NORMAL AUTOMORPHISMS AND THEIR FIXED POINTS

BY

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1. **Introduction.** The elements of the centralizer T of the group of inner automorphisms J of a group G (in the group A of automorphisms of G) are called the normal automorphisms of G. The center Z of G is the set of all elements of G which are fixed by each mapping from J. Likewise, let B be the set of fixed points held in common by the mappings from T. G/B is abelian, and the elements of T which induce either the identity or the involution on G/B form a subgroup W of T. We shall investigate the ascending central series of W. Just as the ascending central series $\{Z_i\}$ is formed over $Z=Z_1$, so an ascending series is formed over B. Elements of G lying in members of this B-series turn out to be fixed points for high powers of normal automorphisms. For automorphisms which induce the identity on G/Z_n, we show that the common fixed points lie in the centralizer of Z_n in G.

The notation is both obvious and conventional. G denotes a group with automorphism group A and inner automorphism group J. The members of the ascending central series of G are the Z_i, and the higher commutator subgroups are the $G^{(i)}$ [3]. If, say, the inner automorphism group of a group H, different from G, is to be denoted, we employ the symbols $J(H)$, and similarly for other groups or subgroups, such as $Z_i(H)$, associated with H. For a subgroup H of G, the centralizer of H in G will be denoted by $C(H; G)$. If $x, y \in G$, then $(x, y)=x^{-1}y^{-1}xy$. For normal subgroups S and T of a group G, $S \div T$ (following R. Baer) will be the *commutator quotient* of S by T, the set of all $x \in G$ such that $(x, t) \in S$ for every $t \in T$. $S \div T$ is a normal subgroup of G. The identity map on a group is indicated by ι, and the identity element of a group is to be e. For a homomorphism f on G, the kernel will be written kern f. $\oplus$ denotes direct summation of groups. A *periodic group* is one in which each element is of finite order, and an abelian periodic group will be called a *torsion group*. If a periodic group G has a uniform order on its elements, then G is said to be *uniform torsion* (*u.t.*), and the least positive uniform order will be called [3] the *exponent* of G. A group is said to be *torsion-free* if it has no nontrivial elements of finite order. A *complete group* is one in which the x^n form a set of generators of G for each positive integer n. The group of integers is to be I; the group of rationals, R; the multiplicative group of nonzero rationals, R^*; and I_n is to be the group of integers modulo n.

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Occasionally, we shall give I_n its usual representation as the group of integral residue classes, modulo n, so that j_n will be the residue class, modulo n, in which the integer j lies.

2. **Normal automorphisms.** Let G be a group, and let H be a normal subgroup of G. Let α be an automorphism of G such that H is admissible under both α and α^{-1}. That is, $\alpha(H)\subset H$ and $\alpha^{-1}(H)\subset H$. Then α induces an automorphism α' on G/H given by $\alpha'(xH)=\alpha(x)H$. In particular, if H is a characteristic subgroup of G, then every automorphism α of G induces an automorphism α' on G/H. If α induces the identity automorphism ι on G/Z_1, α is called a *normal automorphism* (sometimes *center* [3] or *central* [1] automorphism) of G, and $\alpha(x)\equiv x \bmod Z_1$ for every $x\in G$. It is easy to see that $\alpha\in A$ is normal if, and only if, $(x, \alpha(y))=(x, y)$ for every $x, y\in G$. Let T_1 be the set of all normal automorphisms of G. If automorphism composition is interpreted as a multiplication, T_1 becomes a subgroup of the automorphism group A of G with ι as its identity, and T_1 is normal in A. It is well known that $T_1=C(J; A)$ [3]. An endomorphism γ of G for which $\gamma(G)\subset Z_1$ is called a *central endomorphism*. If $\alpha\in T_1$, $\alpha(x)=x\gamma(x)$ for every $x\in G$, where γ is a central endomorphism of G with the further property (A) that to each $y\in G$, there exists a unique $g=g(y;\gamma)\in G$ with $\gamma(g)=g^{-1}y$. We might write $\alpha=\iota+\gamma$. If, conversely, γ is a central endomorphism with (A), then the mapping α, defined by $\alpha=\iota+\gamma$, is in T_1.

Let G be a group for which $Z_2\neq Z_1$. If $u\in Z_2$, $u\notin Z_1$, then the mapping γ_u given by $\gamma_u(x)=(x, u)$ is readily seen to be a central endomorphism of G. These γ_u will be called the *Grün endomorphisms* of G. If $y\in G$, then $\gamma_u(uyu^{-1})=(uyu^{-1})^{-1}y$ so that $\gamma_u(x)=x^{-1}y$ has the solution $x=uyu^{-1}$. If, conversely, x is any solution of $\gamma_u(x)=x^{-1}y$, then $u^{-1}xu=y$ so that $x=uyu^{-1}$; and the solution is unique, establishing (A). Hence α_u, a mapping defined by $\alpha_u(x)=x\gamma_u(x)=u^{-1}xu=\tau_u(x)$, is in T_1. Suppose, conversely, that $\alpha\in T_1\cap J$. Let $\alpha=\tau_u$. Then $u^{-1}xu\equiv x \bmod Z_1$ for every $x\in G$, so that $u\in Z_2$ and $\alpha=\alpha_u$. We state

LEMMA 1. *$T_1\cap J\cong Z_1(J)$, and the elements of the former are in one-to-one correspondence with the Grün endomorphisms of G.*

For endomorphisms γ_α and γ_β satisfying (A), note that $\gamma_\alpha(G)\subset\gamma_\beta(G)$ implies $\gamma_{\alpha\beta}(G)\subset\gamma_\beta(G)$, so that $\gamma_{\alpha^n}(G)\subset\gamma_\alpha(G)$ $(n=1, 2, 3 \cdots)$. γ_ι is the trivial endomorphism ($\gamma_\iota(x)=e$ for every $x\in G$), and $\gamma_\iota(G)\subset\gamma_\alpha(G)$ for every $\alpha\in T_1$.

LEMMA 2. *For $\alpha\in T_1$, $\alpha(\gamma_\alpha(G))=\gamma_\alpha(G)=\alpha(\gamma_{\alpha^{-1}}(G))$.*

Proof. For $x\in G$, $\alpha(\alpha(x))=\alpha(x)\alpha(\gamma_\alpha(x))$, so that $\alpha(\gamma_\alpha(x))\in\gamma_\alpha(G)$. Hence $\alpha(\gamma_\alpha(G))\subset\gamma_\alpha(G)$. $\alpha^{-1}(x)=x\gamma_{\alpha^{-1}}(x)$ implies that $x=\alpha(x)\alpha(\gamma_{\alpha^{-1}}(x))=x\gamma_\alpha(x)\alpha(\gamma_{\alpha^{-1}}(x))$, whence $\gamma_\alpha(x)=\alpha(\gamma_{\alpha^{-1}}(x^{-1}))$; and $\gamma_\alpha(G)\subset\alpha(\gamma_{\alpha^{-1}}(G))$. Replacing x by x^{-1}, we have $\gamma_\alpha(x^{-1})=\alpha(\gamma_{\alpha^{-1}}(x))$. There exists $y\in G$ such that $\alpha(y)=\gamma_\alpha(x^{-1})=y\gamma_\alpha(y)$. Since $\gamma_\alpha(G)$ is a subgroup of G, $y\in\gamma_\alpha(G)$. Thus,

$\alpha(\gamma_\alpha^{-1}(x))=\alpha(y)$ where $y\in\gamma_\alpha(G)$, so that $\alpha(\gamma_\alpha^{-1}(G))\subset\alpha(\gamma_\alpha(G))$.

3. **The common fixed points.** The subgroup Z_1, the center of G, is the set of all elements of G which are fixed by each inner automorphism τ_y of G. For $T_1=C(J; A)$, the set analogous to Z_1 is B_1, where $x\in B_1$ if, and only if, $\alpha(x)=x$ for every $\alpha\in T_1$. If $F(\alpha)$ is the set of the fixed points of $\alpha\in T_1$, then $F(\alpha)$ is a normal subgroup of G. Since $B_1=\bigcap F(\alpha)$, where the cross-cut is taken over all $\alpha\in T_1$, B_1 is likewise a normal subgroup of G. Now $F(\alpha)=\text{kern } \gamma_\alpha$, and $\gamma_\alpha(G)$ is abelian. Thus $F(\alpha)\supset G'$, the derivative of G, for every $\alpha\in T_1$, and $B_1\supset G'$. This shows that G/B_1 is abelian and that if $B_1=(e)$, then G is abelian.

LEMMA 3. $G'\subset B_1\subset C(Z_2; G)$.

Proof. If $x\in B_1$, $\gamma_u(x)=e$ for every Grün endomorphism γ_u, $u\in Z_2$. Consequently x commutes with every such u.

COROLLARY. *If $C(Z_2; G)=Z_1$, then G is of class* 2.

Suppose that H is a characteristic subgroup of G, that $J(H; G)$ is the set of all inner automorphisms τ_v of G where $v\in H$ (where $\tau_v(x)=v^{-1}xv$), and that $Q(H; G)=Q(H)$ is the set of all automorphisms of G such that $\alpha\in Q(H)$ induces the identity automorphism on G/H. For instance, $J(G; G)=J$, and $Q(Z_1; G)=T_1$. $J(H; G)$ is a normal subgroup of $Q(H; G)$. Let $F=F(Q(H; G))$ be the fixed points common to all mappings in $Q(H; G)$, and let $F^*=F(J(H; G))$ be the fixed points common to all mappings in $J(H; G)$. $F^*\supset F$. But $F^*=C(H; G)$, so that $F(Q(H; G))\subset C(H; G)$. This general result will be used later to establish a variation of Lemma 3.

$G=B_1$ if, and only if, G has no proper normal automorphisms. By Lemma 3, $G=B_1$ implies $G=C(Z_2; G)$ so that every element of G commutes with every element of Z_2, and $Z_2\subset Z_1$. Hence the ascending central series of G breaks off with Z_1 if $G=B_1$. Likewise, Lemma 3 has the following obvious

COROLLARY. *G is of class 2 if, and only if, $B_1\subset Z_1$.*

In particular, $G'\subset Z_1$ if, and only if, $B_1\subset Z_1$.

