COMPOUND GROUP EXTENSIONS. III
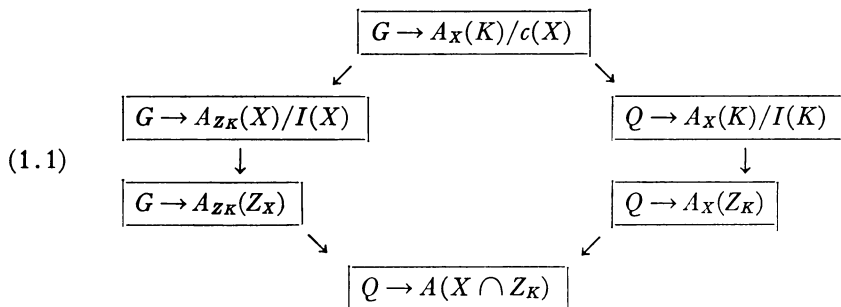
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1. Introduction[1]. Let $\phi\colon K\to G$ be a normal homomorphism, with $X=\phi^{-1}(0)$, $N=\phi(K)$, $Q=G/N$, as in [CGE I]. Any continuation $\bar\phi\colon E\to G$ of $\phi\colon K\to G$ induces several different "operations," which determine each other in accordance with the following scheme:

$$
\boxed{G\to A_X(K)/c(X)}
$$

(1.1)

$$
\boxed{G\to A_{Z_K}(X)/I(X)} \qquad\qquad \boxed{Q\to A_X(K)/I(K)}
$$
$$
\downarrow \qquad\qquad\qquad\qquad\qquad \downarrow
$$
$$
\boxed{G\to A_{Z_K}(Z_X)} \qquad\qquad\qquad \boxed{Q\to A_X(Z_K)}
$$

$$
\boxed{Q\to A(X\cap Z_K)}
$$

where $A_L(M)$ denotes the group of all automorphisms of M which map $L\cap M$ onto itself, and Z_M denotes the center of M.

The role of $H^n(Q,\ X\cap Z_K)$ in [CGE I] results from the adoption of $G\to A_X(K)/c(X)$ as the type of "operation" to be held fixed. In [CGE IV], $G\to A(X)/I(X)$ is held fixed, with the result that continuations are classified in terms of the relative cohomology groups $H^n(G,\ N;\ Z_X)$. In the present paper we explore the effects of adopting $Q\to A_X(K)/I(K)$ as the fixed "operation." We show that in this situation the relevant cohomology groups, as far as continuations are concerned, are the groups $H^n(\phi)$ of the *mapping* $C^n(Q,\ Z_K)\to C^n(Q,\ Z_N)$. Specifically, we find an obstruction in $H^3(\phi)$ to the

existence of a continuation of $\phi: K \to G$ inducing a given $Q \to A_X(K)/I(K)$, and we show that a certain modification $\tilde{H}^2(\phi)$ of $H^2(\phi)$ operates simply transitively on the set of all isomorphism-classes of such continuations. $H^2(\phi)$ itself operates simply transitively on the set of all (appropriately defined) "similarity classes" of such continuations.

We are interested in classifying not only the continuations, but also the modular structures which induce a given $Q \to A_X(K)/I(K)$. Moreover we wish to study these two questions in relation to each other, and to the classification of the extensions of the Q-kernels $Q \to A_X(K)/I(K)$ and $Q \to A(N)/I(N)$. For such purposes it helps to regard the extension (G, μ) of Q by N as a variable rather than a constant. Consequently, we shall begin not with a normal homomorphism $\phi: K \to G$, but merely with two groups Q and K, a normal subgroup X of K, and an arbitrary homomorphism $\tau: Q \to A_X(K)/I(K)$. N will denote K/X, and $\phi: K \to N$ the natural homomorphism. Recalling from [CGE I] the induced homomorphism $\phi_0: A_X(K)/I(K) \to A(N)/I(N)$, we obtain a "derived" Q-kernel $\phi_0\tau: Q \to A(N)/I(N)$. We now redefine a "continuation" as an extension

$$(1.2) \qquad\qquad 0 \to N \to G \overset{\mu}{\to} Q \to 0$$

of the Q-kernel $\phi_0\tau: Q \to A(N)/I(N)$, together with a continuation $\bar{\phi}: E \to G$ of $\phi: K \to G$ in the earlier sense of [CGE I], which induces $\tau: Q \to A_X(K)/I(K)$. In the definition of "isomorphism" of continuations, we insist that the two extensions of the Q-kernel $\phi_0\tau: Q \to A(N)/I(N)$ be not merely isomorphic but identical. (The precise definitions in the correct logical order are given in §3.)

The above results concerning $H^n(\phi)$ are now formulated as follows: $\tilde{H}^2(\phi)$ operates simply, but not transitively, on the set of all isomorphism-classes of continuations, the orbits being the nonempty counter-images of individual extensions of $\phi_0\tau: Q \to A(N)/I(N)$ under the natural mapping

$$(1.3) \qquad \text{(continuations)} \to \text{(extensions of the } Q\text{-kernel } \phi_0\tau);$$

in place of an individual "obstruction element" of $H^3(\phi)$ we now have a "natural mapping" of the set of all extensions of the Q-kernel $\phi_0\tau$ into $H^3(\phi)$, such that the sequence

$$(1.4) \qquad \text{(continuations)} \to \text{(extensions of the } Q\text{-kernel } \phi_0\tau) \overset{\chi}{\to} H^3(\phi)$$

is "exact." Furthermore, we are now able to formulate and establish supplementary facts such as the following: the mapping $\chi: (\text{extensions of } \phi_0\tau) \to H^3(\phi)$ is "equivariant" relative to the natural homomorphism $H^2(Q, Z_N) \to H^3(\phi)$; the image in $H^3(\phi)$ of the set of all extensions of $\phi_0\tau$ (whether empty or not) is equal to the counter-image, under the natural homomorphism $H^3(\phi) \to H^3(Q, Z_K)$, of the Eilenberg-MacLane obstruction [CTAG II] of the Q-kernel $\tau: Q \to A_X(K)/I(K)$.

Let us now define a "modular expansion" as an extension (1.2) of the Q-kernel $\phi_0 \tau : Q \to A(N)/I(N)$ together with a modular structure

$$\theta : G \to A_X(K)/c(X)$$

on $\phi : K \to G$ which induces $\tau : Q \to A_X(K)/I(K)$, i.e. which satisfies

(1.5) $\omega \theta = \tau \mu,$

where $\omega : A_X(K)/c(X) \to A_X(K)/I(K)$ is the natural homomorphism. This concept is somewhat similar to that of a "related crossed module," as defined in [APC]. However, the results of [APC] are not directly applicable here, for three reasons: (1) in [APC] it is assumed that $X \subset Z_K$, a restriction which is no longer needed when "crossed modules" are replaced by "pseudo-modules"; (2) in [APC], the homomorphism $\mu : G \to Q$ is not considered part of the structure—such a homomorphism is merely required to *exist*; hence the mapping (modular expansions)$\to$(related crossed or pseudo-modules) is onto but many-to-one; (3) the results of [APC] are partly false (a counter-example is given in [QKO II], and the results are corrected in [QKO I]).

Nevertheless, [APC] does indicate that the kernel of $H^2(Q, \phi Z_K)$ $\to H^2(Q, Z_N)$ operates simply transitively on the set of all isomorphism-classes of modular expansions which determine a given isomorphism-class of extensions of the Q-kernel $\phi_0 \tau : Q \to A(N)/I(N)$, and that the image of $H^2(Q, \phi Z_K)$ $\to H^2(Q, Z_N)$ operates simply transitively on the set of all isomorphism-classes of extensions of $\phi_0 \tau : Q \to A(N)/I(N)$ which arise from modular expansions. This suggests, but does not prove, that $H^2(Q, \phi Z_K)$ operates simply transitively (in a natural way) on the set of all isomorphism-classes of modular expansions. We show that this is indeed the case, and that both this result and the criterion in [APC] for the *existence* of a modular expansion (minus the restriction $X \subset Z_K$) are in fact special cases, via [CGE II], of the main results of [CGE I].

The main results of the present paper are all stated in §5 (using terminology and notation developed in §§2–4). They include all of the above, together with certain other operations of various groups on certain sets of "continuations" and "modular expansions," related by mappings which are "equivariant" in the sense of §2. Proofs of the major portions of these results are given in Part II (§§7–10). Proofs of the remaining portions, involving for the most part groups of cocycles rather than of cohomology classes, are given in Part III (§§11–16).

The present study was originally undertaken in order to clarify the following phenomena. If $\phi : K \to G$, $\theta : G \to A_X(K)$ is a crossed module in which K is abelian, then the exact sequence

(1.6) $0 \to X \to K \overset{\phi}{\to} N \to 0$

of operator homomorphisms rel. Q induces an exact sequence of cohomology groups

$$(1.7) \quad \cdots \to H^n(Q, X) \to H^n(Q, K) \overset{\phi_*}{\to} H^n(Q, N) \overset{\nabla}{\to} H^{n+1}(Q, X) \to \cdots,$$

and the obstruction of the crossed module is precisely $\nabla(\gamma) \in H^3(Q, X)$, where $\gamma \in H^2(Q, N)$ is the element corresponding to the group extension (1.2) of Q by N. Hence the crossed module is extendible if and only if

$$(1.8) \qquad\qquad \gamma \in \phi_* H^2(Q, K).$$

In attempting to generalize the criterion (1.8) to the nonabelian case, we must replace $\gamma \in H^2(Q, N)$ by $\gamma \in \overline{H}^2(Q, N, \phi_0\tau)$ [CGE I, §5], where $\tau : Q \to A_X(K)/I(K)$ is induced by the given modular structure $\theta : G \to A_X(K)$. Now $\phi : K \to N$ induces $\phi_* : \overline{H}^2(Q, K, \tau) \to H^2(Q, N, \phi_0\tau)$; hence we might try replacing (1.8) by

$$(1.9) \qquad\qquad \gamma \in \phi_* \overline{H}^2(Q, K, \tau).$$

This condition is still necessary for the extendibility of the crossed module, but it is no longer sufficient (actual counter-examples can be constructed). (1.9) is a necessary and sufficient condition for the existence of a continuation of $\phi : K \to G$ which induces $\tau : Q \to A_X(K)/I(K)$; but the modular structure induced by such a continuation need not coincide with the given θ.

However, it turns out that (1.9) suffices for the extendibility of the crossed module in the situation of [CNG], i.e. when all groups are topological, N is the component of the identity in G, and K is a connected covering of N. More generally, condition (1.9) suffices (for pseudo-modules as well as crossed modules) whenever, in the notation of §4, the "iterated connecting homomorphism"

$$(1.10) \qquad\qquad \nabla_1 \nabla_2 : H^1(Q, Z_N/\phi Z_K) \to H^3(Q, X \cap Z_K)$$

is zero. This state of affairs is explained by the results of §5, which imply, as immediate corollaries:

(a) If (1.10) is zero, then any two modular structures on $\phi : K \to G$ which induce the same $\tau : Q \to A_X(K)/I(K)$ have the same obstruction; hence if one such is extendible, so are all others.

(b) If $\nabla_2 : H^1(Q, Z_N/\phi Z_K) \to H^2(Q, \phi Z_K)$ is zero, then any two modular structures on $\phi : K \to G$ which induce the same $\tau : Q \to A_X(K)/I(K)$ are "congruent" (§3), and hence, a fortiori, have the same obstruction.

(c) If $\nabla_2\rho : Z^1(Q, Z_N/\phi Z_K) \to H^2(Q, \phi Z_K)$ has kernel zero, then any two congruent modular structures are equal. Combining this with (b): If

$$Z^1(Q, Z_N/\phi Z_K) = 0$$

then any two modular structures on $\phi : K \to G$ which induce the same $\tau : Q \to A_X(K)/I(K)$ are equal.

(d) If $(\alpha\pi_N)_* : H^3(\phi) \to H^2(Q, Z_N/\phi Z_K)$ is zero, then for every extension (1.2) of the Q-kernel $\phi_0\tau : Q \to A(N)/I(N)$, there exists a modular structure on $\phi : K \to G$ which induces $\tau : Q \to A_X(K)/I(K)$.

In particular, if $\phi Z_K = Z_N$, then for every (G, μ) and τ, there exists one and only one modular structure. Cf. [CNG, Proposition (6.2)].

In a sense, the best generalization to arbitrary pseudo-modules of the abelian formula $\chi(\phi, \theta) = \nabla(\gamma)$ [mentioned in connection with (1.8)] is the statement, included in Theorem 5.3, that $\chi : \mathbf{P} \to H^3(Q, X \cap Z_K)$ is equivariant relative to $\nabla_1 : H^2(Q, \phi Z_K) \to H^3(Q, X \cap Z_K)$, i.e. $\chi(h+p) = \nabla_1 h + \chi p$. The abelian formula is easily derived as a special case of this.

PART I

2. **Equivariant diagrams.** Let the additive group A act additively as a group of left operators on the set X; let the additive group B act similarly on the set Y; let $f : X \to Y$ be a (set-theoretic) mapping, and $\alpha : A \to B$ a homomorphism. We say that f is *equivariant* rel. α if

$$(2.1) \qquad\qquad f(a + x) = \alpha(a) + f(x)$$

for all $a \in A$ and all $x \in X$. If the formula

$$(2.2) \qquad\qquad f(a + x) = -\alpha(a) + f(x)$$

holds in place of (2.1), we say that f is *antivariant* rel. α. If we assert that $f : X \to Y$ is equivariant or antivariant rel. $\alpha : A \to B$, and it happens that $Y = B$, it is to be understood that the operation of B on itself is left translation.

A diagram of set-theoretic mappings

$$
\begin{array}{ccc}
X & \xrightarrow{\ g\ } & Z \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle k} \\
Y & \xrightarrow{\ h\ } & W
\end{array}
$$

(2.3)

will be called *equivariant* relative to the diagram of groups and homomorphisms

$$
\begin{array}{ccc}
A & \xrightarrow{\ \beta\ } & C \\
\downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \delta} \\
B & \xrightarrow{\ \gamma\ } & D
\end{array}
$$

(2.4)

if each group in (2.4) operates on the correspondingly placed set in (2.3), and each mapping in (2.3) is equivariant relative to the corresponding homomorphism in (2.4). The same terminology will be used for more complicated pairs of diagrams, provided the set-diagram and the group-diagram have the same format.

An operation of A on X is called *simple* if $[a \in A$ and there exists $x \in X$ such that $a+x=x]$ implies $[a=0]$; *transitive* if for every pair $(x, x') \in X \times X$ there exists $a \in A$ such that $a+x=x'$; *simply transitive* if it is simple and transitive. The *orbits* in X under A are the equivalence classes of X determined by the relation $[x$ is *equivalent* to x' if there exists $a \in A$ such that $a+x=x']$. It follows from standard conventions that every group operates simply transitively on the empty set, and that the empty set is never counted as an equivalence class.

If $f:X \to Y$ and $g:X \to Z$ are mappings, $f \times g:X \to Y \times Z$ will denote the "cartesian product" of f and g, defined by $(f \times g)(x) = (fx, gx)$. If the mappings are indicated simply by the symbols $X \to Y$ and $X \to Z$ (without symbols such as f, g), $X \to Y \times Z$ will denote the cartesian product of $X \to Y$ and $X \to Z$.

