ON ASYMPTOTIC VALUES OF FUNCTIONS
ANALYTIC IN A CIRCLE(1)

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1. The class of functions $f(z)$, which are analytic and bounded, $|f(z)| < 1$, in the unit circle $U: |z| < 1$ and which have radial limit values of modulus 1 for almost all points $e^{i\theta}$ of $|z| = 1$ is well known; for literature and general properties of these functions we refer the reader to the papers of W. Seidel [16] and A. J. Lohwater [10]. Some of the results mentioned in these papers can be obtained from general theorems in the theory of cluster sets of functions analytic in $U$ (cf. [4] and [13]). In recent papers Lohwater [9; 10; 11] has extended the concept of this class to functions which are meromorphic in $U$ and whose moduli have radial limit 1 for almost all points of some arc $A$ of $|z| = 1$. In particular, we cite the following result [10; 11]: If $f(z)$ is meromorphic in $|z| < 1$ with at most a finite number of zeros and poles and if $\lim_{r \to 1} |f(re^{i\theta})| = 1$ for almost all $e^{i\theta}$ belonging to an arc $A$ of $|z| = 1$, then, unless $f(z)$ is analytic on $A$, there exists at least one curve (called an asymptotic path) terminating at a point of $A$ along which $f(z)$ tends either to 0 or $\infty$. If, in addition, $f(z)$ is of bounded characteristic in $|z| < 1$, there exists at least one radius having this property.

In the present paper, we are motivated by Lohwater’s results to define new boundary cluster sets of functions analytic in $U$ and taking values on an abstract Riemann surface $\mathcal{R}$, and to establish relations between the cluster sets and the asymptotic values of the functions.

2. We begin with the definition of boundary points of an abstract Riemann surface $\mathcal{R}$. Let $\mathfrak{F}$ be a class of filters such that each filter has a base consisting of open sets of $\mathcal{R}$ which have no accumulation points on $\mathcal{R}$. Furthermore we assume that of any two open sets of a base, one is contained in the other; that is, we have a nested base. We obtain a countable sub-base $\{G_n\}$ from the base if we take an exhaustion $\{R_n\}$, $R_n \subset R_{n+1}$, with compact closures, and if we choose an element $G_n$ of the base so that $G_n \cap R_n = \emptyset$ for each $n$. For, given any element $G$ of the base, there is an $R_n$ such that $R_n \cap G \neq \emptyset$ and this shows $G_n \subset G$. Each filter of $\mathfrak{F}$ is defined to be a boundary point of $\mathcal{R}$, and we denote the set of all such boundary points by $\mathfrak{F}_\mathcal{R}$. Let $P_\mathcal{R}$ be a point of $\mathfrak{F}_\mathcal{R}$ with a base $\{G_n\}$, and let $\{P_n\}$ be a sequence of points of $\mathcal{R} + \mathfrak{F}_\mathcal{R}$. If for each $n$ there exists an integer $\nu_0$ such that every $P_{\nu} \geq \nu_0$, or some domain of its base, is contained in $G_n$, we say that $P_\mathcal{R}$ converges to $P_\mathcal{R}$. We keep the original definition of the convergence of points of $\mathcal{R}$. Thus we obtain a topol-
ology for the space \( \mathcal{R} + \mathcal{S}\mathbb{R} \). Boundary points obtained by the completion with respect to a metric in \( \mathcal{R} \) can be reinterpreted in the way above. The ramified boundary points and geodesic boundary points in \([2]\) are examples.

3. Let \( f(z) \) be an analytic function defined in \( U: |z| < 1 \) and taking values on an abstract Riemann surface \( \mathcal{R} \) (with boundary \( \mathcal{S}\mathbb{R} \) if \( \mathcal{R} \) is open). For any set \( E \subset U \) and any point \( z_0 \) on \( C: |z| = 1 \) we define the cluster set \( S_{z_0}^E \) at \( z_0 \) along \( E \) to be the set of all values of \( \mathcal{R} + \mathcal{S}\mathbb{R} \), for each point \( P \) of which there exists a sequence of points \( \{z_n\} \) of \( E \) tending to \( z_0 \) such that \( f(z_n) \to P \) as \( n \to \infty \). We shall write \( S_{z_0}^U \) for \( S_{z_0}^U \), and \( T_{z_0} \) for the cluster set along the radius \( Oz_0 \).