LEMMA 4. (a) *If T_1 is finite and if Z_1 is a torsion group, then G/B_1 is a torsion group.* (b) *If Z_1 is u.t., then G/B_1 is u.t. and* $\exp G/B_1 | \exp Z_1$. (c) *If Z_1 is torsion-free, then so is G/B_1.*

Proof. (a) For $x\in G$ and $\alpha\in T_1$, $\alpha(x)=x\gamma_\alpha(x)$, where $\gamma_\alpha(x)\in Z_1$. There exists a least positive integer $n=n(x; \alpha)$ such that $\gamma_\alpha(x^n)=e$, since Z_1 is a torsion group. Since T_1 is finite, we can form $n(x)$, the least common multiple of all such $n(x; \alpha)$. For $\alpha\in T_1$, $\alpha(x^{n(x)})=x^{n(x)}$ so that $x^{n(x)}\in B_1$, and G/B_1 is a torsion group. (b) has a proof which is an obvious modification of the proof of (a). (c) Suppose that Z_1 is torsion-free and that $x^n\in B_1$. Then for $\alpha\in T_1$, $\alpha(x^n)=x^n$. But $\alpha(x)=x\gamma_\alpha(x)$, so that $\gamma_\alpha(x^n)=e$. Since $\gamma_\alpha(x)$ is not a periodic element, $\gamma_\alpha(x)=e$ and $x\in B_1$. Hence G/B_1 is torsion-free.

LEMMA 5. *If G/B_1 is complete, then each $\gamma_\alpha(G)$ is complete; and if, in addition, Z_1 is torsion-free, G/B_1 and each $\gamma_\alpha(G)$ are direct sums of copies of R, the additive group of the rationals.*

Proof. For $z\in\gamma_\alpha(G)$, there exist $x\in G$ and $\alpha\in T_1$ with $\alpha(x)=xz$. Since G/B_1 is complete, for each positive integer n there exists $y\in G$ with $x\equiv y^n \mod B_1$. $\alpha(y^nx^{-1})=y^nx^{-1}=y^n\gamma_\alpha(y^n)x^{-1}z^{-1}=y^nx^{-1}\gamma_\alpha(y^n)z^{-1}$. Hence $\gamma_\alpha(y^n)=z$, and $[\gamma_\alpha(y)]^n=z$. Since $\gamma_\alpha(G)$ is abelian, this is enough to show that it is complete. If, in addition, Z_1 is torsion-free, then Lemma 4(c) shows that G/B_1 is torsion-free. Also each $\gamma_\alpha(G)$ is torsion-free. But torsion-free, complete abelian groups are direct sums of copies of R.

4. **Automorphisms induced on** G/B_1. Since B_1 is admissible under each normal automorphism of G, each such automorphism induces an automorphism on G/B_1. (See, however, [1] where G/G' for finite G is discussed instead.) If $\alpha\in T_1$ induces the identity on G/B_1, then $\alpha(x)=x\gamma_\alpha(x)\equiv x \mod B_1$ so that $\gamma_\alpha(G)\subset B_1$. Conversely, if $\alpha\in T_1$ and if $\gamma_\alpha(G)\subset B_1$, then the induced automorphism α' has the property $\alpha'(xB_1)=\alpha(x)B_1=x\gamma_\alpha(x)B_1=xB_1$ for every $xB_1\in G/B_1$. Thus, a necessary and sufficient condition that $\alpha\in T_1$ induce ι on G/B_1 is that $\gamma_\alpha(G)\subset B_1$. Let the set of all such $\alpha\in T_1$ be denoted by V_1. By a well known result [3, p. 78] on automorphisms which leave a normal subgroup H and the factor group G/H point-wise fixed, V_1 is an abelian group under automorphism composition. V_1 is a normal subgroup of T_1. For, if $\alpha\in V_1$, $\beta\in T_1$, then $\beta^{-1}\alpha\beta(x)=\beta^{-1}\alpha(x\gamma_\beta(x))=\beta^{-1}(x\gamma_\beta(x)bc)=xbc$ where b, $c\in B_1$, and $\alpha(x)=xb$, $\alpha\gamma_\beta(x)=\gamma_\beta(x)c$. This makes V_1 normal in T_1. Moreover, $\alpha^{-1}\beta^{-1}\alpha\beta(x)=xc$, and we have

LEMMA 6. *If $\alpha\in V_1$ and if $\beta\in T_1$, then $\gamma_{(\alpha,\beta)}=\gamma_\alpha\gamma_\beta$.*

Since G/B_1 is an abelian group, it has the automorphism ω given by $\omega(y)=y^{-1}$ for every $y\in G/B_1$. $\omega^2=\iota$, and ω is called the *involution automorphism.*

LEMMA 7. (a) *If $\alpha\in T_1$ induces the involution automorphism on G/B_1, then α induces the involution automorphism on $\gamma_\alpha(G)$.* (b) *$\alpha\in T_1$ induces ω on G/B_1 if, and only if, $\gamma_\beta\gamma_\alpha(x)=\gamma_\beta(x^{-2})$ for every $x\in G$ and for every $\beta\in T_1$.*

Proof. (a) For $x\in G$, $\alpha(x)=x\gamma_\alpha(x)\equiv x^{-1} \mod B_1$ so that $x^2\gamma_\alpha(x)\in B_1$ and $\gamma_\alpha(x^2)\gamma_\alpha(\gamma_\alpha(x))=e$. Thus $\gamma_\alpha(x)\alpha(\gamma_\alpha(x))=e$, so that $\alpha(\gamma_\alpha(x))=\gamma_\alpha(x^{-1})$, and α induces ω on $\gamma_\alpha(G)$. (b) α induces ω on G/B_1 if, and only if, $x^2\gamma_\alpha(x)\in B_1$ for every $x\in G$. For $\beta\in T_1$, $\gamma_\beta(x^2\gamma_\alpha(x))=e$ so that $\gamma_\beta\gamma_\alpha(x)=\gamma_\beta(x^{-2})$. Conversely, if $\gamma_\beta\gamma_\alpha(x)=\gamma_\beta(x^{-2})$ for every $\beta\in T_1$, then $x^2\gamma_\alpha(x)\in B_1$.

If we let W_1 be the set of all $\alpha\in T_1$ which induce either ι or ω on G/B_1, then W_1 is a group under automorphism composition. Let the set of those elements of W_1 which are not in V_1 be denoted by W_1^*. Assume, for the present, that this set is nonvoid. It is easy to verify that the elements of W_1^* are carried into elements of W_1^* by the inner automorphisms of the group T_1, so that W_1 is a normal subgroup of T_1. The index $[W_1: V_1]=2$, and $W_1/V_1 \cong I_2$; for, if $\alpha, \beta \in W_1^*$, then $\alpha^{-1}\beta(x)=\alpha^{-1}(x^{-1}b)=\alpha^{-1}(x^{-1})b$, where $b \in B_1$. Since $\alpha(x)=x^{-1}c$ (where $c \in B_1$), $\alpha^{-1}(x^{-1})=xc^{-1}$, and $\alpha^{-1}\beta(x)\equiv x \bmod B_1$ so that $\alpha^{-1}\beta \in V_1$.

THEOREM 1. (a) *For a group G, W_1 is j-nilpotent for a given positive integer j, or $Z_j(W_1)$ is included properly in V_1.* (b) *If W_1^* is nonvoid, then $\alpha \in Z_j(W_1) \cap V_1$ if, and only if, $\gamma_\alpha(x^{2^j})=e$ for every $x \in G$ and $\alpha \in V_1$.*

Proof. If W_1^* is void, then $V_1=W_1$ and W_1 is abelian. Let us therefore assume that W_1^* is nonvoid. First suppose that $j=1$, and consider $\alpha \in V_1 \cap Z_1(W_1)$. Choose $\beta \in W_1^*$. Then since $\alpha \in Z_1(W_1)$, $(\alpha, \beta)=\iota$. Now $\gamma_{(\alpha,\beta)}(x) =\gamma_\alpha\gamma_\beta(x)=\gamma_\alpha(x^{-2})$, by Lemmas 6 and 7(b). Since $\gamma_\iota(x)=e$ for every $x \in G$, $\gamma_\alpha(x^{-2})=e$ for every $x \in G$. Conversely, if $\gamma_\alpha(x^2)=e$ for every $x \in G$, then $\gamma_\alpha\gamma_\beta(x)=\gamma_{(\alpha,\beta)}(x)=e$ for every $x \in G$ and for every $\beta \in W_1^*$, by Lemmas 6 and 7(b). But $\gamma_{(\alpha,\beta)}(x)=e$ for every $x \in G$ implies that $(\alpha, \beta)=\iota$ for every $\beta \in W_1^*$. Since V_1 is abelian, and since $\alpha \in V_1$, α is in $Z_1(W_1)$. We have verified (b) in the case $j=1$.

Suppose that there exists $\beta \in Z_1(W_1) \cap W_1^*$. Since $[W_1: V_1]=2$, $\beta \in Z_1(W_1)$ if, and only if, $W_1^* \subset Z_1(W_1)$. If $\alpha \in V_1$, $\beta \in Z_1(W_1) \cap W_1^*$, then $(\alpha, \beta)=\iota$ and $\gamma_{(\alpha,\beta)}(x)=\gamma_\alpha\gamma_\beta(x)=e$ for every $x \in G$, by Lemma 6. By Lemma 7(b), $\beta \in W_1^*$ implies $\gamma_\alpha\gamma_\beta(x)=\gamma_\alpha(x^{-2})=e$. By (b), which has been established for the case $j=1$, $\alpha \in Z_1(W_1)$. Hence $W_1=W_1^* \cup V_1 \subset Z_1(W_1)$. It follows that if $W_1 \neq Z_1(W_1)$ then $Z_1(W_1) \subset V_1$; and this inclusion must be strict. For, if not, $Z_1(W_1)=V_1$ and $W_1/V_1 \cong J(W_1)$. Since $W_1/V_1 \cong I_2$, $J(W_1)$ is cyclic, an impossibility [2]. We have now established (a) for the case $j=1$.

