A GENERALIZATION OF THE RIESZ THEORY OF COMPLETELY CONTINUOUS TRANSFORMATIONS

BY

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The classical theory, due to F. Riesz, of completely continuous transformations deals with a family $T_c = I - cK$ of linear continuous transformations of a Banach space $\mathcal{X}$ into itself, where $I$ is the identity and $K$ is completely continuous. The transformation $T_c$ is then one-to-one, except for the "proper" values of $c$. In this paper we consider cases where the one-to-one-ness no longer holds. The identity transformation $I$ is replaced by a transformation $E$ which maps $\mathcal{X}$ onto the whole of another Banach space $\mathcal{Y}$, but not one-to-one. Then when $K$ is completely continuous, the transformation $T_c = E - cK$ maps $\mathcal{X}$ onto $\mathcal{Y}$ except for a countable set of proper values $c_i$ which have no finite accumulation point. When $\mathcal{W}_1 = T_c \mathcal{X} \neq \mathcal{Y}$, $c$ is said to be a proper value for $K$. We can no longer iterate the transformation $T_c$, but we can apply $T_c$ to $E^{-1} T_c \mathcal{X}$ and obtain a linear closed space $\mathcal{W}_2 \subset \mathcal{W}_1$. After a finite number of repetitions of this process, we find a space $\mathcal{W}_n = \mathcal{W}_{n+1}$. Then $T_c$ transforms $E^{-1} \mathcal{W}_n$ onto $\mathcal{W}_{n}$, which takes the place of the invariant subspace of the Riesz theory.

In the one-to-one case, there is associated with the transformation $I - cK$ a resolvent $R_c$, which has a pole of order $v$ at a proper value $c_0$, and the Laurent expansion of $R_c$ about $c_0$ leads to a decomposition of $K$ into two mutually orthogonal parts. In the many-to-one case, the adjoint transformation $T_c^*$ is one-to-one except when $c$ is a proper value, but the range of $T_c^*$ varies with $c$, so that it is not possible to define the order of a pole of the resolvent in the usual way. However a substitute definition has been found, as well as a decomposition $K = G + H$, where $G$ has only one proper value $c_0$, and $H$ has only the remaining proper values of $K$, and $H$ is semi-orthogonal to $G$ in a suitable sense.

1. Notations and definitions. We shall let $\mathcal{X}$ and $\mathcal{Y}$ denote Banach spaces of infinitely many dimensions, with complex scalars. The adjoint spaces of $\mathcal{X}$ and $\mathcal{Y}$ will be denoted by $\mathcal{X}^*$ and $\mathcal{Y}^*$ respectively. Two subsets $\mathcal{X}_0$ of $\mathcal{X}$ and $\mathcal{X}_0^*$ of $\mathcal{X}^*$ are said to be orthogonal in case $x^* (x) = 0$ for all $x$ in $\mathcal{X}_0$ and $x^*$ in $\mathcal{X}_0^*$. The orthogonal complement of a subset $\mathcal{X}_0$, denoted by $O \mathcal{X}_0$, is the maximal set $\mathcal{X}_0^*$ orthogonal to $\mathcal{X}_0$. It is easily seen that $O \mathcal{X}_0$ always exists (possibly void) and is a linear closed set. A corresponding statement holds for $O \mathcal{X}_0^*$.

A linear transformation $A$ mapping $\mathcal{X}$ into $\mathcal{Y}$ has an adjoint $A^*$ mapping $\mathcal{Y}^*$ into $\mathcal{X}^*$, defined by $x^* = A^* y^*$ where $x^* (x) = y^* A x$ for each $x$. We shall

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deal only with continuous linear transformations $A$. Such a transformation $A$ is said to be completely continuous in case the transform $A(\mathcal{X}_0)$ of every bounded set $\mathcal{X}_0$ is compact (in the sense that every sequence $(y_n)$ chosen from $A(\mathcal{X}_0)$ has a convergent subsequence).

We shall use the letter $E$ to denote a linear continuous transformation mapping $\mathcal{X}$ onto the whole space $\mathcal{Y}$, and shall use $K, G,$ and $H$ to denote completely continuous linear transformations of $\mathcal{X}$ into $\mathcal{Y}$. We also set $T_c = E - cK$, where $c$ is a complex parameter. When it is convenient to set $c = 1$, we shall write $T$ for $T_c$.

A projection $P$ is a linear continuous transformation of $\mathcal{X}$ onto a closed linear subset $\mathcal{X}_0$, which equals the identity on $\mathcal{X}_0$, i.e., $P^2 = P$.

