

# QUADRATIC FUNCTIONALS WITH A SINGULAR END POINT<sup>(1)</sup>

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**Introduction.** This paper completes a major phase in the study of singular quadratic functionals, the initial systematic study of which was made by Morse and Leighton [4]<sup>(2)</sup>. These authors considered functionals

$$(1) \quad J = \int_0^b [ry'^2 + 2qyy' + py^2]dx$$

which were singular at  $x=0$ . Necessary and sufficient conditions for the existence of a minimum limit were found which depended upon the class of admissible curves as well as upon the functional itself. In a later paper, Leighton [2] specialized the functional moderately and obtained necessary and sufficient conditions in terms of the geometry of the functional for the case when  $q(x)=0$  and  $p(x)$  is of one sign near  $x=0$ . In this paper, we consider functionals (1) in which  $q(x)=0$  and give necessary conditions and sufficient conditions for the existence of a minimum limit. In particular, the function  $p(x)$  may oscillate infinitely often neighboring  $x=0$ .

1. **The functional.** This paper is concerned with functionals  $J$  of the type

$$(1.1) \quad J(y) \Big|_e^b = \int_e^b [r(x)y'^2(x) - p(x)y^2(x)]dx \quad (0 < e < b < d),$$

in which  $r(x)$  and  $p(x)$  are continuous functions of the real variable  $x$  on the open interval  $(0, d)$ <sup>(3)</sup>. It is clear that  $x=0$  is, in general, a singular point for the functional. The integrals employed throughout are Lebesgue integrals and their extensions.

## 2. Admissibility: the problem.

*F-admissibility.* A function  $y(x)$  and the curve  $y=y(x)$  will be said to be *F-admissible* on  $[0, b]$  if

1.  $y(x)$  is continuous on  $(0, b]$  and  $y(b)=0$ ;
2.  $y(x)$  is absolutely continuous and  $y'^2$  is integrable Lebesgue on each closed subinterval of  $(0, b]$ .

*F'-admissibility.* A function  $y(x)$  and a curve  $y=y(x)$  will be said to be

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<sup>(2)</sup> The numbers in brackets refer to works listed in the bibliography.

<sup>(3)</sup> We agree that:  $[a, b]$  means the interval  $a \leq x \leq b$ ;  $(a, b]$  means the interval  $a < x \leq b$ ;  $[a, b)$  means the interval  $a \leq x < b$ ;  $(a, b)$  means the interval  $a < x < b$ .

$F'$ -admissible on  $[0, b]$  if

1.  $y(x)$  is  $F$ -admissible on  $[0, b]$ ;
2.  $y(x)$  is bounded on  $[0, b]$ .

$A$ -admissibility. A function  $y(x)$  and the curve  $y=y(x)$  will be said to be  $A$ -admissible on  $[0, b]$  if

1.  $y(x)$  is  $F'$ -admissible on  $[0, b]$ ;
2.  $y(x)$  is continuous on  $[0, b]$ , and  $y(0)=0$ .

One notes that the conditions of admissibility do not imply that either  $y'$  or  $y'^2 \in L[0, b]$ . This is clear from the curve

$$y = x(1 - x) \left( 2 + \sin \frac{1}{x} \right)$$

which possesses admissibility on  $[0, 1]$  of each given type. We also observe that if  $y=y(x)$  is an  $X$ -admissible curve on  $[0, c]$ ,  $0 < c < b$ , the curve  $y=\bar{y}(x)$  defined by the relations

$$\begin{aligned} \bar{y}(x) &= 0 && (c \leq x \leq b), \\ \bar{y}(x) &= y(x) && (0 \leq x \leq c) \end{aligned}$$

is  $X$ -admissible on  $[0, b]$ , where  $X = F, F'$ , or  $A$ . Finally, the classes  $F, F'$ , and  $A$ , in that order, form a descending sequence of sets. In the sequel, we shall mean by  $X$  any one of the symbols  $F, F'$  or  $A$ .

In posing an extremal problem for any functional one can expect both the existence and the nature of solutions of the problem to depend fundamentally on the class of functions admitted. Suppose, for example, we seek the minimum of the functional (1.1) among curves which are  $A$ -admissible on an interval  $[0, b]$  where the functional is regular. The integral (1.1) will then exist as a Lebesgue integral, finite or infinite. A minimum value of  $J$ , if it exists, will necessarily be zero. For, note that if  $y(x)$  is  $A$ -admissible, then  $cy(x)$  is  $A$ -admissible for any real constant  $c$ . Next, we have the relation

$$J(cy) \Big|_e^b = c^2 J(y) \Big|_e^b.$$

Finally, the curve  $y=0$  is  $A$ -admissible and along it

$$J(y) \Big|_e^b = 0.$$

We proceed with a statement of the central problem.

*The problem.* We seek necessary and sufficient conditions that

$$(2.1) \quad \liminf_{x=0} J(y) \Big|_x^b \geq 0$$

for all curves  $y=y(x)$  which are  $X$ -admissible on  $[0, b]$ <sup>(4)</sup>. A functional having this property will be said to possess an  $X$ -*minimum limit* on  $[0, b]$ .

In the next section we summarize briefly some known results on non-singular quadratic functionals.

3. **The regular functional.** Consider the Euler equation (denoted E.E.)

$$(3.1) \quad (ry')' + py = 0$$

associated with the functional  $J$ . It will be understood that a *solution* on the interval  $(0, b]$  is a function  $y(x)$ , or the curve  $y=y(x)$ , which is of class  $C^1$  and satisfies equation (3.1) identically for  $x$  on  $(0, b]$ <sup>(5)</sup>. The function  $y(x)=0$  is termed the *null* solution.

If  $y=y(x)$  is a nonnull solution of equation (3.1) such that

$$y(a) = y(c_0) = 0 \quad (0 < a < c_0 < b),$$

there exists a number  $c$  ( $a < c \leq c_0$ ) such that  $y(c)=0$  and  $y(x) \neq 0$  for  $x$  on  $(a, c)$ . The point  $x=c$  of the  $x$ -axis is called the *first conjugate point* of the point  $x=a$ . Its definition is easily seen to be independent of the particular choice of a nonnull solution vanishing at  $x=a$ . Further, if  $c$  is defined for  $x=a$  it is defined for all  $x$  on the interval  $(0, a)$ , and  $c$  is a continuous, strictly increasing function of  $x$  on this interval.

Using the nomenclature of the present paper, we state without proof the following well-known theorems.

**THEOREM 3.1.** *In order that  $J$  possess an  $A$ -minimum limit on  $[0, b]$  it is necessary that*

$$(3.2) \quad r(x) \geq 0$$

for all  $x$  on  $(0, b)$ .

In the next two theorems<sup>(6)</sup>, it is supposed that  $J$  is *regular*; that is,  $r(x)$  and  $p(x)$  are continuous on the closed interval  $[0, b]$ , and  $r(x)$  is different from zero there.

**THEOREM 3.2.** *In order that a regular functional  $J$  have an  $A$ -minimum limit on  $[0, b]$ , it is necessary and sufficient that  $r(x)$  be positive on  $[0, b]$  and that  $(0, b)$  contain no point conjugate to  $x=0$ .*

We next give conditions under which  $J$  possesses an  $F$ -minimum limit. Let  $x=a$  be a point of  $(0, b)$  and  $y=u(x)$  be a nonnull extremal whose derivative vanishes at  $x=a$ . The point  $x=f$  ( $f > a$ ) of the  $x$ -axis, if it exists, such that  $u(f)=0$  and  $u(x) \neq 0$  for  $x \in (a, f)$  is called the *focal point* of the line

<sup>(4)</sup> All limits taken at  $x=0$  will be right-hand limits.

<sup>(5)</sup> A function  $f(x)$  or a curve  $y=f(x)$  is said to be of class  $C^n$  on an interval if  $f(x)$  and its first  $n$  derivatives are continuous there.

<sup>(6)</sup> For complete treatment of regular functionals, see Morse [3].

$x = a$ . The following result may then be established.

**THEOREM 3.3.** *In order that a regular functional  $J$  have an  $F$ -minimum limit on  $[0, b]$ , it is necessary and sufficient that  $r(x)$  be positive on  $[0, b]$  and that  $(0, b)$  contain no focal point of the line  $x = 0$ .*

Although the integrand of (1.1) is regular whenever  $r(x)$  and  $p(x)$  are continuous, the functional is singular for values of  $x$  for which  $r(x) = 0$ ; that is, at singular points of the Euler equation. Theorem 3.1, therefore, contains the seeds of a theory of singular functionals by suggesting that the condition  $r(x) > 0$  may be relaxed to  $r(x) \geq 0$ . In the theory developed here we shall permit, for example,  $r(0) = 0$  and, further,  $r(x)$  and  $p(x)$  shall be required to be continuous only on the half-closed interval  $(0, b]$ . Therefore, the class of functionals under consideration will also include the regular functionals. A theory, therefore, about singular functionals will contain the regular theory. Though condition (3.2) generalizes at once, the conjugate point condition requires modification. The following section is devoted to a remodeling of the classical Jacobi theory [4].