Now suppose that the theorem holds for the case $j-1$. If $\beta \in W_1^*$ and if $\alpha \in V_1 \cap Z_j(W_1)$, $(\alpha, \beta) \in Z_{j-1}(W_1)$. If $x \in G$, $(\alpha, \beta)(x)=x\gamma_\alpha(\gamma_\beta(x))=x\gamma_\alpha(x^{-2})$. Noting that $(\alpha, \beta) \in V_1$ since $\gamma_\alpha(x^{-2}) \in B_1$, (b) can be applied for the case $j-1$, and $\gamma_\alpha[(x^{2^{j-1}})^{-2}]=e$, whence $\gamma_\alpha(x^{2^j})=e$ for every $x \in G$. Conversely, suppose that $\alpha \in V_1$ and that $\gamma_\alpha(x^{2^j})=e$ for every $x \in G$. Choose $\beta \in W_1^*$. $\gamma_{(\alpha,\beta)}(y) =\gamma_\alpha\gamma_\beta(y)=\gamma_\alpha(y^{-2})$ for every $y \in G$. Let $y=x^{2^{j-1}}$. Then $\gamma_\alpha(y^{-2})=e$ by assumption, and $\gamma_{(\alpha,\beta)}(x^{2^{j-1}})=e$ for every $x \in G$. Since $\alpha \in V_1$ implies $\gamma_\alpha(y^{-2}) \in B_1$, $\gamma_{(\alpha,\beta)}(y) \in B_1$ and $(\alpha, \beta) \in V_1$. By (b) for the case $j-1$, $(\alpha, \beta) \in Z_{j-1}(W_1)$ for every $\beta \in W_1^*$. If $\beta \in V_1$, then the fact that V_1 is abelian allows one to conclude that $(\alpha, \beta)=\iota \in Z_{j-1}(W_1)$. Hence $(\alpha, \beta) \in Z_{j-1}(W_1)$ for every $\beta \in W_1$, and $\alpha \in Z_j(W_1)$. This establishes (b) for the case j.

Since $[W_1; V_1]=2$, the elements of W_1^* all have the form $\beta\alpha$ where $\alpha \in V_1$. Suppose now that $\beta \in Z_j(W_1) \cap W_1^*$ and that α and δ are elements of V_1. $\beta\alpha\beta\delta \equiv \beta\alpha\delta\beta \equiv \beta\delta\alpha\beta \equiv \beta\delta\beta\alpha \bmod Z_{j-1}(W_1)$. Likewise, $\beta\alpha\delta \equiv \delta\beta\alpha \bmod Z_{j-1}(W_1)$. Hence if $Z_j(W_1) \cap W_1^*$ is nonvoid, then $W_1^* \subset Z_j(W_1)$. If $\alpha \in V_1$, $\beta \in Z_j(W_1) \cap W_1{}^*$, then $(\alpha, \beta) \in Z_{j-1}(W_1)$ and $\gamma_{(\alpha,\beta)}(x)=\gamma_\alpha\gamma_\beta(x)=\gamma_\alpha(x^{-2})$. As above,

$\alpha \in V_1$ implies $(\alpha, \beta) \in V_1$ so that (b) for the case $j-1$ applies, and $\gamma_\alpha[x^{(2^{j-1})}]^{-2} = e$. Thus (b) for the case j, established above, places $\alpha \in Z_j(W_1)$. $W_1 = W_1^* \cup V_1 \subset Z_j(W_1)$, and $W_1 = Z_j(W_1)$.

If $W_1 \neq Z_j(W_1)$, the above shows that $Z_j(W_1) \subset V_1$. If the inclusion is not strict, then $Z_j(W_1) = V_1$ and $I_2 \cong W_1/V_1 \cong J(W_1/Z_{j-1}(W_1))$, an impossibility [2]. This completes the proof of the theorem.

Corollary 1. *Let G be a group for which W_1^* is nonvoid.* (a) *If Z_1 is u.t. with exponent dividing 2^j (where $j>1$), then W_1 is nilpotent of class $\leqq j$.* (b) *If Z_1 is torsion-free and if V_1 is nontrivial, then W_1 is non-nilpotent.*

Proof. (a) For α, $\zeta \in V_1$, $(\alpha, \zeta) = \iota$ since V_1 is abelian. Choose $\beta \in W_1^*$. Since $[W_1 : V_1] = 2$, W_1^* is the coset of V_1 in W_1 which contains $\beta\alpha$. For $x \in G$, $\alpha(x) = xb$, $\zeta(x) = xd$, and $\beta(x) = x^{-1}c$, where b, $d \in Z_1 \cap B_1$ and $c \in B_1$. $(\beta\alpha, \zeta)(x) = \alpha^{-1}\beta^{-1}\zeta^{-1}\beta\alpha\zeta(x) = \alpha^{-1}\beta^{-1}\zeta^{-1}\beta(xbd) = \alpha^{-1}\beta^{-1}\zeta^{-1}(x^{-1}cbd) = \alpha^{-1}\beta^{-1}(dx^{-1}cbd) = \alpha^{-1}(dxc^{-1}cbd) = dxb^{-1}bd = xd^2$ so that $(\beta\alpha, \zeta) = \delta_1 \in V_1$. $(\beta\alpha, \beta\zeta)(x) = \alpha^{-1}\beta^{-1}\zeta^{-1}\alpha\beta\zeta(x) = \alpha^{-1}\beta^{-1}\zeta^{-1}\alpha\beta(xd) = \alpha^{-1}\beta^{-1}\zeta^{-1}\alpha(x^{-1}cd) = \alpha^{-1}\beta^{-1}\zeta^{-1}(b^{-1}x^{-1}cd) = \alpha^{-1}\beta^{-1}(b^{-1}dx^{-1}cd) = \alpha^{-1}(b^{-1}dxc^{-1}cd) = b^{-1}dxb^{-1}d = x(b^{-1}d)^2$, so that $(\beta\alpha, \beta\zeta) = \delta_2 \in V_1$. If $\exp Z_1 | 2^j$, then $\delta_i(x^{2^{j-1}}) = x^{2^{j-1}}(b^{1-i}d)^{2^j} = x^{2^{j-1}}$, and $\gamma_{\delta_i}(x^{2^{j-1}}) = e$ for every $x \in G$. By (b) of the theorem, $\delta_i \in Z_{j-1}(W_1)$, so that $W_1' \subset Z_{j-1}(W_1)$, and W_1 is nilpotent of class $\leqq j$. This establishes (a) of the corollary.

(b) Now suppose that Z_1 is torsion-free. By hypothesis, we can find α and $\zeta \in V_1$ and $y \in G$ with $\alpha(y) \neq \zeta(y)$, where $\alpha(y) = yb$ and $\zeta(y) = yd$. Construct δ_i as in part (a) of the corollary. $\delta_i(y^{2^{j-1}}) = y^{2^{j-1}}(b^{1-i}d)^{2^j}$. Since $b \neq d$, we can always adjust our notation so that $d \neq e$. Since Z_1 is torsion-free, $b \neq d$ and $d \neq e$, $(b^{1-i}d)^{2^j} \neq e$ for each positive integer j, so that, by (b) of the theorem, $\delta_i \in Z_{j-1}(W_1)$, and $W_1' \not\subset Z_{j-1}(W_1)$ for all such j, and W_1 is not nilpotent.

Corollary 2. *Let Z_1 be torsion-free, V_1 be nontrivial, W_1^* be nonvoid and let G be complete. Then W_1 has a trivial center.*

Proof. By Corollary 1(b), W_1 is not nilpotent. By (a) of the theorem, $Z_1(W_1)$ is a proper subgroup of V_1. By (b) of the theorem, $\alpha \in Z_1(W_1)$ implies $x^2 \in F(\alpha)$, the set of all fixed points of α, for every $x \in G$. Since G is complete, $G = F(\alpha)$, and $\alpha = \iota$.

It is fairly obvious that α and $\beta \in T_1$ induce the same automorphism on G/B_1 if, and only if, $\alpha \equiv \beta \bmod V_1$; and an equivalent condition is that $\gamma_\alpha(x) \equiv \gamma_\beta(x) \bmod B_1$ for every $x \in G$. It follows that if $\alpha \equiv \beta \bmod V_1$, then there exists an endomorphism $\lambda_{\alpha,\beta}$ on G into $B_1 \cap Z_1$ such that (1) the kernel of $\lambda_{\alpha,\beta}$ is just $F(\alpha^{-1}\beta) = F(\beta^{-1}\alpha)$; (2) $\gamma_\alpha(x) = \gamma_\beta(x)\lambda_{\alpha,\beta}(x)$; and (3) for $g \in G$, $\lambda_{\alpha,\beta}(x) = \beta(x^{-1})g$ has a unique solution $x = x(g) \in G$. Conversely, if λ is an endomorphism of G into $B_1 \cap Z_1$, if $\beta \in T_1$ and if $\lambda(x) = \beta(x^{-1})g$ has a unique solution $x = x(g)$ for every $g \in G$, then the mapping α defined by $\alpha(x) = \beta(x)\lambda(x)$ is a normal automorphism of G such that $\alpha \equiv \beta \bmod V_1$ and such that $\lambda = \lambda_{\alpha,\beta}$. We restate as follows:

LEMMA 8. *If* $\beta \in T_1$ *and if* λ *is an endomorphism of* G *into* $B_1 \cap Z_1$, *then* $\beta + \lambda \in T_1$ *with* $\beta + \lambda \equiv \beta \bmod V_1$ *if, and only if,* $\iota + \beta^{-1}\lambda \in T_1$.

Recall that $\tau_x(y) = x^{-1}yx$.

