AUTOMORPHISMS OF THE SYMPLECTIC MODULAR GROUP

BY

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1. Introduction. Let $\Omega_n$ denote the unimodular group consisting of all $n \times n$ integral matrices of determinant $\pm 1$, and let $I^{(n)}$ be the identity matrix in $\Omega_n$. We shall use 0 to denote a null matrix whose size is determined by the context, $X'$ for the transpose of $X$, and $X \oplus Y$ for the direct sum of $X$ and $Y$. We call an integral matrix primitive if the greatest common divisor of its maximal size minors is 1.

Define

$$\mathcal{F} = \begin{pmatrix} 0 & I^{(n)} \\ -I^{(n)} & 0 \end{pmatrix},$$

and let the symplectic group $Sp_{2n}$ consist of all rational $2n \times 2n$ matrices $M$ satisfying

$$M \mathcal{F} M' = \mathcal{F}.$$  

We define the symplectic modular group $\Gamma_{2n}$ to be the group of integral matrices in $Sp_{2n}$. Although we shall not do so in this paper, it is sometimes more convenient to work with the factor group of $\Gamma_{2n}$ over its center $\pm \mathcal{F}$; see [1; 2; 3] (1). We may also define an extended group $\Delta_{2n}$ consisting of all integral matrices $M$ for which $M \mathcal{F} M' = \pm \mathcal{F}$.

The automorphisms of $Sp_{2n}$ (over any field) have previously been determined [5], as have the automorphisms of $\Gamma_2$ (see [4]). The object of this paper is to determine all automorphisms of $\Gamma_{2n}$. Let us call a homomorphism of $\Gamma_{2n}$ into $\{ \pm 1 \}$ a character. Then we shall prove that every automorphism $\tau$ of $\Gamma_{2n}$ is given by

$$\mathcal{X}^\tau = \psi(\mathcal{X}) \mathcal{X} \mathcal{X}^{-1}$$

for all $\mathcal{X} \in \Gamma_{2n}$, where $\psi$ is a character, and $\mathcal{X} \in \Delta_{2n}$. We may remark at this point that the mapping $\sigma$ defined by

$$\mathcal{X}^\sigma = \mathcal{X}^{-1}$$

for all $\mathcal{X} \in \Gamma_{2n}$

is obviously an automorphism. As we shall see, however, it is an inner automorphism.

Let us set

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(1) Numbers in brackets refer to the bibliography at the end of this paper.
where $A$, $B$, $C$, $D$ are integral $n \times n$ matrices. Then $\mathfrak{M} \in \Gamma_{2n}$ if and only if the following conditions are satisfied:

(4) \[ AB' \text{ symmetric, } CD' \text{ symmetric, } AD' - BC' = I. \]

We single out for future use certain types of elements of $\Gamma_{2n}$:

1. **Translations:**
   \[ \mathfrak{T} = \begin{pmatrix} I & S \\ 0 & I \end{pmatrix} \text{ or } \begin{pmatrix} I & 0 \\ S & I \end{pmatrix}, \]
   \[ S \text{ symmetric.} \]

2. **Rotations:**
   \[ \mathfrak{R} = \begin{pmatrix} U & 0 \\ 0 & U^{-1} \end{pmatrix}, \]
   \[ U \in \Omega_n. \]

3. **Semi-involutions:**
   \[ \mathfrak{S} = \begin{pmatrix} J & I - J \\ J - I & J \end{pmatrix}, \]
   \[ J \text{ diagonal with diagonal elements 0's and 1's.} \]

Further, if $\mathfrak{M}$ given by (3) is in $\Gamma_{2n}$, then

(5) \[ \mathfrak{M}^{-1} = \begin{pmatrix} D' & -B' \\ -C' & A' \end{pmatrix}. \]