**4. The Jacobi condition.** Let there exist a number  $a$  on  $(0, d)$  such that the first conjugate point  $x = c$  of  $x = a$  exists. Then

$$\lim_{a=0} c = c_0$$

exists and is non-negative. The point  $x = c_0$  of the  $x$ -axis is called the *first conjugate point* of the point  $x = 0$  [4]. If  $(0, d)$  contains no pairs of conjugate points,  $x = 0$  will be said to have no conjugate point on  $[0, d)$ . It is clear that if the point  $x = c_0$  exists, it may, in fact, coincide with  $x = 0$ . Further, we note that if the E.E. is regular at  $x = 0$ , the first conjugate point of  $x = 0$  is the same for the classical and for the modified definitions.

We state without proof the analogue of the Jacobi condition.

**THEOREM 4.1.** *In order that  $[0, b]$  afford a minimum limit to  $J$  among  $A$ -admissible curves, it is necessary that  $[0, b)$  contain no point conjugate to  $x = 0$ .*

For a proof of this and the following theorem, the reader is referred to Morse and Leighton [4].

**THEOREM 4.2.** *If the first conjugate point  $c_0$  of  $x = 0$  exists and  $c_0 \neq 0$ , there exists a solution  $u(x)$  of the E.E. such that  $u(c_0) = 0$  and  $u(x) \neq 0$  on  $(0, c_0)$ . If there is no point on  $[0, b]$  conjugate to  $x = 0$ , there exists a solution  $v(x) \neq 0$  on  $(0, b]$ .*

We proceed with an extension of the concept of the focal point of the line  $x = 0$ . To that end, let every sufficiently small interval about the origin contain a point  $x = a$  such that the line  $x = a$  has a focal point  $x = f$  on  $(0, d)$ . We define the *focal point*  $f_0$  of the line  $x = 0$  by the relation

$$f_0 = \liminf_{x=0} f.$$

If  $x=0$  has a neighborhood  $(0, e)$  such that no line  $x=a$ ,  $a \in (0, e)$ , has a focal point on  $(0, d]$ , the focal point of the line  $x=0$  will be said not to exist.

**5. Principal solutions.** We continue with the following theorem.

**THEOREM 5.1**(<sup>7</sup>). *If  $x=0$  is not its own first conjugate point, there exists a nonnull solution  $w(x)$  of the E.E. such that for every solution  $u(x)$  which is linearly independent of  $w(x)$*

$$(5.1) \quad \lim_{x=0} \frac{w(x)}{u(x)} = 0.$$

It may be seen that the condition (5.1) is equivalent to the pair of conditions

$$(5.2) \quad \int_0^b \frac{dx}{rw^2} = \infty,$$

$$(5.3) \quad \int_0^b \frac{dx}{ru^2} < \infty.$$

This is an easily verified consequence of the following well-known relation between a pair of solutions  $u(x)$  and  $w(x)$ :

$$(5.4) \quad w(x) = u(x) \left( c_1 + c_2 \int_x^b \frac{dx}{ru^2} \right).$$

A solution  $w(x)$  of the above theorem has been termed a *principal* solution [2]. One sees that any two principal solutions are linearly dependent. Conversely, a nonnull solution which is linearly dependent on a principal solution is a principal solution.

We state without proof the following theorem which provides another characterization of the principal solution (cf. [4]).

**THEOREM 5.2.** *If  $x=0$  is not its own first conjugate point and if  $w(x)$  is a principal solution, the first positive zero of  $w(x)$ , if it exists, is the first conjugate point of  $x=0$ . Conversely, the first conjugate point of  $x=0$ , if it exists, is the first positive zero of  $w(x)$ .*

**6. The singularity condition.** It will be convenient to introduce the following notation:

$$E[x, y, w] = r(x) \left( y' - \frac{w}{r(x)} y \right)^2, \quad R[x, w] = w' + \frac{w^2}{r(x)} + p(x).$$

We may prove the following theorem on a representation of the functional.

(<sup>7</sup>) Morse and Leighton [4].

THEOREM 6.1. *If  $f(x)$  is absolutely continuous on  $(0, b]$  and if*

$$(6.1) \quad f(x) = O\left(\frac{1}{b-x}\right)$$

as  $x$  approaches  $b^-$ , then

$$(6.2) \quad J(y) \Big|_x^b = -y^2(x)f(x) + \int_x^b E[x, y, f]dx - \int_x^b R[x, f]y^2(x)dx,$$

where  $y(x)$  is  $F$ -admissible on  $[0, b]$ .

Let  $0 < c < b$ . We have

$$J(y) \Big|_x^c = \int_x^c \left[ ry'^2 - 2fyy' + \frac{f^2}{r}y^2 \right] dx - \int_x^c \left[ -2fyy' + \frac{f^2}{r}y^2 + py^2 \right] dx.$$

If the term  $-2fyy'$  above is integrated by parts, one obtains

$$(6.3) \quad \begin{aligned} J(y) \Big|_x^c &= y^2(x)f(x) \Big|_x^c + \int_x^c r \left( y' - \frac{f}{r}y \right)^2 dx - \int_x^c \left( f' + \frac{f^2}{r} + p \right) y^2 dx \\ &= y^2(x)f(x) \Big|_x^c + \int_x^c E[x, y, f]dx - \int_x^c R[x, f]y^2 dx. \end{aligned}$$

Now since  $y(x)$  is  $F$ -admissible,  $y'^2(x) \in L$  near  $x=b$  and thus by the Schwarz inequality

$$y^2(x) = \left( \int_x^b y'(x)dx \right)^2 \leq (b-x) \int_x^b y'^2(x)dx = o(b-x)$$

as  $x$  tends to  $b^-$ . It follows that

$$y^2(x)f(x) = o(b-x)O(1/(b-x)) = o(1)$$

as  $x$  approaches  $b^-$ ; consequently equation (6.2) may be obtained from equation (6.3) by letting  $c \rightarrow b^-$ .

The next theorem is an important consequence of formula (6.2).

THEOREM 6.2. *If  $u(x)$  is a solution of the E.E. which does not vanish on  $(0, b)$ , then  $v(x) = r(x)u'(x)/u(x)$  is a solution of the Riccati equation*

$$(6.4) \quad R[x, v] = 0,$$

and

$$(6.5) \quad J(y) \Big|_x^b = \int_x^b E \left[ x, y(x), r(x) \frac{u'(x)}{u(x)} \right] dx - y^2(x)r(x) \frac{u'(x)}{u(x)}.$$

That  $v(x)$  satisfies (6.4) is well known. Formula (6.5) follows at once

from (6.2) and (6.4).

Formula (6.5) may also be obtained by consideration of the Hilbert integral. Using it Morse and Leighton obtained the following fundamental theorem.

**THEOREM 6.3.** *In order that the interval  $[0, b]$  afford a minimum limit to the functional  $J$  among  $A$ -admissible curves, it is necessary and sufficient that  $[0, b]$  contain no point conjugate to  $x=0$  and that*

$$(6.6) \quad \liminf_{x=0} \left[ -y^2(x)r(x) \frac{u'(x)}{u(x)} \right] \geq 0$$

for every  $A$ -admissible curve  $y=y(x)$  along which

$$(6.7) \quad \liminf_{x=0} J(y) \Big|_x^b < +\infty,$$

where  $u(x)$  is any nonnull solution of the E.E. which vanishes at  $x=b$ .

We observe that if in the statement of Theorem 6.3, the class of  $A$ -admissible curves is replaced by either the class of  $F$ - or  $F'$ -admissible curves, the conclusion of the theorem remains valid. Accordingly we are provided with a necessary and sufficient condition that  $J$  have an  $X$ -minimum limit.

The condition of the previous theorem is called the *singularity condition* belonging to  $x=b$ . The singularity condition is not contained in the conjugate point condition [4, p. 263].

We state, without proof, a theorem which is useful in the applications. The theorem is due, for the case  $X=A$ , to Morse and Leighton [4].

**THEOREM 6.4.** *If every subinterval  $[0, c]$ ,  $0 < c < b$ , of  $[0, b]$  affords a minimum limit to  $J$  among  $X$ -admissible curves, then  $J$  has an  $X$ -minimum limit on  $[0, b]$ .*

The singularity condition possesses two properties with which we shall be concerned. First, it shares jurisdiction with the conjugate point condition by virtue of its formal dependence on the size of the interval  $[0, b]$ . We shall show that the singularity condition and the conjugate point condition are independent. Secondly, the singularity condition depends formally on the class of admissible curves. We shall continue a study conducted by Leighton [2] of necessary and sufficient conditions for minimum limits of the types  $A$  and  $F$  which are expressed in terms of  $r(x)$ ,  $p(x)$ , and the solutions of the Euler equation. In addition we shall consider properties of the functional as they are related to solutions of the Riccati equation.