LEMMA 9. (a) $\alpha \in W_1^*$ *implies that* $\tau_x \alpha^{-1}(x) = \alpha(x)$ *for every* $x \in G$. (b) $\alpha \in T_1$ *and* $\alpha^2 = \iota$ *imply that* α *induces* ω *on* $\gamma_\alpha(G)$. *If, in addition,* $\alpha \in Z_1(W_1) \cap V_1$ *and if* W_1^* *is nonvoid, then* $\gamma_\alpha(G) \subset \text{kern } \gamma_\alpha$.

Proof. (a) is immediate. As for (b), $\alpha \in T_1$ implies that $\alpha(x) = x\gamma_\alpha(x)$ and $\alpha^{-1}(x) = x\alpha^{-1}(\gamma_\alpha(x^{-1})) = x\gamma_{\alpha^{-1}}(x)$. Hence $\alpha^{-1}(\gamma_\alpha(x^{-1})) = \gamma_{\alpha^{-1}}(x)$, or $\alpha(\gamma_\alpha(x)) = \gamma_\alpha(x^{-1})$, since $\alpha^{-1} = \alpha$, and α induces ω on $\gamma_\alpha(G)$. From this, $\gamma_\alpha(x)\gamma_\alpha^2(x) = \gamma_\alpha(x^{-1})$, and $\gamma_\alpha(x^{-2}) = \gamma_\alpha^2(x)$. By Theorem 1(b), $\gamma_\alpha(x^{-2}) = e$, so that $\gamma_\alpha^2(x) = e$ and $\gamma_\alpha(G) \subset \text{kern } \gamma_\alpha$.

5. **The B-series.** $G/B_1(G)$ is an abelian group so that all of its automorphisms are normal. Define $B_2(G)$ as the complete inverse image in G of $B_1(G/B_1(G))$ under the natural homomorphism of G onto $G/B_1(G)$. In general, suppose that $B_j(G)$ is defined. Then $B_{j+1}/B_j \cong B_1(G/B_j)$. We let $B_0(G) = (e)$. Each B_j is a normal subgroup of G, and $i \leqq j$ implies that $B_i \subset B_j$, so that the B-series ascends monotonically in its index. Each G/B_j is abelian $(j>0)$, and B_{j+1}/B_j is the set of elements of G/B_j which are each fixed by all automorphisms of G/B_j $(j>0)$. If $B_{j+1} = B_j$, then for all $k \geqq j$, $B_k = B_j$.

LEMMA 10. *The B-series breaks off at* B_1 *if any one of the following holds*:

(a) G/B_1 *has no elements of order* 2.

(b) Z_1 *is torsion-free, or* Z_1 *has no elements of order* 2 *or no* $\gamma_\alpha(G)$, *for* $\alpha \in T_1$, *has elements of order* 2.

(c) *To each* xB_1 *in* G/B_1, *there exists an automorphism* $\theta = \theta_x$, *such that* $\theta(xB_1) \neq xB_1$.

(d) *To each* $x \in G$, *there exists* $\alpha = \alpha_x \in A$ *such that* α *induces an automorphism on* G/B_1, *and* $\alpha(x) \not\equiv x \bmod B_1$.

(e) *The equation* $\xi^2 = \alpha$, *for* $\alpha \in T_1$, *always has a solution in* T_1.

Proof. (a) Since G/B_1 is abelian, it has the involution automorphism ω. If $gB_1 \in B_2/B_1$, then $\omega(gB_1) = gB_1 = g^{-1}B_1$, and $g^2 \in B_1$. Since G/B_1 has no elements of order 2, $g \in B_1$ and $B_2 \subset B_1$. (b) For $x^2 \in B_1$ and $\alpha \in T_1$, $\alpha(x^2) = x^2 = x^2\gamma_\alpha(x^2)$, and $\gamma_\alpha(x^2) = e$. Since $\gamma_\alpha(G)$ has no elements of order 2, $\gamma_\alpha(x) = e$ and $x \in B_1$. Hence G/B_1 has no elements of order 2, and (a) applies. (c) There is no fixed point common to all automorphisms of G/B_1, so that B_2/B_1 is trivial, and $B_2 = B_1$. (d) α induces α', an automorphism on G/B_1. $\alpha'(xB_1) \neq xB_1$ so that (c) can now be applied. (e) If $g \in B_2$, then, as we saw in the proof of (a), $g^2 \in B_1$. For $\beta \in T_1$ there exists an induced automorphism β' on G/B_1. Since $g \in B_2$, $\beta'(gB_1) = gB_1 = \beta(g)B_1$. Hence $\beta(g) \equiv g \bmod (Z_1 \cap B_1)$. $\beta^2(g) = \beta(g\gamma_\beta(g)) = \beta(g)\gamma_\beta(g) = g\gamma_\beta(g^2)$, since $\gamma_\beta(g) \in B_1 \cap Z_1$. But $g^2 \in B_1$ implies that $\gamma_\beta(g^2) = e$, so that $\beta^2(g) = g$. Since every $\alpha \in T_1$ is, by hypothesis, a square, $g \in B_1$, and $B_2 \subset B_1$.

6. **The case** $G=B_2$. It is obvious that $G=B_2$ if, and only if, $B_1(G/B_1)=G/B_1$; that is, if, and only if, the identity is the only normal automorphism of G/B_1. Since G/B_1 is abelian, we see that $G=B_2$ if, and only if, G/B_1 has no proper automorphism. But this is equivalent [2; p. 101] to

LEMMA 11. *$G=B_2$ if, and only if $G/B_1\cong I_2$.*

Since $G/B_1\cong I_2$ in this case, choose $u\in B_2$, $u\notin B_1$. Then to each $x\in G$, $x\notin B_1$, there exists $b_x\in B_1$ with $x=ub_x$. For $\alpha\in T_1$, $\alpha(x)=u\gamma_\alpha(u)b_x=x\gamma_\alpha(u)$. Hence $\alpha(x)=x\gamma_\alpha(u)$ if $x\notin B_1$, $=x$ if $x\in B_1$. We note that $\gamma_\alpha(u)\in B_1\cap Z_1$, by the proof of Lemma 10(e). Since $u^2\in B_1$ (by the proof of Lemma 10(a)), $\gamma_\alpha(u^2)=e$. It is clear that if α, $\beta\in T_1$ then $\gamma_{\alpha\beta}(u)=\gamma_\alpha(u)\gamma_\beta(u)$. $\gamma_\alpha(u)=e$ if, and only if, $\alpha=\iota$. Moreover, suppose that $c\in Z_1\cap B_1$ and that $c^2=e$. Define α by $\alpha(x)=xc$ if $x\notin B_1$, $=x$ if $x\in B_1$. $\alpha(x)=e$ if, and only if, $x=e$. If $y\in B_1$, $\alpha(y)=y$; and if $y\in G$, $y\notin B_1$, then $\alpha(yc^{-1})=yc^{-1}c=y$. For x, $y\in B_1$, $\alpha(xy)=\alpha(x)\alpha(y)$. If $x\notin B_1$, $x=ub_x$. $\alpha(x)=ub_xc=xc$. For $y\in B_1$, $\alpha(xy)=xyc=\alpha(x)\alpha(y)$. If $y\notin B_1$, then $y=ub_y$ and $\alpha(y)=yc$. $\alpha(xy)=\alpha(u^2b_xb_y)$. But $u^2\in B_1$, and b_x, $b_y\in B_1$. By the case already established for factors in B_1, $\alpha(xy)=u^2b_xb_y=xy=xcyc=\alpha(x)\alpha(y)$, since $c^2=e$. It is thus seen that α is an automorphism of G and that $\alpha\in T_1\cap V_1$ (since α induces the identity on G/Z_1 and on G/B_1). Let K_1 be the subgroup of $B_1\cap Z_1$ generated by the elements of order 2 of that group. We have proved

THEOREM 2. *If $G=B_2$, then T_1 is an elementary abelian group with exponent 2, and $T_1=V_1\cong K_1$.*

COROLLARY. *If $G=B_2$ and if $\alpha\in T_1$, $\alpha\neq\iota$, then $\gamma_\alpha(G)\cong I_2$.*

Proof. By the proof of the theorem, kern $\gamma_\alpha=B_1$. Apply Lemma 11.

7. **Some properties of the** B**-series.**

LEMMA 12. $B_{n+1}(G)/B_1(G)\cong B_n(G/B_1(G))$.

Proof. The lemma is valid for $n=1$. Suppose that it is true for the case $j-1$. Then $B_1((G/B_1)/(B_j/B_1))\cong(B_j(G/B_1))/(B_j/B_1)$ since $B_j/B_1\cong B_{j-1}(G/B_1)$, by the induction hypothesis. But $B_1((G/B_1)/(B_j/B_1))\cong B_1(G/B_j)\cong B_{j+1}/B_j\cong(B_{j+1}/B_1)/(B_j/B_1)$. Hence $B_j(G/B_1)\cong B_{j+1}/B_1$.

We say that G is *B-nilpotent of B-class n* (or *n-B-nilpotent*) if $G=B_n$.

COROLLARY. *Suppose that G is not n-B-nilpotent. The following are equivalent*: (a) *G is $(n+1)$-B-nilpotent.* (b) *G/B_1 is n-B-nilpotent.* (c) *$G/B_n\cong I_2$.*

Proof. The equivalence of (a) and (b) follows from the lemma. $G/B_n\cong I_2$ if, and only if, $B_1(G/B_n)=G/B_n$. But $B_1(G/B_n)\cong B_{n+1}/B_n\cong G/B_n$ if, and only if, $G=B_{n+1}$.

LEMMA 13. *Let G be a group which is $(n+1)$-B-nilpotent but not n-B-nilpotent, and suppose that $Z_k\supset B_n$ and that $Z_{k-1}\not\supset B_n$. Then G is k-nilpotent.*

Proof. $G=B_{n+1}$ implies that $G/B_n\cong I_2$, by Lemma 12, Corollary. Since $G/Z_k\cong(G/B_n)/(Z_k/B_n)$, G/Z_k must be isomorphic to I_2 or (e), the only possible homomorphic images of $G/B_n\cong I_2$. But $G/Z_k\cong J(G/Z_{k-1})$. Since the group of inner automorphisms of a group cannot be a nontrivial cyclic group, $G/Z_k\cong(e)$, and $G=Z_k$.

