon^2}\int_{-\infty}^{\infty}|\phi(s)-\phi(s-u)|K_{k-2}(s)f(s)ds$$
$$\leqq c\max_{-\delta<u<\delta}\int_{-\infty}^{\infty}|\phi(s)-\phi(s-u)|K_{k-2}(s)f(s)ds$$

and the integral can be made arbitrarily small by proper selection of δ.

(d) If ϕ is uniformly continuous in L_∞^{k-2}, then

$$|J_1K_{k-2}(s)| \leqq \frac{\epsilon c}{\pi}\max_{-\delta<u<\delta}|\phi(s)-\phi(s-u)|K_{k-2}(s)\int_{-\delta}^{\delta}\frac{du}{u^2+\epsilon^2}$$
$$\leqq c\max_{-\delta<u<\delta}|\phi(s)-\phi(s-u)|K_{k-2}(s)$$

and this can be made arbitrarily small by proper selection of δ by the definition of uniform continuity in L_∞^{k-2}. Q.E.D.

3. **A theorem of the Hausdorff-Young type.** It is a well known result due to Titchmarsh that if $f(x)\in L_p^0$, $1<p\leqq 2$, then

$$E(0,x) = \frac{1}{(2\pi)^{1/2}}\underset{A\to\infty}{\text{l.i.m.}^{p'}}\int_{-A}^{A}e^{ixt}f(t)dt, \qquad \frac{1}{p}+\frac{1}{p'}=1,$$

exists and

$$\left(\int_{-\infty}^{\infty}|E(0,x)|^{p'}dx\right)^{1/p'} \leqq \frac{1}{(2\pi)^{1/p-1/2}}\left(\int_{-\infty}^{\infty}|f(x)|^p dt\right)^{1/p}.$$

(See [6, Theorem 74].)

This theorem has a direct generalization to the case $k>0$ which can easily be proved using the Riesz Theorem on interpolation of linear operators. Recently a generalization of this theorem with an elegant proof has been presented by Calderón and Zygmund [8]. Their result is as follows:

THEOREM. *Let E_1 and E_2 be two measure spaces, with measures μ and ν respectively. Let T be a normed linear operation defined for all simple functions* *f on E_1. Suppose that T is simultaneously of the types* $(1/\alpha_1, 1/\beta_1)$ *and* $(1/\alpha_2, 1/\beta_2)$, *i.e., that*

$$\|Tf\|_{1/\beta_1} \leqq M_1\|f\|_{1/\alpha_1}, \qquad \|Tf\|_{1/\beta_2} \leqq M_2\|f\|_{1/\alpha_2},$$

the points (α_1, β_1), (α_2, β_2) *belonging to the square*

$$0 \leqq \beta \leqq 1, \qquad 0 \leqq \alpha \leqq 1.$$

Then T is also of type $(1/\alpha, 1/\beta)$ *for all*

$$\begin{aligned} \alpha &= \alpha_1(1-t) + \alpha_2 t, \\ \beta &= \beta_1(1-t) + \beta_2 t, \end{aligned} \qquad 0 < t < 1,$$

with

$$\|Tf\|_{1/\beta} \leqq M_1^{1-t}M_2^t\|f\|_{1/\alpha}. \tag{16}$$

In particular, if $\alpha \neq 0$, *the operation T can be uniquely extended to the whole space* $L^0_{1/\alpha}$ *(with measure* μ*), preserving* (16).

In the above stated theorem

$$\|f\|_r = \left(\int_{E_1} |f|^r d\mu\right)^{1/r}, \tag{17}$$

$$\|Tf\|_s = \left(\int_{E_2} |Tf|^s d\nu\right)^{1/s}. \tag{18}$$

Let $H(x) \in H^k_{p'}$, $2 \leqq p' \leqq \infty$, if

(a) $d^kH/dx^k = h(x)$ exists and

(b) $h(x)$ is the Fourier transform of a function of L^0_p which vanishes outside the interval $(-1, 1)$.

Let $g(x) \in G_{p'}$, $2 \leqq p' \leqq \infty$, if $g(x)$ is the Fourier transform of a function of L^0_p which vanishes in the interval $(-1, 1)$.

Note that g and h each belong to $L^0_{p'}$.