Finally, if

\[ \mathfrak{M}_i = \begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix} \in \Gamma_{2n_i} \quad (i = 1, 2), \]

we define the symplectic direct sum $\mathfrak{M}_1 \cdot \mathfrak{M}_2 \in \Gamma_{2(n_1 + n_2)}$ by

\[ \mathfrak{M}_1 \cdot \mathfrak{M}_2 = \begin{pmatrix} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & B_2 \\ C_1 & 0 & D_1 & 0 \\ 0 & C_2 & 0 & D_2 \end{pmatrix}. \]

We may remark that as $\mathfrak{M}$ ranges over all elements of $\Gamma_{2n}$, the matrix $[-I^{(n)} + I^{(n)}] \mathfrak{M}$ ranges over all elements in $\Delta_{2n} = \{ \mathfrak{x} \in \Delta_{2n} : \mathfrak{x} \in \Gamma_{2n} \}$. Thence $\mathfrak{M}_i \in \Delta_{2n_i} \quad (i = 1, 2)$ implies $\mathfrak{M}_1 \cdot \mathfrak{M}_2 \in \Delta_{2(n_1 + n_2)}$. However, $\mathfrak{M}_1 \in \Gamma_{2n_1}$ and $\mathfrak{M}_2 \in \Delta_{2n_2}$ implies $\mathfrak{M}_1 \cdot \mathfrak{M}_2 \in \Delta_{2(n_1 + n_2)}$.

2. **Involutions in $\Gamma_{2n}$**. It is known [4] that as $x$, $y$, and $z$ range over all non-negative integers such that $2x + y + z = n$, the matrix
(6) \[ W(x, y, z) = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} + \cdots + \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} + (-I)^{(w)} + I^{(s)} \]

(where \(x\) 2x2 blocks occur) gives a complete set of nonconjugate involutions in \(\Omega_n\). By an \([x, y, z]\) involution in \(\Omega_n\) we mean any conjugate of \(W(x, y, z)\) in \(\Omega_n\). Now define

\[ \Phi(x, y, z) = W(x, y, z) + W'(x, y, z) \in \Gamma_{2n}. \]

**Theorem 1.** The matrices \(\Phi(x, y, z)\) with \(2x+y+z=n\) give a complete set of nonconjugate involutions in \(\Gamma_{2n}\).

**Proof.** We use induction on \(n\). The result is trivial for \(n=1\), so now let \(\Phi\) be an involution in \(\Gamma_{2n}, n>1\). From \(\Phi^2 = I_{(2n)}\) we conclude that the characteristic roots of \(\Phi\) are 1's and -1's. Let \(\epsilon\) be a characteristic root of \(\Phi\); then there exists a primitive row vector \(\xi\) such that \(\Phi \xi = \epsilon \xi\). We can then find \([6]\) a matrix \(\Psi \in \Gamma_{2n}\) whose first row is \(\xi\). In that case the first row of \(\Phi_1 = \Psi \Phi \Psi^{-1}\) is \((\epsilon 0 \cdots 0)\). Since \(\Phi_1\) is an involution in \(\Gamma_{2n}\), we obtain

\[
\Phi_1 = \begin{bmatrix}
\epsilon & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
* & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
A_1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
* & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
* & \cdots & \epsilon & 0 & \cdots & 0 & \cdots & \cdots \\
* & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
C_1 & D_1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
* & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
* & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
* & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{bmatrix},
\]

where

\[
\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}
\]

is itself an involution in \(\Gamma_{2(n-1)}\). Continuing this procedure, we see that \(\Phi\) is conjugate in \(\Gamma_{2n}\) to a matrix of the form

\(\Phi_2 = \begin{pmatrix} A & 0 \\ C & D \end{pmatrix}\).