Let \( \{K_n\} \) be an open base of \( \mathcal{R} \), and \( z_0 \) a point of \( C \). If, for a given integer \( n \), there exists at least one open arc \( C_n \) containing \( z_0 \) such that the inner linear measure of the set \( \{z \in C_n; z \neq z_0, T_n \cap K_n \neq \emptyset\} = C_n \) (which may be empty) is zero, we define \( K_n^* \) by setting it equal to \( K_n \); otherwise we put \( K_n^* = \emptyset \). Denote the set \( C - U_n C_n - z_0 \) by \( C^* \), where the summation \( U_n \) is taken over all \( n \), for which \( K_n^* = K_n \). Next we take an open base \( \{K_\alpha\} \) (this is not countable in general) of \( \mathcal{R} + \mathcal{S}\mathbb{R} \) and define \( K_\alpha^* \) in a similar way. We shall denote the set \( \mathcal{R} + \mathcal{S}\mathbb{R} - U_n K_n^* \) by \( ST_{z_0} \). This set is clearly a closed set in \( \mathcal{R} + \mathcal{S}\mathbb{R} \) and may be considered as a sort of boundary cluster set (*).

Let us denote the intersection of any set \( X \) with the circle \( |z - z_0| < \rho \) by \( X_\rho \). The cluster set \( ST_{z_0} \) has a minimal property in the following sense: Taking any set \( H \subset C, z_0 \in H \), of linear measure zero, forming the closure \( M_\rho^{(C-H)} \) of \( U_z \in (C-H) T_z \), and denoting \( \bigcap_{\rho>0} M_\rho^{(C-H)} \) by \( ST_{z_0}^{(C-H)} \), we have the relation \( ST_{z_0} \subset ST_{z_0}^{(C-H)} \). The set \( M_\rho^{(C-H)} \) will be used in the following Theorem 2.

If \( f(z) \to P \in \mathcal{R} + \mathcal{S}\mathbb{R} \) along a curve in \( U \) terminating at \( z_0 \), this curve is called an asymptotic path and the value \( P \) an asymptotic value. The set of points on \( \mathcal{R} \) taken in any neighborhood in \( U \) of \( z_0 \) is called the range of values and denoted by \( R_z(i) \).

4. We first prove the following lemma.

**Lemma.** Let \( T \) be a continuous transformation of \( U \) into a topological space \( X \). Let \( \Delta \) be a domain in \( U \) whose image under \( T \) is contained in a closed set \( F \) in \( X \) and, for almost all \( e^{i\theta} \in \Delta^b \cap C \), where \( \Delta^b \) is the boundary of \( \Delta \), let the image of some end-part of the radius \( Oe^{i\theta} \) be contained in a closed set \( F' \), disjoint from \( F \). If there exists a continuous real-valued function \( g(P) \) in \( X \) which assumes the value 0 on \( F \) and 1 on \( F' \), then \( m(\Delta^b \cap C) = 0 \).

**Proof.** We denote by \( G(z) \) the function obtained by composing the transformation \( T \) with \( g(P) \). By our assumption \( \lim \tau G(re^{i\theta}) = 1 \) at almost all points \( e^{i\theta} \) of \( \Delta^b \cap C \). By Egoroff's theorem, for any integer \( \rho \) there exists a

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(*) If \( \mathcal{R} + \mathcal{S}\mathbb{R} \) is compact, \( S_{z_0}^E \) is never empty whenever \( z_0 \) belongs to the closure of \( E \).

(\*) By means of the theory of functions of real variables, it can be proved that the set \( C_n^* \) is linearly measurable. However, the set corresponding to \( K_\alpha \) may be nonmeasurable in general.

(\#) We used an idea in [4] in the definition of \( ST_{z_0} \).
closed subset \( E_p \) of \( \Delta \cap C \) such that \( m(\Delta \cap C - E_p) < 1/p \) and \( G(e^{i\theta}) \) tends to 1 uniformly for \( e^{i\theta} \in E_p \). Thus we can find \( r_1 < 1 \) such that \( G(z) > 1/2 \) on the set \( Y = \{ re^{i\theta}; r_1 < r < 1, e^{i\theta} \in E_p \} \). We decompose the complement of \( Y \) with respect to the annulus: \( r_1 < r < 1 \) into components \( \{ B_n \} (n = 1, 2, \ldots) \). Let \( \{ B_n \} \) be the components which have points in common with \( \Delta \). Then its number is finite: \( i = 1, 2, \ldots, k \). To prove this, suppose that there are an infinite number of \( \{ B_n \} \) having points in common with \( \Delta \). Since \( \Delta \) is a domain, we can connect a point of \( B_n \cap \Delta \) with a point of each \( B_n \cap \Delta \) (\( i \geq 2 \)) by a curve inside \( \Delta \). This curve must cross the boundary arc of every \( B_n \) on the circle: \( |z| = r_1 \). Any point of accumulation of these points of intersection is a boundary point of \( \Delta \), and, at the same time, a point of \( Y \). This is impossible because, by the continuity of \( G(z) \), \( G(z) = 0 \) on the closure of \( \Delta \) and \( G(z) > 1/2 \) on \( Y \). Therefore \( \Delta \cap C \) is contained in \( \bigcup_{<1} (\bigcup_{<1} B_n) \cap C \). The linear measure of that part of \( \Delta \cap C \) lying in the open intervals of \( \bigcup_{<1} B_n \cap C \) has the same value as \( m(\Delta \cap C) \). But this part is the set \( \Delta \cap C - E_p \) which has linear measure less than \( 1/p \). Hence \( m(\Delta \cap C) < 1/p \). Since \( p \) is an arbitrary integer we see that \( m(\Delta \cap C) = 0 \).

