THE POISSON TRANSFORM[1]

BY

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The Poisson transform is defined by the equation

$$f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{1 + (x - t)^2}\, d\alpha(t). \tag{1}$$

It is assumed that α is of bounded variation in each finite interval, and the integral is interpreted as

$$\int_{-\infty}^{\infty} = \lim_{R\to\infty, S\to\infty} \int_{-S}^{R}.$$

If α is absolutely continuous, (1) takes the form

$$f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\phi(t)}{1 + (x - t)^2}\, dt, \tag{2}$$

where ϕ is Lebesgue integrable in each finite interval. It will be proved that the convergence of (1) or (2) for a single value of x implies its convergence to a function $f(x)$ analytic on the whole real axis.

The main problem with which we shall be concerned is the inversion of (1) and (2) by suitable differential operators. The theory of convolution transforms developed recently by Widder and his students[2] cannot be expected to apply since the bilateral Laplace transform of the kernel has only a line, and not a strip, of convergence. Consequently a new procedure is demanded.

Our inversion formulas for the Poisson transform are motivated by the following argument. Let us consider (2) in the case that ϕ is in

$$L^2(-\infty, \infty).$$

Fourier transformation shows that (2) is inverted by[3] $e^{|H|}f(x) = \phi(x)$, where $H = (1/i)D$, $D = d/dx$. The problem is now to find an interpretation of $e^{|H|}$ which is applicable in the general case. Since

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(2) For the most recent account see Widder and Hirschman, Trans. Amer. Math. Soc. vol. 66 (1949) pp. 135–201.

(3) See, for example, Stone, *Linear transformations in Hilbert space*, p. 446, or Pollard, Duke Math. J. vol. 13 (1946) p. 312.

$$e^{|x|} = \cosh x + \frac{\sinh x}{x} |x|, \qquad -\infty < x < \infty,$$

this suggests the choice

$$e^{|H|} = \cos D + \frac{\sin D}{D} |H|.$$

The natural interpretation in L^2 of $|H|f$ is $\tilde{f}'$, where $\tilde{g}$ denotes the Hilbert transform of g. Hence we are led to conjecture for (2) an inversion formula of the form

$$\phi(x) = (\cos D)f(x) + \frac{\sin D}{D} \tilde{f}'(x). \tag{3}$$

Unfortunately, for a general function of the form (2) the Hilbert transform $\tilde{f}'$ in its usual form[4] need not exist, and a reinterpretation of formula (3) necessary. It will be shown that the formula can be made rigorous as follows. Define the operators $\frown$, $\cos tD$, $(\sin tD)/D$ and T_t by

$$\widehat{g}(x) = -\frac{1}{\pi}\int_{0}^{\infty} u^{-2}[g(x+u) - 2g(x) + g(x-u)]du, \tag{4}$$

$$(\cos tD)g(x) = \sum_{k=0}^{\infty} (-1)^k \frac{t^{2k}}{(2k)!} g^{(2k)}(x), \tag{5}$$

$$\frac{\sin tD}{D} g(x) = \sum_{k=0}^{\infty} (-1)^k \frac{t^{2k+1}}{(2k+1)!} g^{(2k)}(x), \tag{6}$$

$$T_t g(x) = (\cos tD)g(x) + \frac{\sin tD}{D} \widehat{g}(x). \tag{7}$$

Then (1) is inverted for all x by

$$[\alpha(x+) + \alpha(x-)]/2 - [\alpha(0+) + \alpha(0-)]/2 = \lim_{t\to 1-} \int_0^x (T_t f(u))du, \tag{8}$$

and (2) for almost all x by

$$\phi(x) = \lim_{t\to 1-} T_t f(x). \tag{9}$$

The proofs of (8) and (9) constitute the main part of this paper, but we must begin with an exposition of the elementary properties of the Poisson transform.

1. **Properties of the transform.** The basic facts about the Poisson transform are stated in the following theorem.

(4) Titchmarsh, *Theory of Fourier integrals*, p. 120.