**7. Necessary conditions.** We begin with a principal theorem.

**THEOREM 7.1**<sup>(8)</sup>. *In order that the functional  $J$  have an  $A$ -minimum limit*

<sup>(8)</sup> It will be observed that Theorem 7.1 is valid for the general functional (1).

on  $[0, b]$ , it is necessary that

$$(7.1) \quad \liminf_{x=0} J(z) \Big|_x^b > -\infty$$

for every  $F'$ -admissible curve  $y = z(x)$ .

Let  $y = z(x)$  be an  $F'$ -admissible curve such that

$$(7.2) \quad \liminf_{x=0} J(z) \Big|_x^b = -\infty.$$

We distinguish two cases according to whether or not  $x = 0$  is a limit point of zeros of  $z(x)$ . We suppose first that  $x = 0$  is isolated from the zeros of  $z(x)$ . We proceed to construct an  $A$ -admissible curve  $y = y(x)$  along which  $J$  has a negative inferior limit. We define  $y(x)$  as follows. First, let

$$y(x) = z(x) \quad (x_1 \leq x \leq b),$$

where  $x_1$  is chosen such that  $x_1 < b$ ,

$$J(z) \Big|_{x_1}^b < -1,$$

and  $z(x) \neq 0$  for  $0 < x \leq x_1$ . By (7.2),  $x_1$  exists. When  $n$  is odd, let

$$x_{n+1} = \frac{x_n}{n+1},$$

$$y(x) = \frac{y(x_n)}{x_n} x \quad (x_{n+1} \leq x \leq x_n);$$

if  $n$  is even, let

$$y(x) = \frac{y(x_n)}{z(x_n)} z(x) \quad (x_{n+1} \leq x \leq x_n),$$

where  $x_{n+1}$  is chosen such that  $x_{n+1} < x_n$  and

$$J(y) \Big|_{x_{n+1}}^b < -1.$$

By virtue of (7.2),  $x_{n+1}$  exists. Finally we define  $y(0) = 0$ . Thus for every  $x \in [0, b]$ ,  $y(x)$  is defined. Moreover, since  $z(x)$  is  $F'$ -admissible,  $z(x)$  is bounded and the construction yields an  $A$ -admissible function  $y(x)$ . However,

$$\liminf_{x=0} J(y) \Big|_x^b \leq \liminf_{n=\infty} J(y) \Big|_{x_n}^b \leq -1.$$

Accordingly,  $J$  does not possess a minimum limit on  $[0, b]$ .

There remains the case in which  $x=0$  is a limit point of zeros of  $z(x)$ . We define an  $A$ -admissible function  $y(x)$  as follows:

$$y'(x) = z(x) \quad (x_1 \leq x \leq b),$$

where  $x_1$  is chosen such that

$$z(x_1) = 0$$

and such that there exists a number  $x'_1$  ( $x_1 \leq x'_1 \leq b$ ) for which

$$J(y) \Big|_{x'_1}^b < -1.$$

By virtue of (7.2),  $x_1$  is defined. Suppose that  $y(x)$  and  $x_k$  are defined for  $k=1, \dots, n$ ,  $x_n \leq x \leq b$ , such that there exists  $x'_k$  on  $(x_{k+1}, x_k)$  for which

$$J(y) \Big|_{x'_k}^b < -1 \quad (k = 1, 2, \dots, n-1).$$

We define

$$y(x) = \frac{z(x)}{n+1} \quad (x_{n+1} \leq x \leq x_n),$$

where  $x_{n+1}$  is chosen such that  $x_{n+1} < x_n$ ,  $z(x_{n+1})=0$ , and such that for some  $x'_{n+1}$  on  $(x_{n+1}, x_n)$ ,

$$J(y) \Big|_{x'_{n+1}}^b < -1.$$

We complete the definition of  $y(x)$  by setting  $y(0)=0$ . As before,  $y(x)$  is  $A$ -admissible and

$$\liminf_{x=0} J(y) \Big|_x^b \leq \liminf_{n=\infty} J(y) \Big|_{x'_n}^b \leq -1.$$

The proof of the theorem is complete.

We note that the necessary condition of the above theorem is, in fact, a family of necessary conditions. In particular, if we apply the theorem to an  $F'$ -admissible function which is identically equal to one near  $x=0$ , we obtain the following theorem (cf. [2]).

**THEOREM 7.2.<sup>(9)</sup>** *If  $J$  possesses an  $A$ -minimum limit on  $[0, b]$ , then*

$$(7.3) \quad \limsup_{x=0} \int_x^b p(x)dx < +\infty.$$

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<sup>(9)</sup> The proof of this theorem as given in [2] requires supplementation.

We note that conditions (7.1) and (7.2) are independent of the size of the interval  $[0, b]$ . Thus, these conditions are strictly conditions on the singularity at  $x=0$ .

We continue with a lemma.

LEMMA 7.1. *If  $x=0$  is not its own first conjugate point and if the  $y$ -axis contains its own focal point, then*

$$(7.4) \quad \limsup_{x=0} r(x)u(x)u'(x) = +\infty$$

for every nonnull nonprincipal solution  $u(x)$ .

Let  $w(x)$  and  $u(x)$  be respectively principal and nonprincipal solutions. Let  $x_n$  ( $n=1, 2, \dots$ ) be a decreasing sequence of numbers such that  $\lim_{n \rightarrow \infty} x_n = 0$  and such that for each  $n$  the focal point  $x=f(x_n)$  of  $x=x_n$  exists and  $\lim_{n \rightarrow \infty} f(x_n) = 0$ . By hypothesis, such a sequence exists. The solutions

$$(7.5) \quad y_n(x) = u'(x_n)w(x) - w'(x_n)u(x) \quad (n = 1, 2, \dots)$$

will then have the properties that

$$(7.6) \quad y'_n(x_n) = 0, \quad y_n(f(x_n)) = 0$$

and

$$y_n(x) \neq 0, \quad x_n < x < f(x_n),$$

for all  $n$ . From the second equality of (7.6) we have

$$(7.7) \quad u'(x_n)w(f(x_n)) - w'(x_n)u(f(x_n)) = 0.$$

Since  $x=0$  is not its own first conjugate point,  $u(f(x_n)) \neq 0$ , for  $n$  large. Further, since  $w(x)$  and  $u(x)$  are linearly independent,  $u'(x)$  and  $w'(x)$  cannot vanish simultaneously. It follows from (7.7) that  $u'(x_n) \neq 0$ . Accordingly, we may write

$$(7.8) \quad \frac{w'(x_n)}{u'(x_n)} = \frac{w(f(x_n))}{u(f(x_n))}.$$

We have by Abel's formula that

$$(7.9) \quad r(x)[u(x)w'(x) - w(x)u'(x)] = c,$$

or

$$(7.10) \quad \frac{d}{dx} \frac{w(x)}{u(x)} = \frac{c}{r(x)u^2(x)},$$

where  $c$  is a nonzero constant. Without loss of generality, we may assume that  $w(x)$  and  $u(x)$  are of the same sign near  $x=0$ . Since  $w(x)/u(x)$  decreases to zero with  $x$ , it follows that  $c > 0$ . Thus by (7.9) we have, for  $n$  large,

$$(7.11) \quad \frac{w'(x_n)}{u'(x_n)} - \frac{w(x_n)}{u(x_n)} = \frac{c}{r(x_n)u(x_n)u'(x_n)}.$$

Applying (7.8) to (7.11) we obtain the relation

$$(7.12) \quad \frac{w(f(x_n))}{u(f(x_n))} - \frac{w(x_n)}{u(x_n)} = \frac{c}{r(x_n)u(x_n)u'(x_n)}.$$

Now since  $w(x)/u(x)$  decreases steadily to zero with  $x$ , the left-hand side of (7.12) approaches zero through positive values as  $n$  tends to infinity. Thus since  $c > 0$  we must have

$$\lim_{n \rightarrow \infty} r(x_n)u(x_n)u'(x_n) = +\infty.$$

The lemma is proved.

**THEOREM 7.3.** *If  $[0, b]$  affords an  $A$ -minimum limit to  $J$  and if there exists one bounded nonprincipal solution, then all solutions are bounded and the focal point of the  $y$ -axis is positive or nonexistent.*

Let  $w(x)$  be a principal and  $u(x)$  a bounded nonprincipal solution. Then since

$$\lim_{x \rightarrow 0} \frac{w(x)}{u(x)} = 0,$$

and  $u(x)$  is bounded, we have

$$\lim_{x \rightarrow 0} w(x) = 0,$$

so that all solutions are bounded, as asserted.

Further, by hypothesis,  $u(x)$  is  $F'$ -admissible and

$$\liminf_{x \rightarrow 0} J(u) \Big|_x^b = \liminf_{x \rightarrow 0} [r(b)u(b)u'(b) - r(x)u(x)u'(x)] > -\infty.$$

The theorem now follows readily from Lemma 7.1.

Theorem 7.3 contains the following result due to Leighton [2].