5. Our theorems are

**Theorem 1.** Let \( f(z) \) be an analytic function defined in \( U \) and taking values on an abstract Riemann surface \( \mathcal{R} \) (with boundary \( \mathcal{R}_0 \) if \( \mathcal{R} \) is open). Then a point \( P_0 \) of \( S_{z_0} - ST_{z_0} - R_{z_0} \) is an asymptotic value at \( z_0 \) or at points \( z_n \) of \( C \) tending to \( z_0 \) if there exists a path in \( \mathcal{R} \cap S_{z_0} \) converging to \( P_0 \).

**Theorem 2.** Let \( f(z) \) be the same function as in Theorem 1. A point \( P_0 \) of the set \( ST_{z_0} - R_{z_0} \) is an asymptotic value at \( z_0 \) or at points \( z_n \) tending to \( z_0 \) if

(i) there exists a number \( \rho > 0 \) such that there is a path in \( \mathcal{R} \cap (S_{z_0} - M_\rho^{C_n}) \) converging to \( P_0 \), and if

(ii) the set of points on \( |z| = 1 \) where the radial cluster sets \( T_z \) do not contain \( P_0 \) is everywhere dense in a certain open arc \( C' \subset C \) containing \( z_0 \).

We shall prove Theorem 2 for \( P \in \mathcal{R}_0 \). The proof for the case \( P \in \mathcal{R}_0 \) and the proof of Theorem 1 are easily obtained by modifying the proof given below.

Let \( L \) be the path in \( \mathcal{R} \cap (S_{z_0} - M_\rho^{C_n}) \), converging to \( P_0 \). We form two paths on each side of \( L \) and close enough to \( L \) that the domain \( D \) between them is contained in \( \mathcal{R} - M_\rho^{C_n} \). Let \( \{ G_n \} \) be a nested countable base of the filter defining \( P_0 \) and let \( D_n \) be that component of the intersection of \( G_n \) with \( D \) which contains an end-part of \( L \). Obviously, \( D_1 \supset D_2 \supset \cdots \rightarrow P_0 \).

We take two points \( z_1 \) and \( z_2 \) on \( C \) near \( z_0 \) so that \( \text{arg } z_1 < \text{arg } z_0 < \text{arg } z_2 \), \( |z_0 - z_1| < \rho \) and \( |z_0 - z_2| < \rho \), and such that \( T_{z_1} \cup T_{z_2} \) does not contain \( P_0 \). Let \( r' < 1 \) be a number sufficiently near 1. Denote the sector \( \{ re^{i\theta}; r' < r < 1, \text{arg } z_1 < \theta < \text{arg } z_2 \} \) by \( Q \) and its boundary inside \( U \) by \( q \). We may assume that the image of \( q \) lies outside some neighborhood of \( P_0 \). The inverse image of \( D_n \) in \( Q \) is not empty since \( L \subset S_{z_0} \). For \( n \) sufficiently large, some component,
say $\Delta_n$, together with its closure, has no common point with $q$.

Suppose that, in $\Delta_n$, $f(z)$ does not assume values of $D_{n+1}$. Then the closure of the image $f(\Delta_n)$ of $\Delta_n$ is compact in $\mathcal{R}$, and for almost all $z$ of $C_\rho$ the radial cluster sets $T_z$ lie outside the closure of $f(\Delta_n)$. Then by our lemma, the measure $m(\Delta_n^b \cap C) = 0$, the continuous function $g(P)$ of the lemma being defined by the aid of a metric in $\mathcal{R}$. Therefore the harmonic measure of $\Delta_n^b \cap C$ with respect to $\Delta_n$ is zero. We take a small compact Jordan domain $K_0$ inside $D_{n+1}$ and form a harmonic measure function of the boundary of $K_0$ in the domain $D_n - K_0$. If we regard this function as a function defined in $\Delta_n$, it has boundary value 0 except for points of $\Delta_n^b \cap C$ which has harmonic measure zero. By the maximum principle this function must be the constant zero, which is a contradiction.