THEOREM 1.1. *If the transform*

$$f(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{d\alpha(t)}{1+(z-t)^2}, \qquad z = x+iy, \tag{1.1}$$

converges for a single value of z in the strip $|y|<1$ *it converges uniformly in any compact subset of this strip and defines a function analytic there. Therefore differentiation of arbitrary order is permissible under the integral.*

To prove this suppose that the integral in (1.1) converges at $z=z_0$. For each z such that $|y|<1$ define

$$I(R,S) = \frac{1}{\pi}\int_{-S}^{R} \frac{d\alpha(t)}{1+(z-t)^2} = \int_{-S}^{R} \frac{1+(z_0-t)^2}{1+(z-t)^2}\,dF(t), \tag{1.2}$$

where

$$F(t) = \frac{1}{\pi}\int_{-\infty}^{t} \frac{d\alpha(u)}{1+(z_0-u)^2}.$$

Integration by parts of the right-hand member of (1.2) yields

$$I(R,S) = \frac{1+(z_0-R)^2}{1+(z-R)^2}F(R) - \frac{1+(z_0+S)^2}{1+(z+S)^2}F(-S) - 2(z-z_0)\int_{-S}^{R} F(t)\,\frac{1-(t-z)(t-z_0)}{(1+(z-t)^2)^2}\,dt.$$

Since $F(t)$ is bounded it follows that $I(R,S)$ converges as $R\to\infty$, $S\to\infty$ with the uniformity asserted in the theorem. We have incidentally established the fact that for $|y|<1$

$$f(z) = f(z_0) - 2(z-z_0)\int_{-\infty}^{\infty} F(t)\,\frac{1-(t-z)(t-z_0)}{(1+(z-t)^2)^2}\,dt. \tag{1.3}$$

THEOREM 1.2. *If* (1.1) *converges at a point* z_0 *in the strip* $|y|<1$ *then*

$$\alpha(t) = o(t^2), \qquad |t|\to\infty.$$

By the preceding theorem we may assume $z_0=0$. Since

$$\alpha(t) - \alpha(0) = \int_0^t d\alpha(u)$$

we may write

$$\alpha(t) - \alpha(0) = -\int_0^t (1+u^2)\,dF_1(u),$$

where

$$F_1(t) = \frac{1}{\pi}\int_t^\infty \frac{d\alpha(u)}{1+u^2}.$$

Then, integrating by parts, we get

$$\alpha(t) - \alpha(0) = -(1+t^2)F_1(t) + F_1(0) + 2\int_0^t uF_1(u)du.$$

Divide by t^2 and let $t\to+\infty$. Since $F_1(\infty)=0$ we find that $\alpha(t)/t^2\to 0$ as $t\to+\infty$. A similar proof applies for $t\to-\infty$.

2. **The function** $\hat{f}(x)$. The first step in establishing the inversion formulas is the proof of the following theorem.

THEOREM 2.1. *If $f(x)$ is defined by the formula* (1.1), *then the integral*

$$\hat{f}(x) = -\frac{1}{\pi}\int_0^\infty u^{-2}[f(x+u) - 2f(x) + f(x-u)]du$$

converges for all real values of x and

$$\hat{f}(x) = \frac{1}{\pi}\int_{-\infty}^\infty \frac{1-(x-t)^2}{(1+(x-t)^2)^2}d\alpha(t), \qquad -\infty < x < \infty. \tag{2.1}$$

By (1.1)

$$\begin{aligned} u^{-2}&[f(x+u) - 2f(x) + f(x-u)] \\ &= \int_{-\infty}^\infty u^{-2}[k(x+u-t) - 2k(x-t) + k(x-u-t)]d\alpha(t), \end{aligned} \tag{2.2}$$

where $k(x)=(1/\pi)(1/(1+x^2))$. By the same argument used to establish Theorem 1.1 it can be established that for each real x the right-hand side of (2.2) converges uniformly in any interval $-R\leqq u\leqq R$. Consequently we may integrate with respect to u under the integral sign from 0 to R. After a change of variable $x-t=v$ we find that

$$\begin{aligned} -\frac{1}{\pi}&\int_0^R u^{-2}[f(x+u) - 2f(x) + f(x-u)]du \\ &= -\frac{1}{\pi^2}\int_{-\infty}^\infty \frac{1-v^2}{(1+v^2)^2}[\arctan(R-v) + \arctan(R+v)]d_v\alpha(x-v) \\ &\quad + \frac{1}{\pi^2}\int_{-\infty}^\infty \frac{v}{(1+v^2)^2}\log\frac{1+(R+v)^2}{1+(R-v)^2}d_v\alpha(x-v) \\ &\equiv I_1 + I_2, \text{ say.} \end{aligned} \tag{2.3}$$

(The separate convergence of I_1 and I_2 follows from subsequent arguments.) Next we must let $R\to\infty$ in each of the integrals I_1, I_2.