LEMMA 14. *If G is n-B-nilpotent where $n\geqq 2$, then $\alpha(\gamma_\alpha(x))\equiv\gamma_\alpha(x^{-1})$ mod B_{n-2}, for every $\alpha\in T_1$ and for every $x\in G$.*

Proof. First consider the case $n=2$. If $G=B_2$, we see, from the discussion before Theorem 2, that $\alpha\in T_1$ implies that $\gamma_\alpha(x)=\gamma_\alpha(u)$ if $x\notin B_1$, $=e$ if $x\in B_1$. Here, u is a representative of the non-unity coset of B_1 in G. Since $\gamma_\alpha(u^2)=e$ and $\gamma_\alpha(u)\in B_1$, we obtain $\gamma_\alpha(x)\alpha(\gamma_\alpha(x))=\gamma_\alpha(u)\alpha(\gamma_\alpha(u))=[\gamma_\alpha(u)]^2=e$ if $x\notin B_1$. If $x\in B_1$, the calculation still gives e. Recalling that $B_0=(e)$, we see that the lemma is established for $n=2$.

Suppose that the lemma holds for the case $n-1$. Since $G=B_n$, G/B_1 is $(n-1)$-B-nilpotent, by Lemma 12. For $\alpha\in T_1$, consider the induced automorphism α' on the abelian group G/B_1. By the induction assumption, if $\alpha'(xB_1)=(xB_1)(zB_1)$, then $zB_1\alpha'(zB_1)\in B_{n-3}(G/B_1)$. Now $\alpha(x)=x\gamma_\alpha(x)$, so that $\alpha'(xB_1)=x\gamma_\alpha(x)B_1$, and $xz\equiv x\gamma_\alpha(x)$ mod B_1. Hence $z\equiv\gamma_\alpha(x)$ mod B_1 so that $zB_1=\gamma_\alpha(x)B_1$. A substitution shows that $\gamma_\alpha(x)B_1\alpha'(\gamma_\alpha(x)B_1)=\gamma_\alpha(x)\cdot\alpha(\gamma_\alpha(x))B_1\in B_{n-3}(G/B_1)$. But $B_{n-3}(G/B_1)\cong B_{n-2}/B_1$, by Lemma 12. From this we can conclude that $\gamma_\alpha(x)\alpha(\gamma_\alpha(x))\in B_{n-2}$ for every $\alpha\in T_1$ and for every $x\in G$. The lemma is established.

COROLLARY 1. *If $G=B_n$, $n\geqq 2$, and if $\alpha\in T_1$, then $\alpha^2(x)\equiv x$ mod $(Z_1\cap B_{n-2})$ for every $x\in G$.*

Proof. $\alpha(x)=x\gamma_\alpha(x)$ implies that $\alpha^2(x)=x\gamma_\alpha(x)\alpha(\gamma_\alpha(x))$. By the lemma, $\gamma_\alpha(x)\alpha(\gamma_\alpha(x))\in B_{n-2}$.

COROLLARY 2. *If $G=B_n$, $n\geqq 2$, and if $\alpha\in T_1$ induces ω on $\gamma_\alpha(G)$ or on G/B_1, then $\alpha^2=\iota$.*

Proof. If α induces ω on $\gamma_\alpha(G)$, then $\alpha(\gamma_\alpha(x))=\gamma_\alpha(x^{-1})$, so that, by the proof of Corollary 1, $\alpha^2(x)=x$ for every $x\in G$. By Lemma 7(a), if α induces ω on G/B_1, then α induces ω on $\gamma_\alpha(G)$.

COROLLARY 3. *Let $M(\alpha)$ be the largest subgroup of $\gamma_\alpha(G)$ on which α induces the involution automorphism. If $G=B_n$, $n\geqq 2$, then γ_α* kern *$\gamma_{\alpha^2}=M(\alpha)$, and $\gamma_{\alpha^2}(G)$ is an α-admissible subgroup of $\gamma_\alpha(G)\cap B_{n-2}$.*

Proof. $x\in$kern γ_{α^2} if, and only if, $\gamma_{\alpha^2}(x)=\gamma_\alpha(x)\alpha(\gamma_\alpha(x))=e$; that is, equivalently, $\alpha\gamma_\alpha(x)=\gamma_\alpha(x^{-1})$. But the latter is equivalent to $\gamma_\alpha(x)\in M(\alpha)$. By Lemma 2, $\gamma_\alpha(G)$ is α-admissible, so that, for given $x\in G$, $\alpha(\gamma_\alpha(x))=\gamma_\alpha(y)$ for a suitable $y=y(x;\ \alpha)$. Then $\gamma_{\alpha^2}(x)=\gamma_\alpha(x)\gamma_\alpha(y)$, and $\alpha(\gamma_{\alpha^2}(x))=\alpha(\gamma_\alpha(x))\cdot\alpha(\gamma_\alpha(y))=\gamma_\alpha(y)\alpha\gamma_\alpha(y)=\gamma_{\alpha^2}(y)$. This shows that $\gamma_{\alpha^2}(G)$ is α-admissible.

LEMMA 15. *If* $x \in B_n$, $n \geqq 2$, *then* $x^{2^{n-1}} \in B_1$.

Proof. The case $n=2$ was treated in the proof of Lemma 10(a). Suppose that the lemma is valid for $n=j$. If $x \in B_{j+1}$, $xB_1 \in B_{j+1}/B_1 = B_j(G/B_1)$. By the induction assumption, $x^{2^{j-1}}B_1 \in B_1(G/B_1) = B_2/B_1$, and $x^{2^{j-1}} \in B_2$. The case $n=2$ now shows that $[x^{2^{j-1}}]^2 = x^{2^j} \in B_1$.

COROLLARY. B_n/B_1 *is u.t. abelian with* exp $(B_n/B_1) \mid 2^{n-1}$, *so that, for an n-B-nilpotent group G, G/B*$_1$ *is u.t. abelian, and an n-B-nilpotent group with periodic* B_1 *is itself periodic.*

THEOREM 3. *If G is n-B-nilpotent,* $n \geqq 2$, *then* W_1 *is* $(n-1)$*-nilpotent.*

Proof. If W_1^* is void, then $W_1 = V_1$, an abelian group. Suppose that W_1^* is nonvoid. $x \in B_n$ implies $x^{2^{n-1}} \in B_1$, by Lemma 15. If $\alpha \in V_1$, then $\gamma_\alpha(x^{2^{n-1}}) = e$ for every $x \in G$, since $G = B_n$. By Theorem 1 (b), $V_1 \subset Z_{n-1}(W_1)$. By Theorem 1(a), W_1 is $(n-1)$-nilpotent.

COROLLARY. *If G with torsion-free* Z_1 *is n-B-nilpotent, then* $W_1 = V_1$, *or* V_1 *is trivial, and* W_1 *is an elementary abelian group with exponent* 2.

Proof. If W_1^* is nonvoid, then Theorem 1, Corollary 1(b), and the present theorem show that V_1 is trivial. Since $\alpha \in W_1^*$ implies that $\alpha^2 \in V_1$, W_1 is elementary abelian with exponent 2.

LEMMA 16. *Each* B_n *is* T_1*-admissible, and, if* $n \geqq 1$, $\gamma_\alpha(B_n) \subset B_{n-1}$ *for every* $\alpha \in T_1$.

Proof. B_1 is T_1-admissible. Suppose that $B_{n-1}(G)$ is T_1-admissible for every group G. $g \in B_n$ implies that $gB_{n-1} \in B_1(G/B_{n-1})$. For $\alpha \in T_1$, B_{n-1} is both α- and α^{-1}-admissible (by the induction assumption), and α induces an automorphism α' on the abelian group G/B_{n-1}. Since $gB_{n-1} \in B_1(G/B_{n-1})$, $\alpha'(gB_{n-1}) = gB_{n-1} = \alpha(g)B_{n-1}$, and $\alpha(g) \equiv g \mod B_{n-1}$. Hence $\gamma_\alpha(B_n) \subset B_{n-1}$. Since $B_{n-1} \subset B_n$ and $g \in B_n$, $\alpha(g) \in B_n$ so that B_n is α-admissible.

8. **Orbital elements.** An element $x \in G$ is said to be *n-orbital* if $\alpha^n(x) = x$ for every $\alpha \in T_1$. Collecting these n-orbital elements together in a set $L_n = L_n(G)$, we see that L_n is a subgroup of G. Since $T_1 = C(J; A)$, L_n is normal in G. More generally, L_n is $C(T_1; A)$-admissible. (We shall discuss $C(T_1; A)$ below.) $L_1 = B_1$, and $m \mid n$ implies $L_m \subset L_n$. Thus $G' \subset B_1 = L_1 \subset L_n$ for every positive integer n, and G/L_n is abelian. From the proof of Lemma 10(e), we see that $x \in B_2$ implies $\alpha^2(x) = x$ for every $\alpha \in T_1$, so that $B_2 \subset L_2$.

For positive integers $s \geqq t$, let $C(s, t) = s!/t!(s-t)!$. Consider $C = C(2^{n-1}, r)$, where $n \geqq 2$ and $r \leqq n-1$. If r is odd,

$$C = 2^{n-1}s \prod_{k=1}^{(r-1)/2} \left(\frac{2^{n-1} - 2k}{2k} \right)$$

where $s \in R$ is a quotient of odd integers. Let $k = 2^{c_k}d_k$, where c_k is a nonnegative integer, and d_k is an odd integer. Since r is odd and $\leqq 2^{n-1}$, $r\leqq 2^{n-1}-1$ and $k\leqq (r-1)/2\leqq 2^{n-2}-1$. Thus we have $c_k\leqq n-3$, and $(2^{n-1}-2k)/2k = (2^{n-2-c_k}-d_k)/d_k$, a quotient of odd integers. We have proved that r odd implies that $2^{n-1}\mid C(2^{n-1}, r)$. $C(2^{n-1}, r+1) = [(2^{n-1}-r)/(r+1)]C(2^{n-1}, r)$. For odd $r\geqq 5$, the exponent of the highest power of 2 dividing into $r+1$ is $\leqq r-2$, so that $2^{n-r+1}\mid C(2^{n-1}, r+1)$, and $2^{n-(r+1)}\mid C(2^{n-1}, r+1)$. If $r=1$, then $r+1 =2$, and $2^{n-2}=2^{n-(r+1)}\mid C(2^{n-1}, r+1)$. If $r=3$, then $r+1=4$, and $2^{n-3}=2^{n-r}\mid C(2^{n-1}, r+1)$. We summarize in

LEMMA 17. *For* $n\geqq \max(2, r+1)$, $2^{n-r}\mid C(2^{n-1}, r)$.