For a given linear continuous transformation $A$ of $\mathcal{X}$ into $\mathcal{Y}$ we shall set

$$\beta(A, y) = \text{g.l.b.} \left\{ \|x\| \mid Ax = y \right\},$$

for each $y$, and

$$I(A) = \text{l.u.b.} \left\{ \beta(A, y) \mid y \in A\mathcal{X}, \|y\| = 1 \right\}.$$

When the transformation $A$ is one-to-one, obviously $I(A) = \|A^{-1}\|$. (We are admitting $+\infty$ as a value for $\|A^{-1}\|$.)

We shall use $\Re(A)$ to denote the null space of $A$, that is

$$\Re(A) = \{ x \mid Ax = 0 \}.$$

Similarly $\Re(A^*)$ denotes the null space of $A^*$.

2. Preliminary lemmas.

**Lemma 1.** If $\mathcal{X}_0$ is a finite-dimensional linear subspace of $\mathcal{X}$, with basis \{ $x_1, \cdots, x_n$ \}, there exist $n$ elements $x_i^*$ of $\mathcal{X}^*$ such that $x_i^*(x_j) = \delta_{ij}$, and $P(x) = \sum_{i=1}^{n} x_i^*(x)x_i$ is a projection of $\mathcal{X}$ onto $\mathcal{X}_0$. A similar result holds for $\mathcal{X}_0^* \subset \mathcal{X}^*$, with $Q(x^*) = \sum_{i=1}^{n} x^*_i(x_i)x_i^*$.

The first part follows at once from the Hahn-Banach theorem, and the second part follows from more elementary considerations. Note that if the sets \{ $x_1, \cdots, x_n$ \} and \{ $x_1^*, \cdots, x_n^*$ \} are the same in the two cases, then $Q = P^*$.

**Lemma 2.** For each linear closed subspace $\mathcal{X}_0$ of $\mathcal{X}$, $O\mathcal{X}_0 = \mathcal{X}_0$.

This follows from the lemma in Banach [1, p. 57]. The corresponding result for subsets $\mathcal{X}_0^*$ of $\mathcal{X}^*$ does not hold in general, as Banach shows in [1, p. 115]. However, by an easy proof we have

**Lemma 3.** If $\mathcal{X}_0^*$ is a finite-dimensional linear subspace of $\mathcal{X}^*$, then $O\mathcal{X}_0^* = \mathcal{X}_0^*$.

**Lemma 4.** If \{ $x_1^*, \cdots, x_n^*$ \} is a basis for $\mathcal{X}_0^*$, $x_i^*(x_j) = \delta_{ij}$, and
$P(x) = \sum_{i=1}^{n} x_i^*(x)x_i$, then $\mathcal{H}$ is the direct sum of $O\mathcal{H}_0^*$ and the finite-dimensional space $P\mathcal{H}$.

To prove this, we observe that $x - P(x)$ is always in $O\mathcal{H}_0^*$, and that if $x$ lies in $O\mathcal{H}_0^*$, then $Px = 0$.

**Lemma 5.** A linear transformation $A$ of $\mathcal{H}$ into $\mathcal{Y}$ is (completely) continuous if and only if its adjoint $A^*$ is (completely) continuous.

**Proof.** This follows from Theorems 3 and 4 in Banach [1, p. 100]. To show that $A$ is completely continuous whenever $A^*$ is, we note that $A^{**}$, when restricted to $\mathcal{H}$, reduces to $A$, so by Theorem 4 of Banach, $A$ transforms bounded sets in $\mathcal{H}$ into sets in $\mathcal{Y}$ which are compact in $\mathcal{Y}^{**}$. But $\mathcal{Y}$ is closed in $\mathcal{Y}^{**}$.

**Lemma 6.** The range $A\mathcal{H}$ of a linear continuous transformation $A$ of $\mathcal{H}$ into $\mathcal{Y}$ is closed if and only if $I(A) \leq \infty$.

**Proof.** When $A\mathcal{H}$ is closed it constitutes a Banach space, so the necessity of the condition follows from Banach [1, p. 38]. To see that the condition is sufficient, suppose $y_n = Ax_n$, and $\lim y_n = y$. By choosing a subsequence we may suppose that $\|y_{n+1} - y_n\| < 1/2^n$. Then after $x_n$ has been suitably chosen, $x_{n+1}$ may be modified (still satisfying $y_{n+1} = Ax_{n+1}$) so that $\|x_{n+1} - x_n\| \leq I(A)/2^n$. Hence the modified sequence $(x_n)$ has a limit $x$, and $Ax_n$ converges to $Ax = y$.