First we dispose of I_1. Let x be fixed and define

$$G(v) = \frac{1}{\pi}\int_{-\infty}^{v} \frac{1-\xi^2}{(1+\xi^2)^2}\, d_\xi \alpha(x-\xi).$$

Then

$$\begin{aligned} I_1 &= -\frac{1}{\pi}\int_{-\infty}^{\infty} [\arctan (R-v) + \arctan (R+v)]dG(v) \\ &= \int_{-\infty}^{\infty} G(v)[-k(R-v) + k(R+v)]dv \\ &= -\int_{-\infty}^{\infty} G(u+R)k(u)du + \int_{-\infty}^{\infty} G(u-R)k(u)du. \end{aligned}$$

Then by the principle of dominated convergence

$$\lim_{R\to\infty} I_1 = -G(\infty) + G(-\infty),$$

since

$$\int_{-\infty}^{\infty} k(u)du = 1.$$

But $G(-\infty)=0$. Hence

$$\lim_{R\to\infty} I_1 = -G(\infty) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1-(x-t)^2}{(1+(x+t)^2)^2}\, d\alpha(t).$$

Since this is the right-hand side of (2.1), it remains only to show that $\lim_{R\to\infty} I_2 = 0$.

Let x be fixed and define

$$H(v) = \frac{1}{\pi}\int_{\infty}^{v} \frac{d_\xi \alpha(x-\xi)}{1+\xi^2}\, .$$

Then

$$\begin{aligned} \pi I_2 = &-\int_{-\infty}^{\infty} \frac{v}{1+v^2} \log \frac{1+(R+v)^2}{1+(R-v)^2}\, dH(v) \\ &= 2\int_{-\infty}^{\infty} H(v)\, \frac{v}{1+v^2}\, \frac{R+v}{1+(R+v)^2}\, dv + 2\int_{-\infty}^{\infty} H(v)\, \frac{v}{1+v^2}\, \frac{R-v}{1+(R-v)^2}\, dv \\ &+ \int_{-\infty}^{\infty} H(v)\, \frac{1-v^2}{(1+v^2)^2} \log \frac{1+(R+v)^2}{1+(R-v)^2}\, dv \\ &\equiv J_1 + J_2 + J_3, \text{ say.} \end{aligned}$$

Since $H(v)$ is bounded in v we have

$$|J_1| \leqq A_1 \int_{-\infty}^{\infty} \frac{1}{1+|v|} \frac{1}{1+|R+v|} dv.$$

An explicit evaluation of the integral shows that $J_1 \to 0$ as $R \to \infty$. Similarly for J_2.

As for J_3 we have

$$|J_3| \leqq A_2 \int_{-\infty}^{\infty} \frac{1}{1+v^2} \left| \log \frac{1+(v+R)^2}{1+(v-R)^2} \right| dv.$$

Since the integrand is even and $R>0$,

$$\begin{aligned} |J_3| &\leqq 2A_2 \int_0^{\infty} \frac{1}{1+v^2} \log \frac{1+(v+R)^2}{1+(v-R)^2} dv \\ &\leqq 4A_2 \int_0^{\infty} \left[\frac{\pi}{2} - \arctan v\right] \left[\frac{v+R}{1+(v+R)^2} - \frac{v-R}{1+(v-R)^2}\right] dv. \end{aligned}$$

This last integral approaches zero as $R \to \infty$, by the same argument that was used for J_1 and J_2.