From the fact that \(\Phi_2\) is an involution in \(\Gamma_{2n}\), we deduce at once that \(A\) is an involution in \(\Omega_n\), and \(D = A^{(-1)}\). However,

\[
\begin{pmatrix} U & 0 \\ 0 & U^{(-1)} \end{pmatrix} \begin{pmatrix} A & 0 \\ C & D \end{pmatrix} \begin{pmatrix} U^{(-1)} & 0 \\ 0 & U \end{pmatrix} = \begin{pmatrix} UAU^{-1} & 0 \\ \overline{C} & U^{(-1)}DU^{(-1)} \end{pmatrix},
\]
and so by choosing \( U \in \Omega_n \) properly, we find that \( \mathfrak{I} \) is conjugate to \( \mathfrak{I}_3 \) given by

\[
\mathfrak{I}_3 = \begin{pmatrix}
W(x, y, z) & 0 \\
C & W'(x, y, z)
\end{pmatrix}
\]

with a new \( C \). Since \( \mathfrak{I}_3 \in \Gamma_{2n} \) is an involution, we have

(7) \( CW \) symmetric, \( C \) skew-symmetric.

The proof now splits into two cases:

**Case 1.** If either \( y \neq 0 \) or \( z \neq 0 \), we may set \( W(x, y, z) = W_1(\epsilon), \epsilon = \pm 1 \). From (7) we find that

\[
\mathfrak{I}_3 = \begin{pmatrix}
W_1 & 0 & 0 & 0 \\
0 & \epsilon & 0 & 0 \\
C_1 & -\xi' & W_1' & 0 \\
\xi & 0 & 0 & \epsilon
\end{pmatrix},
\]

and that

\[
\mathfrak{B} = \begin{pmatrix}
W_1 & 0 \\
C_1 & W_1'
\end{pmatrix}
\]

is an involution in \( \Gamma_{2(n-1)} \). By the induction hypothesis there exist integers \( x_1, y_1, z_1 \) with \( 2x_1 + y_1 + z_1 = n - 1 \), such that \( \mathfrak{B} \) is conjugate to \( \mathfrak{B}(x_1, y_1, z_1) \). For the moment set \( P = W(x_1, y_1, z_1) \). Then in \( \Gamma_{2n} \), \( \mathfrak{I}_3 \) is conjugate to \( \mathfrak{I}_4 \), where

\[
\mathfrak{I}_4 = \begin{pmatrix}
P & 0 & 0 & 0 \\
0 & \epsilon & 0 & 0 \\
0 & -\xi' & P' & 0 \\
\xi & 0 & 0 & \epsilon
\end{pmatrix}
\]

with a new \( \xi \). But then

\[
\mathfrak{I}_5 = \mathfrak{S} \mathfrak{I}_4 \mathfrak{S}^{-1} = \begin{pmatrix}
P & 0 & 0 & 0 \\
\xi & \epsilon & 0 & 0 \\
0 & 0 & P' & \xi' \\
0 & 0 & 0 & \epsilon
\end{pmatrix}
\]

where \( \mathfrak{S} = \begin{pmatrix}
I^{(n-1)} & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & I^{(n-1)} & 0 \\
0 & -1 & 0 & 0
\end{pmatrix} \).

Since \( \mathfrak{I}_5 \) is now a direct sum \( W + W' \), where \( W \) is an involution in \( \Omega_n \), the result follows upon transforming \( \mathfrak{I}_5 \) by a suitably chosen rotation in \( \Gamma_{2n} \).

**Case 2.** If both \( y \) and \( z \) are 0, we write \( W(x, y, z) = L + W_1 \), where

\[
L = \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}.
\]
Then, as before, \( \mathfrak{x}_4 \) is conjugate to \( \mathfrak{x}_4 \) given by
\[
\mathfrak{x}_4 = \begin{bmatrix}
L & 0 & 0 & 0 \\
0 & W_1 & 0 & 0 \\
0 & b & B & L' \\
-b & 0 & B & L'
\end{bmatrix}.
\]