Let $E^k_{p'} = H^k_{p'} + G_{p'}$ be the sum of the two spaces, i.e., the space formed by pairs of the form $(H+g)$ where $H \in H^k_{p'}$ and $g \in G_{p'}$. For a function $f \in E^k_{p'}$, the decomposition into the form $H+g$ is unique and we may define a norm in $E^k_{p'}$ by

$$\|f\|^{k,p'} = \left(\int_{-\infty}^{\infty} \left|\frac{d^kH}{dx^k} + g\right|^{p'} dx\right)^{1/p'} \qquad \text{for } 2 \leqq p' < \infty \tag{19}$$

and

$$\|f\|^{k,\infty} = \sup\left|\frac{d^kH}{dx^k} + g\right|. \tag{20}$$

If we observe the form of the generalized Fourier transform (6), and the Titchmarsh theorem, then it becomes evident that for $\phi(x)\in L_p^k$, $1<p\leqq 2$, it is possible to define a kth generalized Fourier transform by

$$\begin{aligned} E(k, x) = \frac{1}{(2\pi)^{1/2}}\Big[\int_{-1}^{1} \frac{\phi(t)}{(it)^k}\{e^{ixt} - L_k(t, x)\}dt \\ + \underset{A\to\infty}{\text{l.i.m.}}{}^{p'} \int_{1<|t|<A} \frac{\phi(t)}{(it)^k} e^{ixt}dt\Big], \end{aligned} \tag{21}$$

and that the relation between this and the $k+1$ generalized Fourier transform previously defined is

$$\frac{d}{dx}E(k+1, x) = E(k, x). \tag{22}$$

In equation (7), an integration by parts may now be carried out so that for $\phi\in L_p^{k-1}$, $1<p\leqq 2$, the inversion formula may be expressed in terms of the $E(k-1, x)$ defined in (21).

From equation (21) it is seen that the space of all kth generalized transforms for functions in L_p^k, $1<p\leqq 2$, is exactly the space $E_{p'}^k$. The purpose of the next theorem is to show that this is a bounded transformation and in the case $p=2$ is an isometric transformation.

THEOREM 4. *Let $\phi\in L_p^k$, $1<p\leqq 2$; then*

$$\|E(k, x)\|^{k,p'} \leqq \frac{1}{(2\pi)^{1/p'-1/2}}\|\phi(t)\|_{k,p},$$

the equality holding if $p=2$.

Proof. By the Riesz Theorem it is necessary to investigate only the cases $p=1$ and $p=2$.

If $\phi\in L_1^k$, then $E(k, x)$ is defined by

$$\begin{aligned} E(k, x) &= \frac{1}{(2\pi)^{1/2}}\int_{-1}^{1} \frac{\phi(t)}{(it)^k}\{e^{ixt} - L_k(t, x)\}dt \\ &\quad + \frac{1}{(2\pi)^{1/2}}\int_{|t|>1} \frac{\phi(t)}{(it)^k} e^{ixt}dt = H(x) + g(x), \\ \left|\frac{d^kH(x)}{dx^k} + g(x)\right| &= \frac{1}{(2\pi)^{1/2}}\left|\int_{-1}^{1}\phi(t)e^{ixt}dt + \int_{|t|>1}\frac{\phi(t)}{(it)^k}e^{ixt}dt\right| \\ &\leqq \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty}|\phi(t)|K_k(t)dt = \frac{1}{(2\pi)^{1/2}}\|\phi\|_{k,1}. \end{aligned} \tag{23}$$

If $\phi \in L_2^k$, the second integral of (23) is defined as a limit in mean, so by Plancherel's theorem

$$\int_{-\infty}^{\infty}\left|\frac{d^kH}{dx^k}\right|^2 dx = \int_{-\infty}^{\infty} | h(x) |^2 dx = \int_{-1}^{1} | \phi(t) |^2 dt$$

and

$$\int_{-\infty}^{\infty} | g(x) |^2 dx = \int_{|t|>1}\left|\frac{\phi(t)}{t^k}\right|^2 dt.$$

The equalities

$$\begin{aligned} \|E(k, x)\|^{k,2} &= \left(\int_{-\infty}^{\infty} | h(x) + g(x) |^2 dx\right)^{1/2} \\ &= \left(\int_{-1}^{1} | \phi(t) |^2 dt + \int_{|t|>1}\left|\frac{\phi(t)}{t^k}\right|^2 dt\right)^{1/2} = \|\phi\|_{k,2} \end{aligned}$$

follow from

$$\int_{-\infty}^{\infty} g(x)\overline{h(x)}dx = 0,$$

which is a consequence of Parseval's equality. An application of the Riesz Theorem now finishes the proof.

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