This establishes the formula (2.1). Now interpreting x as a complex variable we can apply to (2.1) the same technique used to establish the properties of (1.1). We find that $\hat{f}(z)$ *is analytic in the strip* $|y|<1$ *and that differentiation of arbitrary order under the integral in* (2.1) *is permissible.*

3. **The differential operators.** Let the operators $\cos tD$, $\sin tD/D$ be defined respectively by (5) and (6) of the introduction. We shall prove that if $f(x)$ is defined by (1.1), and hence $\hat{f}(x)$ by (2.1), we have

$$(3.1)\quad (\cos tD)f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ \frac{1-t}{(1-t)^2+(x-y)^2} + \frac{1+t}{(1+t)^2+(x-y)^2} \right\} d\alpha(y)$$

and

$$(3.2)\quad \frac{\sin tD}{D} \hat{f}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ \frac{1-t}{(1-t)^2+(x-y)^2} - \frac{1+t}{(1+t)^2+(x-y)^2} \right\} d\alpha(y)$$

provided $0 \leqq t < 1$.

We give the details only for (3.2). For (3.1) they are similar and slightly easier.

LEMMA 3.1. *Let n be a fixed integer, and define*

$$L(n, k, \xi) \equiv \int_{-\infty}^{\infty} u^{2k+n} e^{iu\xi} e^{-|u|} \operatorname{sgn} u\, du.$$

Then

$$|L| \leqq \frac{2(2k+n)!}{(1+\xi^2)^{k+1/2+n/2}}.$$

For

$$|L(n, k, \xi)| = 2\left|\int_0^\infty u^{2k+n} \begin{matrix}\sin u\xi \\ \cos u\xi\end{matrix} e^{-u} du\right|,$$

according to the parity of n. Hence

$$|L(n, k, \xi)| \leqq 2\left|\int_0^\infty u^{2k+n} e^{iu\xi} e^{-u} du\right|.$$

An explicit computation of the last integral completes the proof.

Our starting point in proving (3.2) is formula (2.1) which can be written

$$\hat{f}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\alpha(y) \int_{-\infty}^{\infty} e^{iu(x-y)} u e^{-|u|} \operatorname{sgn} u du. \tag{3.3}$$

By the final remarks of §2 it is permissible to differentiate indefinitely under the outer integral of (3.3). This differentiation can be carried under the inside integral to yield

$$\begin{aligned} (-1)^k \hat{f}^{(2k)}(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\alpha(y) \int_{-\infty}^{\infty} e^{iu(x-y)} u^{2k+1} e^{-|u|} \operatorname{sgn} u du \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} L(1, k, x-y) d\alpha(y). \end{aligned} \tag{3.4}$$

According to Theorem 1.2 and the preceding lemma we may integrate by parts to obtain for this last integral the value

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \alpha(y) \frac{d}{dy} L(1, k, x-y) dy.$$

According to the definition of $L(n, k, \xi)$,

$$\frac{d}{d\xi} L(n, k, \xi) = iL(n+1, k, \xi).$$

Hence

$$(-1)^k f^{(2k)}(x) = -\frac{i}{2\pi}\int_{-\infty}^{\infty} \alpha(y) L(2, k, x-y) dy.$$

Formally

$$\frac{\sin tD}{D}\hat{f}(x) = -\frac{i}{2\pi}\sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!}\int_{-\infty}^{\infty} \alpha(y) L(2, k, x-y) dy.$$

By use of Lemma 2.1 for $n=2$ and $k\geqq 1$ it is easily verified that the last expression converges when the integrand is replaced by its absolute value, *provided* $0\leqq t<1$. Consequently

$$\frac{\sin tD}{D}\hat{f}(x) = -\frac{1}{\pi}\int_{-\infty}^{\infty}\alpha(y)\{\cdots\}dy, \tag{3.5}$$

where the expression in brackets is

$$\begin{aligned}\frac{i}{2}\sum_{k=0}^{\infty}&\frac{t^{2k+1}}{(2k+1)!}L(2,k,x-y)\\ &=\frac{i}{2}\int_{-\infty}^{\infty}u(\sinh ut)e^{iu(x-y)}e^{-|u|}\operatorname{sgn} u\,du\\ &=\frac{d}{dy}\int_{0}^{\infty}(\sinh ut)e^{-u}\cos u(x-y)du\\ &=\frac{1}{2}\frac{d}{dy}\left[\frac{\tau}{\tau^2+(x-y)^2}-\frac{\tau_1}{\tau_1^2+(x-y)^2}\right], \qquad \tau=1-t,\ \tau_1=1+t.\end{aligned}$$

Substitute this for the bracket in (3.5) and integrate by parts once again. Since $\alpha(y)=o(y^2)$ the result is formula (3.2).