**COROLLARY 7.1.** *If  $J$  is afforded an  $A$ -minimum limit by an interval  $[0, b]$ , then for every nonprincipal extremal  $u(x)$*

$$(7.13) \quad \limsup_{x \rightarrow 0} |u(x)| > 0.$$

We continue with a theorem which extends the singularity condition.

**THEOREM 7.4.** *In order that  $[0, b]$  afford an  $A$ -minimum limit to  $J$ , it is necessary and sufficient that  $[0, b)$  contain no point conjugate to  $x = 0$  and that*

for any nonnull solution  $u(x)$

$$(7.14) \quad \liminf_{x=0} \left[ -y^2(x)r(x) \frac{u'(x)}{u(x)} \right] \geq 0$$

for every  $A$ -admissible curve  $y=y(x)$  along which

$$(7.15) \quad \liminf_{x=0} J(y) \Big|_x^b < +\infty.$$

The necessity of the conjugate point condition has been stated in Theorem 4.1. We shall prove the necessity of condition (7.14).

If  $[0, b]$  affords an  $A$ -minimum limit to  $J$ , the singularity condition must be satisfied. If, therefore,  $u(x)$  is a non-null solution which vanishes at  $x=b$ , (7.14) is satisfied. If  $x=b$  is the first conjugate point of  $x=0$ ,  $u(x)$  is a principal solution. If  $v(x) \neq 0$  is a nonprincipal solution, then  $v(x)$  vanishes at  $x=c$ ,  $0 < c < b$ . Thus since  $[0, c]$  affords an  $A$ -minimum limit to  $J$ , the singularity condition for  $v(x)$  is satisfied. If  $x=b$  is not the first conjugate point of  $x=0$ ,  $u(x)$  is a nonprincipal solution. In this case a principal solution  $w(x)$  has the property that

$$w(x) \neq 0 \quad (0 < x \leq b).$$

In a neighborhood  $[e, b]$  of  $x=b$  decrease  $r(x)$  and increase  $p(x)$  continuously to obtain  $r_1(x)$ ,  $p_1(x)$ ,  $w_1(x)$ , and  $u_1(x)$  such that

$$w_1(b) = 0, \quad w_1(x) \neq 0, \quad 0 < x < b,$$

and

$$r_1(x) = r(x), \quad p_1(x) = p(x), \quad 0 < x \leq e.$$

Accordingly,

$$u_1(x) = u(x), \quad w_1(x) = w(x), \quad 0 < x \leq e.$$

Furthermore,  $w_1(x)$  is a principal and  $u_1(x)$  a nonprincipal solution. Hence, by Theorem 5.4,  $x=b$  is the first conjugate point of  $x=0$  with respect to the equation

$$(r_1 y)' + p_1 y = 0.$$

Let  $v_1(x)$  be any nonprincipal solution of this equation which is independent of  $u_1(x)$  and let  $v(x)$  be the solution of

$$(r y)' + p y = 0$$

which is identical with  $v_1(x)$  near  $x=0$ . Then

$$r_1(x) [u_1(x)v_1'(x) - v_1(x)u_1'(x)] = c$$

and

$$r(x) [u(x)v'(x) - v(x)u'(x)] = c,$$

where  $c$  is a nonzero constant.

We have, therefore, except possibly in the zeros of  $u(x)$  and  $v(x)$ ,

$$(7.16) \quad r(x) \frac{v'(x)}{v(x)} = r(x) \frac{u'(x)}{u(x)} + \frac{c}{u(x)v(x)}.$$

Now let  $y=y(x)$  be an  $A$ -admissible curve such that (7.15) holds. Since  $J$  has an  $A$ -minimum limit,

$$r^{1/2}(x) \left[ y'(x) - \frac{u'(x)}{u(x)} y(x) \right] = -f(x) \in L^2$$

on the interval  $[0, b]$ . Thus,

$$(7.17) \quad y(x) = u(x) \left[ c_1 + \int_x^a \frac{f}{r^{1/2}u} dx \right].$$

Now since  $u(x)$  is a nonprincipal solution,  $1/r^{1/2}u \in L^2[0, b]$  by Theorem 5.1. Since in addition  $f \in L^2$ , it follows from the Schwarz inequality that  $f/r^{1/2}u \in L$ . The integral  $\int_0^a (f/r^{1/2}u) dx$  consequently converges absolutely. Now by Corollary 7.1

$$\limsup_{x=0} |u(x)| > 0$$

and, therefore,  $c_1$  must satisfy the equation

$$(7.18) \quad c_1 + \int_0^a \frac{f}{r^{1/2}u} dx = 0.$$

Now from (7.16) and (7.17)

$$(7.19) \quad -y^2(x)r(x) \frac{v'(x)}{v(x)} = -y^2(x)r(x) \frac{u'(x)}{u(x)} - c \frac{u(x)}{v(x)} \left[ c_1 + \int_x^a \frac{f}{r^{1/2}u} dx \right]^2.$$

Let  $x$  tend to zero. Since  $u(x)$  and  $v(x)$  are nonprincipal solutions,  $u(x)/v(x)$  approaches a finite limit. It then follows from (7.18) that

$$\liminf_{x=0} \left[ -y^2(x)r(x) \frac{v'(x)}{v(x)} \right] = \liminf_{x=0} \left[ -y^2(x)r(x) \frac{u'(x)}{u(x)} \right] \geq 0.$$

Hence the singularity condition is satisfied for every nonprincipal solution  $u(x)$ . Thus by Theorem 6.4,

$$J_1(y) \Big|_x^b = \int_x^b [r_1 y'^2 - p_1 y^2] dx$$

has an  $A$ -minimum limit on  $[0, b]$ . By Theorem 6.3, the singularity condition for  $w_1(x)$  must be satisfied since  $w_1(b) = 0$ . Thus the singularity condition is satisfied for every solution of the E.E.

Conversely, let the conjugate point condition for  $[0, b]$  and the singularity condition be satisfied for a nonnull solution  $u(x)$ . If  $u(x) \neq 0$  on  $(0, b)$ , then  $J$  has a minimum limit on  $[0, b]$ . Otherwise  $u(x)$  vanishes on  $(0, b)$  and therefore is a nonprincipal solution. Further, by (7.19), we can show that  $J$  has a minimum limit on every subinterval  $[0, c]$ ,  $0 < c < b$ , of  $[0, b]$ . By Theorem 6.4,  $J$  has a minimum limit on  $[0, b]$ .

The theorem is proved.

The above theorem shows that the singularity condition is, in fact, independent of the size of the interval  $[0, b]$ . This property is also expressed by the next theorem with which we conclude this section [4].

**THEOREM 7.5.** *If an interval  $[0, e]$  affords an  $A$ -minimum limit to  $J$ , then the functional will have an  $A$ -minimum limit on any interval  $[0, b]$  for which  $[0, b)$  contains no conjugate point to  $x = 0$ .*

**8. The variable end point problem.** In this section, we shall find necessary and sufficient conditions that an interval  $[0, b]$  afford an  $F$ -minimum limit to the functional  $J$ . The principal results of this section depend upon the definition of the focal point of the  $y$ -axis which was stated in §4.

**THEOREM 8.1.** *If  $[0, b]$  affords a minimum limit to  $J$  among  $F$ -admissible curves, the interval  $[0, b)$  contains no focal point of the  $y$ -axis.*

It is sufficient to show that if  $f_0$  exists, then

$$f_0 \geq a$$

for every number  $a$  on  $(0, b)$ .

Let numbers  $x_1$  and  $x_2$  be chosen from  $(0, b)$  such that

$$x_1 < a < x_2.$$

Let  $v(x)$  be a non-null solution of the E.E. such that

$$v(a) = 0.$$

Let  $u(x)$  be a solution of the E.E. which satisfies the relations

$$\begin{aligned} u(x_1) &= v(x_1), \\ u(x_2) &= 0. \end{aligned}$$

Since  $[0, b)$  is free of conjugate points, a solution  $u(x)$  of this type exists. Now

$$J(u) \Big|_{x_1}^{x_2} = -r(x_1)u(x_1)u'(x_1),$$

and by Sturm's theory

$$r(x_1)u(x_1)u'(x_1) - r(x_1)v(x_1)v'(x_1) > 0;$$

thus,

$$(8.1) \quad -J(u) \Big|_{x_1}^{x_2} + J(v) \Big|_{x_1}^a > 0.$$

We define an  $F$ -admissible function  $y(x)$  as follows:

$$y(x) = \begin{cases} 0 & (x_2 \leq x \leq b), \\ u(x) & (x_1 \leq x \leq x_2), \\ v(x) & (0 < x \leq x_1). \end{cases}$$

But then

$$(8.2) \quad \liminf_{x=0} J(y) \Big|_x^b \geq 0.$$

Now by relations (8.1) and (8.2) we have that

$$(8.3) \quad \begin{aligned} \liminf_{x=0} J(v) \Big|_x^a &= \liminf_{x=0} \left[ J(y) \Big|_x^{x_2} - J(u) \Big|_{x_1}^{x_2} + J(v) \Big|_{x_1}^a \right] \\ &\geq -J(u) \Big|_{x_1}^{x_2} + J(v) \Big|_{x_1}^a > 0. \end{aligned}$$

Hence there exists an interval  $(0, e)$  such that

$$J(v) \Big|_x^a = -r(x)v(x)v'(x) > 0 \quad (0 < x < e).$$

It now follows from Sturm's theorems that if  $x=f(t)$  is the focal point of the line  $x=t$  then

$$f(x) > a \quad (0 < x < e),$$

and thus

$$f_0 \geq a$$

for every number  $a$  on  $(0, b)$ .