**Corollary.** If $A\mathcal{H}$ is closed, and $\lim Ax_n = Ax$, there exists a sequence $(x'_n)$ such that $Ax'_n = Ax_n$ and $\lim x'_n = x$.

**Proof.** From the definition of $I(A)$ we may choose $x'_n$ so that $A(x'_n - x) = A(x_n - x)$ and $\|x'_n - x\| \leq (I(A) + 1)\|A(x_n - x)\|$. (Compare Banach [1, p. 40, Theorem 4].)

**Lemma 7.** If $A$ is linear and continuous, $A\mathcal{H}$ is closed if and only if $A^*\mathcal{Y}^*$ is closed, and then $A^*\mathcal{Y}^* = OR(A)$, $A\mathcal{H} = OR(A^*)$, where $R(A)$ and $R(A^*)$ are the null spaces of $A$ and of $A^*$.

For the proof, see Banach [1, pp. 149, 150, Theorems 8 and 9].

**Lemma 8.** If $A$ is linear and continuous, then $A$ maps $\mathcal{H}$ onto the whole of the space $\mathcal{Y}$ if and only if $A^*$ has a continuous inverse. Likewise $A$ has a continuous inverse if and only if $A^*$ maps $\mathcal{Y}^*$ onto $\mathcal{H}^*$.

For the proof, see Banach [1, pp. 146–148, Theorems 1–4].

**Lemma 9.** If $\mathcal{H}_1$ and $\mathcal{H}_2$ are two closed linear subspaces of $\mathcal{H}$, $\mathcal{H}_2$ being finite-dimensional, then $\mathcal{H}_1 + \mathcal{H}_2$ (the linear space spanned by $\mathcal{H}_1$ and $\mathcal{H}_2$) is also closed.

This is readily proved by induction on the dimension of $\mathcal{H}_2$. 


Lemma 10. Suppose $x_1$ and $x_2$ satisfy the conditions of Lemma 9, and also that $x_1 + x_2 = x$. Let $E$ be a linear continuous transformation of $x$ onto the whole of $\mathcal{Y}$. Then $E(x_1)$ is closed.

Proof. The space $x_3 = E^{-1}E(x_1)$ equals $x_1 + x_4$ where $x_4 \subseteq x_2$, so $x_3$ is closed by Lemma 9. Let $E_3$ denote the restriction of $E$ to $x_3$. Then clearly $I(E_3) \leq I(E)$. By Lemma 6, $I(E) < \infty$, and by the same Lemma 6, $E_3x_3 = E(x_1)$ is closed.

3. Proper values of a completely continuous transformation. In this section we develop some properties of the family of transformations $T_c = E - cK$, described in §1. A complex number $c$ is called a proper value of $K$ in case the range of $T_c$ is a proper subset of $\mathcal{Y}$, or equivalently, in case $T_c$ does not have an inverse.

In case $c = 1$ is a proper value, the null space $\mathcal{N}_1* = \mathcal{N}(T_1*)$ is not null, and the range $\mathcal{W}_1 = TX \neq \mathcal{Y}$. We set $\mathcal{M}_1* = E*\mathcal{N}_1*$, and define $\mathcal{Z}_1$ as the maximal subspace of $x$ satisfying $E_1\mathcal{Z}_1 = \mathcal{W}_1$. This is the start of an iterative definition of four sequences of linear spaces $\mathcal{N}_*, \mathcal{M}_*, \mathcal{W}_*, \mathcal{Z}_*$, by means of the relations

\[
T^*\mathcal{N}_k* \subset \mathcal{M}_{k-1}^* \quad \text{(maximal)},
\]

\[
\mathcal{M}_k* = E^*\mathcal{N}_k*,
\]

\[
\mathcal{W}_k = T\mathcal{Z}_{k-1},
\]

\[
E\mathcal{Z}_k = \mathcal{W}_k \quad \text{(maximal)}.
\]

It is obvious that

\[
\mathcal{N}_k* \supset \mathcal{N}_{k-1}*, \quad \mathcal{M}_k* \supset \mathcal{M}_{k-1}*,
\]

\[
\mathcal{W}_k \subset \mathcal{W}_{k-1}, \quad \mathcal{Z}_k \subset \mathcal{Z}_{k-1}.
\]

Theorem 1. The spaces $\mathcal{N}_k*$ and $\mathcal{M}_k*$ are all finite-dimensional, and $\mathcal{W}_k$ and $\mathcal{Z}_k$ are closed, and $\mathcal{W}_k = OM_k$, $\mathcal{Z}_k = OM_k$.