THEOREM 4. $B_n\subset L_m$, *where* $m=2^{n-1}$.

Proof. Since the earlier cases have been treated, we assume that $n\geqq 3$. Suppose that $g\in B_n$ and that $\alpha\in T_1$. $\alpha^m(g)=\alpha^{m-1}(gb_{n-1})$ where $b_{n-1}\in B_{n-1}$, by Lemma 16. Assume, inductively, that $\alpha^m(g)=\alpha^{m-k}[g\prod_{r=1}^{k} b_{n-r}^{C(k,r)}]$ where $b_{n-r}\in B_{n-r}$, and $\gamma_\alpha(g)=b_{n-1}$, $\gamma_\alpha(b_t)=b_{t-1}$ $(t=n-k+1, n-k+2, \cdots, n-1)$. When $r>n-1$, we take $b_{n-r}=e$. Then

$$\alpha^m(g)=\alpha^{m-(k+1)}\left[gb_{n-1}^{C(k,1)+1}\left(\prod_{r=2}^{k} b_{n-r}^{C(k,r)+C(k,r-1)}\right)b_{n-(k+1)}^{C(k,k)}\right].$$

But $C(k, 1)+1=C(k+1, 1)$, $C(k, r)+C(k, r-1)=C(k+1, r)$, and $C(k, k) =1=C(k+1, k+1)$. Thus,

$$\alpha^m(g) = \alpha^{m-(k+1)}\left[g\prod_{r=1}^{k+1} b_{n-r}^{C(k+1,r)}\right],$$

and the induction is complete. Now take $k=m=2^{n-1}$ and note that $\alpha^0=\iota$. Since b's with nonpositive subscript are e, we can write $\alpha^m(g)=g\prod_{r=1}^{m} b_{n-1}^{C(m,r)}$. By Lemma 15, $b_{n-r}\in\gamma_\alpha(B_{n-r+1})$ implies that $b_{n-r}^{2^{n-r}}=e$. By Lemma 17, however, $2^{n-r}\mid C(m, r)$, so that $\alpha^m(g)=g$.

COROLLARY. *If* $g\in B_n$, $n\geqq 2$, *if* $\alpha\in T_1$, *and if* $m=2^{n-1}$, *then* $\alpha^{m/2}(g) \equiv g \bmod B_1$, *and* $\gamma_{\alpha^{m/2}}(g^2)=e$ *In particular, if* $G=B_n$, *then* $\alpha^{m/2}\in V_1$, *and* T_1/V_1 *is u.t. with exponent dividing* 2^{n-2}.

Proof. By Lemma 12, $gB_1\in B_{n-1}(G/B_1)$. Let α induce α' on G/B_1. By the theorem, $\alpha'^{m/2}(gB_1)=gB_1$, and $\alpha^{m/2}(g)\equiv g \bmod B_1$; that is, $\alpha^{m/2}(g)=gb$ where $b=\gamma_{\alpha^{m/2}}(g)\in B_1$. Also by the theorem, $\alpha^m(g)=g$. But $\alpha^m(g)=\alpha^{m/2}(\alpha^{m/2}(g)) =\alpha^{m/2}(gb)=gb^2$, and $b^2=e$.

LEMMA 18. *Let n be an integer* $\geqq 1$, *and let G be a group for which each automorphism of* G/B_n *can be extended to a normal automorphism of G. Then if* $\gamma_\alpha(g)\in B_n$ *for every* $\alpha\in T_1$, $g\in B_{n+1}$.

Proof. By hypothesis, $\alpha(g)\equiv g \bmod B_n$ for every $\alpha\in T_1$. Let α induce α' on G/B_n. $\alpha'(gB_n)=\alpha(g)B_n=gB_n$. Since the set of induced α' coincides with $A(G/B_n)=T_1(G/B_n)$, $gB_n\in B_1(G/B_n)=B_{n+1}/B_n$, and $g\in B_{n+1}$.

9. **The centralizer of** T_1. Since T_1 is the centralizer of J in A, $U_1 = C(T_1; A) \supset J$, where U_1 is a normal subgroup of A.

LEMMA 19. (a) $B_1(G)$ *is* U_1*-admissible, and if each automorphism of each* G/B_i $(i=1, 2, 3, \cdots)$ *can be extended to a normal automorphism of* G, *then each* B_n, $n \geqq 2$, *is likewise* U_1*-admissible.* (b) $\gamma_\alpha(F(U_1)) \subset F(U_1) \subset Z_1$ *for every* $\alpha \in T_1$. (c) *Each* $\theta \in U_1$ *induces an automorphism on each* $F(\gamma_\alpha)$, $\alpha \in T_1$.

Proof. (a) For $\theta \in U_1$ and $\alpha \in T_1$, $\theta\alpha(x) = \theta(x)\theta\gamma_\alpha(x) = \alpha\theta(x) = \theta(x)\gamma_\alpha\theta(x)$, so that $\theta\gamma_\alpha = \gamma_\alpha\theta$ for every $\alpha \in T_1$. If $g \in B_1$, then $\gamma_\alpha\theta(g) = \theta\gamma_\alpha(g) = \theta(e) = e$ for $\alpha \in T_1$, and $\theta(g) \in B_1$. Now suppose that B_n is U_1-admissible. For $g \in B_{n+1}$, $\gamma_\alpha\theta(g) = \theta\gamma_\alpha(g)$. $\gamma_\alpha(g) \in B_n$, by Lemma 16. By the induction assumption, $\theta\gamma_\alpha(g) \in B_n$. Applying Lemma 18, $\theta(g) \in B_{n+1}$. (b) If $\theta(g) = g$ for every $\theta \in U_1$, then $\gamma_\alpha\theta(g) = \gamma_\alpha(g) = \theta\gamma_\alpha(g)$. Hence $\gamma_\alpha(F(U_1)) \subset F(U_1)$ for every $\alpha \in T_1$. If $\theta(g) = g$ for every $\theta \in U_1$, then $\tau_x(g) = g$ for every $x \in G$, since $J \subset U_1$. But $\tau_x(g) = g$ for every $x \in G$ implies that $g \in Z_1$. (c) If $g \in F(\gamma_\alpha)$, $\gamma_\alpha(g) = g$, and $\theta\gamma_\alpha(g) = \gamma_\alpha\theta(g) = \theta(g)$, so that $\theta(g) \in F(\gamma_\alpha)$. Conversely, if $\theta(g) \in F(\gamma_\alpha)$, then $\theta\gamma_\alpha(g) = \gamma_\alpha\theta(g) = \theta(g)$. Since θ is an automorphism, $\gamma_\alpha(g) = g$, and $g \in F(\gamma_\alpha)$.

THEOREM 5. *Each element of* $C(T_1; A)$ *induces a normal automorphism on* Z_2, *and there exists a homomorphism on* $C(T_1; A)$ *into* $T_1(Z_2)$ *with kernel consisting of all those mappings in* $C(T_1; A)$ *which reduce to the identity on* Z_2.

Proof. $\theta \in U_1$ implies that θ commutes with every Grün automorphism of G. If, therefore, $u \in Z_2$, then $\theta(x^{-1}u^{-1}xu) = \theta(x^{-1})\theta(u^{-1})\theta(x)\theta(u) = \theta(x^{-1})u^{-1}\theta(x)u$ for every $x \in G$. $u\theta(u^{-1})$ is, consequently, in the centralizer of every $\theta(x)$, $x \in G$. Since θ is an automorphism, $u\theta(u^{-1}) \in Z_1(G) \subset Z_1(Z_2(G))$, and $\theta(u) \equiv u \bmod Z_1(Z_2(G))$, so that θ restricted to Z_2 is normal thereon.

COROLLARY. *If* G *is of class* 2, *then* $J \subset C(T_1; A) \subset T_1$, *and* $C(T_1; A) = Z_1(T_1)$.

10. **The higher normal autmorphisms.** If $\alpha \in A$ has the property $\alpha(x) \equiv x \bmod Z_n$ for every $x \in G$, we say that α is an *n-normal automorphism*, and we have described the higher normal automorphisms of G. Let T_n be the set of n-normal automorphisms of G. Under automorphism composition, T_n is a normal subgroup of G, and $m \leqq n$ implies that $T_m \subset T_n$.

THEOREM 6. (a) T_n/T_{n-1} *is isomorphic to a subgroup of* $T_1(G/Z_{n-1})$. (b) T_n/T_1 *is isomorphic to a subgroup of* $T_n(J)$.