Thus we have shown that $f(\Delta_n) \cap D_{n+1} \neq \emptyset$. Consider the inverse image of $D_{n+1}$ in $\Delta_n$ and let $\Delta_{n+1}$ be any component of the image. We can show as above that $f(\Delta_{n+1}) \cap D_{n+1} \neq \emptyset$. In this manner we obtain a sequence of domains $\Delta_n \supset \Delta_{n+1} \supset \cdots$ where $f(\Delta_k) \subset D_k$ ($k = n, n+1, \cdots$). Taking a point $z_k$ in $\Delta_k$ and connecting it with any point $z_{k+1}$ of $\Delta_{k+1}$ by a curve in $\Delta_k$, we get a path $l$ in $Q$ along which $f(z) \to P_0$. By assumption (ii) (we may suppose that the arc $z_1 z_2$ is contained in $C'$), $l$ terminates at a single point of $|z| = 1$. Since $Q$ may be taken arbitrarily near $z_0$ the conclusion of Theorem 2 is obtained.

**Remark.** If we allow a path to oscillate, we may infer the existence of such a path with asymptotic value $P_0$ in any neighborhood of $z_0$, with the following condition replacing (ii):

(ii') there exist points $\zeta$ on $|z| = 1$ on both sides of $z_0$ and arbitrarily close to $z_0$ such that $P_0$ does not belong to $T_{\zeta}$. If we assume only (i), then we know that either there is a path (which may oscillate) in any neighborhood of $z_0$, or there is a sequence of curves which accumulate on a closed arc containing $z_0$, such that $f(z) \to P_0$ uniformly along these curves.$^6$

6. In the theory of cluster sets the difference between a cluster set such as $S_{z_0}$ and a boundary cluster set such as $ST_{z_0}$ is an open set. In the case of an abstract Riemann surface, it is not generally true that $S_{z_0} - ST_{z_0}$ is an open set in $\mathcal{R} + \mathfrak{F}_\mathcal{R}$. On the one hand, suppose that there is a point $P_0 \in \mathcal{R} \cap ((S_{z_0})^b - ST_{z_0})$. Since $ST_{z_0}$ is a closed set, there is a domain $G$ on $\mathcal{R}$, containing $P_0$ and with compact closure corresponding to a parametric circle $|\omega| \leq 1$, such that $G \cap ST_{z_0} = \emptyset$. We denote the open set which is the inverse image in $U$ of $G$ by $\Delta$, and its boundary by $\delta$. It follows from the lemma that, if we take a part $H$ of $\delta \cap C$ sufficiently near $z_0$, its linear measure, and hence its relative harmonic measure with respect to $\Delta$, is zero. The cluster set $S_{z_0}^{(A)}$ of the composed function $\omega(f(z))$ contains the point $\omega = 0$ but does not coincide with the whole $|\omega| \leq 1$, and the boundary cluster set $S_{z_0}^{(A - B)}$ (this is defined by setting

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$^6$ This fact is expressed by the notation $P_0 \in \mathfrak{F}(f)$ in [6].
$E = \delta - H$ in $S_{z_0}^{(a)}$ of §3) is contained in $|\omega| = 1$. This fact contradicts the following theorem which is easily deduced from a theorem in M. Brelot [1]:

Let $\Delta$ be an open set with boundary $\delta$ in the $z$-plane, $z_0$ a nonisolated boundary point of $\Delta$, $H$ a subset of $\delta$, containing $z_0$, of relative harmonic measure zero with respect to $\Delta$, and $f(z)$ analytic in $\Delta$ and on $\delta - H$. Then the difference between the cluster set $S_{z_0}^{(a)}$ and the boundary cluster set $S_{z_0}^{(\delta - H)}$ is an open set.

We state our result in

**Theorem 3:** Under the assumption of Theorem 1, $\Re \cap (S_{z_0} - ST_{z_0})$ is an open set.

On the other hand, however, we shall show that $S_{z_0} - ST_{z_0}$ is not necessarily open. Let $\Re$ be the circle $|w| < 1$ and suppose that $\Re$ consists of only one point $w_0$ on $|w| = 1$. Let $f(z)$ be the identity function: $w = z$ ($w_0 = z_0$). Then $S_{z_0} = \{w_0\}$ but $ST_{z_0} = \emptyset$, so that $S_{z_0} - ST_{z_0} = \{w_0\}$ is a closed set.

7. We shall discuss next some special cases of Theorems 1 and 2. Suppose first that $f(z)$ possesses a radial limit almost everywhere on an open arc $A$ of $|z| = 1$ containing $z_0$. Then $ST_{z_0}$ is the intersection of the closures of certain sets of such limit values. For instance, if the range of values $R_{z_0}$ is compact relatively in $\Re$ and does not cover a set of positive logarithmic capacity, $f(z)$ has this property(7).