However,
\[
\mathcal{M} = \begin{bmatrix}
0 & 0 & I^{(2)} & 0 \\
0 & I^{(n-2)} & 0 & 0 \\
-I^{(2)} & 0 & 0 & 0 \\
0 & 0 & 0 & I^{(n-2)}
\end{bmatrix} \begin{bmatrix}
I^{(n)} & 0 \\
0 & 0^{(n-2)} & I^{(n)}
\end{bmatrix} \in \Gamma_{2n},
\]
and we have
\[
\mathcal{M} \mathfrak{x}_4 \mathcal{M}^{-1} = \begin{pmatrix} L' & B \\
0 & W_1 \end{pmatrix} + \begin{pmatrix} L & 0 \\
B' & W_1' \end{pmatrix}.
\]

The result then follows as in the previous case.

We have thus shown that any involution \( \mathfrak{x} \in \Gamma_{2n} \) is conjugate to some \( \mathfrak{B}(x, y, z) \). On the other hand, if \( \mathfrak{B}(x, y, z) \) and \( \mathfrak{B}(x_0, y_0, z_0) \) were conjugate in \( \Gamma_{2n} \), they would certainly be conjugate in \( \Omega_{2n} \). This implies [4] that \( x = x_0, y = y_0, \) and \( z = z_0 \).

The conjugates of \( \mathfrak{B}(x, y, z) \) in \( \Gamma_{2n} \) will be called \( (x, y, z) \) involutions.

3. Characterization of the \( \pm (0, 1, n-1) \) involutions. In \( Sp_{2n} \), every involution is conjugate to one of the form \( I^{(2p)} \ast -I^{(2q)} \), with \( p + q = n \). Any involution in the class of \( I^{(2p)} \ast -I^{(2q)} \) is said to have signature \( \{ p, q \} \) (see [5]). One easily proves that any \( (x, y, z) \) involution in \( \Gamma_{2n} \) has signature \( \{ x+z, x+y \} \), and that the negative of an \( (x, y, z) \) involution is of type \( (x, y, z) \).

It is known that an abelian set of involutions of signature \( \{ p, q \} \) in \( Sp_{2n} \) cannot contain more than \( C_{n,p} \) elements (see [5, Theorem 2; 7, §19]). We shall use this fact in proving the following basic result:

**Theorem 2.** Under any automorphism of \( \Gamma_{2n} \), the image of a \( (0, 1, n-1) \) involution is either a \( (0, 1, n-1) \) involution or a \( (0, n-1, 1) \) involution.

**Proof.** (i) An abelian set of involutions in \( \Gamma_{2n} \), each of type \( (x, y, z) \), we shall call an \( (x, y, z) \) set. Let \( f(x, y, z) \) be the number of elements in an \( (x, y, z) \) set of largest size. The above-quoted result shows that
\[
f(x, y, z) \leq C_{n, x+z},
\]
so for \( (x, y, z) = \pm (0, 0, n), \pm (0, 1, n-1), \pm (1, 0, n-2) \) we have \( f(x, y, z) \leq n \).
We now show that $f(x, y, z) > n$ except for the 6 cases given above.

From an abelian set $\mathcal{A}$ of $[x, y, z]$ involutions in $\Omega_n$, one obtains an $(x, y, z)$ set in $\Gamma_n$ by taking the set of matrices $U + U'^{-1}$, $U \in \mathcal{A}$. We know, however, that there exist abelian sets of $[x, y, z]$ involutions in $\Omega_n$ containing more than $n$ elements, except for the 6 cases listed above (see [8, §§12 and 13]).

