A GENERALIZATION OF THE RIESZ THEORY OF COMPLETELY CONTINUOUS TRANSFORMATIONS(1)

BY

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The classical theory, due to F. Riesz, of completely continuous transformations deals with a family $T_c = I - cK$ of linear continuous transformations of a Banach space $\mathfrak{X}$ into itself, where I is the identity and K is completely continuous. The transformation T_c is then one-to-one, except for the "proper" values of c. In this paper we consider cases where the one-to-oneness no longer holds. The identity transformation I is replaced by a transformation E which maps $\mathfrak{X}$ onto the whole of another Banach space $\mathfrak{Y}$, but *not* one-to-one. Then when K is completely continuous, the transformation $T_c = E - cK$ maps $\mathfrak{X}$ onto $\mathfrak{Y}$ except for a countable set of proper values c_i which have no finite accumulation point. When $\mathfrak{W}_1 = T_c\mathfrak{X} \neq \mathfrak{Y}$, c is said to be a *proper value* for K. We can no longer iterate the transformation T_c, but we can apply T_c to $E^{-1}T_c\mathfrak{X}$ and obtain a linear closed space $\mathfrak{W}_2 \subset \mathfrak{W}_1$. After a finite number of repetitions of this process, we find a space $\mathfrak{W}_\nu = \mathfrak{W}_{\nu+1}$. Then T_c transforms $E^{-1}\mathfrak{W}_\nu$ onto $\mathfrak{W}_\nu$, which takes the place of the invariant subspace of the Riesz theory.

In the one-to-one case, there is associated with the transformation $I - cK$ a resolvent R_c, which has a pole of order ν at a proper value c_0, and the Laurent expansion of R_c about c_0 leads to a decomposition of K into two mutually orthogonal parts. In the many-to-one case, the adjoint transformation T_c^* is one-to-one except when c is a proper value, but the range of T_c^* varies with c, so that it is not possible to define the order of a pole of the resolvent in the usual way. However a substitute definition has been found, as well as a decomposition $K = G + H$, where G has only one proper value c_0, and H has only the remaining proper values of K, and H is semi-orthogonal to G in a suitable sense.

1. **Notations and definitions.** We shall let $\mathfrak{X}$ and $\mathfrak{Y}$ denote Banach spaces of infinitely many dimensions, with complex scalars. The adjoint spaces of $\mathfrak{X}$ and $\mathfrak{Y}$ will be denoted by $\mathfrak{X}^*$ and $\mathfrak{Y}^*$ respectively. Two subsets $\mathfrak{X}_0$ of $\mathfrak{X}$ and $\mathfrak{X}_0^*$ of $\mathfrak{X}^*$ are said to be *orthogonal* in case $x^*(x) = 0$ for all x in $\mathfrak{X}_0$ and x^* in $\mathfrak{X}_0^*$. The *orthogonal complement* of a subset $\mathfrak{X}_0$, denoted by $O\mathfrak{X}_0$, is the maximal set $\mathfrak{X}_0^*$ orthogonal to $\mathfrak{X}_0$. It is easily seen that $O\mathfrak{X}_0$ always exists (possibly void) and is a linear closed set. A corresponding statement holds for $O\mathfrak{X}_0^*$.

A linear transformation A mapping $\mathfrak{X}$ into $\mathfrak{Y}$ has an *adjoint* A^* mapping $\mathfrak{Y}^*$ into $\mathfrak{X}^*$, defined by $x^* = A^*y^*$ where $x^*(x) = y^*Ax$ for each x. We shall

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deal only with continuous linear transformations A. Such a transformation A is said to be *completely continuous* in case the transform $A(\mathfrak{X}_0)$ of every bounded set $\mathfrak{X}_0$ is compact (in the sense that every sequence (y_n) chosen from $A(\mathfrak{X}_0)$ has a convergent subsequence).

We shall use the letter E to denote a linear continuous transformation mapping $\mathfrak{X}$ onto the whole space $\mathfrak{Y}$, and shall use K, G, and H to denote completely continuous linear transformations of $\mathfrak{X}$ into $\mathfrak{Y}$. We also set $T_c = E - cK$, where c is a complex parameter. When it is convenient to set $c = 1$, we shall write T for T_c.

A *projection* P is a linear continuous transformation of $\mathfrak{X}$ onto a closed linear subset $\mathfrak{X}_0$, which equals the identity on $\mathfrak{X}_0$, i.e., $P^2 = P$.