The proof is complete.

We proceed with two lemmas.

LEMMA 8.1. *If  $(0, b)$  contains no focal point of the  $y$ -axis, then  $(0, b)$  contains no point conjugate to  $x=0$ .*

It is clear that  $x=0$  is not its own first conjugate point. Suppose the first conjugate point of  $x=0$  exists and lies on  $(0, b)$ , say at  $x=c_0$ . Then for  $a$  positive and sufficiently small, a nonnull solution which vanishes on  $(0, a)$  will have a zero on  $(c_0, b)$ . Let  $u(x)$  be a nonnull solution such that  $u(x_0) = u(x_1) = 0$ , where  $x_0 \in (0, a)$ ,  $x_1 \in (c_0, b)$ , and  $u(x) \neq 0$  for  $x_0 < x < x_1$ . Let  $v(x)$  be a nonnull solution such that  $v'(x_0) = 0$ .

Now since

$$r(x) [v(x)u'(x) - u(x)v'(x)] = c \quad (0 < x \leq b),$$

where  $c$  is a nonzero constant, we have at  $x=x_0$ ,

$$v(x_0) = \frac{c}{r(x_0)u'(x_0)},$$

and at  $x=x_1$

$$v(x_1) = \frac{c}{r(x_1)u'(x_1)}.$$

Now since  $u(x) \neq 0$  ( $x_0 < x < x_1$ ),  $u'(x)$  has opposite signs at  $x=x_0$  and  $x=x_1$ . Hence  $v(x)$  vanishes on  $(x_0, x_1)$ . Since  $x=x_0$  is an arbitrary point of  $(0, a)$  it follows that the focal point of  $x=0$  lies on  $[0, c_0]$ . This contradiction proves the lemma.

LEMMA 8.2. *If  $(0, b)$  has no focal point of the  $y$ -axis, then every nonnull solution  $u(x)$  which vanishes on  $(0, b)$  has the property that*

$$r(x)u(x)u'(x) < 0$$

for  $x$  near  $x=0$ .

Let  $x=a$  be a point of  $(0, b)$  and  $u(x)$  a nonnull solution which vanishes at  $x=a$ . Then clearly  $u(x)$  and  $u'(x)$  are not zero near  $x=0$ . If  $r(x)u(x)u'(x) < 0$  near  $x=0$ , the proof is complete. In the remaining case we may assume without loss in generality that  $u(x) > 0$  on  $(0, a)$  and  $u'(x) > 0$  on an interval  $(0, e)$ ,  $0 < e < a$ . Let  $v(x)$  be a solution such that at  $x=x_0$ ,  $0 < x_0 < e$ ,  $v'(x_0) = 0$ , and  $v(x_0) > 0$ . We have

$$r(x) [v(x)u'(x) - u(x)v'(x)] = c \quad (0 < x \leq b),$$

where  $c$  is nonzero constant. At  $x=x_0$ , we have

$$r(x_0)v(x_0)u'(x_0) = c,$$

and thus by our assumption  $c > 0$ . At  $x=a$ , we have

$$r(a)v(a)u'(a) = c.$$

Thus since  $u'(a) < 0, v(a) < 0$  and it follows that  $v(x)$  vanishes on  $(x_0, a)$ . Hence every line  $x = x_0, x_0 \in (0, e)$ , has a focal point on  $(0, a)$  and the  $y$ -axis thus has a focal point on  $[0, a]$ . From this contradiction we infer the truth of the lemma.

We come to a principal theorem.

**THEOREM 8.2.** *If  $[0, b)$  contains no focal point of the  $y$ -axis, the interval  $[0, b]$  affords a minimum limit to  $J$  among  $F$ -admissible curves.*

By Lemma 8.1,  $[0, b)$  contains no point conjugate to  $x = 0$ . If, therefore,  $u(x)$  is a solution which vanishes at  $x = a, 0 < a < b$ , then  $u(x)$  is not zero on  $(0, a)$ . If  $y = y(x)$  is a curve which is  $F$ -admissible on  $[0, a]$ , we have according to Theorem 6.2

$$(8.4) \quad J(y) \Big|_x^a = \int_x^a E\left(x, y, \frac{u'}{u}\right) dx - y^2(x)r(x) \frac{u'(x)}{u(x)}.$$

By Lemma 8.2

$$r(x)u(x)u'(x) < 0$$

for  $x$  near  $x = 0$ , and hence,

$$r(x) \frac{u'(x)}{u(x)} = \frac{r(x)u'(x)u(x)}{u^2(x)} < 0$$

in such a neighborhood. Thus, by (8.4), every interval  $[0, a], 0 < a < b$ , affords an  $F$ -minimum limit to  $J$ . By Theorem 6.4,  $J$  possesses an  $F$ -minimum limit on  $[0, b]$ .

The proof is complete.

Theorem 8.1 and Theorem 8.2 may be combined and stated in the following form.

**THEOREM 8.3.** *In order that  $[0, b]$  afford an  $F$ -minimum limit to  $J$ , it is necessary and sufficient that as  $x \rightarrow 0^+$  every solution  $u(x)$  which vanishes on  $(0, b)$  approach monotonically a limit, finite or infinite, ascending if  $u(x)$  is positive near  $x = 0$  and descending if  $u(x)$  is negative there.*

We may also formulate the sufficient condition in the following way.

**THEOREM 8.4.** *If*

$$(8.5) \quad \liminf_{x=0} J(y) \Big|_x^b > -\infty$$

*for every  $F$ -admissible curve  $y = y(x)$ , then the focal point  $x = f$  of  $x = 0$  is positive or nonexistent, and*

$$\liminf_{x=0} J(y) \Big|_x^f \geq 0$$

for every curve  $y=y(x)$  which is  $F$ -admissible on  $[0, f]$ .

For, by Lemma 7.1, the condition (8.5) implies that the  $y$ -axis does not contain its own focal point<sup>(10)</sup>.

9. **The fixed end point problem.** In this section we discuss the problem of the  $A$ -minimum limit. We begin with a statement of a theorem due to Leighton [2].

**THEOREM 9.1.** *If  $p(x) > 0$ , near  $x=0$ , the functional  $J$  possesses an  $A$ -minimum limit on  $[0, b]$  if and only if there exists no point conjugate to  $x=0$  on  $[0, b]$  and*

$$(9.1) \quad \int_0^b p(x)dx < \infty .$$

Recalling Theorem 7.2, we observe that the necessary half of the above theorem generalizes in the natural way to apply to functions  $p(x)$  which are not of one sign. The question now arises as to the possibility of replacing (9.1) by (7.3) to obtain a sufficient singularity condition. We shall show that this is impossible. In the following example we display a functional  $J$  such that  $[0, b]$  contains no conjugate point of the origin,

$$\lim_{x=0} \int_x^b p(x)dx = -\infty ,$$

and such that there exists an  $F'$ -admissible curve  $y=y(x)$  along which

$$\liminf_{x=0} J(y) \Big|_x^b = -\infty .$$

Hence, this example will show in addition that the necessary condition stated in Theorem 7.1 is independent of previously established necessary conditions.

**EXAMPLE 9.1.** Let

$$r(x) = x^2$$

and

$$p(x) = \frac{4x^{-4} \sin x^{-2}}{2 + \sin x^{-2}} - \frac{2x^{-2} \cos x^{-2}}{2 + \sin x^{-2}} .$$

A principal solution of the E.E. is

$$w(x) = 2 + \sin x^{-2} .$$

---

<sup>(10)</sup> The following extension of the above ideas is immediate. Let  $u(x)$  be a nonnull solution of the E.E.  $(ry' + qy)' - (qy' + py) = 0$ , such that  $r(a)u'(a) + q(a)u(a) = 0$ . The focal point  $f(a)$  of the line  $x=a, a > 0$ , may be defined to be the first zero of  $u(x)$  which is not smaller than  $a$ . If we define the focal point of the  $y$ -axis to be  $\liminf_{a \rightarrow 0} f(a)$ , Theorems 8.1 and 8.2 are then valid for the general functional (1).