(ii) The $\pm (0, 0, n)$ involutions in $\Gamma_{2n}$ are $\pm I^{(2n)}$, so that certainly a $(0, 1, n-1)$ involution cannot be mapped onto a $\pm (0, 0, n)$ involution by an automorphism of $\Gamma_{2n}$. It remains to prove that the image cannot be of type $\pm (1, 0, n-2)$. To begin with, a simple calculation shows that two rotations $U + U'^{-1}$ and $V + V'^{-1}$ are conjugate in $\Gamma_{2n}$ if and only if $U$ and $V$ are conjugate in $\Omega_n$. For $n>2$, there are at least two nonconjugate $[1, 0, n-2]$ sets in $\Omega_n$, each containing $n$ elements; on the other hand, there is a unique (up to conjugacy) abelian set of $n [0, 1, n-1]$ involutions in $\Omega_n$ (see [8, §12]). Hence for $n>2$, the image of a $(0, 1, n-1)$ involution in $\Gamma_{2n}$ must be of type $\pm (0, 1, n-1)$.

(iii) The case $n=1$ is trivial, and so we are left with $n=2$. Now we have

$$I^{(2)} * - I^{(2)} = \left( I^{(2)} * \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right)^2,$$

so any $(0, 1, 1)$ involution in $\Gamma_4$ is the square of some element of $\Gamma_4$. We show that the $(1, 0, 0)$ involutions in $\Gamma_4$ are not squares. For suppose that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^2 = \begin{pmatrix} L & 0 \\ 0 & L' \end{pmatrix},$$

where $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_4$ and $L = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$.

From (5) we then have

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} L & 0 \\ 0 & L' \end{pmatrix} \begin{pmatrix} D' & -B' \\ -C' & A' \end{pmatrix}.$$

This implies that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} a & 0 & 0 & 0 \\ (a - d)/2 & d & 0 & b \\ c & -2c & a & (a + d)/2 \\ -2c & 4c & 0 & -d \end{pmatrix}.$$

Using $AD' - BC' = I$, we find that

$$-d^2 - 4bc = 1,$$

whence $d^2 \equiv -1 \pmod{4}$, since $a$, $b$, $c$, $d$ are integers. This is impossible, and so the theorem is proved.

4. Automorphisms of $\Gamma_4$. As is usually the case with determination of
automorphisms of a group of matrices, the lower the dimension the more difficult the proof. We begin by stating in (i) some earlier results (see [4]) which will be needed.

(i) The group \( \Delta_2 \) coincides with \( \Omega_2 \), and \( \Gamma_2 \) is the subgroup \( \Omega_2^\dagger \) consisting of all elements of \( \Omega_2 \) with determinant +1. For the remainder of this paper we let

\[
S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.
\]

Then \( S \) and \( T \) generate \( \Gamma_2 \), and in any relation \( S^{m_1}T^{n_1}S^{m_2}T^{n_2} \cdots = I \) the sum \( m_1 + n_1 + m_2 + n_2 + \cdots \) is always even. Hence the elements \( X \in \Gamma_2 \) can be classified as even or odd according to the parity of the sum of the exponents when \( X \) is expressed as a product of powers of \( S \) and \( T \). The only nontrivial character of \( \Gamma_2 \) is defined by

\[
\epsilon(X) = \begin{cases} 1, & X \text{ even}, \\ -1, & X \text{ odd}. \end{cases}
\]

Then every automorphism \( \tau \) of \( \Gamma_2 \) is given by

\[
X^\tau = \lambda(X)AXA^{-1}
\]

for all \( X \in \Gamma_2 \), where \( \lambda \) is a character, and \( A \in \Omega_2 \).

(ii) Now let \( \tau \) be any automorphism of \( \Gamma_4 \). After changing \( \tau \) by a suitable inner automorphism, we may assume that \( \Psi \tau = \pm \Psi \), where

\[
\Psi = I^{(3)} \cdot -I^{(3)}.
\]