For a given linear continuous transformation A of $\mathfrak{X}$ into $\mathfrak{Y}$ we shall set

$$\beta(A, y) = \text{g.l.b.} \left[\|x\| \mid Ax = y\right]$$

for each y, and

$$I(A) = \text{l.u.b.}\left[\beta(A, y) \mid y \in A\mathfrak{X}, \|y\| = 1\right].$$

When the transformation A is one-to-one, obviously $I(A) = \|A^{-1}\|$. (We are admitting $+\infty$ as a value for $\|A^{-1}\|$.)

We shall use $\mathfrak{N}(A)$ to denote the null space of A, that is

$$\mathfrak{N}(A) = \left[x \mid Ax = 0\right].$$

Similarly $\mathfrak{N}(A^*)$ denotes the null space of A^*.

2. **Preliminary lemmas.**

LEMMA 1. *If $\mathfrak{X}_0$ is a finite-dimensional linear subspace of $\mathfrak{X}$, with basis $\{x_1, \cdots, x_n\}$, there exist n elements x_i^* of $\mathfrak{X}^*$ such that $x_i^*(x_j) = \delta_{ij}$, and $P(x) = \sum_{i=1}^n x_i^*(x)x_i$ is a projection of $\mathfrak{X}$ onto $\mathfrak{X}_0$. A similar result holds for $\mathfrak{X}_0^* \subset \mathfrak{X}^*$, with $Q(x^*) = \sum_{i=1}^n x^*(x_i)x_i^*$.*

The first part follows at once from the Hahn-Banach theorem, and the second part follows from more elementary considerations. Note that if the sets $\{x_1, \cdots, x_n\}$ and $\{x_1^*, \cdots, x_n^*\}$ are the same in the two cases, then $Q = P^*$.

LEMMA 2. *For each linear closed subspace $\mathfrak{X}_0$ of $\mathfrak{X}$, $OO\mathfrak{X}_0 = \mathfrak{X}_0$.*

This follows from the lemma in Banach [1, p. 57]. The corresponding result for subsets $\mathfrak{X}_0^*$ of $\mathfrak{X}^*$ does not hold in general, as Banach shows in [1, p. 115]. However, by an easy proof we have

LEMMA 3. *If $\mathfrak{X}_0^*$ is a finite-dimensional linear subspace of $\mathfrak{X}^*$, then $OO\mathfrak{X}_0^* = \mathfrak{X}_0^*$.*

LEMMA 4. *If $\{x_1^*, \cdots, x_n^*\}$ is a basis for $\mathfrak{X}_0^*$, $x_i^*(x_j) = \delta_{ij}$, and*

$P(x)=\sum_{i=1}^{n} x_i^*(x)x_i$, then $\mathfrak{X}$ is the direct sum of $O\mathfrak{X}_0^*$ and the finite-dimensional space $P\mathfrak{X}$.

To prove this, we observe that $x-P(x)$ is always in $O\mathfrak{X}_0^*$, and that if x lies in $O\mathfrak{X}_0^*$, then $Px=0$.

LEMMA 5. *A linear transformation A of $\mathfrak{X}$ into $\mathfrak{Y}$ is (completely) continuous if and only if its adjoint A^* is (completely) continuous.*

Proof. This follows from Theorems 3 and 4 in Banach [1, p. 100]. To show that A is completely continuous whenever A^* is, we note that A^{**}, when restricted to $\mathfrak{X}$, reduces to A, so by Theorem 4 of Banach, A transforms bounded sets in $\mathfrak{X}$ into sets in $\mathfrak{Y}$ which are compact in $\mathfrak{Y}^{**}$. But $\mathfrak{Y}$ is closed in $\mathfrak{Y}^{**}$.

LEMMA 6. *The range $A\mathfrak{X}$ of a linear continuous transformation A of $\mathfrak{X}$ into $\mathfrak{Y}$ is closed if and only if $I(A)<\infty$.*

Proof. When $A\mathfrak{X}$ is closed it constitutes a Banach space, so the necessity of the condition follows from Banach [1, p. 38]. To see that the condition is sufficient, suppose $y_n=Ax_n$, and $\lim y_n=y$. By choosing a subsequence we may suppose that $\|y_{n+1}-y_n\|<1/2^n$. Then after x_n has been suitably chosen, x_{n+1} may be modified (still satisfying $y_{n+1}=Ax_{n+1}$) so that $\|x_{n+1}-x_n\| \leqq I(A)/2^n$. Hence the modified sequence (x_n) has a limit x, and Ax_n converges to $Ax=y$.

COROLLARY. *If $A\mathfrak{X}$ is closed, and* $\lim Ax_n=Ax$, *there exists a sequence (x_n') such that $Ax_n'=Ax_n$ and* $\lim x_n'=x$.