Proof. (a) $\alpha \in T_n$ induces an automorphism α' on G/Z_{n-1}. For every $x \in G$, $xZ_{n-1} \in G/Z_{n-1}$, and $\alpha'(xZ_{n-1}) = \alpha(x)Z_{n-1} = xzZ_{n-1}$ where $z \in Z_n$. Then $zZ_{n-1} \in Z_1(G/Z_{n-1})$, so that α' is normal on G/Z_{n-1}. It is not difficult to see that if $\alpha, \beta \in T_n$, then $(\alpha\beta)' = \alpha'\beta'$, so that $(')$ is a homomorphism on T_n into $T_1(G/Z_{n-1})$. Suppose that $\alpha' = \iota$. Then $\alpha'(xZ_{n-1}) = xZ_{n-1}$ for every $x \in G$, and

$\alpha(x) \equiv x \bmod Z_{n-1}$. Hence α induces the identity on G/Z_{n-1}, and $\alpha \in T_{n-1}$. Conversely, if $\alpha \in T_{n-1}$, $\alpha'(xZ_{n-1}) = \alpha(x)Z_{n-1} = xZ_{n-1}$, and then $\alpha' = \iota$ on G/Z_{n-1}. Therefore, kern $(') = T_{n-1}$. (b) $\alpha \in T_{n+1}$ induces an automorphism α'' on $G/Z_1 \cong J$, given by $\alpha''(xZ_1) = \alpha(x)Z_1 = x\gamma(x)Z_1$, where $\gamma(x) \in Z_{n+1}$. Hence $\alpha''(xZ_1) = xZ_1 \bmod (Z_{n+1}/Z_1)$. Now $Z_n(J) \cong Z_n(G/Z_1) \cong Z_{n+1}/Z_1$, as an induction will show. Hence α'' is, effectively, in $T_n(J)$. α induces ι if, and only if, $\alpha(x) \equiv x \bmod Z_1$ for every $x \in G$, and kern $('') = T_1$.

COROLLARY 1. *Let α' be a nontrivial normal automorphism of G/Z_n which can be extended to a higher normal automorphism α of G. Then $\alpha \notin T_n$.*

COROLLARY 2. *Each $\alpha \in T_n$ induces a homomorphism of G and an endomorphism of G/Z_{n-1} into $Z_n/Z_{n-1} = Z_1(G/Z_{n-1})$.*

Proof. The endomorphism is $\gamma_{\alpha'}$, and the homomorphism is obtained by following the natural mapping ϕ_{n-1} of G onto G/Z_{n-1} by $\gamma_{\alpha'}$. Moreover, $\gamma_{\alpha'}\phi_{n-1}(x) = x^{-1}\alpha(x)Z_{n-1}$ for every $x \in G$.

Let S be a set of automorphisms of G and let $N(S)$ be the set of all $g \in G$ such that $\alpha(g) \equiv g \bmod Z_1$ for every $\alpha \in S$.

LEMMA 20. *If K is a subgroup of A, then $C(K; A) \cap J = J(N(K); G)$. In particular, $Z_1(A) \cap J = J(N(A); G)$.*

Proof. If $\tau_g\alpha = \alpha\tau_g$ for every $\alpha \in K$, then $g^{-1}\alpha(x)g = \alpha(g^{-1})\alpha(x)\alpha(g)$ for every $x \in G$, so that $g\alpha(g^{-1})$ is in the centralizer of every $\alpha(x)$. Since α is an automorphism, $g\alpha(g^{-1}) \in Z_1$, and $\alpha(g) \equiv g \bmod Z_1$, so that $g \in N(K)$ and $\tau_g \in J(N(K); G)$. The proof can be read in reverse to obtain the converse.

LEMMA 21. *The following are equivalent*: (a) $J \subset Z_1(A)$. (b) $A = T_1$. *Either of these conditions implies that G is of class* 2.

Proof. $A = T_1$ if, and only if, $G = N(A)$. But if the latter holds, $J(N(A); G) = J$; and conversely, if $J(N(A); G) = J$, $x \in G$ implies the existence of $y \in N(A)$ with $\tau_x = \tau_y$. Then $x \equiv y \bmod Z_1$, so that, if $\alpha \in A$, $\alpha(x) \equiv \alpha(y) \equiv y \equiv x \bmod Z_1$. This shows that $x \in N(A)$ and that $N(A) = G$. By Lemma 20, $J \cap Z_1(A) = J(N(A); G) = J$, so that $J \subset Z_1(A)$, and (a) implies (b). A slight rearrangement of the above argument shows that (b) implies (a). Now if every automorphism of G is a normal automorphism, $x^{-1}yx \equiv y \bmod Z_1$ for every $x, y \in G$. This implies that $G' \subset Z_1$, and G is of class 2.

A similar result is contained in

THEOREM 7. *Let G be a group with the properties* (1) *$J \subset T_n$ and* (2) *each $\alpha \in A$ induces ι on each Z_{j+1}/Z_j $(j = 1, 2, \cdots, n)$. Then $J \subset Z_n(A)$.*

Proof. First, we establish three lemmas:

(R) For a group G, $J \subset T_n$ if, and only if, G is of class $n+1$.

(S) For a group G, $J \cap Z_n(A) \subset J(Z_{n+1}; G)$.

(T) A group G with property (2) has the further property that $J\cap Z_n(A) = J(Z_{n+1}; G)$ (for the n of property (2)).

To prove (R), use the proof of the last statement of Lemma 21 as a model. As for (S), take $n=0$. Then $J\cap Z_n(A)$ consists of ι alone, and the inclusion is trivially valid. Suppose that it holds for $n=k$. $\tau_g\in J\cap Z_{k+1}(A)$ implies that $(\alpha, \tau_g)\in Z_k(A)$ for every $\alpha\in A$. A brief computation shows that $(\alpha, \tau_g)=\tau_h$, where $h=g\alpha^{-1}(g^{-1})$. By the induction assumption, $g\alpha^{-1}(g^{-1})\in Z_{k+1}$, and this is to be valid for every $\alpha\in A$. If we take $\alpha=\tau_x$, $x\in G$, then $gxg^{-1}x^{-1}\in Z_{k+1}$ for every $x\in G$, and $g\in Z_{k+2}$. But this means that $J\cap Z_{k+1}(A)\subset J(Z_{k+2}; G)$.

To prove (T), let $\{\alpha_i\}$ $(i=1, 2, \cdots, n)$ be any finite set of elements of A. For a fixed $g\in G$, define $g_1=g\alpha_1^{-1}(g^{-1})$. If g_k is defined, let $g_{k+1}=g_k\alpha_{k+1}^{-1}(g_k^{-1})$. A different finite set of elements of A, or even the same set in a different order, may very well lead to a different finite sequence $\{g_i\}$ on g. Let $G_i(g) = G_i$ be the set of all g_i obtained in this fashion for fixed g and fixed positive integer i. By Lemma 20, $\tau_g\in Z_1(A)$ if, and only if, $g\in N(A)$. But $g\in N(A)$ if, and only if, $\alpha(g)\equiv g \bmod Z_1$ for every $\alpha\in A$. The latter condition is equivalent to $G_1\subset Z_1$. Now suppose that $\tau_g\in Z_k(A)$ if, and only if, $G_k\subset Z_1$. $\tau_g \in Z_{k+1}(A)$ if, and only if, $J(G_1(g); G)\subset Z_k(A)$. By the induction assumption, this is equivalent to $G_k(h)\subset Z_1$ for every $h\in G_1(g)$. Since $\bigcup G_k(h)=G_{k+1}(g)$, where the set union is taken over all $h\in G_1(g)$, $\tau_g\in Z_{k+1}(A)$ if, and only if, $G_{k+1}(g)\subset Z_1$.

Now suppose that $\tau_g\in J(Z_{n+1}; G)$. Then $g\in Z_{n+1}$ and $G_1(g)\subset Z_n$, since each $\alpha\in A$ induces the identity on Z_{n+1}/Z_n. Assume, inductively, that $G_k(g) \subset Z_{n-k+1}$. Since each member of A induces the identity on Z_{n-k+1}/Z_{n-k}, $G_{k+1}(g)\subset Z_{n-k}$. In particular, $G_n(g)\subset Z_1$. By the above, $J(Z_{n+1}; G)\subset Z_n(A)$. Along with (S), this is enough to establish (T).

To prove the theorem, note that $J\subset T_n$ implies, by (R), that G is of class $n+1$. Therefore, in (T), replace Z_{n+1} by G. Since $J(G; G)=J$, the theorem is proved.

For a subgroup K of A, it is clear that $F(K)\subset N(K)$, that $Z_1\subset N(K)$, and that $N(K)$ is K-admissible. We prove a preliminary result on $Q(H; G)$ for a characteristic subgroup H of G.

LEMMA 22. *Let H be a characteristic subgroup of G.* (a) $Q(H; G)\cap J = J(H\div G; G)$. (b) *$\alpha$ and β induce the same automorphism on G/H if, and only if, $\alpha\equiv\beta$* mod $Q(H; G)$. *In particular, $\tau_x\equiv\tau_y$ if, and only if, $x\equiv y$* mod $H\div G$.

Proof. (a) $\tau_g\in Q(H; G)$ if, and only if, $\tau_g(x)\equiv x \bmod H$ for every $x\in G$. But this latter condition is equivalent to $(g, x^{-1})\in H$ for every $x\in G$, and this is true if, and only if, $g\in H\div G$. (b) is obvious.

LEMMA 23. (a) $T_n\div J=T_{n+1}$. (b) $(T_n, J)\subset J(Z_n; G)$. (c) $T_n\cap J=J(Z_{n+1}; G)$.

Proof. (a) and (b) can be established by routine arguments. To verify (c), replace H by Z_n in Lemma 22(a), and note that $Q(Z_n;\ G)=T_n$ and that $Z_n \div G = Z_{n+1}$.

COROLLARY. (a) *For a subgroup K of A and for a positive integer n, $N(K) = Z_n$ if, and only if, $C(K;\ A)\cap J = T_{n-1}\cap J$.* (b) *$N(K)=Z_1$ if, and only if, $C(K;\ A)\cap J$ is trivial.* (c) *If G is n-nilpotent, and if K is a subgroup of A, then $K\subset T_1$ if, and only if, $C(K;\ A)\cap J = T_{n-1}\cap J$.*

Proof. (a) If $N(K)=Z_n(G)$, then $J(Z_n;\ G)=J(N(K);\ G)=C(K;\ A)\cap J$, by Lemma 20. Since $J(Z_n;\ G)=T_{n-1}\cap J$ (by (c) of the lemma), half the statement is established. Conversely, suppose that $C(K;\ A)\cap J=T_{n-1}\cap J = J(Z_n;\ G)$. One can readily check the equivalence of the following statements: (1) $x\in Z_n$. (2) $\tau_x\alpha=\alpha\tau_x$ for every $\alpha\in K$. (3) $\tau_x\alpha\alpha^{-1}(y)=\alpha\tau_x\alpha^{-1}(y)$ for every $y\in G$, $\alpha\in K$. (4) $x^{-1}yx=\alpha(x^{-1})y\alpha(x)$ for every $y\in G$ and every $\alpha\in K$. (5) $\alpha(x)x^{-1}\in Z_1$ for every $\alpha\in K$. (6) $x\in N(K)$. (b) follows from (a) by taking $n=1$. (c) $K\subset T_1$ if, and only if, $N(K)=G$. Since $G=Z_n$, (a) is applicable.