Since \( \Psi \) and \( -\Psi \) are conjugate in \( \Gamma_4 \), assume in fact that \( \Psi \tau = \Psi \). Then any element of \( \Gamma_4 \) which commutes with \( \Psi \) maps into another such element, so that

\[
(Y_1 \star Z_1)^\tau = Y_2 \star Z_2,
\]

where \( Y_1, Y_2, Z_1, Z_2 \in \Gamma_2 \). Let us set

\[
(Y \star I)^\tau = Y^\alpha \star Y^\beta \quad \text{for } Y \in \Gamma_2,
\]

\[
(I \star Z)^\tau = Z^\gamma \star Z^\delta \quad \text{for } Z \in \Gamma_2.
\]

Then \( \alpha, \beta, \gamma, \delta \) are all homomorphisms of \( \Gamma_2 \) into itself, since

\[
(Y_1 \star Z_1)(Y_2 \star Z_2) = Y_1Y_2 \star Z_1Z_2.
\]

Further, since \( Y \star I \) and \( I \star Z \) commute, so do \( Y^\alpha \) and \( Z^\gamma \) for all pairs of elements \( Y, Z \in \Gamma_2 \); also, every element of \( \Gamma_2 \) is a product \( Y^\alpha Z^\gamma \) for some such pair. Since \( S \in \Gamma_2 \), there exists an element \( X \in \Gamma_2 \) such that \( SX^{-1} \in \Gamma_2^\alpha \) and \( X \in \Gamma_2^\gamma \). But then \( X \) commutes with \( SX^{-1} \), whence \( X = \pm I \) or \( \pm S \). Therefore either \( S \in \Gamma_2^\alpha \) or \( S \in \Gamma_2^\gamma \).
Suppose now that \( S \in \Gamma_2^g \); since every element of \( \Gamma_2^g \) commutes with \( S \), we see that \( \Gamma_2^g \subset \{ \pm 1, \pm S \} \). However, \( S \in \Gamma_2^g \) would imply the finiteness of \( \Gamma_2^g \), whence \( \Gamma_2^g = \Gamma_2^g \Gamma_2^g \) could not be true. Therefore \( \Gamma_2^g \subset \{ \pm 1 \} \), and then certainly \( \Gamma_2^g = \Gamma_2 \). Similarly, one of \( \Gamma_2^g, \Gamma_2^g \) is \( \Gamma_2 \), and the other is included in \( \{ \pm 1 \} \).

Now we use the fact that \((- \beta)^r = - \beta \), that is
\[
(-I \cdot I)^r = -I \cdot I.
\]
Therefore \((-I)^a = -I \); but if \( \Gamma_2^g \subset \{ \pm 1 \} \), the fact that \(-I = S^2 \) would imply \((-I)^a = I \). Hence \( \Gamma_2^g = \Gamma_2 \), \( \Gamma_2^g \subset \{ \pm 1 \} \), and therefore \( \Gamma_2^g \subset \{ \pm 1 \} \), \( \Gamma_2^g = \Gamma_2 \).

Next we prove that \( \alpha \) is an automorphism; we need merely prove that \( Y^a = I \) implies \( Y = I \). But if \( Y^a = I \), then \( (Y \cdot I)^r = I \cdot \pm I \). Since \( (I \cdot I)^r = I \cdot I \) and \( (I \cdot -I)^r = I \cdot -I \), this implies that \( Y = I \). By the same reasoning, \( \delta \) is also an automorphism.