Proof. Let $\mathcal{B}$ be a bounded subset of $\mathcal{N}_*$. Then $E^*\mathcal{B} = K^*\mathcal{B}$ is compact, since $K^*$ is completely continuous. Since $E^*$ has a bounded inverse, $\mathcal{B}$ is compact. Thus $\mathcal{N}_*^*$ is finite-dimensional, since every bounded subset is compact. (See Banach [1, p. 84, Theorem 8].) If $\mathcal{M}_{k-1}*$ is finite-dimensional, then $\mathcal{N}_k*$ is also. To show that $\mathcal{W}_1 = T\mathcal{X}$ is closed, we shall show that $T^*\mathcal{Y}$ is closed, and apply Lemma 7. Since $\mathcal{N}_1*$ is finite-dimensional, there is a projection $P$ of $\mathcal{Y}$ onto $\mathcal{N}_*$. If $T^*y_0*$ tends to $x_0*$, we may assume $Py_0* = 0$. If the sequence $(y_0*)$ is bounded, we may (by choice of a subsequence) require that $K^*y_0*$ converges, say to $x_1*$. Then $E^*y_0*$ tends to $x_0* + x_1*$ which must equal $E^*y_0*$ for some $y_0*$, since the range of $E^*$ is closed by Lemma 7. Then by Lemma 8, $y_0*$ tends to $y_0*$, so $K^*y_0* = x_1*$, and $T^*y_0* = x_0*$. The sequence $(y_0*)$ must be bounded, since if $\|y_0*\|$ tends to infinity, the sequence $y_{in}^* = y_0*/\|y_0*\|$ satisfies the preceding conditions with $x_0* = 0$. Hence we would have $y_{in}$ approaching $y_0*$ with $\|y_0*\| = 1$, $P(y_0*) = 0$, $T^*y_0* = 0$, and so $y_0* = P(y_0*)$, which is a con-
contradiction. Thus $T^*y^*$ is closed, so $\mathcal{B}_1 = T\mathcal{X}$ is closed, and $\mathcal{B}_1 = E^{-1}\mathcal{B}_1$ is also closed. Lemma 7 yields the further information that $\mathcal{B}_1 = O\mathcal{M}_1^*$, and it follows readily that $\mathcal{B}_1 = O\mathcal{M}_1^*$.

Now suppose that $\mathcal{B}_k = O\mathcal{M}_k^*$. Then $\mathcal{B}_k$ is closed, and since $\mathcal{M}_k^*$ is finite-dimensional, $\mathcal{X}$ is the direct sum of $\mathcal{B}_k$ and a finite dimensional space, by Lemma 4. Now $T$ maps $\mathcal{X}$ onto the closed space $\mathcal{B}_1$, so $\mathcal{B}_k = T\mathcal{B}_k$ is closed, by Lemma 10 with $\mathcal{X}_1$ replaced by $\mathcal{B}_k$, $\mathcal{E}$ by $T$, and $\mathcal{Y}$ by $\mathcal{B}_1$. Then $\mathcal{B}_{k+1} = E^{-1}\mathcal{B}_{k+1}$ is also closed and by a sequence of steps the relations $\mathcal{N}_{k+1} = O\mathcal{B}_{k+1}$, $\mathcal{B}_{k+1} = O\mathcal{N}_{k+1}$, $\mathcal{B}_{k+1} = O\mathcal{M}_{k+1}$ are readily obtained. This completes the induction.

**Theorem 2.** There exists a least integer $v$ such that $\mathcal{B}_{v+1} = \mathcal{B}_v$, and then $\mathcal{B}_{k+v} = \mathcal{B}_v$ for all $k$.

**Proof.** From the preceding theorem and the fact that $\mathcal{N}_k^*$ and $\mathcal{M}_k^*$ are in one-to-one correspondence, it is clear that $\mathcal{B}_{v+1} = \mathcal{B}_v$, $\mathcal{B}_{v+1} = \mathcal{B}_v$, $\mathcal{N}_{v+1} = \mathcal{N}_v^*$, $\mathcal{M}_{v+1} = \mathcal{M}_v^*$ all happen for the same index $v$. We make the proof by considering the nondecreasing sequence of finite-dimensional spaces $\mathcal{M}_k^*$. If $\mathcal{M}_{k+1} \neq \mathcal{M}_k^*$ for every $k$, there exists a unit vector $x_{k+1} \in \mathcal{M}_{k+1}^*$ whose distance from $\mathcal{M}_k^*$ is unity. For each $k$ there is a point $y_k^* \in \mathcal{M}_k^*$ with $E^*y_k^* = x_k^*$, $\|y_k^*\| \leq I(E^*)$. Then when $m > k$, $x_k^* - x_k^* = E^*y_m^* - E^*y_k^* = T^*y_m^* - T^*y_k^* + K^*y_m^* - K^*y_k^*$, and $T^*y_m^* \in \mathcal{M}_{k+1}$, $T^*y_k^* \in \mathcal{M}_{k+1} \subset \mathcal{M}_{m-1}$, so $\|K^*y_m^* - K^*y_k^*\| \geq 1$ while $\|y_k^*\|$ is bounded, which contradicts the complete continuity of $K^*$.