We have next the following corollary to Theorem 1:

**Corollary 1:** Let $R_{z_0}$ be conformally equivalent neither to the Riemann sphere punctured at most two points nor to a torus. Then any value $P_0$ of $\Re$ belonging to $S_{z_0} - ST_{z_0} - R_{z_0}$ is a radial limit either at $z_0$ or at points $z_n$ tending to $z_0$.

First we observe that there is a path in $\Re \cap S_{z_0}$ terminating at $P_0$ since a neighborhood of $P_0$ is contained in $S_{z_0}$ by Theorem 3. By Theorem 1 we can then obtain a path terminating at a point $z'$ of $C$ near $z_0$, with the asymptotic value $P_0$. We can prove by the generalization of Lindelöf's theorem that $f(z)$ has the same asymptotic value $P_0$ along the radius with the end point $z'$.

We consider the class of functions studied by Lohwater [11]: A function $f(z)$, meromorphic in $|z| < 1$, is said to belong to class $(U^*)$ if there exists an arc $A$ of $C$ such that $\lim_{r \to 1} |f(re^{i\theta})| = 1$ for almost all $e^{i\theta}$ of $A$.

**Corollary 2 [10; 11].** Let $w = f(z)$ be a function of class $(U^*)$. If $f(z)$ is not analytic on the arc $A$ and if it possesses at most a finite number of zeros and poles in a neighborhood of $A$, then there exists at least one curve, terminating at a point of $A$, along which $f(z) \to 0$ or $\infty$.

From an extension of Schwarz's symmetry principle [4] it follows that, at any singular point $z_0$ of $A$, at least one of 0 and $\infty$ belongs to $S_{z_0}$. Since

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(4) This was proved in [4] and [13] in the case when $\Re$ is the extended $w$-plane.
(7) See Theorem 3.3 in [14].
(6) The case when $R$ is the whole plane and $S_{z_0}$ is not was discussed in [13].
$ST_{z_0} \subset \{|w| = 1\}$, and since $S_{z_0} - ST_{z_0}$ is an open set by Theorem 3, the assumptions of Theorem 1 are satisfied for $P_0 = 0$ or $\infty$; hence Corollary 2 is established.

In connection with Theorem 2, we remark that if $R_{z_0}$ satisfies the conditions of Corollary 1, it cannot happen that $f(z)$ tends to a value in $\mathfrak{R}$ uniformly on a sequence of curves accumulating on an arc of $C$ near $z_0$. Therefore for such a function condition (ii) in Theorem 2 is not necessary if the point $P_0$ belongs to $\mathfrak{R}$. However, we do not know whether condition (ii) is necessary in general, even for the functions of class $(U^*)(9)$, although condition (i) is fulfilled for these functions.

We shall prove

**Corollary 3 (Calderón-Domíngues-Zygmund [3], cf. also [8])**. Let $w = f(z)$ be a bounded analytic function defined in $|z| < 1$. Let $f(z)$ have a radial limit of modulus one almost everywhere on an arc $A$ of $C$. Then if $f(z)$ is not analytic on $A$, every value of $|w| = 1$ is a radial limit at infinitely many points of $A$.

Let $z_0 \in A$ be a singular point. By Theorem 2 and Lindelöf's theorem any point $w$ of $|w| = 1$ is the radial limit at $z_0$ or at $z_n$ tending to $z_0$. If such $\{z_n\}$ exists, the corollary is already proved. Also if there are singular points on $C$ tending to $z_0$, then our corollary follows. Hence suppose that $f(z)$ were analytic on $C$ near $z_0$, except at $z_0$, and $f(z) \neq w$ there. In this situation $f(z)$ would have limit values $w_1$ and $w_2$ respectively as $z \in C$ moves toward $z_0$ from both sides. Since $f(z)$ is bounded, $w_1 = w_2$ and $f(z)$ would tend to this value uniformly as $z$ approaches $z_0$ from the inside of $U$ by Lindelöf's theorem. Then $f(z)$ would be analytic at $z_0$ and this is a contradiction. Thus the corollary is proved.

8. In this section we shall consider meromorphic functions of bounded characteristic. Such a function has radial limit almost everywhere, and the set of points of $C$ where the function has the same radial limit has linear measure zero by Riesz-Nevanlinna's theorem. Therefore condition (ii) of Theorem 2 is not necessary. However, we can show by an example that at a point where an asymptotic path terminates, the function does not always have a radial limit of the same value. We know that this is true for a meromorphic function with at least three exceptional values. Hence let us remark that the following theorem is not a special case of Corollary 1 if and only if $f(z)$ omits only two values.