Proof. From the definition of $I(A)$ we may choose x_n' so that $A(x_n'-x)=A(x_n-x)$ and $\|x_n'-x\|\leqq (I(A)+1)\|A(x_n-x)\|$. (Compare Banach [1, p. 40, Theorem 4].)

LEMMA 7. *If A is linear and continuous, $A\mathfrak{X}$ is closed if and only if $A^*\mathfrak{Y}^*$ is closed, and then $A^*\mathfrak{Y}^*=O\mathfrak{N}(A)$, $A\mathfrak{X}=O\mathfrak{N}(A^*)$, where $\mathfrak{N}(A)$ and $\mathfrak{N}(A^*)$ are the null spaces of A and of A^*.*

For the proof, see Banach [1, pp. 149, 150, Theorems 8 and 9].

LEMMA 8. *If A is linear and continuous, then A maps $\mathfrak{X}$ onto the whole of the space $\mathfrak{Y}$ if and only if A^* has a continuous inverse. Likewise A has a continuous inverse if and only if A^* maps $\mathfrak{Y}^*$ onto $\mathfrak{X}^*$.*

For the proof, see Banach [1, pp. 146–148, Theorems 1–4].

LEMMA 9. *If $\mathfrak{X}_1$ and $\mathfrak{X}_2$ are two closed linear subspaces of $\mathfrak{X}$, $\mathfrak{X}_2$ being finite-dimensional, then $\mathfrak{X}_1+\mathfrak{X}_2$ (the linear space spanned by $\mathfrak{X}_1$ and $\mathfrak{X}_2$) is also closed.*

This is readily proved by induction on the dimension of $\mathfrak{X}_2$.

LEMMA 10. *Suppose* $\mathfrak{X}_1$ *and* $\mathfrak{X}_2$ *satisfy the conditions of* Lemma 9, *and also that* $\mathfrak{X}_1+\mathfrak{X}_2=\mathfrak{X}$. *Let* E *be a linear continuous transformation of* $\mathfrak{X}$ *onto the whole of* $\mathfrak{Y}$. *Then* $E\mathfrak{X}_1$ *is closed.*

Proof. The space $\mathfrak{X}_3=E^{-1}E\mathfrak{X}_1$ equals $\mathfrak{X}_1+\mathfrak{X}_4$ where $\mathfrak{X}_4\subset\mathfrak{X}_2$, so $\mathfrak{X}_3$ is closed by Lemma 9. Let E_3 denote the restriction of E to $\mathfrak{X}_3$. Then clearly $I(E_3) \leqq I(E)$. By Lemma 6, $I(E)<\infty$, and by the same Lemma 6, $E_3\mathfrak{X}_3=E\mathfrak{X}_1$ is closed.

3. **Proper values of a completely continuous transformation.** In this section we develop some properties of the family of transformations $T_c=E-cK$, described in §1. A complex number c is called a *proper value* of K in case the range of T_c is a proper subset of $\mathfrak{Y}$, or equivalently, in case T_c^* does not have an inverse.

In case $c=1$ is a proper value, the null space $\mathfrak{N}_1^*=\mathfrak{N}(T^*)$ is not null, and the range $\mathfrak{W}_1=T\mathfrak{X}\neq\mathfrak{Y}$. We set $\mathfrak{M}_1^*=E^*\mathfrak{N}_1^*$, and define $\mathfrak{Z}_1$ as the maximal subspace of $\mathfrak{X}$ satisfying $E\mathfrak{Z}_1=\mathfrak{W}_1$. This is the start of an iterative definition of four sequences of linear spaces $\mathfrak{N}_k^*$, $\mathfrak{M}_k^*$, $\mathfrak{W}_k$, $\mathfrak{Z}_k$, by means of the relations

$$\begin{aligned} T^*\mathfrak{N}_k^* &\subset \mathfrak{M}_{k-1}^* && (\mathfrak{N}_k^* \text{ maximal}),\\ \mathfrak{M}_k^* &= E^*\mathfrak{N}_k^*, &&\\ \mathfrak{W}_k &= T\mathfrak{Z}_{k-1}, &&\\ E\mathfrak{Z}_k &= \mathfrak{W}_k && (\mathfrak{Z}_k \text{ maximal}). \end{aligned}$$

It is obvious that

$$\begin{aligned} \mathfrak{N}_k^* &\supset \mathfrak{N}_{k-1}^*, & \mathfrak{M}_k^* &\supset \mathfrak{M}_{k-1}^*,\\ \mathfrak{W}_k &\subset \mathfrak{W}_{k-1}, & \mathfrak{Z}_k &\subset \mathfrak{Z}_{k-1}. \end{aligned}$$

THEOREM 1. *The spaces* $\mathfrak{N}_k^*$ *and* $\mathfrak{M}_k^*$ *are all finite-dimensional, and* $\mathfrak{W}_k$ *and* $\mathfrak{Z}_k$ *are closed, and* $\mathfrak{W}_k=O\mathfrak{N}_k^*$, $\mathfrak{Z}_k=O\mathfrak{M}_k^*$.