We call the diagram (2.3) *regular* if, for all $(y, z) \in Y \times Z: (y, z) \in (f \times g)(X)$ if and only if $h(y) = k(z)$. Every regular diagram is commutative, but not conversely. We call (2.3) *strict* if $f \times g:X \to Y \times Z$ is 1–1 into; *strictly regular* if it is strict and regular.

A sequence of mappings $X \to^f Y \to^g A$, where X and Y are sets and A is a group, will be called *exact* if $f(X) = g^{-1}(0)$.

If A operates simply on X, and x, x' are elements of the same orbit in X under A, then $x-x'$ will denote the element $a \in A$ such that $a+x'=x$.

Let $f:X \to Y$ be a set-theoretic mapping. By *the fibers of* $f:X \to Y$ we mean the equivalence classes of X determined by the relation $[x$ is *equivalent* to x' if $f(x) = f(x')]$. In other words, the fibers of a mapping are the nonempty counter-images of individual points.

The following propositions are easily verified:

(2.5) *Let $f:X \to Y$ be equivariant rel. $\alpha:A \to B$; let the operation of B on Y be simple, and let the orbits in X under A be the fibers of $X \to T$. Then the orbits in X under $\alpha^{-1}(0)$ are the fibers of $X \to T \times Y$.*

(2.6) *Let A operate simply on X; let B and C be subgroups of A. If the orbits in X under B and C are the fibers of $X \to S$ and $X \to T$ respectively, then the orbits under $B \cap C$ are the fibers of $X \to S \times T$.*

(2.7) *Let $X \to^f Y \to^g C$ be equivariant relative to $A \to B \to C$ (where X and Y are sets and A, B, C are groups). If X is nonempty, the operation of A on X is transitive, the operation of B on Y is simple, and $X \to Y \to C$ is exact, then so is $A \to B \to C$. "Conversely," if $(gf)^{-1}(0)$ is nonempty, the operations of A and B on X and Y respectively are transitive, and $A \to B \to C$ is exact, then so is $X \to Y \to C$.*

(2.8) *Let $f:X \to Y$ be equivariant rel. $\alpha:A \to B$. If S is an orbit in X under A, then $f(S)$ is an orbit in Y under $\alpha(A)$. If $f(X) = Y$, then each orbit in Y under $\alpha(A)$ is the image of some orbit in X under A; hence in this case, "the images of the orbits are the orbits under the image."*

(2.9) *Let (2.3) and (2.4) be diagrams of set-theoretic mappings and homomorphisms respectively. Let B, C, D operate on Y, Z, W respectively, so that*

$Y \to W$ is equivariant rel. $B \to D$, and $Z \to W$ is equivariant rel. $C \to D$. If (2.3) is strictly regular and (2.4) is commutative, then there exists one and only one operation of A on X such that (2.3) is equivariant rel. (2.4).

(2.10) Let (2.3) be equivariant rel. (2.4). If the operations of B and C on Y and Z respectively are simple, and (2.4) is strict, then the operation of A on X is simple.

(2.11) Let (2.3) be equivariant rel. (2.4). Let the orbits in Y under B be the fibers of $Y \to S$; let the orbits in Z under C be the fibers of $Z \to T$. If (2.3) is strict and commutative, and (2.4) is regular, then the orbits in X under A are the fibers of $X \to S \times T$.

(2.12) Let the set-theoretic map $f : X \to Y$ and the homomorphism $\alpha : A \to B$ be onto. Let A operate on X so that the orbits in X under $\alpha^{-1}(0)$ are the fibers of $X \to Y$. Then there exists one and only one operation of B on Y so that f is equivariant rel. α. If the operation of A on X is simple, then so is the operation of B on Y. If $g : Y \to T$ is a mapping such that the orbits in X under A are the fibers of $gf : X \to T$, then the orbits in Y under B are the fibers of $g : Y \to T$.

(2.13) Let $X \to Y$ be equivariant rel. $A \to B$. Let the mapping $Y \to Z$ and the homomorphism $B \to C$ be such that $X \to Z$ is equivariant rel. $A \to C$. If $X \to Y$ and $A \to B$ are onto, then $Y \to Z$ is equivariant rel. $B \to C$.

(2.14) Let (2.3) be regular, and equivariant rel. (2.4); let D be abelian, and let the operation of D on W be simply transitive. Define $\chi : Y \times Z \to D$ by $\chi(y, z) = h(y) - k(z)$. Define $\rho : B + C \to D$ by $\rho(b, c) = \gamma(b) - \delta(c)$. Let $B + C$ operate on $Y \times Z$ by $(b, c) + (y, z) = (b+y, c+z)$. Then

$$ X \xrightarrow{f \times g} Y \times Z \xrightarrow{\chi} D $$

is exact, and equivariant relative to

$$ A \xrightarrow{\alpha \times \beta} B + C \xrightarrow{\rho} D. $$

(2.15) Let (2.3) be equivariant relative to (2.4), and let (2.4) be regular. If X is nonempty, the operations of B and C on Y and Z respectively are transitive, the operation of D on W is simple, and (2.3) is commutative, then (2.3) is regular.

3. **The sets** CON, EXT(τ), EXT$(\phi_0\tau)$, P. It is suggested that the reader consult diagram (5.4) while reading this section.

Throughout the paper we assume given two fixed groups Q and K, a normal subgroup X of K, and a homomorphism $\tau : Q \to A_X(K)/I(K)$. We denote K/X by N, and the natural homomorphism by $\phi : K \to N$. We shall use all the symbols introduced on pp. 110–111 of [CGE I], not involving G or μ.

A *continuation* is a group extension (1.2) of Q by N, together with a group E containing K as a subgroup, and a homomorphism $\bar\phi : E \to G$ satisfying

(3.1) $\bar\phi \,|\, K = \phi,$ $\bar\phi(E) = G,$ $\bar\phi^{-1}(0) = X,$ $\tau\mu\bar\phi = \nu C_K.$

The last condition means that

(3.2) $$0 \to K \to E \overset{\mu\bar{\phi}}{\to} Q \to 0$$

is an extension of the Q-kernel $\tau: Q \to A_X(K)/I(K)$. It follows that (1.2) is an extension of the Q-kernel $\phi_0\tau: Q \to A(N)/I(N)$.

Two continuations $(G, \mu, E, \bar{\phi})$ and $(G', \mu', E', \bar{\phi}')$ will be called *isomorphic* if $(G, \mu) = (G', \mu')$ and there exists a homomorphism $W: E \to E'$ satisfying

(3.3) $$W \mid K = \text{identity}, \qquad \bar{\phi}'W = \bar{\phi}.$$

By the "five-lemma," such a homomorphism is automatically an iso-morphism of E onto E'.

Two continuations $(G, \mu, E, \bar{\phi})$ and $(G', \mu', E', \bar{\phi}')$ will be called *similar* if $(G, \mu) = (G', \mu')$ and there exist $n_0 \in Z_N$ and a homomorphism $W: E \to E'$ satis-fying

(3.4) $$W \mid K = \text{identity}, \qquad n_0 + \bar{\phi}e = \bar{\phi}'We + n_0$$

for all $e \in E$. Such a W is automatically an isomorphism of E onto E'. Condi-tion (3.4b) may be written $\bar{\phi}'W = (c_Gn_0)\bar{\phi}$.

CON, **CON**, and $\mathfrak{CON}$ will denote, respectively, the set of all continua-tions, the set of all isomorphism-classes of continuations, and the set of all similarity-classes of continuations. Since any two isomorphic continuations are similar, we have natural mappings $\text{CON} \to \textbf{CON} \to \mathfrak{CON}$.

$\text{EXT}(\tau)$ and $\textbf{EXT}(\tau)$ will denote, respectively, the set of all extensions of the Q-kernel $\tau: Q \to A_X(K)/I(K)$ and the set of all isomorphism-classes of such. $\text{EXT}(\phi_0\tau)$ and $\textbf{EXT}(\phi_0\tau)$ will denote, respectively, the set of all exten-sions of the Q-kernel $\phi_0\tau: Q \to A(N)/I(N)$ and the set of all isomorphism-classes of such. The passage from $(G, \mu, E, \bar{\phi})$ to $(E, \mu\bar{\phi})$ is a mapping $\text{CON} \to \text{EXT}(\tau)$, which induces a mapping $\mathfrak{CON} \to \textbf{EXT}(\tau)$.

A *modular expansion* is a group extension (1.2) of Q by N, together with a modular structure $\theta: G \to A_X(K)/c(X)$ on the normal homomorphism $\phi: K \to G$ satisfying (1.5). P will denote the set of all modular expansions. The passage from (G, μ, θ) to (G, μ) is a mapping $\text{P} \to \text{EXT}(\phi_0\tau)$.

Two modular expansions (G, μ, θ) and (G', μ', θ') will be called *isomorphic* if there exists a homomorphism $W: G \to G'$ satisfying

(3.5) $$W \mid N = \text{identity}, \qquad \mu'W = \mu, \qquad \theta'W = \theta.$$

Such a W is automatically an isomorphism of G onto G'. Two modular expan-sions (G, μ, θ) and (G', μ', θ') will be called *congruent* if they are isomorphic and $(G, \mu) = (G', \mu')$. $\mathfrak{P}$ and **P** will denote, respectively, the set of all congruence classes of modular expansions and the set of all isomorphism-classes of modu-lar expansions. We have natural mappings $\text{P} \to \mathfrak{P} \to \textbf{P}$. Furthermore, $\text{P} \to \text{EXT}(\phi_0\tau)$ induces mappings $\mathfrak{P} \to \text{EXT}(\phi_0\tau)$ and $\textbf{P} \to \textbf{EXT}(\phi_0\tau)$.

Given a continuation $(G, \mu, E, \bar{\phi})$, define $\theta: G \to A_X(K)/c(X)$ by $\theta\bar{\phi} = \lambda C_K$ (cf. [CGE I], §3). Then the passage from $(G, \mu, E, \bar{\phi})$ to (G, μ, θ) is a mapping $\text{CON} \to \text{P}$, which induces mappings $\textbf{CON} \to \text{P}$ and $\mathfrak{CON} \to \mathfrak{P}$.

Given an extension

(3.6) $0 \to K \to E \overset{\rho}{\to} Q \to 0$

of the Q-kernel $\tau: Q \to A_X(K)/I(K)$, let $G = E/X$, and let $\bar{\phi}: E \to G$ denote the natural homomorphism. Then $\rho: E \to Q$ induces a homomorphism $\mu: G \to Q$, defined by $\mu\bar{\phi} = \rho$; and $(G, \mu, E, \bar{\phi})$ is a continuation. This construction defines a mapping $\mathrm{EXT}(\tau) \to \mathrm{CON}$, which is a right inverse of $\mathrm{CON} \to \mathrm{EXT}(\tau)$, showing that the latter mapping and hence also $\mathfrak{CON} \to \mathbf{EXT}(\tau)$ are onto. Since $\mathrm{EXT}(\tau) \to \mathrm{CON}$ does not commute with all the mappings defined previously, it will, for the most part, be ignored. However, the mapping $\mathrm{EXT}(\tau) \to \mathbf{P}$ obtained from the composition $\mathrm{EXT}(\tau) \to \mathrm{CON} \to \mathrm{P} \to \mathfrak{P} \to \mathbf{P}$ does commute with the other mappings, and will be included in our investigation. $\mathrm{EXT}(\tau) \to \mathbf{P}$ induces a mapping $\mathbf{EXT}(\tau) \to \mathbf{P}$.

If $(G, \mu) \in \mathrm{EXT}(\phi_0\tau)$, and $T = \mathrm{CON}$, $\mathbf{CON}$, $\mathfrak{CON}$, P, or $\mathfrak{P}$, then $T_{(G,\mu)}$ will denote the counter-image of (G, μ) under the mapping $T \to \mathrm{EXT}(\phi_0\tau)$.

4. Cohomology groups. The following symbols will denote the indicated natural homomorphisms:

(4.1) $i: X \cap Z_K \to Z_K$; $j: \phi Z_K \to Z_N$; $\alpha: N \to N/\phi Z_K$.

The maps $Z_K \to \phi Z_K$ and $Z_N \to Z_N/\phi Z_K$ obtained by restricting ϕ and α respectively will be denoted by the same symbols ϕ and α.

Given any cochain complex C, $\rho: Z^n \to H^n$ will denote the natural mapping of the group of n-dimensional cocycles onto the n-dimensional cohomology group.

$\alpha: N \to N/\phi Z_K$ induces homomorphisms

$\alpha_\sharp: A_{\phi Z_K}(N) \to A(N/\phi Z_K)$, $\alpha_0: A_{\phi Z_K}(N)/I(N) \to A(N/\phi Z_K)/I(N/\phi Z_K)$

analogous to those induced by $\phi: K \to N$ [CGE I, p. 110]. Clearly $\phi_0\tau(Q) \subset A_{\phi Z_K}(N)/I(N)$.

If $Y = X \cap Z_K$, Z_K, ϕZ_K, Z_N, $Z_N/\phi Z_K$, $Z_{\alpha N}$, or $Z_{\alpha N}/\alpha Z_N$, the cohomology groups $H^n(Q, Y)$ are to be understood as defined in terms of the operation of Q and Y induced by $\tau: Q \to A_X(K)/I(K)$, $\phi_0\tau: Q \to A(N)/I(N)$, or $\alpha_0\phi_0\tau: Q \to A(\alpha N)/I(\alpha N)$.

Let $C^n(\phi)$ denote the direct sum $C^n(Q, Z_K) + C^{n-1}(Q, Z_N)$. In particular, $C^0(\phi) = Z_K$, and $C^n(\phi) = 0$ for $n < 0$. Define $D: C^n(\phi) \to C^{n+1}(\phi)$ by

(4.2) $D(f, g) = [\delta f, \delta g + (-1)^{n-1}\phi f]$

(cf. [E–S, p. 155]). Then $DD = 0$. Let

$Z^n(\phi) = C^n(\phi) \cap D^{-1}(0)$, $B^n(\phi) = D[C^{n-1}(\phi)]$, $H^n(\phi) = Z^n(\phi)/B^n(\phi)$.

Define

$$\text{(4.3)} \quad \epsilon_K : C^n(Q, Z_K) \to C^n(\phi), \qquad \epsilon_N : C^{n-1}(Q, Z_N) \to C^n(\phi),$$
$$\pi_K : C^n(\phi) \to C^n(Q, Z_K), \qquad \pi_N : C^n(\phi) \to C^{n-1}(Q; Z_N)$$

by

$$\text{(4.4)} \quad \epsilon_K(f) = (f, 0), \qquad \epsilon_N(g) = (0, g), \qquad \pi_K(f, g) = f, \qquad \pi_N(f, g) = g.$$

Then the homomorphisms

$$\text{(4.5)} \quad \epsilon_K i : C^n(Q, X \cap Z_K) \to C^n(\phi), \qquad \epsilon_N : C^{n-1}(Q, Z_N) \to C^n(\phi),$$
$$\pi_K : C^n(\phi) \to C^n(Q, Z_K), \qquad \alpha\pi_N : C^n(\phi) \to C^{n-1}(Q, Z_N/\phi Z_K)$$

are cochain mappings. We obtain a diagram

$$\text{(4.6)}$$

in which all rows and columns are exact, and each small rectangle is commutative, except for the one in the upper right-hand corner, which is anti-commutative for n odd: that is, for n odd, the sum of the maps $(\epsilon_K i)_* \nabla_1 :$ $H^n(Q, \phi Z_K) \to H^{n+1}(\phi)$ and $(\epsilon_N)_* j_* : H^n(Q, \phi Z_K) \to H^{n+1}(\phi)$ is zero.