(iii) Now define
\[
Y_1 \circ Y_2 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \circ \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} 0 & a_1 & 0 & b_1 \\ a_2 & 0 & b_2 & 0 \\ c_1 & 0 & d_1 & 0 \\ c_2 & 0 & d_2 & 0 \end{pmatrix}.
\]
Then \( Y_1 \circ Y_2 \in \Gamma_4 \) if and only if \( Y_1, Y_2 \in \Gamma_2 \). The elements of \( \Gamma_4 \) which anticommute with \( \beta \) are of the form \( Y_1 \circ Y_2 \), and we have
\[
\begin{align*}
(A \cdot B)(C \circ D) &= AC \circ BD, \\
(A \circ B)(C \cdot D) &= AD \circ BC, \\
(A \circ B)(C \circ D) &= AD \cdot BC.
\end{align*}
\]
Suppose now that \( (I \circ I)^r = U \circ V \). Since \( (I \circ I)^2 = I \cdot I \), we have \( (U \circ V)^2 = UV \cdot VU = I \cdot I \), so \( V = U^{-1} \). But now let
\[
\Phi^r = (U^{-1} \cdot I) \Phi^r (U \cdot I).
\]
Then \( \Phi^r = \beta \), \( \sigma \) and \( \tau \) differ by an inner automorphism, and \( (I \circ I)^r = I \circ I \). Changing notation, we henceforth assume \( \Phi^r = \beta \) and \( (I \circ I)^r = I \circ I \). From
\[
(I \circ I)(Y \cdot Z)(I \circ I) = Z \cdot Y
\]
we deduce
\[
(I \circ I)(Z \cdot Y^a \cdot Y^b Z^b)(I \circ I) = Y^a Z^a \cdot Z^b Y^b.
\]
Therefore
\[
Z \cdot Y^a = Z^b Y^b
\]
for all \( Y, Z \in \Gamma_2 \). Hence \( \beta = \gamma, \alpha = \delta \). We have thus shown that for any \( Y, Z \in \Gamma_2 \), we have
where $\lambda$ is a character, and $\alpha$ is an automorphism of $\Gamma_2$.

(iv) From the discussion in part (i) of this section, we know that there exists a character $\mu$ and an element $A \in \Delta_2$ such that $X^\alpha = \mu(X) A X A^{-1}$ for all $X \in \Gamma_2$. We remark next that if $\xi \in \Delta_{2n}$, the map $\phi$ defined by $\xi^\alpha = \xi \xi^{-1}$ for each $\xi \in \Gamma_{2n}$ is clearly an automorphism of $\Gamma_{2n}$. In particular, let us define an automorphism $\sigma$ of $\Gamma_4$ by

$$\xi^\alpha = (A^{-1} \ast A^{-1}) \xi^\alpha (A \ast A)$$

for all $\xi \in \Gamma_4$.

Calling this new automorphism $\tau$ instead of $\sigma$, we then know that

$$(Y \ast Z)^\tau = (Y \ast Z)^\alpha (I \circ I) (\sigma \ast A) = I \circ I.$$ 

Thence we have

$$(Y \circ Z)^\tau = (Y \circ Z)^\tau (I \circ I)^\tau = \lambda (Z) \mu (Y) Y \ast \lambda (Y) \mu (Z) Z.$$ 

(v) We apply the above results to the 4 generators of $\Gamma_4$, which are given by (see [3])

$$\begin{align*}
K_1 &= I \circ I, & K_2 &= T + T^{-1}, & S_0 &= S \ast I, & T_0 &= T \ast I
\end{align*}$$

(Where $S$ and $T$ are defined by (8)). We have at once

$$\begin{align*}
K_1^\tau &= K_1, & S_0^\tau &= \pm S \ast \pm I, & T_0^\tau &= \pm T \ast \pm I, & S_0 T_0^\tau &= S_0 T_0
\end{align*}$$

(the last equation holding because $S_0 T_0$ is a square).

We use now (and again later) an argument due to Hua [5] to find the possible images $K_2^\tau$.

Observe that

$$\begin{bmatrix}
I^{(2)} & 2n & 0 \\
0 & 0 & 0
\end{bmatrix}
\text{ and }
\begin{bmatrix}
I^{(2)} & 0 & 0 \\
0 & 0 & I^{(2)}
\end{bmatrix}$$

are elements of $\Gamma_4$ which are invariant under $\tau$; their product is also invariant. Hence the group of all elements of $\Gamma_4$ which commute element-wise with the set of matrices of the form

$$\begin{bmatrix}
I^{(2)} & \lambda_1 & 0 \\
0 & 0 & \lambda_2
\end{bmatrix},$$

$\lambda_1, \lambda_2$ even integers,

is mapped onto itself by $\tau$. This group is readily found to consist of all elements of $\Gamma_4$ of the form
\[
\begin{pmatrix}
E & B \\
0 & E
\end{pmatrix},
\text{ where } E = \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \text{ and } EB' = BE.
\]