Proof. Let $\mathfrak{B}$ be a bounded subset of $\mathfrak{N}_1^*$. Then $E^*\mathfrak{B}=K^*\mathfrak{B}$ is compact, since K^* is completely continuous. Since E^* has a bounded inverse, $\mathfrak{B}$ is compact. Thus $\mathfrak{N}_1^*$ is finite-dimensional, since every bounded subset is compact. (See Banach [1, p. 84, Theorem 8].) If $\mathfrak{M}_{k-1}^*$ is finite-dimensional, then $\mathfrak{N}_k^*$ is also. To show that $\mathfrak{W}_1=T\mathfrak{X}$ is closed, we shall show that $T^*\mathfrak{Y}^*$ is closed, and apply Lemma 7. Since $\mathfrak{N}_1^*$ is finite-dimensional, there is a projection P of $\mathfrak{Y}^*$ onto $\mathfrak{N}_1^*$. If $T^*y_n^*$ tends to x_0^*, we may assume $Py_n^*=0$. If the sequence (y_n^*) is bounded, we may (by choice of a subsequence) require that $K^*y_n^*$ converges, say to x_1^*. Then $E^*y_n^*$ tends to $x_0^*+x_1^*$ which must equal $E^*y_0^*$ for some y_0^*, since the range of E^* is closed by Lemma 7. Then by Lemma 8, y_n^* tends to y_0^*, so $K^*y_0^*=x_1^*$, and $T^*y_0^*=x_0^*$. The sequence (y_n^*) must be bounded, since if $\|y_n^*\|$ tends to infinity, the sequence $y_{1n}^*=y_n^*/\|y_n^*\|$ satisfies the preceding conditions with $x_0^*=0$. Hence we would have y_{1n}^* approaching y_0^* with $\|y_0^*\|=1$, $P(y_0^*)=0$, $T^*y_0^*=0$, and so $y_0^*=P(y_0^*)$, which is a contradiction. Thus $T^*\mathfrak{Y}^*$ is closed, so $\mathfrak{W}_1 = T\mathfrak{X}$ is closed, and $\mathfrak{Z}_1 = E^{-1}\mathfrak{W}_1$ is also closed. Lemma 7 yields the further information that $\mathfrak{W}_1 = O\mathfrak{N}_1^*$, and it follows readily that $\mathfrak{Z}_1 = O\mathfrak{M}_1^*$.

Now suppose that $\mathfrak{Z}_k = O\mathfrak{M}_k^*$. Then $\mathfrak{Z}_k$ is closed, and since $\mathfrak{M}_k^*$ is finite-dimensional, $\mathfrak{X}$ is the direct sum of $\mathfrak{Z}_k$ and a finite dimensional space, by Lemma 4. Now T maps $\mathfrak{X}$ onto the closed space $\mathfrak{W}_1$, so $\mathfrak{W}_{k+1} = T\mathfrak{Z}_k$ is closed, by Lemma 10 with $\mathfrak{X}_1$ replaced by $\mathfrak{Z}_k$, E by T, and $\mathfrak{Y}$ by $\mathfrak{W}_1$. Then $\mathfrak{Z}_{k+1} = E^{-1}\mathfrak{W}_{k+1}$ is also closed and by a sequence of steps the relations $\mathfrak{N}_{k+1}^* = O\mathfrak{W}_{k+1}$, $\mathfrak{W}_{k+1} = O\mathfrak{N}_{k+1}^*$, $\mathfrak{Z}_{k+1} = O\mathfrak{M}_{k+1}^*$ are readily obtained. This completes the induction.