The presence of 0 in (4.6) gives rise to an "involution" of the diagram, in which the same nontrivial exact sequences occur in a different arrangement. Each small rectangle in the involuted diagram is commutative, except for

$$\text{(4.7)}$$

which is anti-commutative for n odd.

It follows from these assertions that the sequences

(4.8)
$$\cdots \to H^n(Q, X \cap Z_K) + H^{n-1}(Q, Z_N) \xrightarrow{(\epsilon_K i)_* + (-1)^n(\epsilon_N)_*} H^n(\phi)$$
$$\xrightarrow{\phi_*(\pi_K)_*} H^n(Q, \phi Z_K) \xrightarrow{\nabla_1 + j_*} H^{n+1}(Q, X \cap Z_K) + H^n(Q, Z_N) \to \cdots$$

(4.9)
$$\cdots \to H^n(Q, Z_K) + H^{n-1}(Q, Z_N/\phi Z_K) \xrightarrow{\phi_* + (-1)^{n-1}\nabla_2} H^n(Q, \phi Z_K)$$
$$\xrightarrow{(\epsilon_N)_* j_*} H^{n+1}(\phi) \xrightarrow{(\pi_K)_* + (\alpha\pi_N)_*} H^{n+1}(Q, Z_K) + H^n(Q, Z_N/\phi Z_K) \to \cdots$$

are exact. They remain exact if $\phi_*(\pi_K)_*$ is replaced by $\nabla_2(\alpha\pi_N)_*$, and/or $(\epsilon_N)_* j_*$ is replaced by $(\epsilon_K i)_* \nabla_1$.

Let $\tilde{B}^n(\phi) = D\epsilon_K[C^{n-1}(Q, Z_K)]$, $\tilde{H}^n(\phi) = Z^n(\phi)/\tilde{B}^n(\phi)$. Let $\beta: \tilde{H}^n(\phi) \to H^n(\phi)$ denote the natural homomorphism. $\epsilon_K i: C^n(Q, X \cap Z_K) \to C^n(\phi)$ maps $B^n(Q, X \cap Z_K)$ into $\tilde{B}^n(\phi)$, and therefore induces a homomorphism $(\epsilon_K i)_{\#}: H^n(Q, X \cap Z_K) \to \tilde{H}^n(\phi)$. $\alpha\pi_N: C^n(\phi) \to C^{n-1}(Q, Z_N/\phi Z_K)$ maps $\tilde{B}^n(\phi)$ to zero, and therefore induces a homomorphism $(\alpha\pi_N)_{\#}: \tilde{H}^n(\phi) \to Z^{n-1}(Q, Z_N/\phi Z_K)$. $\epsilon_N: C^{n-1}(Q, Z_N) \to C^n(\phi)$ carries the subgroup $\phi Z^{n-1}(Q, Z_K)$ of $C^{n-1}(Q, Z_N)$ into $\tilde{B}^n(\phi)$, and therefore induces a homomorphism

$$(\epsilon_N)_{\#}: Z^{n-1}(Q, Z_N)/\phi Z^{n-1}(Q, Z_K) \to \tilde{H}^n(\phi).$$

Commutativity holds throughout the diagram

(4.10)

where the unlabelled horizontal homomorphism is induced by $(\epsilon_N)_*: H^{n-1}(Q, Z_N) \to H^n(\phi)$. For each n, the sequences

(4.11)
$$0 \to Z^{n-1}(Q, Z_N)/\phi Z^{n-1}(Q, Z_K) \xrightarrow{(\epsilon_N)_{\#}} \tilde{H}^n(\phi) \xrightarrow{(\pi_K)_* \beta} H^n(Q, Z_K)$$
$$\xrightarrow{j_* \phi_*} H^n(Q, Z_N) \xrightarrow{(\epsilon_N)_*} H^{n+1}(\phi) \to \cdots$$

(4.12)
$$0 \to H^n(Q, X \cap Z_K) \xrightarrow{(\epsilon_K i)_{\#}} \tilde{H}^n(\phi) \xrightarrow{(\alpha\pi_N)_{\#}} Z^{n-1}(Q, Z_N/\phi Z_K)$$
$$\xrightarrow{\nabla_1 \nabla_2 \rho} H^{n+1}(Q, X \cap Z_K) \xrightarrow{(\epsilon_K i)_*} H^{n+1}(\phi) \to \cdots$$

are exact. With the finite number of exceptions shown, (4.11) and (4.12) coincide with sequences from (4.6).

All of the above holds, of course, if the exact sequence

$$(4.13) \quad 0 \to C^n(Q, X \cap Z_K) \to C^n(Q, Z_K) \xrightarrow{j\phi} C^n(Q, Z_N) \to C^n(Q, Z_N/\phi Z_K) \to 0$$

is replaced by an arbitrary exact sequence (of the same length) of cochain complexes.

5. **The main results.** We recall from [CTAG II], and Theorem 5.12 of [CGE I], that $H^2(Q, Z_K)$ and $H^2(Q, Z_N)$ operate in a natural way as simply transitive transformation groups on **EXT**(τ) and **EXT**$(\phi_0\tau)$ respectively. By Theorem 10.2 of [CGE I], $H^2(Q, X \cap Z_K)$ operates naturally as a simply transitive transformation group on each fiber of **CON**→P; hence we may say that $H^2(Q, X \cap Z_K)$ operates simply on **CON**, the orbits being the fibers of **CON**→P. The notion of the "obstruction of a pseudo-module," defined in [CGE I], provides a natural mapping $\chi: P \to H^3(Q, X \cap Z_K)$. It is easily verified that any two isomorphic modular expansions have the same obstruction; hence $\chi: P \to H^3(Q, X \cap Z_K)$ induces a mapping $\chi: \mathbf{P} \to H^3(Q, X \cap Z_K)$. By Corollary 10.3 of [CGE I], the sequence

$$(5.1) \qquad\qquad \mathbf{CON} \to P \xrightarrow{\chi} H^3(Q, X \cap Z_K)$$

is exact; hence so is

$$(5.2) \qquad\qquad \mathbf{EXT}(\tau) \to \mathbf{P} \xrightarrow{\chi} H^3(Q, X \cap Z_K).$$

THEOREM 5.3. *There exist a natural mapping* $\chi: \mathbf{EXT}(\phi_0\tau) \to H^3(\phi)$, *natural operations of* $Z^1(Q, \phi Z_K)$ *and* $Z^1(Q, Z_N)$ *on* CON, *and natural operations of* $\bar{H}^2(\phi)$, $H^2(\phi)$, $Z^1(Q, Z_N/\phi Z_K)$, $\nabla_2 H^1(Q, Z_N/\phi Z_K)$, $H^2(Q, \phi Z_K)$ *on* **CON**, $\mathfrak{CON}$, P, $\mathfrak{P}$, **P** *respectively, such that the diagram*

$$(5.4)$$

is *equivariant relative to the diagram*

$$\begin{array}{ccccc}
Z^1(Q, \phi Z_K) & \xrightarrow{\ j\ } & Z^1(Q, Z_N) & \xrightarrow{\hspace{4cm}} & 0 \\
\downarrow \nabla_1 \rho & & \downarrow \rho \epsilon_N & & \downarrow \\
H^2(Q, X \cap Z_K) \xrightarrow{(\epsilon_K i)\sharp} & \tilde{H}^2(\phi) & \xrightarrow{\ \beta\ } H^2(\phi) & \xrightarrow{(\pi_K)_*} & H^2(Q, Z_K)
\end{array}$$

$$(5.5)$$

$$\begin{array}{ccc}
\downarrow (\alpha\pi_N)\sharp & \downarrow \nabla_2(\alpha\pi_N)_* & \downarrow \phi_* \\
Z^1(Q, Z_N/\phi Z_K) \xrightarrow{\nabla_{2\rho}} \nabla_2 H^1(Q, Z_N/\phi Z_K) \to H^2(Q, \phi Z_K) \xrightarrow{\nabla_1} H^3(Q, X \cap Z_K) \\
\downarrow \qquad\qquad \downarrow j_* \qquad\qquad \downarrow (\epsilon_K i)_* \\
0 \xrightarrow{\hspace{2cm}} H^2(Q, Z_K) \xrightarrow{(\epsilon_N)_*} H^3(\phi),
\end{array}$$

except that CON→**CON** is antivariant rel. $\nabla_1\rho: Z^1(Q, \phi Z_K) \to H^2(Q, X \cap Z_K)$. All the operations are simple. The operation of $H^2(Q, \phi Z_K)$ on **P** is simply transitive. The orbits in CON under $Z^1(Q, \phi Z_K)$ and $Z^1(Q, Z_N)$ are the fibers of CON →**P**×EXT(τ) and CON→EXT($\phi_0\tau$)×EXT(τ) respectively. The orbits in **CON** under $\tilde{H}^2(\phi)$ are the fibers of **CON**→EXT($\phi_0\tau$). If $S = \mathfrak{CON}$, P, or $\mathfrak{P}$, then the orbits in S under the corresponding group are the fibers of S→EXT($\phi_0\tau$).

We note that diagrams (5.4) and (5.5) are commutative, except for the upper left-hand rectangle of (5.5), which is anti-commutative.

THEOREM 5.6. *In diagram (5.4), the rectangles from $\mathfrak{CON}$ to **P** and from **P** to $H^3(\phi)$ are regular. The rectangle from $\mathfrak{P}$ to **EXT**($\phi_0\tau$) is strictly regular. The rectangles*

$$(5.7)$$

$$\begin{array}{ccc}
\textbf{CON} \longrightarrow \textbf{EXT}(\tau) & \quad & \textbf{CON} \longrightarrow \textbf{EXT}(\tau) \\
\downarrow \qquad\qquad \downarrow & & \downarrow \qquad\qquad \downarrow \\
\text{P} \longrightarrow \text{P} & & \text{EXT}\,(\phi_0\tau) \to \textbf{EXT}\,(\phi_0\tau)
\end{array}$$

are regular.

The assertions concerning (5.7a) and the rectangle from $\mathfrak{P}$ to **EXT**($\phi_0\tau$) may be verified directly from the definitions in §3. Since P→$\mathfrak{P}$ is onto, the regularity of the rectangle from $\mathfrak{CON}$ to **P**, and hence of (5.7b), follows automatically. Consequently the only part of Theorem 5.6 which we shall prove in the sequel is the regularity of the rectangle from **P** to $H^3(\phi)$.

THEOREM 5.8. *The sequences*

$$(5.9) \qquad\qquad \textbf{EXT}(\tau) \to \textbf{EXT}(\phi_0\tau) \xrightarrow{\chi} H^3(\phi)$$

$$(5.10) \qquad\qquad \text{P} \to \textbf{EXT}(\phi_0\tau) \xrightarrow{(\alpha\pi_N)_*\chi} H^2(Q, Z_N/\phi Z_K)$$

are exact.

The fact that all the unlabelled horizontal maps in (5.4) are onto yields many equivalent formulations of Theorems 5.6 and 5.8. In particular, the exactness of (5.9) is equivalent to the exactness of (1.4).

$\chi(\tau) \in H^3(Q, Z_K)$ and $\chi(\phi_0\tau) \in H^3(Q, Z_N)$ will denote, respectively, the obstructions [CTAG II] of the Q-kernels $\tau: Q \to A_X(K)/I(K)$ and $\phi_0\tau: Q \to A(N)/I(N)$.

THEOREM 5.11. *The subset* $\chi(\mathbf{P})$ *of* $H^3(Q, X \cap Z_K)$ *is equal to* $i_*^{-1}[\chi(\tau)]$. *The subset* $\chi[\mathbf{EXT}(\phi_0\tau)]$ *of* $H^3(\phi)$ *is equal to* $(\pi_K)_*^{-1}[\chi(\tau)]$.

Theorem 5.11 holds whether the sets in question are empty or not. Thus the "empty" case of Theorem 5.11a asserts that there exists a modular expansion if and only if $\chi(\tau) \in i_* H^3(Q, X \cap Z_K)$, i.e. if and only if the element $\phi_*\chi(\tau)$ of $H^3(Q, \phi Z_K)$ vanishes (cf. [APC, Theorem 1]). The "empty" case of Theorem 5.11b asserts that the Q-kernel $\phi_0\tau: Q \to A(N)/I(N)$ is extendible if and only if $\chi(\tau) \in (\pi_K)_* H^3(\phi)$, i.e. if and only if $j_*\phi_*\chi(\tau) = 0$. Actually, it is easily verified that

$$j_*\phi_*\chi(\tau) = \chi(\phi_0\tau)$$

(cf. [APC]). Hence, in the sequel, we shall prove 5.11b only in the nonempty case.

6. **Corollaries.** Applying (2.14) to Theorem 5.6, we find that

(6.1) $\mathfrak{CON} \to \mathfrak{P} \times \mathbf{EXT}(\tau) \to H^2(Q, \phi Z_K)$,

(6.2) $\mathbf{P} \to \mathbf{EXT}(\phi_0\tau) \times H^3(Q, X \cap Z_K) \to H^3(\phi)$,

(6.3) $\bar{\mathfrak{P}} \to \mathrm{EXT}(\phi_0\tau) \times \mathbf{P} \to H^2(Q, Z_N)$,

(6.4) $\mathbf{CON} \to P \times \mathbf{EXT}(\tau) \to H^2(Q, \phi Z_K)$,

(6.5) $\mathbf{CON} \to \mathrm{EXT}(\phi_0\tau) \times \mathbf{EXT}(\tau) \to H^2(Q, Z_N)$

form exact sequences, equivariant relative to

(6.6) $H^2(\phi) \to \nabla_2 H^1(Q, Z_N/\phi Z_K) + H^2(Q, Z_K) \to H^2(Q, \phi Z_K)$,

(6.7) $H^2(Q, \phi Z_K) \to H^2(Q, Z_N) + H^3(Q, X \cap Z_K) \to H^3(\phi)$,

(6.8) $\nabla_2 H^1(Q, Z_N/\phi Z_K) \to H^2(Q, \phi Z_K) \to H^2(Q, Z_N)$,

(6.9) $\tilde{H}^2(\phi) \to Z^1(Q, Z_N/\phi Z_K) + H^2(Q, Z_K) \to H^2(Q, \phi Z_K)$,

(6.10) $\tilde{H}^2(\phi) \to H^2(Q, Z_K) \to H^2(Q, Z_N)$

respectively. The exact sequence (6.1) [equivalently, (6.4)] yields, in a sense, a cohomological criterion for the existence of a continuation which determines simultaneously a given modular expansion and a given extension of the Q-kernel $\tau: Q \to A_X(K)/I(X)$. The other sequences may be interpreted similarly.