**Theorem 4.** Let $w = f(z)$ be a meromorphic function of bounded characteristic in $U$, and suppose that it does not take two values $w_0$ and $w_1$ near a point $z_0$ of $|z| = 1$. If $w_0$ belongs to $S_{z_0} - ST_{z_0}$, then $w_0$ is a radial limit at $z_0$ or at points $z_n$ tending to $z_0$.

(9) In a recent letter Professor Lohwater remarked to the author that his unpublished proof of the first part of [9] is not complete and that that aspect of the question is still open.
Proof. Without loss of generality we may suppose that $w_0 = 0$, $w_1 = \infty$, and $ST_{\alpha}$ lies outside $|w| \leq 1$. Then $f(z)$ has a representation (see [12])

$$f(z) = \frac{\Omega_1(z)}{\Omega_2(z)} \exp \left[ \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\phi} + z}{e^{i\phi} - z} d\mu(\phi) + i\lambda \right],$$

where $\Omega_1(z)$ and $\Omega_2(z)$ are finite Blaschke products, $\lambda$ is a real constant, and $\mu(\theta)$ is a function of bounded variation in $[0, 2\pi]$ such that $\mu(\theta) = \left\{ \mu(\theta^+) + \mu(\theta^-) \right\}/2$. In order to prove our theorem it is sufficient to prove that

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{1 + r^2 - 2r \cos (\phi - \theta)} d\mu(\phi)$$

$$(z = re^{i\theta})$$

tends to $-\infty$ along a radius at $z_0$ or at a point arbitrarily near $z_0$.

Let $C_\rho$ be an arc such that the set $M_\rho$ defined in §3 lies outside $|w| \leq 1$, and let us decompose $u(z)$ into the integral $u_1(z)$ on $C_\rho$ and the integral $u_2(z)$ on its complement.

We denote the set function corresponding to $\mu(\theta)$ by $\mu^*(X)$. Then by de la Vallée Poussin's decomposition theorem (see [15, p. 127]) we have

$$\mu^*(X) = \mu^*(X \cap E_{+\infty}) + \mu^*(X \cap E_{-\infty}) + \int_X \mu'(\phi) d(\phi)$$

for any Borel set $X$ consisting of points of continuity of $\mu(\theta)$, where $E_{+\infty}$ and $E_{-\infty}$ represent the sets of points at which $\mu(\theta)$ has derivatives equal to $+\infty$ and $-\infty$ respectively. According to Fatou's theorem, $u(re^{i\theta}) \rightarrow \mu'(\theta)$ as $r \rightarrow 1$ for almost all $\theta$. By hypothesis, $|f(re^{i\theta})| = \exp \left[ u(re^{i\theta}) \right]$ tends to $\exp \left[ \mu'(\theta) \right] > 1$ as $r \rightarrow 1$ for almost all $re^{i\theta}$ of $C_\rho$. Hence $\mu'(\theta) > 0$ for almost all $e^{i\theta}$ of $C_\rho$. Now it is a result of Lohwater [10, Lemma] that if there is a negative jump of $\mu(\theta)$ on $C$, $u(z)$ tends to $-\infty$ radially at this point. We now suppose that $\mu(\theta)$ has no negative jump on $C_\rho$. Let $Y$ be any Borel subset of $C_\rho$ which does not contain points of discontinuity of $\mu(\theta)$. If $E_{-\infty} \cap C_\rho = \emptyset$, then the positive-ness of $\mu^*(Y)$ follows from the above equality because $\mu^*(Y \cap E_{+\infty})$ is always non-negative (see lemma in [15, p. 126]). Let $X$ be any Borel subset of $C_\rho$, and $\{a_n\}$ be the points of discontinuity of $\mu(\theta)$ on $X$; $\{a_n\}$ coincide with the jumps of $\mu(\theta)$. Then $\mu^*(X) = \mu^*(X - \{a_n\}) + \sum_n \mu^*(a_n)$ and $\mu^*(a_n)$ is equal to the saltus at $a_n$. Since both terms of the right side are positive, $\mu^*(X) > 0$. Therefore $u_1(z) > 0$ and hence $u(z) > m > -\infty$ near $z_0$. Hence $|f(z)| > m_1 > 0$ near $z_0$. This contradicts the assumption that $0 \in S_{\xi_0}$. Thus there is at least one point $e^{i\theta}$ of $E_{-\infty}$ on $C_\rho$. At this point $u(z)$ has a radial limit $-\infty$. On account of the arbitrariness of $C_\rho$, the theorem is concluded.

As mentioned in §1, a special case of this theorem was proved in [10]. Whether the existence of $w_1 \in R_{\xi_0}$ in our theorem is necessary or not is not yet determined. Also notice that we have no such result corresponding to Theo-

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(10) The writer owes the idea of the proof to [10].
9. In order to determine more completely the asymptotic values of an analytic function, we shall define a smaller boundary cluster set. Up to now we have used the radial cluster set \( T_{z_0} \) to define the boundary cluster set \( ST_{z_0} \). In many cases the set \( T_{z_0} \) is too large; we shall find it more suitable to use, instead, the set \( \Gamma_{z_0} \) of all asymptotic values at \( z_0 \), in some cases.