THEOREM 2. *There exists a least integer ν such that $\mathfrak{W}_{\nu+1} = \mathfrak{W}_\nu$, and then $\mathfrak{W}_{\nu+k} = \mathfrak{W}_\nu$ for all k.*

Proof. From the preceding theorem and the fact that $\mathfrak{N}_k^*$ and $\mathfrak{M}_k^*$ are in one-to-one correspondence, it is clear that $\mathfrak{W}_{\nu+1} = \mathfrak{W}_\nu$, $\mathfrak{Z}_{\nu+1} = \mathfrak{Z}_\nu$, $\mathfrak{N}_{\nu+1}^* = \mathfrak{N}_\nu^*$, $\mathfrak{M}_{\nu+1}^* = \mathfrak{M}_\nu^*$ all happen for the same index ν. We make the proof by considering the nondecreasing sequence of finite-dimensional spaces $\mathfrak{M}_k^*$. If $\mathfrak{M}_{k+1}^* \neq \mathfrak{M}_k^*$ for every k, there exists a unit vector $x_{k+1}^* \in \mathfrak{M}_{k+1}^*$ whose distance from $\mathfrak{M}_k^*$ is unity. For each k there is a point $y_k^* \in \mathfrak{N}_k^*$ with $E^*y_k^* = x_k^*$, $\|y_k^*\| \leq I(E^*)$. Then when $m > k$, $x_m^* - x_k^* = E^*y_m^* - E^*y_k^* = T^*y_m^* - T^*y_k^* + K^*y_m^* - K^*y_k^*$, and $T^*y_m^* \in \mathfrak{M}_{m-1}^*$, $T^*y_k^* \in \mathfrak{M}_{k-1}^* \subset \mathfrak{M}_{m-1}^*$, so $\|K^*y_m^* - K^*y_k^*\| \geq 1$ while $\|y_k^*\|$ is bounded, which contradicts the complete continuity of K^*.

We note that the space $\mathfrak{Z}_\nu$ is mapped onto $\mathfrak{W}_\nu$ by both E and T. This pair of subspaces replaces the invariant subspace of the classical theory of Riesz.

We remark also that if we consider the sequence of spaces defined by the relations:

$$\mathfrak{N}_1 = \mathfrak{N}(T), \qquad \mathfrak{M}_k = E\mathfrak{N}_k,$$
$$T\mathfrak{N}_{k+1} \subset \mathfrak{M}_k, \qquad\qquad (\mathfrak{N}_{k+1} \text{ maximal}),$$

we may have $\mathfrak{N}_{k+1} \neq \mathfrak{N}_k$ for every k. For example, let $\mathfrak{X}$ and $\mathfrak{Y}$ be classical Hilbert space, and let the transformation $y = Ex$ be defined by $y_i = x_{i+3}$ while $y = Kx$ is defined by

$$y_1 = x_4, \qquad y_2 = x_5 - x_2, \qquad y_{i+2} = x_{i+4}/(i + 4).$$

Then a basis for $\mathfrak{N}_k$ is composed of $\delta_1, \delta_3, \delta_4, \alpha_1, \cdots, \alpha_k$, where δ_i has only its ith component different from zero, α_1 has its ith component equal to $1/(i-1)!$ except the first four, which are zero, and $E\alpha_j = T\alpha_{j+1}$.

THEOREM 3. *Let (c_j) be a sequence (finite or infinite) of distinct proper values of K. Let $\mathfrak{N}^{*(j)}$ denote the null space of $T_{c_j}^*$, and let $\mathfrak{W}^{(j)} = T_{c_j}\mathfrak{X}$. Let $[y_{ij}^* | i = 1, \cdots, k_j]$ be a basis for $\mathfrak{N}^{*(j)}$. Then the functionals y_{ij}^* are finitely linearly independent, and there exists a system $[y_{ij}]$ of elements of $\mathfrak{Y}$ such that:*

(a) *for each j, $\mathfrak{W}^{(j)}$ and $[y_{ij} | i = 1, \cdots, k_j]$ span the space $\mathfrak{Y}$;*

(b) $y_{il}^*(y_{kj}) = \delta_{ik}\delta_{lj}$ *for* $l \leqq j$;
(c) y_{kj} *lies in* $\mathfrak{W}^{(l)}$ *for* $l < j$;
(d) *the elements* y_{ij} *are finitely linearly inuependent.*