Applying (2.5) and (2.6) to Theorem 5.3, we find that:

The orbits in	under the subgroup	of	are the fibers of
CON	$\phi Z^1(Q, Z_K)$	$Z^1(Q, Z_N)$	CON→CON×EXT(τ)
CON	$Z^1(Q, \phi Z_K)$	$Z^1(Q, Z_N)$	CON→P×EXT(τ)
CON	$\rho\epsilon_N Z^1(Q, Z_N)$	$\tilde{H}^2(\phi)$	**CON**→EXT($\phi_0\tau$)×**EXT**(τ)
CON	$(\epsilon_K i)_\# \mathbf{H}^2(Q, X\cap Z_K)$	$\tilde{H}^2(\phi)$	**CON**→P
CON	$[\phi_*(\pi_K)_*\beta]^{-1}(0)$	$\tilde{H}^2(\phi)$	**CON**→$\mathfrak{P}$
CON	$(\epsilon_K i)_\# \nabla_1 H^1(Q, \phi Z_K)$	$\tilde{H}^2(\phi)$	**CON**→P×**EXT**(τ)
EXT(τ)	$i_* H^2(Q, X\cap Z_K)$	$H^2(Q, Z_K)$	**EXT**(τ)→P
EXT(τ)	$(\pi_K)_* H^2(\phi)$	$H^2(Q, Z_K)$	**EXT**(τ)→**EXT**($\phi_0\tau$)
P	$\alpha Z^1(Q, Z_N)$	$Z^1(Q, Z_N/\phi Z_K)$	P→$\mathfrak{P}$
P	$(\alpha\pi_N)_\# \tilde{H}^2(\phi)$	$Z^1(Q, Z_N/\phi Z_K)$	P→EXT($\phi_0\tau$)×$H^3(Q, X\cap Z_K)$
$\mathfrak{P}$	$\nabla_2 H^1(Q, Z_N/\phi Z_K)\cap\phi_* H^2(Q, Z_K)$	$\nabla_2 H^1(Q, Z_N/\phi Z_K)$	$\mathfrak{P}$→EXT($\phi_0\tau$)×$H^3(Q, X\cap Z_K)$
P	$\phi_* H^2(Q, Z_K)$	$H^2(Q, \phi Z_K)$	P$\xrightarrow{\chi}H^3(Q, X\cap Z_K)$
P	$\nabla_2 H^1(Q, Z_N/\phi Z_K)$	$H^2(Q, \phi Z_K)$	P→**EXT**($\phi_0\tau$)
P	$\nabla_2 H^1(Q, Z_N/\phi Z_K)\cap\phi_* H^2(Q, Z_K)$	$H^2(Q, \phi Z_K)$	P→**EXT**($\phi_0\tau$)×$H^3(Q, X\cap Z_K)$
EXT($\phi_0\tau$)	$j_*\phi_* H^2(Q, Z_K)$	$H^2(Q, Z_N)$	**EXT**($\phi_0\tau$)$\xrightarrow{\chi}H^3(\phi)$
EXT($\phi_0\tau$)	$j_* H^2(Q, \phi Z_K)$	$H^2(Q, Z_N)$	**EXT**($\phi_0\tau$)$\xrightarrow{(\alpha\pi_N)_*\chi}H^2(Q, Z_N/\phi Z_K)$

In the table on the following page we assert that the domain (and hence the image) of the mapping in the second column is nonempty if and only if the condition in the first column is satisfied; and that in the nonempty case, the image of the mapping is an orbit under the subgroup in the third column of the group in the fourth column. (G, μ) is an arbitrary element of EXT($\phi_0\tau$); $\chi(G, \mu)$ is the image of (G, μ) under the composition EXT($\phi_0\tau$)→**EXT**($\phi_0\tau$) $\rightarrow^\chi H^3(\phi)$.

In the last three rows, of course, "orbit" means "coset." The assertions concerning the images are applications of (2.8). The third row of the table generalizes Theorems 1 and 2 of [APC].

$\chi(\tau)=0$	$\mathbf{EXT}(\tau)\to\mathbf{P}$	$\phi_*H^2(Q, Z_K)$	$H^2(Q, \phi Z_K)$
$\chi(\tau)=0$	$\mathbf{EXT}(\tau)\to\mathbf{EXT}(\phi_0\tau)$	$j_*\phi_*H^2(Q, Z_K)$	$H^2(Q, Z_N)$
$\phi_*\chi(\tau)=0$	$\mathbf{P}\to\mathbf{EXT}(\phi_0\tau)$	$j_*H^2(Q, \phi Z_K)$	$H^2(Q, Z_N)$
$\phi_*\chi(\tau)=0$	$\mathbf{P}\to\mathbf{EXT}(\phi_0\tau)\times H^3(Q, X\cap Z_K)$	all (a, b) such that $(\epsilon_N)_*(a)=(\epsilon_K i)_*(b)$	$H^2(Q, Z_N)+H^3(Q, X\cap Z_K)$
$\chi(G, \mu)=0$	$\mathbf{CON}_{(G,\mu)}\to\mathbf{P}_{(G,\mu)}$	$(\alpha\pi_N)_{\#}\tilde{H}^2(\phi)$	$Z^1(Q, Z_N/\phi Z_K)$
$\chi(G, \mu)=0$	$\mathbf{CON}_{(G,\mu)}\to\mathfrak{P}_{(G,\mu)}$	$\nabla_2 H^1(Q, Z_N/\phi Z_K)$ $\cap\phi_*H^2(Q, Z_K)$	$\nabla_2 H^1(Q, Z_N/\phi Z_K)$
$\chi(G, \mu)=0$	$\mathbf{CON}_{(G,\mu)}\to\mathbf{P}_{(G,\mu)}\times\mathbf{EXT}(\tau)$	all (a, b) such that $\nabla_2\rho(a)=\phi_*(b)$	$Z^1(Q, Z_N/\phi Z_K)+H^2(Q, Z_K)$
$\chi(G, \mu)=0$	$\mathbf{CON}_{(G,\mu)}\to\mathfrak{P}_{(G,\mu)}\times\mathbf{EXT}(\tau)$	all (a, b) such that $a=\phi_*(b)$	$\nabla_2 H^1(Q, Z_N/\phi Z_K)+H^2(Q, Z_K)$
$\phi_*\chi(\tau)=0$	$\mathbf{P}\xrightarrow{\chi_3}(\bar{H}Q, X\cap Z_K)$	$\nabla_1 H^2(Q, \phi Z_K)$	$H^3(Q, X\cap Z_K)$
$\phi_*\chi(\tau)=0$	$\mathbf{P}\xrightarrow{(\epsilon_K i)*\chi}H^3(\phi)$	$(\epsilon_K i)_*\nabla_1 H^2(Q, \phi Z_K)$	$H^3(\phi)$
$(\alpha\pi_N)_*\chi(G, \mu)$ $=0$	$\mathbf{P}_{(G,\mu)}\to H^3(Q, X\cap Z_K)$	$\nabla_1\nabla_2 H^1(Q, Z_N/\phi Z_K)$	$H^3(Q, X\cap Z_K)$

PART II

7. The role of $H^n(\phi)$ and $\tilde{H}^2(\phi)$. In this section we shall define the operations of $\tilde{H}^2(\phi)$ and $H^2(\phi)$ on $\mathbf{CON}$ and $\mathfrak{CON}$ respectively, and the map $\chi:\mathbf{EXT}(\phi_0\tau)\to H^3(\phi)$. We shall prove: the operations of $\tilde{H}^2(\phi)$ and $H^2(\phi)$ are simple, and their orbits are as in Theorem 5.3; the maps $\mathbf{CON}\to\mathbf{CON}\to\mathfrak{CON}$ $\to\mathbf{EXT}(\tau)$, $\chi:\mathbf{EXT}(\phi_0\tau)\to H^3(\phi)$ are equivariant relative to the appropriate homomorphisms; the rectangle in (5.4) from $\mathbf{P}$ to $H^3(\phi)$ is commutative; sequence (5.9) is exact; the second assertion of Theorem 5.11 holds.

Consider $(G, \mu)\in\mathbf{EXT}(\phi_0\tau)$. By a *weak structural cochain* in (G, μ) we shall mean a triple (f, v, t), where $f\in C^2(Q, K)$, $v\in C^1(Q, G)$, $t\in C^1[Q, A_X(K)]$, satisfying

$$(7.1) \qquad \mu v = 1, \qquad \phi_{\#}t = C_N v, \qquad c_K f = \operatorname{dev} t, \qquad vt = \tau.$$

This differs from the concept of a "structural cochain in the normal homomorphism $\phi:K\to G$" in the sense of §12 of [CGE I] in that the condition $\phi f=\operatorname{dev} v$ has been dropped and the condition $vt=\tau$ added.

$S_{(G,\mu)}$ will denote the set of all weak structural cochains in (G, μ).

(7.2) *For every* $(G, \mu) \in \mathbf{EXT}(\phi_0\tau)$, $S_{(G,\mu)}$ *is nonempty.*

Proof. Choose $v' \in C^1(Q, G)$ and $t \in C^1[Q, A_X(K)]$ satisfying (7.1a) and (7.1d). Recall that $\bar{\nu}: A(N) \to A(N)/I(N)$ denotes the natural homomorphism. Then $\bar{\nu}\phi_{\sharp}t = \phi_0\nu t = \phi_0\tau = \phi_0\tau\mu v' = \bar{\nu}C_N v'$, hence $\phi_{\sharp}t - C_N v' \subset I(N)$. Therefore there exists $d \in C^1(Q, N)$ such that $\phi_{\sharp}t = c_N d + C_N v'$. Let $v = d + v'$. Since dev $t \subset I(K)$, f may be chosen to satisfy (7.1c). Q.E.D.

Given $h \in C^1(Q, K)$, $(a, b) \in C^2(\phi)$, $(f, v, t) \in S_{(G,\mu)}$, define

$$(7.3) \qquad\qquad h + (f, v, t) = (h_t \cdot f, \phi h + v, c_K h + t),$$

$$(7.4) \qquad\qquad (a, b) + (f, v, t) = (a + f, b + v, t).$$

It is easily verified that $C^1(Q, K)$ and $C^2(\phi)$ now act as groups of operators on $S_{(G,\mu)}$. Let $\overline{S}_{(G,\mu)}$ denote the set of all orbits in $S_{(G,\mu)}$ under $C^1(Q, K)$.

THEOREM 7.5. *There exists one and only one operation of $C^2(\phi)/\tilde{B}^2(\phi)$ on $\overline{S}_{(G,\mu)}$ such that $S_{(G,\mu)} \to \overline{S}_{(G,\mu)}$ is equivariant relative to the natural homomorphism $\rho: C^2(\phi) \to C^2(\phi)/\tilde{B}^2(\phi)$. The operation is simply transitive.*

Proof. Let $\gamma: S_{(G,\mu)} \to \overline{S}_{(G,\mu)}$ denote the natural mapping. Consider $(a, b) \in C^2(\phi)$ and $(f, v, t) \in S_{(G,\mu)}$. To establish the first assertion, it clearly suffices to show that the definition $\rho(a, b) + \gamma(f, v, t) = \gamma[(a, b) + (f, v, t)]$ is "independent of the choice of representatives."

So consider also $(c, d) \in \tilde{B}^2(\phi)$ and $h \in C^1(Q, K)$; we must show that $(a, b) + (f, v, t)$ and $[(a, b) + (c, d)] + [h + (f, v, t)]$ lie in the same orbit under $C^1(Q, K)$.

Choose $g \in C^1(Q, Z_K)$ such that $(c, d) = D\epsilon_K g = (\delta g, \phi g)$. Then $g + h \in C^1(Q, K)$, and $(g + h) + [(a, b) + (f, v, t)] = (g + h) + (a + f, b + v, t) = [(g + h)_t \cdot (a + f), \phi(g + h) + b + v, c_K(g + h) + t] = [a + (g + h)_t \cdot f, \phi g + \phi h + b + v, c_K h + t] = (a + \delta g + h_t \cdot f, b + \phi g + \phi h + v, c_K h + t) = (a + c + h_t \cdot f, b + d + \phi h + v, c_K h + t) = (a + c, b + d) + (h_t \cdot f, \phi h + v, c_K h + t) = [(a, b) + (c, d)] + [h + (f, v, t)]$, completing the first portion of the proof.

Now consider any (f, v, t) and $(f', v', t') \in S_{(G,\mu)}$. $\nu(t' - t) = 0$; hence we may choose $h \in C^1(Q, K)$ such that $t' - t = c_K h$. Let $(a, b) = (h_t \cdot f - f', \phi h + v - v') \in C^2(Q, K) + C^1(Q, N)$. Then $c_K a = c_K(h_t \cdot f) - c_K f' = \text{dev}(c_K h + t) - \text{dev } t' = 0$, and $c_N b = c_N \phi h + C_N v - C_N v' = \phi_{\sharp}(c_K h + t - t') = 0$; therefore $(a, b) \in C^2(\phi)$. Now $\rho(a, b) + \gamma(f', v', t') = \gamma(a + f', b + v', t') = \gamma(h_t \cdot f, \phi h + v, c_K h + t) = \gamma[h + (f, v, t)] = \gamma(f, v, t)$, showing that the new operation is transitive.

Suppose, finally, that $\rho(a, b) + \gamma(f, v, t) = \gamma(f, v, t)$. This means there exists $h \in C^1(Q, K)$ such that $(a + f, b + v, t) = h + (f, v, t) = (h_t \cdot f, \phi h + v, c_K h + t)$. From $t = c_K h + t$ we infer $h \in C^1(Q, Z_K)$; hence $(a + f, b + v) = (\delta h + f, \phi h + v)$, hence $(a, b) = (\delta h, \phi h) = D\epsilon_K h \in \tilde{B}^2(\phi)$, hence $\rho(a, b) = 0$; therefore the operation is simple. Q.E.D.

REMARK. If R denotes the group of all $[(a, b), h] \in C^2(\phi) + C^1(Q, Z_K)$ such that $(a, b) + D\epsilon_K h = 0$, it may be shown that $[C^2(\phi) + C^1(Q, K)]/R$ operates simply transitively on $S_{(G,\mu)}$. We shall not need this fact, however.

Given $(f, v, t) \in S_{(G,\mu)}$, define

(7.6) $\delta(f, v, t) = (\delta_t f, \operatorname{dev} v - \phi f) \in C^3(Q, K) + C^2(Q, N).$

(7.7) $\delta[S_{(G,\mu)}] \subset Z^3(\phi).$

Proof. By Theorem 5.17 of [CGE I] (derived from [CTAG II]), $\delta_t f \in Z^3(Q, Z_K)$. $c_N(\operatorname{dev} v - \phi f) = \operatorname{dev} (C_N v) - \phi_\# c_K f = \operatorname{dev} (\phi_\# t) - \phi_\# \operatorname{dev} t = 0$. Therefore $\delta(f, v, t) \in C^3(\phi)$, and $\pi_K D\delta(f, v, t) = 0$. $\pi_N D\delta(f, v, t) = \pi_N D(\delta_t f, \operatorname{dev} v - \phi f) = \delta(\operatorname{dev} v - \phi f) + \phi(\delta_t f) = \delta(\operatorname{dev} v - \phi f) + \delta_{\phi_\# t}(\phi f) = \delta_{\phi_\# t}(\operatorname{dev} v - \phi f + \phi f) = \delta_{c_N v}(\operatorname{dev} v) = 0$. Therefore $D\delta(f, v, t) = 0$. Q.E.D.