Thus the first conjugate point of  $x=0$  is nonexistent. Moreover  $w(x)$  is  $F'$ -admissible near  $x=0$  and

$$\begin{aligned} \liminf_{x=0} J(w) \Big|_x^b &= \liminf_{x=0} r(x)w(x)w'(x) \Big|_x^b \\ &= \liminf_{x=0} [-2x^{-1}(2 + \sin x^{-2}) \cos x^{-2}] \Big|_x^b \\ &= -\infty. \end{aligned}$$

Therefore by Theorem 7.1, the functional fails to have a minimum limit. Finally, we assert that

$$(9.2) \quad \lim_{x=0} \int_x p(x)dx = -\infty.$$

Let  $v=x^{-2}$ . Then

$$\begin{aligned} \int_x p(x)dx &= \int^v \frac{2v^{1/2} \sin v}{2 + \sin v} dv - \int^v \frac{v^{-1/2} \cos v}{2 + \sin v} dv \\ &= 2I_1 - I_2. \end{aligned}$$

We show first that

$$(9.3) \quad \lim_{v \rightarrow \infty} I_1 = -\infty$$

and then that

$$(9.4) \quad I_2 = O(1) \quad (0 < v < \infty).$$

In order to establish (9.3) it will be sufficient to show that for the "maximizing" sequence  $v_n = (2n+1)\pi$ ,

$$\lim_{n \rightarrow \infty} I_1 = -\infty.$$

We have for  $v = (2n+1)\pi$ ,

$$\begin{aligned} (9.5) \quad I_1 &= \sum_k^n \int_{(2k-1)\pi}^{2k\pi} + \int_{2k\pi}^{(2k+1)\pi} \\ &\leq \sum_k^n [(2k-1)\pi]^{1/2} \int_{(2k-1)\pi}^{2k\pi} \frac{\sin v}{2 + \sin v} dv \\ &\quad + [(2k+1)\pi]^{1/2} \int_{2k\pi}^{(2k+1)\pi} \frac{\sin v}{2 + \sin v} dv \\ &= \sum_k^n [(2k+1)\pi]^{1/2} \left[ \frac{(2k-1)^{1/2}}{2k+1} \int_{(2k-1)\pi}^{2k\pi} + \int_{2k\pi}^{(2k+1)\pi} \right]. \end{aligned}$$

Now since

$$-\int_{(2k-1)\pi}^{2k\pi} \frac{\sin v}{2 + \sin v} dv > \int_{2k\pi}^{(2k+1)\pi} \frac{\sin v}{2 + \sin v} dv > 0,$$

for every  $k = 1, 2, \dots$ , and since these integrals have the same value, equation (9.3) follows from (9.5). As for (9.4) we have, on integrating by parts,

$$I_2 = v^{-1/2} \log(2 + \sin v) \Big|_0^v + \int_0^v \frac{v^{-3/2}}{2} \log(2 + \sin v) dv = O(1).$$

We continue with two lemmas.

LEMMA 9.1. *If  $x=0$  is not its own first conjugate point, if  $u(x)$  is a nonprincipal solution which is positive near  $x=0$ , and if*

$$(9.6) \quad \limsup_{x \rightarrow 0} r(x)u'(x) < \infty,$$

then

$$(9.7) \quad \liminf_{x \rightarrow 0} u(x) > 0.$$

Suppose first that there exists a nonprincipal solution  $v(x)$  such that  $v'(x)$  does not change sign near  $x=0$ . Then  $\lim_{x \rightarrow 0} v(x)$  exists finite or infinite. From the relation

$$(9.8) \quad u(x) = v(x) \left( c_1 + c_2 \int_x^b \frac{dx}{r(x)v^2(x)} \right),$$

it follows that  $\lim_{x \rightarrow 0} u(x)$  exists for every nonprincipal solution. If

$$(9.9) \quad \lim_{x \rightarrow 0} u(x) > 0$$

the theorem follows. If

$$(9.10) \quad \lim_{x \rightarrow 0} u(x) = 0,$$

then  $\lim v(x) = 0$  since  $u(x)$  is a nonprincipal solution. Every solution, therefore, which vanishes on  $(0, b)$  satisfies condition (9.10). Accordingly, the  $y$ -axis contains its own focal point, and by Lemma 7.1

$$(9.11) \quad \limsup_{x \rightarrow 0} r(x)u(x)u'(x) = +\infty.$$

But from (9.10), equation (9.11) implies that

$$\limsup_{x \rightarrow 0} r(x)u'(x) = +\infty,$$

contrary to hypothesis. We conclude that if  $v(x)$  exists,  $\lim_{x \rightarrow 0} u(x) > 0$ .

In the remaining case, there exists no nonprincipal solution whose derivative is of one sign near  $x=0$ . Let  $v(x)$  be a nonprincipal solution such that  $v(x) > 0$  near  $x=0$  and

$$(9.12) \quad r(x)[v(x)u'(x) - u(x)v'(x)] = c \quad (0 < x \leq b),$$

where  $c$  is a positive constant. Suppose now that  $u(x)$  does not satisfy (9.7). Then

$$(9.13) \quad \liminf_{x=0} u(x) = 0; \quad \liminf_{x=0} v(x) = 0.$$

Since  $v'(x)$  is not of one sign we may find a sequence of minimum points  $x = x_n$ ,  $n = 1, 2, \dots$ , which tend to zero and for which

$$(9.14) \quad \lim_{n=\infty} v(x_n) = 0, \quad v'(x_n) = 0, \quad n = 1, 2, \dots$$

Upon setting  $x = x_n$  in equation (9.12), we have by (9.14)

$$(9.15) \quad \lim_{n=\infty} r(x_n)u'(x_n) = \lim_{n=\infty} \frac{c}{v(x_n)} = \infty.$$

Thus we have obtained a contradiction, and the lemma is proved.

LEMMA 9.2. *If  $q(x)$  is absolutely continuous on  $[0, e]$  and if  $p(x)$  satisfies the condition*

$$(9.16) \quad \int_x^e p(x)dx = O(1)$$

as  $x$  tends to zero, then

$$(9.17) \quad \int_x^e p(x)q(x)dx = O(1)$$

as  $x$  tends to zero.

By (9.16) we have

$$(9.18) \quad \left| \int_x^e p(x)dx \right| \leq M \quad (0 < x < e),$$

where  $M$  is independent of  $x$ .

After an integration by parts, we obtain

$$(9.19) \quad \int_x^e p(x)q(x)dx = q(x) \int_x^e p(x)dx + \int_x^e q'(t) \int_t^e p(s)dsdt.$$

Now since  $q(x)$  is absolutely continuous on  $[0, e]$ ,  $q(x)$  is bounded there and  $q'(x) \in L[0, e]$ . Therefore by (9.18) and (9.19)

$$\left| \int_x^e p(x)q(x)dx \right| \leq M \max_{0 \leq x \leq e} |q(x)| + M \int_x^e |q'(x)|dx = O(1) \quad (0 < x < e).$$

The proof is complete.

We come to a principal theorem.

THEOREM 9.2. *If  $[0, b]$  contains no conjugate point of  $x=0$  and if*

$$(9.20) \quad \int_x^b p(x)dx = O(1) \quad (0 < x \leq b),$$

*the functional  $J$  is afforded an  $A$ -minimum limit by  $[0, b]$ .*

By Theorem 7.4, it is sufficient to show that

$$w(x) = r(x) \frac{u'(x)}{u(x)}$$

is bounded near  $x=0$  for a nonprincipal solution  $u(x)$ . By Theorem 6.2 we have

$$w'(x) + \frac{w^2(x)}{r(x)} + p(x) = 0 \quad (0 < x < a \leq b),$$

where  $a$  is a sufficiently small number. Thus

$$(9.21) \quad w(x) = w(a) + \int_x^a \frac{w^2(x)}{r(x)} dx + \int_x^a p(x)dx,$$

and the theorem will follow if we show that the first integral in (9.21) is bounded as  $x$  tends to zero. Suppose the contrary, that is

$$(9.22) \quad \int_0^a \frac{w^2(x)}{r(x)} dx = \infty.$$

Then by (9.20) and (9.21) it follows that

$$(9.23) \quad \lim_{x \rightarrow 0} w(x) = \lim_{x \rightarrow 0} r(x) \frac{u'(x)}{u(x)} = \infty.$$

If  $u(x) > 0$  near  $x=0$ , then  $u'(x) > 0$  on an interval  $[0, e]$ ; consequently,  $u(x)$  is absolutely continuous on  $[0, e]$ . It follows that

$$(9.24) \quad \lim_{x \rightarrow 0} u(x) = 0.$$

For, from the E.E. we have

$$(9.25) \quad r(x)u'(x) = r(e)u'(e) + \int_x^e p(x)u(x)dx.$$

Now since  $u(x)$  possesses the same properties as  $q(x)$  in Lemma 9.2, it follows that

$$(9.26) \quad r(x)u'(x) = O(1) \quad (0 < x \leq b),$$

and (9.24) follows from (9.23). Recall that  $u(x)$  is a nonprincipal solution of the E.E. and note that by virtue of (9.24) it fails to satisfy the conclusion (9.7) of Lemma 9.1. We then have

$$(9.27) \quad \limsup_{x=0} r(x)u'(x) = \infty.$$

Thus (9.26) and (9.27) contradict each other, and the theorem is proved.

In the following corollary, the hypothesis of the previous theorem is assumed.