Proof. Suppose

$$\sum_{j=1}^{n} \sum_{i=1}^{k_j} a_{ij} y_{ij}^* = 0.$$

Then

$$0 = \sum_{j=1}^{n-1} \sum_{i}^{*} (E - c_n K^*) a_{ij} y_{ij}^*$$

$$= E^* \sum_{j=1}^{n-1} \left(1 - \frac{c_n}{c_j}\right) \sum_i a_{ij} y_{ij}^*,$$

and since E^* is one-to-one, this gives a relation involving only $n-1$ of the spaces $\mathfrak{N}^{*(j)}$. Hence by descent we arrive at a contradiction. By Theorem 1 for $k=1$, and Lemmas 1 and 4 we obtain (a) and (b) for $j=1$, $l=1$. Suppose that the system $[y_{ij}]$ has been determined so as to satisfy (a), (b), and (c) for $j<n$. Then the intersection $\mathfrak{V}^{(n-1)}$ of $\mathfrak{W}^{(1)}, \cdots, \mathfrak{W}^{(n-1)}$ is the orthogonal complement of the union of $\mathfrak{N}^{*(1)}, \cdots, \mathfrak{N}^{*(n-1)}$ and since the y_{ij}^* are linearly independent, there are points y_{in} in $\mathfrak{V}^{(n-1)}$ such that (b) holds for $l=j=n$. Property (a) holds for $j=n$, by the same argument as before. Property (d) follows at once from (b).

Theorem 4. *The proper values of K have no finite accumulation point.*

Proof. Suppose (c_n) is a bounded infinite sequence of distinct proper values. Let $\mathfrak{S}_n^*$ be the space spanned by $\mathfrak{N}^{*(1)}, \cdots, \mathfrak{N}^{*(n)}$. Then $(\mathfrak{S}_n^*)$ is a properly increasing sequence of finite-dimensional linear spaces, so each $\mathfrak{S}_n^*$ contains a unit vector v_n^* at unit distance from $\mathfrak{S}_{n-1}^*$. The sequence $(c_n v_n^*)$ is bounded, so $(c_n K^* v_n^*)$ is compact. If y^* is in $\mathfrak{N}^{*(j)}$, $T_c^* y^* = E^*(1 - c/c_j) y^*$, which is zero if $c = c_j$, so

$$T_{c_p}^* \mathfrak{S}_p^* \subset E^* \mathfrak{S}_{p-1}^*. \tag{1}$$

Now for $p > q$, let

$$d_{pq} = \left\| c_p K^* v_p^* - c_q K^* v_q^* \right\| = \left\| E^* v_p^* - T_{c_p}^* v_p^* - E^* v_q^* + T_{c_q}^* v_q^* \right\|,$$

so by (1)

$$d_{pq} = \left\| E^* v_p^* - E^* u^* \right\|, \quad \text{where} \quad u^* \text{ is in } \mathfrak{S}_{p-1}^* \quad \text{and} \quad \left\| v_p^* - u^* \right\| = 1.$$

Since the inverse of E^* is bounded, by Lemma 8, this contradicts the compactness of $(c_n K^* v_n^*)$.

Since the range of T_c^* may vary with c, we state properties of the inverse of T_c^* as follows.

THEOREM 5. *If c_0 is not a proper value of K, and (c_n) and (y_n^*) are sequences such that* $\lim c_n = c_0$, $\lim T_{c_n}^* y_n^* = x_0^*$, *then y_n^* tends to the unique solution y_0^* of $T_{c_0}^* y^* = x_0^*$. Consequently if $T_c^* y^*(c) = x^*(c)$, and $x^*(c)$ is continuous (differentiable) at c_0, then $y^*(c)$ is continuous (differentiable) at c_0.*

Proof. Suppose first that (y_n^*) is bounded. Then $\lim (T_{c_n}^* y_n^* - T_{c_0}^* y_n^*) = 0$ so $\lim T_{c_0}^* y_n^* = x_0^*$, and hence $\lim y_n^* = R x_0^*$, where R is the inverse of $T_{c_0}^*$. If $\lim \|y_n^*\| = \infty$, set $v_n^* = y_n^*/\|y_n^*\|$. Then $\lim T_{c_n}^* v_n^* = 0$, so by the first case $\lim v_n^* = 0$, which is a contradiction.

With the help of the last theorem we see that a necessary and sufficient condition for c_0 to be a proper value of K is that there exist sequences (c_n) and (y_n^*) such that:

(a) $\lim c_n = c_0$,

(b) $\|y_n^*\| = 1$,

(c) $\lim T_{c_n}^* y_n^* = 0$.

This suggests the following definitions:

(I) $c = \infty$ is a *singular value* for K in case there exist sequences (c_n) and (y_n^*) such that (a′), (b), and (c) hold, where (a′) means: $\lim c_n = \infty$.

(II) c_0 is a *proper value of order* μ for K in case μ is the maximum order of vanishing of $T_c^* y^*(c)$, where $y^*(c)$ is a polynomial in c, not vanishing at c_0.

THEOREM 6. *If $c = 1$ is a proper value for K, it is a proper value of order ν, where ν is the integer given by Theorem* 2.