(7.8) If $h \in C^1(Q, K)$ and $(f, v, t) \in S_{(G,\mu)}$, then $\delta[h + (f, v, t)] = \delta(f, v, t)$. (*"Any two equivalent weak structural cochains have the same coboundary."*)

Proof. By Lemma 5.16 of [CGE I], $\delta_{c_K h + t}(h_t \cdot f) = \delta_t f$. By (7.7), $\operatorname{dev} v - \phi f \in C^2(Q, Z_N)$. Hence, using formulae (4.8) and (4.5) of [CGE I], $\phi(h_t \cdot f) = (\phi h)_{\phi_\# t} \cdot (\phi f) = (\phi h)_{c_N v} \cdot [(\phi f - \operatorname{dev} v) + \operatorname{dev} v] = (\phi f - \operatorname{dev} v) + (\phi h)_{c_N v} \cdot (\operatorname{dev} v) = \phi f - \operatorname{dev} v + \operatorname{dev}(\phi h + v)$. Therefore $\delta[h + (f, v, t)] = [\delta_{c_K h + t}(h_t \cdot f), \operatorname{dev}(\phi h + v) - \phi(h_t \cdot f)] = (\delta_t f, \operatorname{dev} v - \phi f) = \delta(f, v, t)$. Q.E.D.

Hence $\delta : S_{(G,\mu)} \to Z^3(\phi)$ induces a mapping $\delta : \overline{S}_{(G,\mu)} \to Z^3(\phi)$.

(7.9) $\delta : S_{(G,\mu)} \to Z^3(\phi)$ *is equivariant rel.* $D : C^2(\phi) \to Z^3(\phi)$.

Proof. $\delta[(a, b) + (f, v, t)] = [\delta_t(a + f), \operatorname{dev}(b + v) - \phi(a + f)] = (\delta a + \delta_t f, \delta b - \phi a + \operatorname{dev} v - \phi f) = D(a, b) + \delta(f, v, t)$. Q.E.D.

Therefore, by (2.13),

(7.10) $\delta : \overline{S}_{(G,\mu)} \to Z^3(\phi)$ *is equivariant rel.* $D_\# : C^2(\phi)/\tilde{B}^2(\phi) \to Z^3(\phi)$.

Applying (2.8) to (7.10), we find that $\delta[\overline{S}_{(G,\mu)}]$ is a coset of

$$D_\#[C^2(\phi)/\tilde{B}^2(\phi)] = D[C^2(\phi)]$$

in $Z^3(\phi)$. In other words, $\delta[S_{(G,\mu)}]$ is a coset of $B^3(\phi)$ in $Z^3(\phi)$. We define $\chi(G, \mu) = \delta[S_{(G,\mu)}] \in H^3(\phi)$. Clearly $\chi : \operatorname{EXT}(\phi_0 \tau) \to H^3(\phi)$ is constant over each isomorphism-class; hence it induces a mapping

$$\chi : \operatorname{EXT}(\phi_0 \tau) \to H^3(\phi).$$

(7.11) *In the sense of §5 of* [CGE I], *let* (f, t) *be a structural cochain in the* Q-*kernel* $\tau : Q \to A_X(K)/I(K)$, *and let* $(g, \phi_\# t)$ *be a structural cocycle in the* Q-*kernel* $\phi_0 \tau : Q \to A(N)/I(N)$, *representing* $e \in \operatorname{EXT}(\phi_0 \tau)$. *Then* $(\delta_t f, g - \phi f) \in Z^3(\phi)$ *represents* $\chi e \in H^3(\phi)$.

Proof. Choose (G, μ) representing e. Then there exists $v \in C^1(Q, G)$ such that $\mu v = 1$, $\operatorname{dev} v = g$, $C_N v = \phi_\# t$. Now (f, v, t) is a weak structural cochain in (G, μ), and χe is represented by $\delta(f, v, t) = (\delta_t f, g - \phi f)$. Q.E.D.

(7.12) $\chi : \operatorname{EXT}(\phi_0 \tau) \to H^3(\phi)$ *is equivariant rel.* $(\epsilon_N)_* : H^2(Q, Z_N) \to H^3(\phi)$.

Proof. Given $a \in Z^2(Q, Z_N)$ and $e \in \operatorname{EXT}(\phi_0 \tau)$. Choose a structural cochain (f, t) in the Q-kernel $\tau : Q \to A_X(K)/I(K)$; then e may be represented by a structural cocycle of the form $(g, \phi_\# t)$, and the structural cocycle $(a + g, \phi_\# t)$

represents $\rho a + e$. Therefore, by (7.11), $\chi(\rho a + e) = \rho(\delta_t f, a + g - \phi f) = \rho[\epsilon_N(a)$ $+ (\delta_t f, g - \phi f)] = \rho \epsilon_N(a) + \rho(\delta_t f, g - \phi f) = (\epsilon_N)_* \rho(a) + \chi e$. Q.E.D.

(7.13) *The rectangle in* (5.4) *from* **P** *to* $H^3(\phi)$ *is commutative.*

Proof. Given a modular expansion $p = (G, \mu, \theta) \in \mathbf{P}$. Choose a structural cochain (f, v, t) in the pseudo-module (ϕ, θ), in the sense of [CGE I], §10. Then $(f, v, t) \in S_{(G,\mu)}$ and $\phi f = \text{dev } v$. Therefore $(\epsilon_K i)_* \chi p = (\epsilon_K i)_* \rho(\delta_t f)$ $= \rho(\delta_t f, 0) = \rho(\delta_t f, \text{dev } v - \phi f) = \chi(G,\mu)$. Q.E.D.

A *structural cocycle in* $(G, \mu) \in \text{EXT}(\phi_0 \tau)$ is an element $(f, v, t) \in S_{(G,\mu)}$ whose coboundary is zero. This is the same thing as a structural cocycle in $\phi: K \to G$, in the sense of §12 of [CGE I], satisfying the additional condition $vt = \tau$. Hence by §12 of [CGE I],

(7.14) *Given* $(G, \mu) \in \text{EXT}(\phi_0 \tau)$. *Then* (G, μ) *comes from an element of* CON *if and only if* (G, μ) *admits a structural cocycle.*

It follows at once that (1.4), and hence (5.9), is exact.

Using §11 of [CGE I], it is easy to show that (G, μ) comes from an element of P if and only if (G, μ) admits a weak structural cochain (f, v, t) such that $\phi f = \text{dev } v$. This yields a proof of the exactness of (5.10); but we prefer the proof given in §10.

PROOF OF THEOREM 5.11b. By the remark at the end of §5, we may assume that $\chi[\text{EXT}(\phi_0 \tau)]$ and $(\pi_K)_*^{-1}[\chi(\tau)]$ are nonempty. Then, by (7.12) and the exactness of the second row of (4.6), both are cosets of $(\epsilon_N)_* H^2(Q, Z_N)$ in $H^3(\phi)$. To show that they are the same coset, it suffices to establish

(7.15) *If* $e \in \text{EXT}(\phi_0 \tau)$ *then* $(\pi_K)_* \chi e = \chi(\tau)$.

This, however, follows immediately from (7.11). Q.E.D.

We return to the consideration of a fixed $(G, \mu) \in \text{EXT}(\phi_0 \tau)$. Let $S^0_{(G,\mu)}$ and $\overline{S}^0_{(G,\mu)}$ denote the (possibly empty) kernels of $\delta: S_{(G,\mu)} \to Z^3(\phi)$ and $\delta: \overline{S}^0_{(G,\mu)} \to Z^3(\phi)$ respectively. We note that $\overline{S}^0_{(G,\mu)}$ may also be regarded as the set of all orbits in $S^0_{(G,\mu)}$ under $C^1(Q, K)$, which operates on $S^0_{(G,\mu)}$ because of (7.8).

Applying (2.5) to (7.10), we find that the orbits in $\overline{S}_{(G,\mu)}$ under $D_{\not=}^{-1}(0)$ $= \tilde{H}^2(\phi)$ are the fibers of $\delta: \overline{S}_{(G,\mu)} \to Z^3(\phi)$. In particular, $\tilde{H}^2(\phi)$ operates simply transitively on $\overline{S}^0_{(G,\mu)}$. But by §12 of [CGE I], $\overline{S}^0_{(G,\mu)}$ is in a natural 1-1 correspondence with $\mathbf{CON}_{(G,\mu)}$. We may say that $\tilde{H}^2(\phi)$ operates simply on all of $\mathbf{CON}$, the orbits being the fibers of $\mathbf{CON} \to \text{EXT}(\phi_0 \tau)$.

If $(f, v, t) \in S^0_{(G,\mu)}$ represents the continuation $\bar{\phi}: E \to G$, then (f, t) is a structural cocycle in the Q-kernel $\tau: Q \to A_X(K)/I(K)$, representing $(E, \mu\bar{\phi})$ $\in \text{EXT}(\tau)$. Consequently, a comparison of (7.4) with formulae (10.8) and (5.11) of [CGE I] yields immediately

(7.16) $\mathbf{CON} \to \mathbf{CON} \to \text{EXT}(\tau)$ *is equivariant relative to*

$$H^2(Q, X \cap Z_K) \xrightarrow{(\epsilon_K i)_\#} \tilde{H}^2(\phi) \xrightarrow{(\pi_K)_* \beta} H^2(Q, Z_K).$$

We record for reference the corollary

(7.17) **CON**→**EXT** (τ) *is equivariant rel.* $i_*: H^2(Q, X \cap Z_K) \to H^2(Q, Z_K)$.

Since any two isomorphic continuations are similar (§3), the elements of **CON**$_{(G,\mu)}$ divide into "similarity classes," which are the fibers of **CON**$_{(G,\mu)}$ →$\mathfrak{CON}_{(G,\mu)}$.

(7.18) *The similarity classes of* **CON**$_{(G,\mu)}$ *are the orbits under the subgroup* $\beta^{-1}(0) = B^2(\phi)/\bar{B}^2(\phi)$ *of* $\tilde{H}^2(\phi)$.

Proof. Consider two elements $\bar\phi: E \to G$ and $\bar\phi': E' \to G$ of CON$_{(G,\mu)}$. Choose $t \in C^1[Q, A_X(K)]$ such that $\nu t = \tau$. We know that $\bar\phi: E \to G$ and $\bar\phi': E' \to G$ may be represented by structural cocycles of the form (f, v, t) and (f', v', t) respectively. Let $\gamma: S^0_{(G,\mu)} \to \bar{S}^0_{(G,\mu)}$ denote the natural mapping, and let $x = \gamma(f', v', t) - \gamma(f, v, t) \in \tilde{H}^2(\phi)$. We have to show that $\bar\phi: E \to G$ and $\bar\phi': E' \to G$ are similar if and only if $\beta x = 0$, i.e. if and only if $(f' - f, v' - v) \in B^2(\phi)$.

Choose $d \in C^1(Q, E)$ and $d' \in C^1(Q, E')$ such that $(\text{dev } d, \bar\phi d, C_K d) = (f, v, t)$ and $(\text{dev } d', \bar\phi' d', C_K d') = (f', v', t)$. In the presence of (3.4a), condition (3.4b) holds automatically for $e \in K$, and is therefore equivalent to $\bar\phi' W d = (c_G n_0) d\bar\phi d$, i.e. $\bar\phi' W d = (c_G n_0) v$. By the identification $Z_N = C^0(Q, Z_N)$ and the definition of $\delta: C^0(Q, Z_N) \to C^1(Q, Z_N)$, we have, for each $n_0 \in Z_N$,

$$(7.19) \qquad\qquad \delta n_0 = (1 - c_G n_0) v.$$

Therefore $\bar\phi: E \to G$ and $\bar\phi': E' \to G$ are similar if and only if there exist a homomorphism $W: E \to E'$ and an element $n_0 \in Z_N$ such that $W|K = 1$ and

$$(7.20) \qquad\qquad v = \delta n_0 + \bar\phi' W d.$$

By Lemma 5.22 of [CGE I], the formula $Wd = h + d'$ establishes a 1-1 correspondence between homomorphisms $W: E \to E'$ satisfying

$$(7.21) \qquad\qquad W \mid K = 1, \qquad \mu\bar\phi' W = \mu\bar\phi,$$

and cochains $h \in C^1(Q, Z_K)$ satisfying $f = h_t \cdot f'$ $(= \delta h + f')$. Therefore $\bar\phi: E \to G$ and $\bar\phi': E' \to G$ are similar if and only if there exists $(h, n_0) \in C^1(\phi)$ such that $f = \delta h + f'$ and $v = \delta n_0 + \bar\phi'(h + d')$ $[= \delta n_0 + \phi h + v']$. Therefore the continuations are similar if and only if there exists $(h, n_0) \in C^1(\phi)$ such that $(f - f', v - v') = (\delta h, \delta n_0 + \phi h) = D(h, n_0)$. Q.E.D.

Hence, by (2.12),

(7.22) *There exists one and only one operation of* $H^2(\phi)$ *on* $\mathfrak{CON}$ *such that* **CON**→$\mathfrak{CON}$ *is equivariant rel.* $\beta: \tilde{H}^2(\phi) \to H^2(\phi)$. *The operation is simple, and the orbits are the fibers of* $\mathfrak{CON}$→EXT ($\phi_0 \tau$).

By (2.13) and the second half of (7.16), $\mathfrak{CON}$→**EXT** (τ) is equivariant rel. $(\pi_K)_*: H^2(\phi) \to H^2(Q, Z_K)$.

8. Differentiation; the graph. In this section we shall apply the results of [CGE I] and [CGE II] to obtain the simply transitive operation of $H^2(Q, \phi Z_K)$ on **P** and a criterion for the existence of a modular expansion. We shall also show that **P**→**EXT** ($\phi_0 \tau$) is equivariant rel. $j_*: H^2(Q, \phi Z_K) \to H^2(Q, Z_N)$, and we shall prepare the ground for several of the proofs in later sections.