Let \( f(z) \), \( \{K_n\} \), \( \{K_\alpha\} \) and \( z_0 \) be the same as in §3. We shall use the same notation as in §3, whenever it causes no confusion. If, for a given integer \( n \), there is an open arc \( C_n \) containing \( z_0 \) such that the inner linear measure of the set \( \{z \in C_n; z \neq z_0, \Gamma_z \cap K_n \neq \emptyset\} = C_n^* \) is zero, we define \( K_n^* \) by \( K_n \); otherwise we set \( K_n^* = \emptyset \). We denote the set \( C - \bigcup_a C_n^* - z_0 \) by \( C^* \). We define \( C^* \) in a similar fashion for \( \{K_\alpha\} \) and set \( ST_{z_0} = \mathbb{R} + \mathbb{R}^+ - \bigcup_a K_\alpha^* \). This set \( ST_{z_0} \) is the smallest set in the following sense: Let \( H \subset C, z_0 \in H \), be a set of linear measure zero, and denote the closure of \( \bigcup_{C_C; C \subset C - H} \Gamma_z \) by \( N_{\rho}^{(C-H)} \) and the intersection \( \bigcap_{\rho \geq 0} N_{\rho}^{(C-H)} \) by \( ST_{z_0}^{(C-H)} \). Then \( ST_{z_0} \subset ST_{z_0}^{(C-H)} \).

The following theorems correspond respectively to Theorems 1 and 2:

**Theorem 5.** Let \( f(z) \) be an analytic function defined in \( |z| < 1 \) and taking values of an abstract Riemann surface \( \mathbb{R} \) (with boundary \( \mathbb{R} \) if \( \mathbb{R} \) is open). Then a point \( P_0 \) of \( S_{z_0} - ST_{z_0} - R_{z_0} \) is an asymptotic value either at \( z_0 \) or at points \( z_n \) tending to \( z_0 \) if (i) there exists a path in \( \mathbb{R} \setminus S_{z_0} \) converging to \( P_0 \), and if (ii) there exists a set \( E \) of points on \( |z| = 1 \), dense in some neighborhood of \( z_0 \), such that, for each \( \xi \in E \), there is a path \( l_\xi \) on \( |z| < 1 \) terminating at \( \xi \) with the property that the cluster set of \( f(z) \) along \( l_\xi \) does not contain \( P_0 \).

**Theorem 6.** Let \( f(z) \) be the same as in Theorem 5. A point \( P_0 \) of \( ST_{z_0} - R_{z_0} \) is an asymptotic value either at \( z_0 \) or at points \( z_n \) tending to \( z_0 \) if (i) there exists a path in \( \mathbb{R} \setminus (S_{z_0} - N_{\rho}^{(C')}) \) converging to \( P_0 \), where \( \rho \) is a certain positive number, and if condition (ii) of Theorem 5 is satisfied.

Notice that condition (ii) is required even in Theorem 5. If we lift this requirement in Theorems 5 and 6, then the same remark as in §5 is given.

The proofs for these theorems are similar to but simpler than those for Theorems 1 and 2 and are omitted here. We shall explain Theorem 5 in a special case.

Let \( w = f(z) \) be a meromorphic function defined in \( |z| < 1 \) and suppose that \( f(z) \) omits at least three values in a certain neighborhood of a point \( z_0 \) on \( |z| = 1 \). Then the points of \( |z| = 1 \) which have radial limits are everywhere dense in an open arc containing \( z_0 \) (see [5] and [13]). Therefore condition (ii) of Theorem 5 is not necessary. If \( S_{z_0} \) is the whole \( w \)-plane, condition (i) in Theorem 5 is clearly fulfilled, while, if \( S_{z_0} \) is not the whole \( w \)-plane, Theorem 5 is contained in Theorem 1. Therefore, for our function \( f(z) \), Theorem 5 may be stated without conditions (i) and (ii) (this theorem was stated in [13]). The
modular function shows that Theorem 5 is not contained in Theorem 1.

We remark that $9cH(SH - STH)$ is an open set. This is trivial if $9cC^0$, and if $9c(T_S*0$ then $dirSTzll = dirSTH(11)$ and our assertion follows from Theorem 3.