We note that the space $\mathcal{B}_v$ is mapped onto $\mathcal{B}_v$ by both $E$ and $T$. This pair of subspaces replaces the invariant subspace of the classical theory of Riesz.

We remark also that if we consider the sequence of spaces defined by the relations:

$$
\mathcal{N}_1 = \mathcal{N}(T), \quad \mathcal{M}_k = EN_k,
$$

$$
T\mathcal{M}_{k+1} \subset \mathcal{M}_k, \quad (\mathcal{M}_{k+1} \text{ maximal}),
$$

we may have $\mathcal{N}_{k+1} \neq \mathcal{N}_k$ for every $k$. For example, let $\mathcal{X}$ and $\mathcal{Y}$ be classical Hilbert space, and let the transformation $y = Ex$ be defined by $y_1 = x_1 + x_2$, $y_2 = x_3 - x_2$, $y_{i+2} = x_{i+4}/(i+4)$.

Then a basis for $\mathcal{N}_k$ is composed of $\delta_1, \delta_3, \delta_4, \alpha_1, \ldots, \alpha_k$, where $\delta_i$ has only its $i$th component different from zero, $\alpha_i$ has its $i$th component equal to $1/(i-1)!$ except the first four, which are zero, and $E\alpha_j = T\alpha_{j+1}$.

**Theorem 3.** Let $(c_j)$ be a sequence (finite or infinite) of distinct proper values of $K$. Let $\mathcal{N}^{(i)}$ denote the null space of $T^{*j}$, and let $\mathcal{X}^{(i)} = T_{c_i}\mathcal{X}$. Let $\{y^*_i, i=1, \ldots, k_j\}$ be a basis for $\mathcal{N}^{(j)}$. Then the functionals $y^*_i$ are finitely linearly independent, and there exists a system $\{y_{ij}\}$ of elements of $\mathcal{Y}$ such that:

(a) for each $j$, $\mathcal{W}^{(j)}$ and $\{y_{ij}, i=1, \ldots, k_j\}$ span the space $\mathcal{Y}$;
(b) \( y_{kl}(y_{kj}) = \delta_{ik}\delta_{lj} \) for \( l \leq j \);
(c) \( y_{kj} \) lies in \( \mathcal{B}^{(1)} \) for \( l < j \);
(d) the elements \( y_{ij} \) are finitely linearly independent.

**Proof.** Suppose
\[
\sum_{j=1}^{n} \sum_{i=1}^{k_j} a_{ij} y_{ij}^* = 0.
\]
Then
\[
0 = \sum_{j=1}^{n-1} \sum_{i} (E - c_n K^*) a_{ij} y_{ij}^* = E^* \sum_{j=1}^{n-1} \left(1 - \frac{c_n}{c_j}\right) \sum_{i} a_{ij} y_{ij}^*,
\]
and since \( E^* \) is one-to-one, this gives a relation involving only \( n-1 \) of the spaces \( \mathfrak{M}^{(i)} \). Hence by descent we arrive at a contradiction. By Theorem 1 for \( k = 1 \), and Lemmas 1 and 4 we obtain (a) and (b) for \( j = 1, l = 1 \). Suppose that the system \([y_{ij}]\) has been determined so as to satisfy (a), (b), and (c) for \( j < n \). Then the intersection \( \mathfrak{B}^{(n-1)} \) of \( \mathfrak{B}^{(1)}, \ldots, \mathfrak{B}^{(n-1)} \) is the orthogonal complement of the union of \( \mathfrak{M}^{(1)}, \ldots, \mathfrak{M}^{(n-1)} \) and since the \( y_{ij}^* \) are linearly independent, there are points \( y_{in} \) in \( \mathfrak{B}^{(n-1)} \) such that (b) holds for \( l = j = n \).

Property (a) holds for \( j = n \), by the same argument as before. Property (d) follows at once from (b).