In the present paper, our given data consist of the groups Q, K, X and the homomorphism $\tau: Q \to A_X(K)/I(K)$. Now the groups Q, N, ϕZ_K and the homomorphism $\phi_0 \tau: Q \to A(N)/I(N)$ constitute a similar set of data, to which all our results may be applied. We call the system $(Q, N, \phi Z_K, \phi_0 \tau)$ the *derivative* of the system (Q, K, X, τ). By the *derivative* of any concept or statement formulated in terms of (Q, K, X, τ) we mean the same concept or statement as applied to $(Q, N, \phi Z_K, \phi_0 \tau)$, and perhaps re-interpreted as a new concept or statement about (Q, K, X, τ). For instance, denoting differentiation by primes, we have

$$Q' = Q, \quad K' = N, \quad X' = (X \cap Z_K)' = \phi Z_K, \quad \tau' = \phi_0 \tau,$$
(8.1) $\quad (Z_K)' = Z_N, \quad \phi' = \alpha, \quad (\phi Z_K)' = \alpha(Z_N) = Z_N/\phi Z_K,$
$$N' = N/\phi Z_K, \quad [\mathrm{EXT}(\tau)]' = \mathrm{EXT}(\phi_0 \tau), \quad i' = j, \quad (\nabla_1)' = \nabla_2.$$

The derived system has certain special properties not generally shared by the original system. For example, $X' \subset (Z_K)'$. More significantly, since $\chi(\phi_0 \tau) = j_* \phi_* \chi(\tau) \in j_* H^3(Q, \phi Z_K)$, we have $[\phi_* \chi(\tau)]' = \alpha_* \chi(\phi_0 \tau) = 0$. By the derivative of the "empty" case of Theorem 5.11a (which we have not yet proved), this suggests that P' is always nonempty. It turns out, in fact, that the derived system admits a certain "natural" or "preferred" modular expansion. For let $\psi_\#: N/\phi Z_K \approx I(K)/c(X)$ denote the isomorphism defined by commutativity in

(8.2)

$$
\begin{array}{c}
N \\
\quad\Big\downarrow{\scriptstyle\alpha} \quad \searrow{\scriptstyle\psi} \\
N/\phi Z_K \xrightarrow{\ \psi_\#\ } I(K)/c(X).
\end{array}
$$

(See [CGE I, formulae 2.9, 2.10])

Then by Theorem 2.3 of [CGE II], the diagram

(8.3)

$$
\begin{array}{ccc}
0 \to N/\phi Z_K & & Q \longrightarrow 0 \\
\Big\downarrow{\scriptstyle\psi_\#} & & \Big\downarrow{\scriptstyle\tau} \\
0 \to I(K)/c(X) \to A_X(K)/c(X) \xrightarrow{\ \omega\ } A_X(K)/I(K) \to 0
\end{array}
$$

admits a completion

(8.4)

$$
\begin{array}{ccccc}
0 \to N/\phi Z_K & \xrightarrow{\ \epsilon\ } & \Gamma & \xrightarrow{\ \pi_1\ } & Q \longrightarrow 0 \\
\Big\downarrow{\scriptstyle\psi_\#} & & \Big\downarrow{\scriptstyle\pi_2} & & \Big\downarrow{\scriptstyle\tau} \\
0 \to I(K)/c(X) & \to & A_X(K)/c(X) & \xrightarrow{\ \omega\ } & A_X(K)/I(K) \to 0,
\end{array}
$$

and any two such are isomorphic in one and only one way. We call $(\Gamma, \epsilon, \pi_1, \pi_2)$ the *graph* of the given system (Q, K, X, τ); it is the same as the graph of $\tau : Q \to A_X(K)/I(K)$ rel. $\omega : A_X(K)/c(X) \to A_X(K)/I(K)$, in the terminology of §6 of [CGE I], if we identify $N/\phi Z_K$ with $I(K)/c(X)$ via $\psi_\#$.

THEOREM 8.5. *The homomorphism* $\phi_*\pi_2 : \Gamma \to A(N)$ *is a modular structure on the normal homomorphism* $\epsilon\alpha : N \to \Gamma$. *P is in a natural 1–1 correspondence with the set of all extensions of the crossed module* $(\epsilon\alpha, \phi_*\pi_2)$; *P is in a natural 1–1 correspondence with the set of all isomorphism-classes of extensions of the crossed module* $(\epsilon\alpha, \phi_*\pi_2)$.

Proof. We noted at the end of §2 of [CGE I] that $\phi_* : A_X(K)/c(X) \to A(N)$ is a modular structure on the normal homomorphism $\psi : N \to A_X(K)/c(X)$. Therefore Theorem 5.4 of [CGE II], applied to the diagram

$$(8.6)$$

$$
\begin{array}{ccc}
0 \to N & & Q \longrightarrow 0 \\
\quad \downarrow \psi & & \quad \downarrow \tau \\
0 \to I(K)/c(X) \to A_X(K)/c(X) \xrightarrow{\omega} A_X(K)/I(K) \to 0,
\end{array}
$$

tells us: $\phi_*\pi_2 : \Gamma \to A(N)$ is a modular structure on the normal homomorphism $\epsilon\alpha : N \to \Gamma$; the extensions of the crossed module $(\epsilon\alpha, \phi_*\pi_2)$ are in a natural 1–1 correspondence with those completions

$$(8.7)$$

$$
\begin{array}{ccccc}
0 \to N & \longrightarrow & G & \xrightarrow{\mu} & Q \longrightarrow 0 \\
\quad \downarrow \psi & & \downarrow \theta & & \quad \downarrow \tau \\
0 \to I(K)/c(X) & \longrightarrow & A_X(K)/c(X) & \xrightarrow{\omega} & A_X(K)/I(K) \to 0
\end{array}
$$

of (8.6) which satisfy the appropriate version of formula (5.3) of [CGE II], namely $\phi_*\theta = C_N$; ditto for isomorphism-classes. But we notice that by definition, an element of P is precisely a completion (8.7) satisfying $\phi_*\theta = C_N$; and isomorphism of "modular expansions" means the same as isomorphism of "completions of (8.6)." Q.E.D.

The derivative of $\omega : A_X(K)/c(X) \to A_X(K)/I(K)$ is

$$\bar{\nu} : A_{\phi Z_K}(N) \to A_{\phi Z_K}(N)/I(N).$$

Now $(\phi_0\tau)\pi_1 = \phi_0\omega\pi_2 = \bar{\nu}(\phi_*\pi_2)$. Thus (Γ, π_1, π_2) satisfies the derivative of condition (1.5); hence $(\Gamma, \pi_1, \pi_2) \in P'$.

The kernel of the normal homomorphism $\epsilon\alpha : N \to \Gamma$ is ϕZ_K. It is easily verified that the operation of Q on ϕZ_K resulting from the modular structure $\phi_*\pi_2 : \Gamma \to A(N)$ coincides with the operation described in §4 of the present paper. Therefore the cohomology groups $H^n(Q, X)$ of §§8 and 9 of [CGE I]

become, in the present application, $H^n(Q, \phi Z_K)$. Hence, combining Theorem 8.5 with [Theorem 8.5 and Corollary 9.5 of CGE I], we obtain

(8.8) $H^2(Q, \phi Z_K)$ *operates in a natural way as a simply transitive transformation group on* **P**. **P** *is nonempty if and only if the obstruction in* $H^3(Q, \phi Z_K)$ *of the crossed module* $(\epsilon\alpha, \phi_*\pi_2)$ *vanishes.*

Theorem 8.5 focuses our attention on those continuations of the derived system which are extensions of the particular crossed module $(\epsilon\alpha, \phi_*\pi_2)$. We may say that P and **P** are the derivatives, respectively, of "a particular fiber of CON→P" and "a particular fiber of **CON**→**P**." Then the operation of $H^2(Q, \phi Z_K)$ on **P** is, by construction, the derivative of the operation of $H^2(Q, X \cap Z_K)$ on any particular fiber of **CON**→**P**. Therefore, by differentiating (7.17), we obtain

(8.9) **P**→**EXT**$(\phi_0\tau)$ *is equivariant rel.* $j_*: H^2(Q, \phi Z_K) \to H^2(Q, Z_N)$.

9. **Modified structural cochains.** In this section we shall show that **EXT**(τ)→**P** and $\chi: \mathbf{P} \to H^3(Q, X \cap Z_K)$ are equivariant relative to the corresponding homomorphisms; we shall also prove Theorem 5.11a.

DEFINITION. A *modified structural cochain* is an ordered pair (f, s), where $f \in C^2(Q, N)$ and $s \in C^1[Q, A_X(K)/c(X)]$, such that

(9.1) $$\omega s = \tau, \qquad \psi f = \operatorname{dev} s.$$

One easily verifies

(9.2) *The passage from* (f, u) *to* $(f, \pi_2 u)$ *is a 1–1 mapping of the set of all structural cochains in the crossed module* $(\epsilon\alpha, \phi_*\pi_2)$ *(in the sense of* [CGE I], *§9) onto the set of all "modified structural cochains."*

We define "coboundary," "equivalence," etc. for modified structural cochains by composing the definitions in [CGE I], §§7–9 [as applied to the crossed module $(\epsilon\alpha, \phi_*\pi_2)$] with the 1-1 correspondence given by (9.2). We say that a modified structural cocycle (f, s) *represents* a modular expansion $p \in \mathbf{P}$ if (f, u) represents e, where (f, u) corresponds to (f, s) via (9.2) and e is the extension of $(\epsilon\alpha, \phi_*\pi_2)$ which corresponds to p via Theorem 8.5. Then we find:

(9.3) $$\delta(f, s) = \delta_{\phi_*} f \in Z^3(Q, \phi Z_K);$$

two modified structural cochains (f, s) and (f', s') are equivalent if and only if there exists $h \in C^1(Q, N)$ such that

(9.4) $$f' = h_{\phi_*} \cdot f, \qquad s' = \psi h + s;$$

a modified structural cocycle (f, s) represents a modular expansion (G, μ, θ) if and only if there exists $v \in C^1(Q, G)$ such that

(9.5) $$\mu v = 1, \qquad \operatorname{dev} v = f, \qquad \theta v = s;$$

the operation of $H^2(Q, \phi Z_K)$ on the set of equivalence classes of modified structural cocycles (and hence on **P**) is induced by

(9.6) $$a + (f, s) = (a + f, s)$$

where (f, s) is a modified structural cocycle and $a \in Z^2(Q, \phi Z_K)$.

Let M denote the set of all modified structural cochains, and S_τ the set of all structural cochains in the Q-kernel $\tau: Q \to A_X(K)/I(K)$. With either symbol, the superscript 0 changes "cochain" to "cocycle."

(9.7) $\phi \times \lambda : C^2(Q, K) + C^1[Q, A_X(K)] \to C^2(Q, N) + C^1[Q, A_X(K)/c(X)]$
maps S_τ onto M and S_τ^0 into M^0.

Proof. Given $(g, t) \in S_\tau$, $\omega \lambda t = \nu t = \tau$, and $\psi \phi g = \lambda c_{Kg} = \lambda$ dev $t = $ dev (λt); hence $(\phi g, \lambda t) \in M$. Conversely, given $(f, s) \in M$, choose $t \in C^1[Q, A_X(K)]$ such that $\lambda t = s$. Then $\psi f = $ dev $s = \lambda$ dev t; hence by Lemma 6.4 of [CGE I], there exists $g \in C^2(Q, K)$ such that $\phi g = f$ and $c_{Kg} = $ dev t. $\nu t = \omega \lambda t = \omega s = \tau$; therefore $(g, t) \in S_\tau$ and $(\phi \times \lambda)(g, t) = (f, s)$. Finally, if $(g, t) \in S_\tau^0$, then $\delta[(\phi \times \lambda)(g, t)] = \delta_{\phi_* \lambda} \phi g = \delta_{\phi_* t}(\phi g) = \phi(\delta_t g) = 0$. Q.E.D.

(9.8) *If $s \in S_\tau^0$ represents $e \in \mathrm{EXT}(\tau)$, and $p \in \mathbf{P}$ is the image of e under the mapping $\mathrm{EXT}(\tau) \to \mathbf{P}$ of §3, then $(\phi \times \lambda)(s)$ represents p.*

Proof. Let $(g, t) = s$, $(E, \rho) = e$, $(G, \mu, \theta) = p$ as usual. Choose $d \in C^1(Q, E)$ such that $\rho d = 1$, dev $d = g$, $C_K d = t$. Let $v = \bar\phi d \in C^1(Q, G)$, where $\bar\phi : E \to E/X = G$ is the natural homomorphism, as in §3. Then $\mu v = \mu \bar\phi d = \rho d = 1$; dev $v = $ dev $(\bar\phi d) = \bar\phi$ dev $d = \phi g$; $\theta v = \theta \bar\phi d = \lambda C_K d = \lambda t$, verifying the appropriate version of (9.5). Q.E.D.

Since $Z^2(Q, Z_K)$ and $Z^2(Q, \phi Z_K)$ operate on S_τ^0 and M^0 respectively by means of the first coordinate in each case, it is obvious that $(\phi \times \lambda) : S_\tau^0 \to M^0$ is equivariant rel. $\phi : Z^2(Q, Z_K) \to Z^2(Q, \phi Z_K)$. Combining this with (9.8), we obtain

(9.9) $\mathbf{EXT}(\tau) \to \mathbf{P}$ *is equivariant rel.* $\phi_* : H^2(Q, Z_K) \to H^2(Q, \phi Z_K)$.

(9.10) *Let $(g, t) \in S_\tau$ be such that $(\phi g, \lambda t) \in M^0$; let (G, μ, θ) be a modular expansion represented by $(\phi g, \lambda t)$. Then $\chi(G, \mu, \theta) = \rho(\delta_t g) \in H^3(Q, X \cap Z_K)$.*

Proof. Choose $v \in C^1(Q, G)$ satisfying (9.5), with (f, s) replaced by $(\phi g, \lambda t)$. Then (g, v, t) is a structural cochain in the pseudo-module $\phi : K \to G$, $\theta : G \to A_X(K)/c(X)$, in the sense of §10 of [CGE I]. Q.E.D.

(9.11) $\chi : \mathbf{P} \to H^3(Q, X \cap Z_K)$ *is equivariant rel.* $\nabla_1 : H^2(Q, \phi Z_K) \to H^3(Q, X \cap Z_K)$.

Proof. Given $\phi a \in Z^2(Q, \phi Z_K)$ [where $a \in C^2(Q, Z_K)$] and $p \in \mathbf{P}$. Choose $(f, s) \in M^0$ representing the modular expansions in the isomorphism-class p; by (9.7), choose $(g, t) \in S_\tau$ such that $(\phi g, \lambda t) = (f, s)$. Then the modular expansions in the isomorphism-class $\rho \phi a + p$ are represented by the modified structural cocycle $\phi a + (f, s) = [\phi(a + g), \lambda t]$. Therefore by (9.10), $\chi(\rho \phi a + p) = \rho[\delta_t(a + g)] = \rho \delta a + \rho(\delta_t g) = \nabla_1 \rho \phi a + \chi p$. Q.E.D.

(9.12) *The obstruction of the crossed module $(\epsilon \alpha, \phi_* \pi_2)$ is equal to $\phi_* \chi(\tau)$ $\in H^3(Q, \phi Z_K)$.*

Proof. Choose $(g, t) \in S_\tau$; then by (9.7), $(\phi g, \lambda t) \in M$. Therefore $\chi(\epsilon \alpha, \phi_* \pi_2)$ $= \rho \delta(\phi g, \lambda t) = \rho \delta_{\phi_* \lambda t}(\phi g) = \rho \delta_{\phi_* t}(\phi g) = \rho \phi(\delta_t g) = \phi_* \rho(\delta_t g) = \phi_* \chi(\tau)$. Q.E.D.

Combining (9.12) with (8.8b), we obtain

(9.13) *There exists a modular expansion if and only if* $\phi_* \chi(\tau) = 0$ (cf. [APC]).