4. **The inversion formulas.** According to formulas (3.1), (3.2) and the definition (7) of T_t we have

$$T_tf(x) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\tau d\alpha(y)}{\tau^2+(x-y)^2}, \qquad \tau = 1-t, \tag{4.1}$$

for any function of the form (1.1).

LEMMA 4.1. *If* (1) *converges and* R, ρ *are positive numbers such that* $R>\rho>0$, *we have*

$$\left(\int_{R}^{\infty}+\int_{-\infty}^{-R}\right)\frac{\tau}{\tau^2+(x-y)^2}d\alpha(y) = o(1), \qquad \tau\to 0+, \tag{4.2}$$

uniformly in $-\rho\leqq x\leqq\rho$.

Let

$$I = \int_{R}^{\infty}\frac{\tau}{\tau^2+(x-y)^2}d\alpha(y).$$

Then

$$I = \int_{R}^{\infty}\frac{\tau(1+y^2)}{\tau^2+(x-y)^2}dF(y),$$

where

$$F(y) = \int_R^y \frac{d\alpha(\xi)}{1+\xi^2}\cdot$$

Therefore

$$I = \tau F(\infty) - 2\tau \int_R^\infty F(y)\, \frac{\tau^2 y + (x-y)(xy+1)}{(\tau^2 + (x-y)^2)^2}\, dy.$$

Since F is bounded in y and x satisfies $|x| \leqq \rho$, the integrand is dominated for $\tau^2 < 1$ by

$$\text{constant}\ \frac{y + (y+\rho)(\rho y + 1)}{(y-\rho)^4}\cdot$$

Hence $I = O(\tau)$, $\tau \to 0$. A similar argument applies to the integral $\int_{-\infty}^{-R}$ and this proves (4.2).

Combining (4.1) and (4.2) we have proved the following lemma.

LEMMA 4.2. *If $f(x)$ is defined by* (1), *then*

$$T_t f(x) = \frac{1}{\pi}\int_{-R}^{R} \frac{\tau}{\tau^2 + (x-y)^2}\, d\alpha(y) + o(1), \qquad \tau \to 0+, \tag{4.3}$$

uniformly in $-\rho \leqq x \leqq \rho$, provided $R > \rho > 0$.

According to this lemma we may integrate both sides of (4.3) to obtain

$$\int_0^x T_t f(u)du = \frac{1}{\pi}\int_{-R}^{R}\left(\arctan\frac{x-y}{\tau} + \arctan\frac{y}{\tau}\right) d\alpha(y) + o(1), \qquad \tau \to 0+,$$

provided $R > |x|$. If we integrate by parts in the right-hand side we find that the integrated part is $o(1)$ as $\tau \to 0+$. Consequently

$$\begin{aligned} &\int_0^x T_t f(u)du \\ &\quad = \frac{1}{\pi}\int_{-R}^{R} \alpha(y)\left\{\frac{\tau}{\tau^2+(x-y)^2} - \frac{\tau}{\tau^2+y^2}\right\} dy + o(1), \qquad \tau \to 0+. \end{aligned} \tag{4.4}$$

Since $R > |x|$, standard arguments[6] show that as $\tau \to 0+$ the integral on the right-hand side of (4.4) approaches

$$[\alpha(x+) + \alpha(x-)]/2 - [\alpha(0+) + \alpha(0-)]/2.$$

This completes the proof that the Poisson transform (1) is inverted by the formula (8).

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[6] Titchmarsh, *Theory of Fourier integrals*, pp. 28–31.

THEOREM 4.1. (1) *is inverted for all real x by* (8).

The treatment of (2) is similar. Now (4.3) takes the form

$$T_t f(x) = \frac{1}{\pi}\int_{-R}^{R} \frac{\tau}{\tau^2 + (x-y)^2}\phi(y)dy + o(1), \qquad \tau \to 0+,$$

provided $R > |x|$. The uniformity is not needed. It is known[5] that the integral on the right approaches $\phi(x)$ for almost all x.

THEOREM 4.2. (2) *is inverted for almost all real x by* (9).

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