**Theorem 4.** The proper values of \( K \) have no finite accumulation point.

**Proof.** Suppose \((c_n)\) is a bounded infinite sequence of distinct proper values. Let \( \mathfrak{S}^*_n \) be the space spanned by \( \mathfrak{M}^{(1)}, \ldots, \mathfrak{M}^{(n)} \). Then \((\mathfrak{S}^*_n)\) is a properly increasing sequence of finite-dimensional linear spaces, so each \( \mathfrak{S}^*_n \) contains a unit vector \( v_n^* \) at unit distance from \( \mathfrak{S}^*_{n-1} \). The sequence \((c_n v_n^*)\) is bounded, so \((c_n K^* v_n^*)\) is compact. If \( y^* \) is in \( \mathfrak{M}^{(i)} \), \( T_{c_p} y^* = E^* (1-c/c_j) y^* \), which is zero if \( c = c_j \), so
\[
T_{c_p} \mathfrak{S}^*_p \subset E^* \mathfrak{S}^*_p-1.
\]
Now for \( p > q \), let
\[
d_{pq} = \|c_p K^* v_p^* - c_q K^* v_q^*\| = \|E^* v_p^* - T_{c_p} v_p^* - E^* v_q^* + T_{c_q} v_q^*\|,
\]
so by (1)
\[
d_{pq} = \|E^* v_p^* - E^* u^*\|, \text{ where } u^* \text{ is in } \mathfrak{S}^*_{p-1} \text{ and } \|v_p^* - u^*\| = 1.
\]
Since the inverse of \( E^* \) is bounded, by Lemma 8, this contradicts the compactness of \((c_n K^* v_n^*)\).
Since the range of $T^*_{c}$ may vary with $c$, we state properties of the inverse of $T^*_{c}$ as follows.

**Theorem 5.** If $c_0$ is not a proper value of $K$, and $(c_n)$ and $(y_n^*)$ are sequences such that $\lim c_n = c_0$, then $y_n^*$ tends to the unique solution $y_0^*$ of $T_{c_0}^*y^* = x_0^*$. Consequently if $T_{c_0}^*y^*(c) = x^*(c)$, and $x^*(c)$ is continuous (differentiable) at $c_0$, then $y^*(c)$ is continuous (differentiable) at $c_0$.

**Proof.** Suppose first that $(y_n^*)$ is bounded. Then $\lim (T_{c_n}^*y_n^* - T_{c_0}^*y_n^*) = 0$ so $\lim T_{c_n}^*y_n^* = x_0^*$, and hence $\lim y_n^* = R x_0^*$, where $R$ is the inverse of $T_{c_0}^*$. If $\lim \|y_n^*\| = \infty$, set $v_n^* = y_n^*/\|y_n^*\|$. Then $\lim T_{c_n}^*v_n^* = 0$, so by the first case $\lim v_n^* = 0$, which is a contradiction.

With the help of the last theorem we see that a necessary and sufficient condition for $c_0$ to be a proper value of $K$ is that there exist sequences $(c_n)$ and $(y_n^*)$ such that:

1. $\lim c_n = c_0$,
2. $\|y_n^*\| = 1$,
3. $\lim T_{c_n}^*y_n^* = 0$.

This suggests the following definitions:

1. $c = \infty$ is a singular value for $K$ in case there exist sequences $(c_n)$ and $(y_n^*)$ such that $(a')$, $(b)$, and $(c)$ hold, where $(a')$ means $\lim c_n = \infty$.

2. $c_0$ is a proper value of order $\mu$ for $K$ in case $\mu$ is the maximum order of vanishing of $T_{c_0}^*y^*(c)$, where $y^*(c)$ is a polynomial in $c$, not vanishing at $c_0$.

**Theorem 6.** If $c = 1$ is a proper value for $K$, it is a proper value of order $\nu$, where $\nu$ is the integer given by Theorem 2.

**Proof.** It is readily verified by induction that $K^*$ (as well as $E^*$) maps $\mathbb{R}_x^*$ onto $\mathbb{R}_x^*$ one-to-one. Denoting by $D$ the operation of differentiation with respect to $c$, we have

$$D^kT^*_cy^*(c) = T^*_cD^ky^*(c) - kK^*D^{k-1}y^*(c).$$

Hence if $c = 1$ is a proper value of order $\mu$, there is a polynomial $y^*(c)$ such that $0 \neq y^*(1) \in \mathbb{R}_x^*$, and by induction $D^k y^*(1) \in \mathbb{N}_{x_{k+1}} - \mathbb{R}_x^*$ for $k < \mu$. Hence $\mu \leq \nu$. To show that $\mu = \nu$, we set $y^*(c) = \sum_{i=1}^{\nu} (c-1)^{i-1} \eta_i^*$, where $\eta_i^* \in \mathbb{N}_x^* - \mathbb{R}_x^*$ and $T_i^*\eta_i^* = K^*\eta_{i-1}^*$, so that no $\eta_i^* = 0$, and $T_{c_0}^*y^*(c) = -(c-1)K^*\eta_1^*$.