10. Finally we shall examine the assumptions of Theorems 1 and 2. Let $9c$ be a simply-connected domain in the $w$-plane which spirals down on $|w| = 1$ from the outside and suppose that $\varphi_{\varphi}$ consists of only one point $w_0$ of $|w| = 1$ with the ordinary topology. We map $9c$ onto $|z| < 1$ and denote by $z_0$ the point on $|z| = 1$ which corresponds to $w_0$. Then $S_{z_0} = \{w_0\}$ and $ST_{z_0} = \emptyset$.

Clearly $w_0$ is never an asymptotic value; here, the condition in Theorem 1 is not satisfied. However, if we take $P_0 \in S_{z_0} - ST_{z_0}$ in $9c$, then the required curve is obtained by Theorem 3.

The following examples show that condition (i) is necessary in Theorem 2 even if we take the point $P_0$ in $9c$.

**Example 1.** Take the circles $U_w$: $|w| < 1$ and $V_n$: $|w - 1| \leq 1/n$ ($n = 1, 2, \ldots$). Set $U_w - V_n = G_n$ and connect $G_n$ and $G_{n+1}$ by a small strip domain $S_n$ near the point $w = -1$ so that $S_n \rightarrow -1$ as $n \rightarrow \infty$ and $G_1 \cup S_1 \cup G_2 \cup S_2 \cup \ldots$ is a simply-connected Riemann surface $9c$. Map $9c$ onto $U$: $|z| < 1$ conformally. Then by Koebe's theorem the image of $S_n$ and hence the image of $G_n$ tends to a point, say $z_0$, of $C$: $|z| = 1$, and $z_0$ is the only point which is not an image of any boundary point of $\{G_n\}$ and $\{S_n\}$. For the function $w = f(z)$ mapping $U$ into the $w$-plane through $9c$, the cluster sets are $S_{z_0} = \{|w| \leq 1\}$ and $ST_{z_0} = ST_{z_0} = \{|w| = 1\}$. The point $w = 1$ is neither taken by $f(z)$ nor is an asymptotic value; the path which is required in (i) of Theorem 2 actually does not exist.

**Example 2.** Take the circles $U_w$: $|w| < 1$ and $V_n$: $|w - 1| \leq 1/n$ ($n = 1, 2, \ldots$). Set $U_w - V_n = G_n$ and connect $G_n$ and $G_{n+1}$ by a small strip domain $S_n$ near the point $w = -1$ so that $S_n \rightarrow -1$ as $n \rightarrow \infty$ and $A_1 \cup S_1 \cup B_1 \cup S'_1 \cup A_2 \cup S_2 \cup B_2 \cup \ldots$ is a simply-connected Riemann surface $9c$. We map $9c$ onto $U$ conformally and denote the function corresponding to the mappings $U \rightarrow 9c \rightarrow w$-plane by $f(z)$. We shall construct $\{S_n\}$ so that the $z$-images of $A_n, B_n, S_n$, and $S'_n$ tend to a point of $C$ as $n \rightarrow \infty$.

Suppose that each $S_n$ contains a part $S_n^*$ which is mapped conformally onto a rectangle: $0 < u < a_n$, $0 < v < b_n$ such that the sides with length $a_n$ correspond to a part of the boundary of $9c$. Consider, on $S_n^*$, the function which maps $9c$ onto $U$, we transform it into the function defined on the rectangle and denote it by $z = g(u + iv)$. This is a schlicht function, and we have, by Schwarz's inequality,

(11) We can prove this as for Theorem 3.3 of [14].

(12) For instance, take the part of $1/n + 1 < |w + 1| < 1/n$ outside $U_w$ as $S_n$.

(13) The writer owes some technique in the construction of this example to [7].
If the images of $A_n$, $S_n$, $B_n$, and $S'_n$ do not tend to a point, they must tend to some arc, say $z_1z_2$. Then, denoting the area of the image of $S_n$ by $s_n$, we get from the above inequality

$$(|z_1 - z_2|^2 - a_n) s_n \leq a_n b_n$$

whence

$$0 < |z_1 - z_2|^2 \leq s_n b_n / a_n.$$  

Since $s_n \to 0$ as $n \to \infty$, there arises a contradiction if we assume $b_n/a_n < M < \infty$. Therefore under the assumption that $S_n$ contains such a part $S^*_n$ (this means that $S^*_n$ is "narrow") it is proved that $A_n$, $S_n$, $B_n$, and $S'_n$ tend to a point, say $z_0$, of $C$. In this example $w=1$ is the only one exceptional value and $STz_0 = STz_1$ coincides with $|w| = 1$. The point $w=1$ is not an asymptotic value; actually condition (i) in Theorem 2 is not fulfilled.(14).

Whether condition (ii) in Theorem 2 is really necessary or not is not yet known, as already stated in §7.

**Bibliography**


(14) This example, in which the unique exceptional value $w=1$ is not an asymptotic value, gives a negative answer to the question raised in p. 120 of [6].

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