This proves the "empty" case of Theorem 5.11a. Assume now that $\chi(\mathbf{P})$ and $i_*^{-1}[\chi(\tau)]$ are nonempty. Then, by (9.11) and the exactness of the top row of (4.6), both are cosets of $\nabla_1 H^2(Q, \phi Z_K)$ in $H^3(Q, X \cap Z_K)$. To show that they are the *same* coset, it suffices to establish

(9.14) *If* $p \in \mathrm{P}$, *then* $i_* \chi(p) = \chi(\tau)$ (cf. [APC]).

This, however, follows at once from (9.7) and (9.10).

10. **Proofs by formal devices.** In this section we conclude the proofs of Theorems 5.6 and 5.8 by showing that (5.10) is exact and that the rectangle in (5.4) from **P** to $H^3(\phi)$ is regular.

Let $\eta : \mathbf{P} \to \mathbf{EXT}(\phi_0 \tau)$ denote the natural mapping.

(10.1) *If* $p \in \mathbf{P}$ *then* $(\alpha \pi_N)_* \chi \eta p = 0$.

Proof. By (7.13), $(\alpha \pi_N)_* \chi \eta p = (\alpha \pi_N)_* (\epsilon_K i)_* \chi p = 0$. Q.E.D.

(10.1) establishes half of the exactness of (5.10). To prove the other half, suppose $e \in \mathbf{EXT}(\phi_0 \tau)$ and $(\alpha \pi_N)_* \chi e = 0$. Then by (7.15) and the anticommutativity of (4.7) (with $n = 3$), $\phi_* \chi(\tau) = \phi_* (\pi_K)_* \chi e = - \nabla_2 (\alpha \pi_N)_* \chi e = 0$. Therefore by (9.13), **P** is nonempty. Therefore by (10.1), $[(\alpha \pi_N)_* \chi \eta]^{-1}(0)$ is nonempty. But now (5.10) is exact by (2.7b).

Now suppose $e \in \mathbf{EXT}(\phi_0 \tau)$, $h \in H^3(Q, X \cap Z_K)$, and $\chi e = (\epsilon_K i)_* h$. Then by (7.15), $\phi_* \chi(\tau) = \phi_* (\pi_K)_* \chi e = \phi_* (\pi_K)_* (\epsilon_K i)_* h = \phi_* i_* h = 0$. Therefore by (9.13), **P** is nonempty. The regularity of the rectangle from *P* to $H^3(\phi)$ now follows from (7.13) and (2.15), because it is easily verified that the corresponding rectangle of homomorphisms is regular.

Part III

11. **Automorphism groups.** Given an extension (1.2) of the Q-kernel $\phi_0 \tau : Q \to A(N)/I(N)$, let $A^N(G)$ denote the group of all automorphisms W of G satisfying

(11.1) $W \mid N = 1, \qquad \mu W = \mu.$

$C^1(G, G)$, the set of all 1-cochains on G in G, forms an additive group by addition of values in G. Elements of $C^1(G, G)$ may also be multiplied, by composition. $C^1(G, G)$ is then a "near-ring," satisfying all the axioms for a ring with identity except the commutativity of addition and the left distributive law $x(y+z) = xy + xz$.

(11.1) *Given* $d \in C^1(Q, G)$ *such that* $\mu d = 1$, *and* $r \in C^1(Q, N)$. *Then there exists at most one* $W \in A^N(G)$ *such that* $Wd = r + d$. *There exists such a* W *if and only if* $r \in Z^1(Q, Z_N)$.

Proof. Transform the notation of Lemma (5.24) of [CGE I] as follows: $(f, t, Q, K, E, \rho, d, f', t', Q', K', E', \rho', d', \phi, \theta, r, \Phi) \to (f, t, Q, N, G, \mu, d, f, t, Q, N, G, \mu, d, 1, 1, r, W)$. Then the lemma tells us: there exists at most one homomorphism $W : G \to G$ satisfying (11.1) and $Wd = r + d$; there exists such

a homomorphisn if and only if $f = r_t \cdot f$ and $t = c_N r + t$, where $f = \mathrm{dev}\ d$ and $t = C_N d$. But by the "five-lemma," any homomorphism $W : G \to G$ satisfying (11.2) is automatically an isomorphism and hence an element of $A^N(G)$; and by formula (4.6) of [CGE I], the conditions $f = r_t \cdot f$ and $c_N r = 0$ are together equivalent to $r \in Z^1(Q, Z_N)$. Q.E.D.

It follows that for fixed $d \in C^1(Q, G)$ such that $\mu d = 1$, the set of all $(r, W) \in Z^1(Q, Z_N) \times A^N(G)$ such that $Wd = r + d$ is the graph of a function $\sigma_d : Z^1(Q, Z_N) \to A^N(G)$ which is 1-1 onto. Given $n \in N$, $q \in Q$, and $r \in Z^1(Q, Z_N)$, we have $(r\mu + 1)(n + dq) = r\mu(n + dq) + n + dq = r(\mu n + \mu dq) + n + dq = rq + n + dq = n + rq + dq = n + (r + d)q = [\sigma_d(r)](n) + [\sigma_d(r)](dq) = [\sigma_d(r)](n + dq)$. Therefore $\sigma_d(r) = r\mu + 1$. Hence σ_d is independent of d, and we have proved that the function $\sigma : Z^1(Q, Z_N) \to A^N(G)$ defined by $\sigma(r) = r\mu + 1$ is 1-1 onto. But $(\sigma r)(\sigma r') = (r\mu + 1)(r'\mu + 1) = r\mu(r'\mu + 1) + r'\mu + 1 = r(\mu r'\mu + \mu) + r'\mu + 1 = r\mu + r'\mu + 1 = (r + r')\mu + 1 = \sigma(r + r')$. Therefore

(11.3) *The formula* $\sigma(r) = r\mu + 1$ *defines an isomorphism of the additive group* $Z^1(Q, Z_N)$ *onto the multiplicative group* $A^N(G)$ (cf. [AEG]).

It follows, of course, that $A^N(G)$ is always commutative, and that up to isomorphism, $A^N(G)$ is independent of the particular extension (G, μ) of the Q-kernel $\phi_0 \tau : Q \to A(N)/I(N)$.

We shall be interested in the subgroups $\sigma[\phi Z^1(Q, Z_K)] \subset \sigma[Z^1(Q, \phi Z_K)] \subset A^N(G)$. We note:

(11.4) $A^N(G)/\sigma[Z^1(Q, \phi Z_K)] \approx \alpha Z^1(Q, Z_N) \subset Z^1(Q, Z_N/\phi Z_K)$,

(11.5) $\sigma[Z^1(Q, \phi Z_K)]/\sigma[\phi Z^1(Q, Z_K)] \approx H^1(Q, \phi Z_K)/\phi_* H^1(Q, Z_K)$

$$\approx \nabla_1 H^1(Q, \phi Z_K) \subset H^2(Q, X \cap Z_K).$$

Applying the above to the particular extension of the Q-kernel $\alpha_0 \phi_0 \tau : Q \to A(\alpha N)/I(\alpha N)$ given by the top row of (8.4), we obtain, as the "derivatives" of (11.3) and (11.5),

(11.6) $\sigma' : Z^1(Q, Z_{\alpha N}) \approx A^{\epsilon \alpha N}(\Gamma)$ by $\sigma'(r) = \epsilon r \pi_1 + 1$,

(11.7) $\sigma'[Z^1(Q, Z_N/\phi Z_K)]/\sigma'[\alpha Z^1(Q, Z_N)] \approx H^1(Q, Z_N/\phi Z_K)/\alpha_* H^1(Q, Z_N)$

$$\approx \nabla_2 H^1(Q, Z_N/\phi Z_K) \subset H^2(Q, \phi Z_K).$$

For each $(G, \mu) \in \mathrm{EXT}(\phi_0 \tau)$ we obtain, from (11.4),

(11.8) $A^N(G)/\sigma[Z^1(Q, \phi Z_K)] \approx \sigma' \alpha Z^1(Q, Z_N) \subset A^{\epsilon \alpha N}(\Gamma)$.

It may be verified that the homomorphism $\xi : A^N(G) \to A^{\epsilon \alpha N}(\Gamma)$ which induces (11.8) is characterized by

(11.9) $(\xi W)\gamma = \epsilon \alpha(Wg - g) + \gamma$

where $W \in A^N(G)$, $\gamma \in \Gamma$, $g \in G$, and $\mu g = \pi_1 \gamma$.

12. **The operation of** $Z^1(Q, Z_N)$ **on CON.** Given $(G, \mu) \in \mathrm{EXT}(\phi_0 \tau)$, let $\mathrm{CON}_{(G, \mu)}$ denote the counter-image of (G, μ) under the mapping CON

→EXT($\phi_0\tau$). Given $(E,\ \bar\phi)\in$CON$_{(G,\mu)}$ and $W\in A^N(G)$, define $W(E,\ \bar\phi)$ $=(E,\ W\bar\phi)\in$CON$_{(G,\mu)}$. Then $A^N(G)$ operates as a left multiplicative transformation group on CON$_{(G,\mu)}$. It is readily verified that the operation is simple, and that the orbits are the fibers of CON$_{(G,\mu)}\to$EXT(τ). Combining this with (11.3), we obtain a simple operation of $Z^1(Q, Z_N)$ on CON$_{(G,\mu)}$, with the same orbits. Since $Z^1(Q, Z_N)$ is independent of (G, μ), we may now say

(12.1) $Z^1(Q, Z_N)$ *operates simply on* CON, *the orbits being the fibers of* CON$\to$EXT$(\phi_0\tau)\times$EXT(τ).

(12.2) *Given* $(G, \mu)\in$EXT$(\phi_0\tau)$ *and* $b\in Z^1(Q, Z_N)$. *Let* (f, v, t) *be a structural cocycle in* (G, μ) (§7) *representing* $(E, \bar\phi)\in$CON$_{(G,\mu)}$. *Then* $(f, b+v, t)$ *is a structural cocycle in* (G, μ) *representing* $b+(E, \bar\phi)$.

Proof. By definition, $b+(E, \bar\phi)=[E, (b\mu+1)\bar\phi]$. Choose $d\in C^1(Q, E)$ such that $\mu\bar\phi d=1$ and (dev d, $\bar\phi d$, $C_K d)=(f, v, t)$. Then $[\text{dev } d, (b\mu+1)\bar\phi d, C_K d]$ $=(\text{dev } d, b+\bar\phi d, C_K d)=(f, b+v, t)$. Therefore $(f, b+v, t)$ represents $b+(E, \bar\phi)$; it follows automatically that $(f, b+v, t)$ is a structural cocycle. Q.E.D.

Comparison of (12.2) with (7.4) shows that CON$\to$**CON** is equivariant rel. $\rho\epsilon_N : Z^1(Q, Z_N)\to\tilde H^2(\phi)$.

13. **The operations of** $Z^1(Q, Z_N/\phi Z_K)$ **on P.** In this section we shall describe two operations of $Z^1(Q, Z_N/\phi Z_K)$ on P, which differ in sign. The "first" operation is the one referred to in Theorem 5.3; with respect to it, **CON**$\to$P $\to$P is equivariant relative to

$$\tilde H^2(\phi)\xrightarrow{(\alpha\pi_N)_\sharp} Z^1(Q, Z_N/\phi Z_K)\xrightarrow{\nabla_2\rho} H^2(Q, \phi Z_K).$$

Moreover, for each $(G, \mu)\in$EXT$(\phi_0\tau)$, let P$_{(G,\mu)}$ denote the counter-image of (G, μ) under the mapping P$\to$EXT$(\phi_0\tau)$. Then we shall describe a natural operation of $A^N(G)$ on P$_{(G,\mu)}$ such that, using the "first" operation of $Z^1(Q, Z_N/\phi Z_K)$ on P$_{(G,\mu)}$, the identity mapping P$_{(G,\mu)}\to$P$_{(G,\mu)}$ is equivariant relative to the composition $\alpha\sigma^{-1} : A^N(G)\to Z^1(Q, Z_N/\phi Z_K)$.

However, we shall show in §15 that the "second" operation is the "derivative" of the simply transitive operation (not yet established) of $Z^1(Q, \phi Z_K)$ on each fiber of CON$\to$P$\times$EXT(τ).

In this section, to provide a clear basis for comparison, we shall describe both operations in terms of cochains. For the most part we shall consider a fixed $(G, \mu)\in$EXT$(\phi_0\tau)$. We note that P$_{(G,\mu)}$ may be identified with each of the following sets:

(a) the set of all modular structures $\theta : G\to A_X(K)/c(X)$ on the normal homomorphism $\phi : K\to G$ which satisfy (1.5);

(b) the set of all extensions of the crossed module $\epsilon\alpha : N\to\Gamma$, $\phi_*\pi_2 : \Gamma$ $\to A(N)$ which determine $(G, \mu)\in$EXT$(\phi_0\tau)$ (§8);

(c) the set of all equivalence classes of those frames (u, s) [CGE I, §11] in the normal homomorphism $\phi : K\to G$ which satisfy $\omega s=\tau$.

The condition $\omega s=\tau$ will be implicitly assumed whenever we speak of "frames."

Given a frame (u, s), and $b \in C^1(Q, G)$, define

(13.1) $$b + (u, s) = (b + u, s).$$

(13.2) $b+(u, s)$ *is a frame if and only if* $b \in C^1(Q, Z_N)$ *and* $\alpha \delta b = 0$.

Proof. Suppose $b+(u, s)$ is a frame. Then $1 = \mu(b+u) = \mu b + 1$, hence $\mu b = 0$, i.e. $b \in C^1(Q, N)$. $c_N b + C_N u = C_N(b+u) = \phi_* s = C_N u$, hence $c_N b = 0$, i.e. $b \in C^1(Q, Z_N)$. $\psi \delta b = \psi(\delta b + \mathrm{dev}\ u - \mathrm{dev}\ u) = \psi[\mathrm{dev}\ (b+u) - \mathrm{dev}\ u] = \mathrm{dev}\ s - \mathrm{dev}\ s = 0$; hence $\alpha \delta b = 0$. The argument is clearly reversible. Q.E.D.

Let $\tilde{Z}^1(Q, Z_N/\phi Z_K) = C^1(Q, Z_N) \cap (\alpha \delta)^{-1}(0)$. It follows from (13.2) that (13.1) defines a simple operation of $\tilde{Z}^1(Q, Z_N/\phi Z_K)$ on the set of all frames [in $\phi: K \to G$, satisfying $\omega s = \tau$]. If the frames (u, s) and (u', s') are equivalent in the sense of [CGE I, §11], and $b \in \tilde{Z}^1(Q, Z_N/\phi Z_K)$, then since $u - u' \in C^1(Q, N)$, we have $\psi[(b+u) - (b+u')] = \psi[b + (u-u') - b] = \psi(u-u') = s - s'$, hence $b+(u, s)$ and $b+(u', s')$ are equivalent. Therefore the operation of $\tilde{Z}^1(Q, Z_N/\phi Z_K)$ on the set of all frames induces an operation of

$$\tilde{Z}^1(Q, Z_N/\phi Z_K)$$

on the set of all equivalence classes of frames.