Proof. It is readily verified by induction that K^* (as well as E^*) maps $\mathfrak{N}_k^*$ onto $\mathfrak{M}_k^*$ one-to-one. Denoting by D the operation of differentiation with respect to c, we have

$$D^k T_c^* y^*(c) = T_c^* D^k y^*(c) - k K^* D^{k-1} y^*(c).$$

Hence if $c = 1$ is a proper value of order μ, there is a polynomial $y^*(c)$ such that $0 \neq y^*(1) \in \mathfrak{N}_1^*$, and by induction $D^k y^*(1) \in \mathfrak{N}_{k+1}^* - \mathfrak{N}_k^*$ for $k < \mu$. Hence $\mu \leqq \nu$. To show that $\mu = \nu$, we set $y^*(c) = \sum_{j=1}^{\nu} (c-1)^{j-1} \eta_j^*$, where $\eta_\nu^* \in \mathfrak{N}_\nu^* - \mathfrak{N}_{\nu-1}^*$ and $T_1^* \eta_j^* = K^* \eta_{j-1}^*$, so that no $\eta_j^* = 0$, and $T_c^* y^*(c) = -(c-1)^\nu K^* \eta_\nu^*$.

We next consider a decomposition of K relative to a proper value, which we take for convenience to be $c = 1$. In view of the definition of the spaces $\mathfrak{N}_i^*$, there exists a basis $[\theta_{ij}^* | j = 1, \cdots, p_i; i = 1, \cdots, \nu]$ for $\mathfrak{N}_\nu^*$ such that $[\theta_{ij}^* | j = 1, \cdots, p_i; i = 1, \cdots, k]$ is a basis for $\mathfrak{N}_k^*$, and moreover

$$(2) \quad (E^* - K^*)\theta_{i+1,j}^* = E^* \theta_{ij}^* \quad \text{for} \quad j = 1, \cdots, p_{i+1}, \; i = 1, \cdots, \nu - 1.$$

Note that when the basis $[\theta_{ij}^*]$ is displayed by columns, the index i ranges from 1 to q_j, and j ranges from 1 to p_1. Choose η_{ij} from $\mathfrak{Y}$ so that

$$\theta^*_{ij}(\eta_{kl}) = \delta_{ik}\delta_{jl}, \tag{3}$$

and set

$$Q_j(y) = \sum_{i=1}^{q_j} \theta^*_{ij}(y)\eta_{ij}, \qquad Q = \sum_{j=1}^{p_1} Q_j,$$

$$Q^*_j(y^*) = \sum_{i=1}^{q_j} y^*(\eta_{ij})\theta^*_{ij},$$

$$G_j = Q_j K, \qquad G = \sum_{j=1}^{p_1} G_j, \qquad H = K - G. \tag{4}$$

Then clearly Q^* projects $\mathfrak{Y}^*$ onto $\mathfrak{N}_\nu^*$, and Q_j^* projects $\mathfrak{Y}^*$ onto the subspace of $\mathfrak{N}_\nu^*$ corresponding to a single proper vector θ^*_{1j} by means of the relations (2).

THEOREM 7. *The decomposition of K given in* (4) *has the following properties*:

(a) *each G_j has only one independent proper vector θ^*_{1j}, which corresponds to the proper value $c=1$*;

(b) *G has only the proper value $c=1$, and the null-spaces $\mathfrak{N}_k^*$ for G are the same as those for K*;

(c) *$c=\infty$ is not a singular value for G*;

(d) *$c=1$ is not a proper value for H*;

(e) *$G_l^* y^*=0$ for every y^* satisfying $E^*y^*=G_j^*y_1^*$ for some y_1^* and some $j\neq l$*;

(f) *$H^*y^*=0$ for every y^* satisfying $E^*y^*=G^*y_1^*$ for some y_1^**;

(g) *if c is a proper value for K and $c\neq 1$, then c is a proper value for H, and conversely.*