THEOREM 8. $G^{(n)}\subset F(T_n)\subset C(Z_n;\ G)$.

Proof. If $\alpha\in T_n$, then $\alpha(x^{-1}y^{-1}xy)=t^{-1}x^{-1}u^{-1}y^{-1}xtyu$ where $t,\ u\in Z_n$. Hence $\alpha(x^{-1}y^{-1}xy)\equiv x^{-1}y^{-1}xy \bmod Z_{n-1}$, and α induces the identity on $G'/(Z_{n-1}\cap G')$. Suppose, inductively, that $\alpha\in T_n$ induces the identity on $G^{(k)}/(Z_{n-k}\cap G^{(k)})$. A set of generators of $G^{(k+1)}$ is all $(x,\ y)$ where $x,\ y\in G^{(k)}$. $\alpha(x^{-1}y^{-1}xy)=t^{-1}x^{-1}u^{-1}y^{-1}xtyu$, where $t,\ u\in Z_{n-k}$. Hence $\alpha(x^{-1}y^{-1}xy)\equiv x^{-1}y^{-1}xy \bmod Z_{n-k-1}$; and our induction shows that $\alpha\in T_n$ induces the identity on each $G^{(k)}/(Z_{n-k}\cap G^{(k)})$. Now take $k=n$ so that $Z_{n-k}=(e)$. That is, each $\alpha\in T_n$ induces the identity on $G^{(n)}$, whence $G^{(n)}\subset F(T_n)$. By the discussion after Lemma 3, $F(T_n)\subset C(Z_n;\ G)$.

COROLLARY 1. $G^{(n)}\subset F(J(Z_{n+1};\ G))$.

Proof. $T_n\supset J(Z_{n+1};\ G)$, by Lemma 23(c).

COROLLARY 2. *If $F(T_n)=(e)$ or if $F(J(Z_{n+1};\ G))=(e)$ for some positive integer n, then G is solvable* [3].

COROLLARY 3. $J(G^{(n)};\ G)\subset C(T_{n+1};\ A)$.

Proof. $\alpha\in T_{n+1}$, and $g\in G^{(n)}$ imply that $\alpha(g)\equiv g \bmod Z_1$, by the proof of the theorem. Hence $J(G^{(n)};\ G)\subset J(N(T_{n+1});\ G)=C(T_{n+1};\ A)\cap J$, by Lemma 20.

If we let $U_n=C(T_n;\ A)$, then, by Lemma 20, $J(N(T_n);\ G)\subset U_n$. As in Lemma 19(a), $F(T_n)$ is U_n-admissible.

11. **Examples.** (A) For positive integers $n>2$, let D_n denote the nth dihedral group, the group of isometries of a regular n-gon. D_n is the semidirect product of I_n and of I_2 with the multiplication rules $(x_n,\ 0_2)(y_n,\ z_2) = (x_n+y_n,\ z_2)$ and $(x_n,\ 1_2)(y_n,\ z_2)=(x_n-y_n,\ 1_2+z_2)$. For $n>2$, there is a nontrivial element in the center if, and only if, n is even; and in this case, the

center consists of two elements, $(0_n, 0_2)$ and $(h_n, 0_2)$, where h is an integer such that $2h=n$. Since D_n is a group with two generators, there are three nontrivial possibilities for central endomorphisms. The verification of the following results is easy: $T_1(D_{4k})$ is isomorphic to the Klein four group, $I_2 \oplus I_2$. Let us denote the four group by $\mathfrak{V}$. $B_1(D_{4k})$ consists of all $(x_{4k}, 0_2)$ where x is even, so that $B_1(D_{4k}) \cong I_{2k}$. Likewise, $B_1(D_{4k})=D'_{4k}$, and, in fact, $D_{4k}/B_1(D_{4k}) \cong \mathfrak{V}$. It follows that the B-series breaks off at $B_1(D_{4k})$. $T_1(D_{4k+2}) \cong I_2$. $B_1(D_{4k+2})$ consists of all $(x_{4k+2}, 0_2)$, so that $B_1(D_{4k+2}) \cong I_n$. $D_{4k+2}/B_1(D_{4k+2}) \cong I_2$ so that, by Lemma 11, D_{4k+2} is 2-B-nilpotent. If $n=4$, then $D_4/Z_1(D_4)$ is isomorphic to $\mathfrak{V}$ whence D_4 is of class 2. Then $T_2(D_4)=A(D_4)$, and it can be readily verified that $A(D_4) \cong D_4$ and that $F(T_2(D_4))=B_1(D_4)=Z_1(D_4)$.

(B) Let G be a group of type (2^∞). G is isomorphic to the additive group, modulo 1, of the rationals $k/2^n$, where k is an odd integer or 0. Since G is abelian, $T_1(G)=A(G)$. G has a nontrivial automorphism $\alpha(k/2^n)=1-(k/2^n)$ corresponding to the conjugation automorphism on the representation of G on the unit circle. The only fixed points are $1=0$ and $1/2$. Conversely, if β is any automorphism of G, $2\beta(1/2)=\beta(1)=1=0$, so that $\beta(1/2)=1/2$ or 0. Thus $B_1 \cong I_2$. Since $G/B_1 \cong G$, B_2 consists of 0, 1/4, 1/2, and 3/4, and $B_2 \cong I_4$. In general, $B_n \cong I_{2^n}$. $G=\bigcup B_n$ where the union is taken over all positive integral values of n.

(C) Let G be the multiplicative group of all nonsingular 2 by 2 matrices over the field of rationals, R. It is well known that Z_1 consists of all

$$\begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix}, \qquad u \neq 0,$$

and that $Z_2=Z_1$. By Lemma 1, we have an example of a group for which $T_1 \cap J$ is trivial. Let μ be an endomorphism of the multiplicative group of nonzero rationals R^* where $x\mu(x^2a)=1$ has a unique solution $x=x(a;\mu)$ for every $a \in R^*$. Let d_1 and d_2 be integers with the restriction $|d_i|=1$. Define a mapping $\alpha=\alpha(\mu; d_1, d_2)$ on G by

$$\alpha\begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} r\mu(r) & 0 \\ 0 & \mu(r) \end{pmatrix} \qquad \text{for every } r \in R^*,$$

$$\alpha\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & d_1 \\ d_1 & 0 \end{pmatrix} \quad \text{and} \quad \alpha\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} d_2 & d_2 \\ 0 & d_2 \end{pmatrix}.$$

Then it is possible to prove that α is a normal automorphism of G, and each normal automorphism of G is such an $\alpha(\mu; d_1, d_2)$. A matrix $M \in G$ is in B_1 if, and only if, it can be factored (without regard to the order of the factors) into a product of an even number of factors

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

an even number of factors

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and a set of factors

$$\begin{pmatrix} a_i & 0 \\ 0 & 1 \end{pmatrix} \qquad (i = 1, 2, \cdots, n),$$

where $a_1a_2 \cdots a_n = 1$. V_1 turns out to be the set of all $\alpha(\mu; d_1, d_2)$ with $\mu(x) = \pm 1$ for every $x \in R^*$. W_1^* consists of all α with $\mu(x) = \pm(1/x)$, and $[T_1: V_1]$ is equal to the number of normal subgroups of index 2 in G which contain

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

W_1 for this group is abelian. Now W_1^* is nonvoid, V_1 is nontrivial and Z_1 has the periodic element

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Therefore, this example shows that we cannot drop the hypothesis of aperiodicity for Z_1 in Theorem 1, Corollary 1(b). $G/B_1 \cong \mathfrak{B} \oplus R^*$ whence $B_2 = B_1$. Let μ be an endomorphism of R^* such that, to each positive prime p, there exists a positive prime q with $q | \mu(p) | = 1 = q | \mu(q) |$. Then $\alpha(\mu; d_1, d_2) \in T_1$, so that, for this group, T_1 is far from trivial and $T_1 \neq W_1$. Negatively, one can show, for instance, that if μ is an endomorphism of R^* for which $|\mu(p)|$ is always a product of k positive primes (or a product of the reciprocals of $k+1$ positive primes) for every positive prime p, then the corresponding α is not an automorphism.

(D) Let G be a group with generators a, b, and c, where $a^2 = e$, $ab = ba$, $ac = ca$, and $bc = cba$. Then every element of G can be written uniquely as a product $a^ib^jc^k$ where i is 0 or 1, and j and k range over the integers. $Z_1 \cong I_2$ and $G/Z_1 \cong R \oplus R$ so that G is nilpotent of class 2. One can verify that $T_1 \cong \mathfrak{B}$. An element is in B_1 if, and only if, j and k are both even. Under any automorphism α each center element, a^i, is fixed. There is an automorphism β which changes the sign of j in each term. Its set of fixed points is precisely all elements with $j = 0$. There is an automorphism δ which changes the sign of k in each term, and the corresponding fixed points are all elements with $k = 0$. The cross-cut of these two sets of fixed points is Z_1, so that $Z_1 = F(A) = F(T_2)$, and this latter set is included in $F(T_1) = B_1$ properly. (In the example of D_4 above, $F(T_1) = F(T_2)$ for the class 2 group D_4.)

Presumably, by extending the group of this example or by considering n by n triangular matrices with a diagonal of unities, one could exhibit groups with significant T_n, for $n>2$.

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