$b+(u, s)$ is equivalent to (u, s) if and only if $s - s = \psi(b+u-u)$, i.e. if and only if $b \in C^1(Q, \phi Z_K)$. Since

(13.3) $$0 \to C^1(Q, \phi Z_K) \to \tilde{Z}^1(Q, Z_N/\phi Z_K) \xrightarrow{\alpha} Z^1(Q, Z_N/\phi Z_K) \to 0$$

is exact, it follows that the operation of $\tilde{Z}^1(Q, Z_N/\phi Z_K)$ on the set of all equivalence classes of frames induces a simple operation of $Z^1(Q, Z_N/\phi Z_K)$ on the same set, and hence on $P_{(G,\mu)}$. This is our "first operation."

Let T denote the set of all ordered pairs (u, s) such that $u \in C^1(Q, G)$, $s \in C^1[Q, A_X(K)/c(X)]$, and $\mu u = 1$. The concept of "equivalence" between two elements (u, s) and (u', s') of T, namely $s - s' = \psi(u - u')$, makes sense whether the elements of T are frames or not. If (u, s) and (u', s') are equivalent, it is easily verified that (u', s') is a frame if and only if (u, s) is a frame.

Given $(u, s) \in T$ and $b \in C^1(Q, N)$, define

(13.4) $$b +' (u, s) = (u, \psi b + s).$$

(13.5) $b+'(u, s)$ *is equivalent to* $-b+(u, s)$.
Proof. $(\psi b + s) - s = \psi[u - (-b+u)]$. Q.E.D.
Hence, by (13.2),
(13.6) *If* (u, s) *is a frame, then:* $b+'(u, s)$ *is a frame if and only if* $b \in C^1(Q, Z_N)$ *and* $\alpha \delta b = 0$.

Using (13.5) at each step, it follows that (13.4), as well as (13.1), induces a simple operation of $Z^1(Q, Z_N/\phi Z_K)$ on $P_{(G,\mu)}$, and that for each $h \in Z^1(Q, Z_N/\phi Z_K)$ and each $\theta \in P_{(G,\mu)}$, $h +' \theta = -h + \theta$. The operation induced by (13.4) will be called the "second operation" and denoted with primes.

(13.7) *Using the first operation of* $Z^1(Q, Z_N/\phi Z_K)$ *on* $P_{(G,\mu)}$, $\mathbf{CON}_{(G,\mu)}$
$\to P_{(G,\mu)}$ *is equivariant rel.* $(\alpha\pi_N)_{\sharp}: \tilde{H}^2(\phi)\to Z^1(Q, Z_N/\phi Z_K)$.

Proof. Let F denote the set of all frames (u, s) in $\phi: K\to G$ such that
$\omega s = \tau$; let $\overline{F}$ denote the set of all equivalence classes of such frames. Recall
from §7 that $S^0_{(G,\mu)}$ denotes the set of all structural cocycles in (G, μ), and that
$\overline{S}^0_{(G,\mu)}$ denotes the set of all equivalence classes of such. Let $\xi: \mathbf{CON}_{(G,\mu)}$
$\to P_{(G,\mu)}$ denote the natural mapping. By §12 of [CGE I], the formula
$\xi(f, v, t) = (v, \lambda t)$ defines a mapping $\xi: S^0_{(G,\mu)}\to F$, which induces a mapping
$\xi: \overline{S}^0_{(G,\mu)}\to \overline{F}$ such that the diagram

$$\mathbf{CON}_{(G,\mu)} \xrightarrow{\xi} P_{(G,\mu)}$$

(13.8)

$$\overline{S}^0_{(G,\mu)} \xrightarrow{\xi} \overline{F}$$

is commutative. It therefore suffices to show that $\xi: S^0_{(G,\mu)}\to F$ is equivariant
rel. $\pi_N: Z^2(\phi)\to \tilde{Z}^1(Q, Z_N/\phi Z_K)$. But $\xi[(a, b)+(f, v, t)] = \xi(a+f, b+v, t)$
$= (b+v, \lambda t) = b+(v, \lambda t) = \pi_N(a, b)+\xi(f, v, t)$. Q.E.D.

The equivariance of $P\to \mathbf{P}$ rel. $\nabla_2\rho: Z^1(Q, Z_N/\phi Z_K)\to H^2(Q, \phi Z_K)$ will be
proved in §15, where we shall also investigate the orbits.

Now consider $\theta\in P_{(G,\mu)}$ and $W\in A^N(G)$. Then it is easily verified that
$\theta W^{-1}: G\to A_X(K)/c(X)$ satisfies the required conditions, so that θW^{-1}
$\in P_{(G,\mu)}$. Hence the formula $W\theta = \theta W^{-1}$ defines a left operation of $A^N(G)$ on
$P_{(G,\mu)}$. By definition of "congruence," the orbits under this operation are the
fibers of $P_{(G,\mu)}\to \mathfrak{P}$.

(13.9) *Using the first operation of* $Z^1(Q, Z_N/\phi Z_K)$ *on* $P_{(G,\mu)}$, *the identity
map* $P_{(G,\mu)}\to P_{(G,\mu)}$ *is equivariant rel.* $\alpha\sigma^{-1}: A^N(G)\to Z^1(Q, Z_N/\phi Z_K)$.

Proof. Given $W\in A^N(G)$ and $\theta\in P_{(G,\mu)}$; let $r=\sigma^{-1}(W)\in Z^1(Q, Z_N)$, and
choose a frame (u, s) associated with θ. Then $(W\theta)(r+u) = \theta W^{-1}(r+u)$
$= \theta r + \theta(-r\mu+1)u = \theta u = s$. Therefore $r+(u, s) = (r+u, s)$ is a frame associ-
ated with $W\theta$. Therefore $W\theta = \alpha r + \theta = \alpha\sigma^{-1}W+\theta$. Q.E.D.

14. The operation of $Z^1(Q, \phi Z_K)$ **on** $\mathbf{CON}$. We define the operation of
$Z^1(Q, \phi Z_K)$ on $\mathbf{CON}$ by the condition that the identity mapping $\mathbf{CON}\to \mathbf{CON}$
shall be equivariant relative to the inclusion $j: Z^1(Q, \phi Z_K)\to Z^1(Q, Z_N)$.

Combining (13.7) with the last statement of §12, we have

(14.1) *Using the first operation of* $Z^1(Q, Z_N/\phi Z_K)$ *on* P, $\mathbf{CON}\to P$ *is equi-
variant rel.* $\alpha: Z^1(Q, Z_N)\to Z^1(Q, Z_N/\phi Z_K)$.

Hence, by (12.1) and (2.5),

(14.2) *The orbits in* $\mathbf{CON}$ *under* $Z^1(Q, \phi Z_K)$ *are the fibers of* $\mathbf{CON}\to \mathbf{P}$
$\times \mathrm{EXT}(\phi_0\tau)\times \mathrm{EXT}(\tau)$, *i.e. the fibers of* $\mathbf{CON}\to \mathbf{P}\times \mathrm{EXT}(\tau)$.

(14.3) $\mathbf{CON}\to \mathbf{CON}$ *is antivariant rel.* $\nabla_1\rho: Z^1(Q, \phi Z_K)\to H^2(Q, X\cap Z_K)$.

Proof. This follows immediately from the following established facts about
the upper left-hand rectangle of (5.4): the other three maps in the rectangle

are equivariant; the two horizontal maps are identity maps; the corresponding rectangle of homomorphisms is anti-commutative.

15. The operations of $Z^1(Q, Z_N/\phi Z_K)$ **on** P **(concluded).** By (14.2), $Z^1(Q, \phi Z_K)$ operates simply transitively on each fiber of $\mathrm{CON}{\to}\mathrm{P}{\times}\mathrm{EXT}(\tau)$. Differentiating, we obtain a simply transitive operation of $Z^1(Q, Z_N/\phi Z_K)$ on each fiber of $\mathrm{P}{\to}\mathrm{EXT}(\phi_0\tau)$.

(15.1) *For each* $(G, \mu){\in}\mathrm{EXT}(\phi_0\tau)$, *the operation of* $Z^1(Q, Z_N/\phi Z_K)$ *on* $\mathrm{P}_{(G,\mu)}$ *obtained by differentiating the operation of* $Z^1(Q, \phi Z_K)$ *on* CON *coincides with the "second" operation, i.e. the one induced by* (13.4).

Proof. Given $\theta{\in}\mathrm{P}_{(G,\mu)}$ and $\alpha b{\in}Z^1(Q, Z_N/\phi Z_K)$, where $b{\in}\tilde{Z}^1(Q, Z_N/\phi Z_K)$ $= C^1(Q, Z_N)\cap(\alpha\delta)^{-1}(0)$. Since $\omega\theta=\tau$, there exists one and only one homomorphism $\bar{\theta}:G{\to}\Gamma$ such that $\pi_1\bar{\theta}=\mu$ and $\pi_2\bar{\theta}=\theta$; $\theta{\in}\mathrm{P}_{(G,\mu)}$ corresponds, via Theorem 8.5, to the extension $\bar{\theta}:G{\to}\Gamma$ of the crossed module $\epsilon\alpha:N{\to}\Gamma$, $\phi_*\pi_2:\Gamma{\to}A(N)$. Let $W=\sigma'(\alpha b){\in}A^{\epsilon\alpha N}(\Gamma)$. Then, in the sense of the "derivative," $\alpha b+\theta$ is the modular expansion corresponding to the extension $W\bar{\theta}:G$ $\to\Gamma$ of the crossed module $(\epsilon\alpha, \phi_*\pi_2)$. That is, in the sense of the "derivative," $\alpha b+\theta$ is the modular structure $\pi_2 W\bar{\theta}:G{\to}A_X(K)/c(X)$ on the normal homomorphism $\phi:K{\to}G$.

Choose $u{\in}C^1(Q, G)$ such that $\mu u=1$. Then $(u, \theta u)$ is a frame associated with the modular structure $\theta:G{\to}A_X(K)/c(X)$. Therefore, in the sense of (13.4), $\alpha b+'\theta$ is the modular structure associated with the frame $b+'(u, \theta u)$ $=(u, \psi b+\theta u)$. But $\pi_2 W\bar{\theta}u=\pi_2(\epsilon\alpha b\pi_1+1)\bar{\theta}u=\pi_2\epsilon\alpha b\pi_1\bar{\theta}u+\pi_2\bar{\theta}u=\psi_*\alpha b\mu u+\theta u$ $=\psi b+\theta u$. Therefore $\alpha b+'\theta=\pi_2 W\bar{\theta}$. Q.E.D.

Therefore

(15.2) *The orbits in* P *under the "second" operation by* $Z^1(Q, Z_N/\phi Z_K)$ *(and hence also under the "first" operation) are the fibers of* $\mathrm{P}{\to}\mathrm{EXT}(\phi_0\tau)$.

Differentiating (14.3), and using the fact that the "first" and "second" operations differ in sign, we obtain

(15.3) *Using the second (first) operation of* $Z^1(Q, Z_N/\phi Z_K)$ *on* P, $\mathrm{P}{\to}\mathrm{P}$ *is antivariant (equivariant) rel.* $\nabla_2\rho:Z^1(Q, Z_N/\phi Z_K){\to}H^2(Q, \phi Z_K)$.

16. The operation of $\nabla_2 H^1(Q, Z_N/\phi Z_K)$ **on** $\mathfrak{P}$. Henceforth we shall refer only to the *first* operation of $Z^1(Q, Z_N/\phi Z_K)$ on P.

By (15.2), (15.3), and (2.5), the orbits in P under the kernel $\alpha Z^1(Q, Z_N)$ of $\nabla_2\rho:Z^1(Q, Z_N/\phi Z_K){\to}H^2(Q, \phi Z_K)$ are the fibers of $\mathrm{P}{\to}\mathrm{EXT}(\phi_0\tau){\times}\mathrm{P}$, i.e. the fibers of $\mathrm{P}{\to}\mathfrak{P}$. [This also follows from (13.9) and the remark preceding (13.9).] Therefore, by (2.12) and (15.2),

(16.1) *There exists one and only one operation of* $\nabla_2 H^1(Q, Z_N/\phi Z_K)$ *on* $\mathfrak{P}$ *such that* $\mathrm{P}{\to}\mathfrak{P}$ *is equivariant rel.* $\nabla_2\rho:Z^1(Q, Z_N/\phi Z_K){\to}\nabla_2 H^1(Q, Z_N/\phi Z_K)$. *The operation is simple, and the orbits are the fibers of* $\mathfrak{P}{\to}\mathrm{EXT}(\phi_0\tau)$.

By (15.3) and (2.13), $\mathfrak{P}{\to}\mathrm{P}$ is equivariant relative to the inclusion $\nabla_2 H^1(Q, Z_N/\phi Z_K){\to}H^2(Q, \phi Z_K)$.

By (13.7) and (16.1), $\mathbf{CON}{\to}\mathfrak{P}$ is equivariant rel. $(\nabla_2\rho)(\alpha\pi_N)_{\#}:\tilde{H}^2(\phi)$ $\to\nabla_2 H^1(Q, Z_N/\phi Z_K)$. But $(\nabla_2\rho)(\alpha\pi_N)_{\#}=[\nabla_2(\alpha\pi_N)_*]\beta$, and $\mathbf{CON}{\to}\mathfrak{CON}$ is

equivariant rel. $\beta: \tilde{H}^2(\phi) \to H^2(\phi)$. Therefore, by (2.13), $\mathfrak{COM} \to \mathfrak{P}$ is equivariant rel. $\nabla_2(\alpha\pi_N)_* : H^2(\phi) \to \nabla_2 H^1(Q, Z_N/\phi Z_K)$.

BIBLIOGRAPHY

[AEG] R. Baer, *Automorphismen von Erweiterungsgruppen*, Actualités Scientifiques et Industrielles, No. 205, Paris, 1935.

[APC] A. P. Cobbe, *Some algebraic properties of crossed modules*, Quart. J. Math. Oxford Ser. (2) vol. 2 (1951) pp. 269–285.

[CFB] R. L. Taylor, *On the classification of fibre bundles*, Ph.D. thesis, Oxford, 1950.

[CGE I] ———, *Compound group extensions* I, Trans. Amer. Math. Soc. vol. 75 (1953) pp. 106–135.

[CGE II] ———, *Compound group extensions* II, Trans. Amer. Math. Soc. vol. 75 (1953) pp. 304–310.

[CGE IV] M. Auslander, *Compound group extensions* IV, forthcoming.

[CNG] R. L. Taylor, *Covering groups of nonconnected topological groups*, Proc. Amer. Math. Soc. vol. 5 (1954) pp. 753–768.

[CTAG I] S. Eilenberg and S. MacLane, *Cohomology theory in abstract groups* I, Ann. of Math. vol. 48 (1947) pp. 51–58.

[CTAG II] ———, *Cohomology theory in abstract groups* II, Ann. of Math. vol. 48 (1947) pp. 326–341.

[CTAG III] S. MacLane, *Cohomology theory in abstract groups* III, Ann. of Math. vol. 50 (1949) pp. 736–761.

[E-S] S. Eilenberg and N. Steenrod, *Foundations of algebraic topology*, Princeton University Press, 1952.

[QKO I] A. P. Cobbe and R. L. Taylor, *On Q-kernels with operators*, I, forthcoming.

[QKO II] R. L. Taylor, *On Q-kernels with operators* II, forthcoming.

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