Proof. If $(E^*-cG_j^*)y^*=0$, it follows from (2) and the fact that E^* is one-to-one that y^* is a linear combination of $\theta^*_{1j}, \cdots, \theta^*_{q_j,j}$, and by another application of (2) that $c=1$ and y^* is a multiple of θ^*_{1j}. This proves (a). To prove (b), we observe that Q^* projects $\mathfrak{Y}^*$ onto $\mathfrak{N}_\nu^*$, and from (2) that K^* maps $\mathfrak{N}_\nu^*$ onto $\mathfrak{M}_\nu^*$ one-to-one, so G^* maps $\mathfrak{Y}^*$ onto $\mathfrak{M}_\nu^*$, and $G^*=K^*$ on $\mathfrak{N}_\nu^*$. Thus if $(E^*-cG^*)y^*=0$, y^* is in $\mathfrak{N}_\nu^*$, and $(c-1)E^*y^*=c(E^*-G^*)y^*=c(E^*-K^*)y^*$, so if $c\neq 1$, y^* is in $\mathfrak{N}_{\nu-1}^*$. Proceeding by descent, we finally find $y^*=0$. The remainder of (b) follows readily. If we suppose that $c=\infty$ is a singular value for G, there must exist sequences (c_n) and (y_n^*) with $\lim c_n=\infty$, $\|y_n^*\|=1$, $\lim (E^*-c_nG^*)y_n^*=0$. Since the sequence $(E^*y_n^*)$ is bounded, $(c_nG^*y_n^*)$ is likewise bounded and lies in the finite-dimensional space $\mathfrak{M}_\nu^*$, so we may suppose that it converges to a point x^* in $\mathfrak{M}_\nu^*$. Then $x^*=E^*y^*$ where y^* is in $\mathfrak{N}_\nu^*$, and since E^* has a continuous inverse, $\lim y_n^*=y^*$, so $\|y^*\|=1$. But $\lim G^*y_n^*=0=G^*y^*$, and since G^* gives a one-to-one correspondence between $\mathfrak{N}_\nu^*$ and $\mathfrak{M}_\nu^*$, we have $y^*=0$, which is a contradiction. This proves (c). To secure (d) we observe that if $0=(E^*-H^*)y^*=(E^*-K^*)y^*+G^*y^*$, then y^* is in $\mathfrak{N}_\nu^*$ since G^*y^* is in $\mathfrak{M}_\nu^*$, so $H^*y^*=0$, $E^*y^*=0$,

$y^*=0$. We next prove (e) and (f). If $E^*y^*=G_j^*y_1^*$, we have from (2) (setting $\theta_{0j}^*=0$)

$$y^* = \sum_{i=1}^{q_j} y_1^*(\eta_{ij})(\theta_{ij}^* - \theta_{i-1,j}^*), \tag{5}$$

since E^* is one-to-one, and

$$G_j^* y_1^* = \sum_{i=1}^{q_j} y_1^*(\eta_{ij}) K^* \theta_{ij}^*. \tag{6}$$

So by (5) and (6) and (3),

$$G^* y^* = \sum_{i=1}^{q_j} \sum_{m=1}^{q_l} y_1^*(\eta_{ij}) [\theta_{ij}^*(\eta_{ml}) - \theta_{i-1,j}^*(\eta_{ml})] K^* \theta_{ml}^* = \delta_{jl} K^* y^*,$$

and hence $H^*y^*=0$. Property (f) follows immediately. Finally, to prove (g), we observe that if $c\neq 1$, $y^*\neq 0$, and $(E^*-cK^*)y^*=0$, $G^*y^*=E^*y_1^*$, where y_1^* is in $\mathfrak{N}_\nu^*$, and so $H^*y_1^*=0$, by (f). Hence

$$\begin{aligned} 0 &= E^*y^* - cG^*y^* - cH^*y^* = E^*y^* - cE^*y_1^* - cH^*y^* + c^2H^*y_1^* \\ &= (E^* - cH^*)(y^* - cy_1^*). \end{aligned}$$

Now if $y^*=cy_1^*$ we would have $0=(E^*-cK^*)y_1^*=(E^*-cG^*)y_1^*$, and this cannot happen for $c\neq 1$ by part (b) of the theorem. So c is a proper value of H. For the converse, let $(E^*-cH^*)y^*=0$ with $y^*\neq 0$. Then y^* is not in $\mathfrak{N}_\nu^*$, and there is a finite sequence $y_1^*, \cdots, y_k^*$, such that $G^*y^*=E^*y_1^*$, $(E^*-K^*)y_i^*=E^*y_{i+1}^*$ for $i=1, \cdots, k-1$, each y_i^* is in $\mathfrak{N}_\nu^*$, y_k^* is in $\mathfrak{N}_1^*$, and so

$$(E^* - cK^*)\left(y^* + \sum_{i=1}^{k} \alpha_i y_i^*\right) = E^*\left[- cy_1^* + (1 - c) \sum_{i=1}^{k} \alpha_i y_i^* + c \sum_{i=1}^{k-1} \alpha_i y_{i+1}^*\right].$$

The expression in square brackets vanishes if $(1-c)\alpha_1=c$, $(1-c)\alpha_i=-c\alpha_{i-1}$ for $i=2, \cdots, k$.

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