COROLLARY 9.1. *If*

$$(9.28) \quad \int_0^b \frac{dx}{r(x)} < \infty,$$

*then the focal point  $x=f_0$  of the  $y$ -axis is positive or nonexistent and  $J$  possesses an  $F$ -minimum limit on  $[0, f_0]$ .*

Let  $u(x)$  be a nonprincipal solution of the E.E. Then according to the proof of Theorem 9.2,  $r(x)u'(x)/u(x)$  is bounded on an interval  $(0, a]$ , and

$$(9.29) \quad \log u(x) \Big|_x^a = \int_x^a \frac{1}{r(x)} \left[ r(x) \frac{u'(x)}{u(x)} \right] dx.$$

Therefore  $\lim_{x=0} u(x)$  exists, is finite, and  $\neq 0$ . It follows that

$$r(x)u(x)u'(x) = u^2(x)r(x) \frac{u'(x)}{u(x)} = O(1).$$

By Lemma 7.1, it is now clear that  $f_0 > 0$ , if it exists.

The following example shows that condition (9.28) of the above corollary cannot be removed.

EXAMPLE 9.2. Let  $a, b, c$ , and  $k$  be constants to be specified later. We take

$$r(x) = x^b; \quad u(x) = x^a(k + \sin \log x),$$

where  $u(x)$  is a solution of the E.E. Further

$$\begin{aligned} u'(x) &= x^{a-1} [\cos \log x + a \sin \log x + ak] \\ &= x^{a-1} [(1 + a^2)^{1/2} \sin(\log x + c) + ak], \end{aligned}$$

where  $c$  satisfies the relations

$$\sin c = (1 + a^2)^{-1/2}; \quad \cos c = a(1 + a^2)^{-1/2}.$$

Now if  $w(x) = r(x)u'(x)/u(x)$ , then in the above notation

$$w(x) = x^{b-1}(k + \sin \log x)^{-1}[(1 + a^2)^{+1/2} \sin(\log x + c) + ak].$$

A solution  $v(x)$  which vanishes at  $x=d$  is given by

$$v(x) = u(x) \int_x^d \frac{dx}{r(x)u^2(x)},$$

and thus

$$v'(x) = -\frac{1}{r(x)u(x)} + u'(x) \int_x^d \frac{dx}{r(x)u^2(x)}.$$

Now choose  $a, b, c$ , and  $k$  so that  $b > 1$ ,  $a + b < 0$ , and such that  $k > 1$  and  $k$  is small enough to satisfy the inequality

$$(1 + a^2)^{1/2} > |ak|.$$

Then there exists no point conjugate to  $x=0$  on  $[0, e]$  if  $e$  is sufficiently small. Further,

$$\int_0^e \frac{dx}{r(x)} = \infty, \quad w(x) = O(x^{b-1}),$$

$$\int_x^e p(x)dx = -\int_x^e \frac{w^2(x)}{r(x)} dx - w(x) \Big|_x^e = O(1) \quad (0 < x \leq e),$$

and, hence,  $[0, e]$  affords an  $A$ -minimum limit to  $J$ . Also

$$\limsup_{x=0} u'(x) = \infty,$$

$$\liminf_{x=0} u'(x) = -\infty,$$

and

$$\lim_{x=0} r(x)u(x) = \infty.$$

Therefore, the solution  $v(x)$  which vanishes at  $x=d$  has a derivative which vanishes infinitely often near  $x=0$ . Thus  $x=0$  is the focal point of the  $y$ -axis.

COROLLARY 9.2. *If  $x=0$  is not its own first conjugate point and if*

$$(9.30) \quad \int_0^b \frac{dx}{r(x)} < \infty,$$

then

$$w(x) = r(x) \frac{u'(x)}{u(x)}$$

is bounded near  $x=0$  for a nonprincipal solution  $u(x)$  if and only if

$$(9.31) \quad \int_x^b p(x)dx = O(1) \quad (0 < x \leq b).$$

Suppose that (9.30) holds and that  $w(x)$  is bounded. Let  $a$  be chosen such that  $u(x) \neq 0$  for  $x$  on  $(0, a]$ . Then by (9.21)

$$\int_x^a p(x)dx = w(x) - w(a) - \int_x^a \frac{w^2(x)}{r(x)} dx,$$

and (9.31) follows. Conversely, if (9.31) holds, it follows from Theorem 9.2 that  $w(x)$  is bounded.

We continue with the following theorem.

**THEOREM 9.3.** *If  $[0, b]$  contains no point conjugate to  $x=0$ , if there exists a bounded nonprincipal solution  $u(x)$  then  $[0, b]$  affords an  $A$ -minimum limit to  $J$  if and only if the focal point of the  $y$ -axis is positive or nonexistent. Hence a subinterval of  $[0, b]$  affords an  $F$ -minimum limit to  $J$  if  $[0, b]$  affords an  $A$ -minimum limit to  $J$ .*

Since  $u(x)$  is bounded, it is  $F'$ -admissible near  $x=0$ . Hence by Theorem 7.1

$$\liminf_{x=0} J(u) \Big|_x^b = \liminf_{x=0} r(x)u(x)u'(x) \Big|_x^b > -\infty.$$

From Lemma 7.1, it follows that the  $y$ -axis does not contain its own focal point. Hence by Theorem 8.2, there exists a subinterval of  $[0, b]$  which affords an  $F$ -minimum limit to  $J$ .

**THEOREM 9.4.** *If  $[0, b]$  affords an  $A$ -minimum limit to  $J$  there exists a bounded nonprincipal solution of the E.E. if*

$$\int_0^b \frac{dx}{r(x)} < \infty$$

and

$$\int_x^b p(x)dx = O(1) \quad (0 < x \leq b).$$

We state the following theorem due to Leighton [2].

**THEOREM 9.5.** *If  $p(x) > 0$  near  $x=0$  and if an interval  $[0, b]$  affords an  $A$ -minimum limit to  $J$ , then the focal point  $f_0$  of the  $y$ -axis is positive or nonexistent. Thus  $J$  has an  $F$ -minimum limit on  $[0, f_0]$ .*

When  $\int_0^b dx/r(x) < \infty$  this theorem is contained in Corollary 9.1. In the remaining case we may state the slightly sharper result.

THEOREM 9.6. *If  $p(x) > 0$  near  $x = 0$  if  $[0, b)$  contains no point conjugate to  $x = 0$ , and if*

$$\int_0^b p(x)dx < \infty, \quad \int_0^b \frac{dx}{r(x)} = \infty,$$

*then  $J$  has an  $F$ -minimum limit on  $[0, b]$ .*

We shall show that every solution  $w(x)$  of the Riccati equation increases monotonically to zero as  $x$  approaches zero. Thus  $w(x)$  is negative near  $x = 0$  and the theorem follows from Theorem 6.3. Since

$$(9.32) \quad w'(x) = -\frac{w^2(x)}{r(x)} - p(x) \quad (0 < x < a)$$

it follows from the hypothesis that  $w'(x) > 0$  for  $x$  sufficiently near to  $x = 0$ . Thus, as  $x$  approaches zero,  $w(x)$  increases monotonically to a limit  $L$ , finite or infinite. We have from (9.32)

$$-\frac{w'(x)}{w^2(x)} = \frac{1}{r(x)} + \frac{p(x)}{w^2(x)}.$$

Thus for  $a$  sufficiently close to zero

$$(9.33) \quad \frac{1}{w(a)} - \frac{1}{w(x)} = \int_x^a \frac{dx}{r(x)} + \int_x^a \frac{p(x)}{w^2(x)} dx.$$

Now let  $x$  tend to zero. If  $L$  were  $+\infty$ , the right-hand side of (9.33) would approach  $+\infty$  while the left-hand side would approach  $1/w(a)$ . Thus  $L < \infty$ . Suppose now  $L \neq 0$ . From (9.32) we have

$$(9.34) \quad w(x) = w(a) + \int_x^a \frac{w^2(x)}{r(x)} dx + \int_x^a p(x)dx.$$

The left-hand side remains finite as  $x$  tends to 0; consequently, so does the right-hand side. Thus, by the hypothesis,  $\int_0^a (w^2(x)/r(x))dx$  is finite. It follows that  $L = 0$ .

THEOREM 9.7. *If  $[0, b)$  contains no point conjugate to  $x = 0$  with respect to the equation*

$$(9.35) \quad (ry)' + ry = 0,$$

*if*

$$(9.36) \quad \int_0^b r(x)dx < \infty$$

*and*

$$(9.37) \quad r(x) - p(x) \geq 0 \quad (0 < x \leq b),$$

then  $[0, b]$  affords an  $A$ -minimum limit to  $J$ .

We observe first that (9.36) and (9.37) imply by Theorem 9.2 that  $[0, b]$  affords a minimum limit to the functional

$$J_1(y) \Big|_x^b = \int_x^b [r(x)y'^2 - r(x)y^2] dx.$$

Then by (9.37) we have for each  $A$ -admissible curve  $y=y(x)$  and every  $x$  ( $0 < x \leq b$ )

$$J(y) \Big|_x^b \geq J_1(y) \Big|_x^b.$$

Thus

$$\liminf_{x=0} J(y) \Big|_x^b \geq \liminf_{x=0} J_1(y) \Big|_x^b \geq 0$$

and the theorem follows.