We next consider a decomposition of $K$ relative to a proper value, which we take for convenience to be $c = 1$. In view of the definition of the spaces $\mathbb{N}_x^*$, there exists a basis $[\theta_{ij}^*|j = 1, \ldots, \rho_i; i = 1, \ldots, \nu]$ for $\mathbb{N}_x^*$ such that $[\theta_{ij}^*|j = 1, \ldots, \rho_i; i = 1, \ldots, k]$ is a basis for $\mathbb{N}_x^*$, and moreover

$$\begin{align*}
(1) & \quad (E^* - K^*)\theta^*_{i+1, j} = E^*\theta^*_{ij} \quad \text{for} \quad j = 1, \ldots, \rho_{i+1}, \quad i = 1, \ldots, \nu - 1.
\end{align*}$$

Note that when the basis $[\theta_{ij}^*]$ is displayed by columns, the index $i$ ranges from 1 to $q$, and $j$ ranges from 1 to $p_i$. Choose $\eta_{ij}$ from $\mathbb{N}_x$ so that

$$\begin{align*}
(2) & \quad (E^* - K^*)\theta^*_{i+1, j} = E^*\theta^*_{ij} \quad \text{for} \quad j = 1, \ldots, \rho_i, \quad i = 1, \ldots, \nu - 1.
\end{align*}$$
\[ \theta_{ij}(\eta_{kl}) = \delta_{ik}\delta_{jl}, \]

and set

\[ Q_j(y) = \sum_{i=1}^{q_j} \theta_{ij}^*(y)\eta_{ij}, \quad Q = \sum_{j=1}^{p_1} Q_j, \]

\[ Q_j^*(y^*) = \sum_{i=1}^{q_j} y^*(\eta_{ij})\theta_{ij}^*, \]

\[ G_j = Q_jK, \quad G = \sum_{j=1}^{p_1} G_j, \quad H = K - G. \]

Then clearly \( Q^* \) projects \( \mathcal{Y}^* \) onto \( \mathcal{M}^* \), and \( Q_j^* \) projects \( \mathcal{Y}^* \) onto the subspace of \( \mathcal{M}_j^* \) corresponding to a single proper vector \( \theta_{ij}^* \) by means of the relations (2).

**Theorem 7.** The decomposition of \( K \) given in (4) has the following properties:

(a) each \( G_j \) has only one independent proper vector \( \theta_{ij}^* \), which corresponds to the proper value \( c = 1 \);

(b) \( G \) has only the proper value \( c = 1 \), and the null-spaces \( \mathcal{M}_j^* \) for \( G \) are the same as those for \( K \);

(c) \( c = \infty \) is not a singular value for \( G \);

(d) \( c = 1 \) is not a proper value for \( H \);

(e) \( G_j^*y^* = 0 \) for every \( y^* \) satisfying \( E^*y^* = G_j^*y_l^* \) for some \( y_l^* \) and some \( j \neq l \);

(f) \( H^*y^* = 0 \) for every \( y^* \) satisfying \( E^*y^* = G^*y_l^* \) for some \( y_l^* \);

(g) if \( c \) is a proper value for \( K \) and \( c \neq 1 \), then \( c \) is a proper value for \( H \), and conversely.