We proceed with a theorem of a different type.

**THEOREM 9.8.** *If  $[0, b]$  contains no point conjugate to  $x=0$ , and if*

$$(9.38) \quad |r(x)p(x)| \leq M \quad (0 < x \leq b)$$

where  $M$  is a constant, then  $[0, b]$  affords an  $A$ -minimum limit to  $J$ .

We consider a number of cases. Let  $u(x)$  be a nonprincipal solution of the E.E. which is positive near  $x=0$ . Suppose first that

$$(9.39) \quad w(x) = r(x) \frac{u'(x)}{u(x)}$$

is of one sign near  $x=0$ . If

$$w(x) \leq 0 \quad (0 < x \leq c \leq b),$$

the proof is complete. Otherwise, we have

$$(9.40) \quad w(x) \geq 0 \quad (0 < x \leq c \leq b)$$

and thus

$$(9.41) \quad u'(x) \geq 0 \quad (0 < x \leq c \leq b).$$

Thus

$$(9.42) \quad \lim_{x=0} u(x) = L$$

exists, is finite and nonnegative. We assert that

$$(9.43) \quad r(x)u'(x) = O(1) \quad (0 < x \leq b)$$

and that this implies

$$(9.44) \quad L > 0.$$

From the E.E.

$$(r(x)u'(x))' + p(x)u(x) = 0 \quad (0 < x \leq b)$$

we have

$$(9.45) \quad \begin{aligned} 2r(x)u'(x)(r(x)u'(x))' &= -2r(x)p(x)u(x)u'(x) \\ &= -r(x)p(x)[u^2(x)]' \end{aligned} \quad (0 < x \leq b).$$

If both members of this identity are integrated between the limits  $x$  and  $c$ , one obtains

$$(9.46) \quad [r(x)u'(x)]^2 = [r(c)u'(c)]^2 + \int_x^c r(x)p(x)[u^2(x)]' dx.$$

From this equation, (9.38), and (9.41) it follows that

$$\begin{aligned} [r(x)u'(x)]^2 &\leq [r(c)u'(c)]^2 + M \int_x^c [u^2(x)]' dx \\ &= [r(c)u'(c)]^2 + M[u^2(c) - u^2(x)] \\ &\leq [r(c)u'(c)]^2 + M[u^2(c) - L^2]. \end{aligned}$$

This establishes (9.43). But then by Lemma 9.1 it follows that (9.44) is satisfied and the assertion is proved. We have then that

$$(9.47) \quad r(x) \frac{u'(x)}{u(x)} = O(1) \quad (0 < x \leq b),$$

and the theorem follows in this case by Theorem 7.4. In the remaining case  $w(x)$  changes sign infinitely often neighboring  $x=0$ . There then exists a sequence  $x_n$  ( $n=1, 2, \dots$ ) tending to zero such that

$$(9.48) \quad w'(x_n) = 0 \quad (n = 1, 2, \dots)$$

and

$$(9.49) \quad \limsup_{x \rightarrow 0} w^2(x) = \lim_{n \rightarrow \infty} w^2(x_n).$$

From the Riccati relation

$$w'(x) + \frac{w^2(x)}{r(x)} + p(x) = 0 \quad (0 < x \leq b)$$

and (9.48) we have at  $x = x_n$

$$w^2(x_n) = -r(x_n)p(x_n).$$

Thus by (9.38)

$$w^2(x_n) = |r(x_n)p(x_n)| \leq M.$$

This relation combined with (9.49) yields (9.47), and the proof is complete.

We note in the proof of the above theorem that if  $w'(x)$  is not strictly positive near  $x=0$ , then at the points  $x_n$  of relative maxima and minima

$$w^2(x_n) = -r(x_n)p(x_n) \quad (n = 1, 2, \dots)$$

and thus

$$p(x_n) \leq 0 \quad (n = 1, 2, \dots).$$

Hence in this case, (9.38) may be replaced by the weaker condition

$$(9.50) \quad r(x)p(x) \geq M \quad (0 < x \leq b).$$

If  $w(x)$  is of one sign near  $x=0$ , then  $r(x)p(x) \leq M$  ( $0 < x \leq b$ ) may replace (9.38). In the next theorem we consider the extent to which (9.38) is a necessary singularity condition.

LEMMA 9.3. *If  $[0, b]$  affords an  $A$ -minimum limit to  $J$  and if there exists a nonprincipal solution  $u(x)$  such that*

$$w(x) = r(x) (u'(x)/u(x))$$

*is of one sign near  $x=0$ , then*

$$(9.51) \quad \limsup_{x=0} w(x) < \infty.$$

If  $w(x) \leq 0$  near  $x=0$ , (9.51) is clearly satisfied. Otherwise we have  $w(x) \geq 0$  for  $x$  near the origin. Since  $[0, b)$  contains no point conjugate to  $x=0$ , we may suppose without loss of generality that  $u(x)$  and  $u'(x)$  are non-negative near the origin. Then  $u(x)$  decreases to a nonnegative limit as  $x$  tends to 0. By Corollary 7.1

$$(9.52) \quad \lim_{x=0} u(x) > 0.$$

Further since  $u(x)$  is  $F'$ -admissible near  $x=0$ , we have

$$\liminf_{x=0} J(u) \Big|_x^b = \liminf_{x=0} r(x)u'(x)u(x) \Big|_x^b > -\infty,$$

which implies, by (9.52), that

$$(9.53) \quad \limsup_{x=0} r(x)u'(x) < \infty.$$

The lemma now follows from (9.52) and (9.53).

**THEOREM 9.9.** *If  $[0, b)$  contains no point conjugate to  $x=0$  and if*

$$w(x) = r(x) \frac{u'(x)}{u(x)}$$

*increases monotonically as  $x$  decreases to zero, then in order that  $[0, b]$  afford an  $A$ -minimum limit to  $J$ , it is necessary that*

$$(9.54) \quad r(x)p(x) \geq M \quad (0 < x \leq b),$$

*while it is sufficient that*

$$(9.55) \quad r(x)p(x) \leq N \quad (0 < x \leq b)$$

*for some constants  $M$  and  $N$ .*

The hypothesis implies that

$$w'(x) \leq 0$$

for  $x$  near to  $x=0$  and therefore, by the Riccati equation

$$w'(x) + \frac{w^2(x)}{r(x)} + p(x) = 0 \quad (0 < x \leq b)$$

implies in turn by Lemma 9.3 that  $r(x)p(x) \geq -w^2(x) \geq M$  ( $0 < x \leq b$ ), for an appropriate constant  $M$ .

The remainder of the theorem follows easily from the proof of Theorem 9.8.

We note that if

$$(9.56) \quad \int_0^b \frac{dx}{r(x)} < \infty,$$

then  $\int_0^b p(x)dx$  exists and converges absolutely. This statement follows at once from (9.38) and the relation

$$\int_x^b p(x)dx = \int_x^b \frac{r(x)p(x)}{r(x)} dx.$$

Accordingly Theorem 9.9 is contained in Theorem 9.2 for the class of functionals satisfying (9.56). This is not to be unexpected since Theorem 9.2 is the best of its kind for this class of functionals. If  $\int_0^b dx/r(x) = \infty$  the above statements do not apply as the following example shows.

**EXAMPLE 9.3.** Let

$$r(x) = e^{-1/x}; \quad p(x) = -e^{+1/x} \sin^2 x^{-1} + x^{-2} \cos x^{-1}.$$

A solution of the Riccati equation

$$(9.57) \quad w' + \frac{w^2}{r(x)} + p(x) = 0$$

is given by

$$w(x) = \sin x^{-1}$$

and thus  $[0, \infty]$  has no point conjugate to  $x=0$ . Therefore  $J$  has an  $A$ -minimum limit on every interval  $[0, b]$ . Now

$$r(x)p(x) = -\sin^2 x^{-1} + e^{-1/x}x^{-2} \cos x^{-1} = O(1) \quad (0 < x \leq b).$$

However

$$\int_0^b \frac{w^2(x)}{r(x)} dx = \infty$$

so that since  $w(x)$  is bounded, it follows by (9.57) that

$$\lim_{x=0} \int_x^b p(x) dx = -\infty.$$

From Theorem 9.9, one may derive the following oscillation criterion.

COROLLARY 9.3. *If*

$$(9.58) \quad \limsup_{x=0} \int_x^b p(x) dx = \infty,$$

$$(9.59) \quad |r(x)p(x)| \leq M \quad (0 < x \leq b),$$

then  $x=0$  is its own first conjugate point.

For, if there exists an interval  $(0, e)$  which contains no point conjugate to  $x=0$ , then, by (9.59) and Theorem 9.9,  $[0, e]$  affords an  $A$ -minimum limit to  $J$ . The corollary follows at once.

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