**Proof.** If \( (E^*-cG_j^*)y^* = 0 \), it follows from (2) and the fact that \( E^* \) is one-to-one that \( y^* \) is a linear combination of \( \theta_{ij}^*, \cdots, \theta_{ij}^* \), and by another application of (2) that \( c = 1 \) and \( y^* \) is a multiple of \( \theta_{ij}^* \). This proves (a). To prove (b), we observe that \( Q^* \) projects \( \mathcal{Y}^* \) onto \( \mathcal{M}_j^* \), and from (2) that \( K^* \) maps \( \mathcal{M}_j^* \) onto \( \mathcal{M}_j^* \) one-to-one, so \( G^* \) maps \( \mathcal{Y}^* \) onto \( \mathcal{M}_j^* \), and \( G^* = K^* \) on \( \mathcal{M}_j^* \). Thus if \( (E^*-cG_j^*)y^* = 0 \), \( y^* \) is in \( \mathcal{M}_j^* \), and \( (c-1)E^*y^* = c(E^*-G_j^*)y^* = c(E^*-K^*)y^* \), so if \( c \neq 1 \), \( y^* \) is in \( \mathcal{M}_j^* \). Proceeding by descent, we finally find \( y^* = 0 \). The remainder of (b) follows readily. If we suppose that \( c = \infty \) is a singular value for \( G \), there must exist sequences \( (c_n) \) and \( (y_n^*) \) with \( \lim c_n = \infty \), \( \|y_n^*\| = 1 \), \( \lim (E^*-c_nG_j^*)y_n^* = 0 \). Since the sequence \( (E^*y_n^*) \) is bounded, \( (c_nG_j^*y_n^*) \) is likewise bounded and lies in the finite-dimensional space \( \mathcal{M}_j^* \), so we may suppose that it converges to a point \( x^* \) in \( \mathcal{M}_j^* \). Then \( x^* = E^*y^* \) where \( y^* \) is in \( \mathcal{M}_j^* \), and since \( E^* \) has a continuous inverse, \( \lim y_{n^*} = y^* \), so \( \|y^*\| = 1 \). But lim \( G^*y_{n^*} = 0 = G^*y^* \), and since \( G^* \) gives a one-to-one correspondence between \( \mathcal{M}_j^* \) and \( \mathcal{M}_j^* \), we have \( y^* = 0 \), which is a contradiction. This proves (c). To secure (d) we observe that if \( 0 = (E^*-H^*)y^* = (E^*-K^*)y^* + G^*y^* \), then \( y^* \) is in \( \mathcal{M}_j^* \) since \( G^*y^* \) is in \( \mathcal{M}_j^* \), so \( H^*y^* = 0 \), \( E^*y^* = 0 \),
$y^* = 0$. We next prove (e) and (f). If $E^*y^* = G_j^*y_j^*$, we have from (2) (setting $	heta_{0j}^* = 0$)

$$y^* = \sum_{i=1}^{q} y_i^*(\theta_{ij}^* - \theta_{i-1,j}^*),$$

since $E^*$ is one-to-one, and

$$G_j^*y_j = \sum_{i=1}^{q} y_i^*(\eta_{ij}^*)K_i^*\theta_{ij}^*.$$

So by (5) and (6) and (3),

$$G^*y^* = \sum_{i=1}^{q} \sum_{m=1}^{q} y_i^*(\eta_{ij}^*)(\theta_{i}^* - \theta_{i-1,j}^*)K_{ij}^* = \delta_{ji}K^*y^*,$$

and hence $H^*y^* = 0$. Property (f) follows immediately. Finally, to prove (g), we observe that if $c \neq 1$, $y^* \neq 0$, and $(E^* - cK^*)y^* = 0$, $G^*y^* = E^*y_1^*$, where $y_1^*$ is in $\mathfrak{N}_*$, and so $H^*y_1^* = 0$, by (f). Hence

$$0 = E^*y^* - cG^*y^* - cH^*y^* = E^*y^* - cE^*y_1^* - cH^*y^* + c^2H^*y_1^*$$

$$= (E^* - cH^*)(y^* - cy_1^*).$$

Now if $y^* = cy_1^*$ we would have $0 = (E^* - cK^*)y_1^* = (E^* - cG^*)y_1^*$, and this cannot happen for $c \neq 1$ by part (b) of the theorem. So $c$ is a proper value of $H$. For the converse, let $(E^* - cH^*)y^* = 0$ with $y^* \neq 0$. Then $y^*$ is not in $\mathfrak{N}_*$, and there is a finite sequence $y_1^*, \ldots, y_k^*$, such that $G^*y^* = E^*y_1^*$, $(E^* - K^*)y_i^* = E^*y_{i+1}^*$ for $i = 1, \ldots, k-1$, each $y_i^*$ is in $\mathfrak{N}_*$, and so

$$(E^* - cK^*)\left(y^* + \sum_{i=1}^{k} \alpha_i y_i^*\right) = E^* \left[-cy_1^* + (1 - c) \sum_{i=1}^{k} \alpha_i y_i^* + c \sum_{i=1}^{k-1} \alpha_i y_{i+1}^*\right].$$

The expression in square brackets vanishes if $(1-c)\alpha_1 = c$, $(1-c)\alpha_i = -c\alpha_{i-1}$ for $i = 2, \ldots, k$.

**Bibliography**


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