The squares of these elements are the matrices of \(\Gamma_4\) given by

\[
\begin{pmatrix}
I & M \\
0 & I
\end{pmatrix},
\]

where \(M\) is symmetric and all elements of \(M\) are even. Hence

\[
\begin{pmatrix}
I & M \\
0 & I
\end{pmatrix}^* = \begin{pmatrix} I & M_1 \\ 0 & I \end{pmatrix}
\]

for even symmetric \(M\), and \(M_1\) is also even and symmetric.

Next observe that

\[
\begin{pmatrix}
0 & I \\
-I & 0
\end{pmatrix}\begin{pmatrix} I & M \\ 0 & I \end{pmatrix}\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} = \begin{pmatrix} I & 0 \\ -M & I \end{pmatrix}.
\]

Since

\[
\begin{pmatrix}
0 & I \\
-I & 0
\end{pmatrix}^* = (S\cdot S)^* = \pm S\cdot \pm S,
\]

we see that for even symmetric \(N\) we have

\[
\begin{pmatrix} I & 0 \\ N & I \end{pmatrix}^* = \begin{pmatrix} I & 0 \\ N_1 & I \end{pmatrix},
\]

with \(N_1\) even and symmetric.

Now let \(\Sigma\) be the group of matrices of the form (9) with \(M\) even and symmetric, and let \(\Sigma'\) be the group of matrices given by (10) with even symmetric \(N\). Then \(\tau\) maps both \(\Sigma\) and \(\Sigma'\) onto themselves, and so any element commuting with both \(\Sigma\) and \(\Sigma'\) maps into another such element. However, these elements are precisely the rotations in \(\Gamma_4\). Hence for each \(U \in \Omega_2\) we have

\[
\begin{pmatrix} U & 0 \\ 0 & U^{-1} \end{pmatrix}^* = \begin{pmatrix} U^\sigma & 0 \\ 0 & (U^\sigma)^{-1} \end{pmatrix}.
\]

The map \(U \rightarrow U^\sigma\) is an automorphism \(\sigma\) of \(\Omega_2\), and we already know from \(\Psi^\sigma = \Psi\) and \(\Psi_1^\sigma = \Psi_1\) that \(S^\sigma = S\). Consequently (see [4]) there are only 4 possibilities for \(T^\sigma\), given by

\[
\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix}.
\]

(vi) We next apply \(\tau\) to both sides of the equation
\[(S \cdot I) \begin{pmatrix} T^2 & 0 \\ 0 & T^{-2} \end{pmatrix} (S \cdot I)^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ -2 & 0 & 0 & 1 \end{bmatrix}\]

and use equation (10). This shows that
\[(T^2)^o = \begin{pmatrix} 1 & \pm 2 \\ 0 & 1 \end{pmatrix},\]

and so either
\[T^o = T \text{ or } T^o = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} = T_1 (\text{say}).\]

Now we show that \(S_0^o = \pm S_0, T_0^o = \pm T_0.\) For, \(S_0^o = S_0, T_0^o = T_0,\) or \(S_0^o = T_1 + T_1^{-1},\) and it is easily verified that \(\pm (T \cdot I)\) does not commute with either of these two possible images of \(T_0.\) Therefore \(T_0^o = \pm (T \cdot I),\) whence \(S_0^o = \pm (S \cdot I).\)

Next suppose that \(S_2^o = T_1 + T_1^{-1};\) then define \(\tau_1\) by \(X^o = \mathbb{F} X \mathbb{F}^{-1}.\) Then \(S_0^o = S_0, T_0^o = T_0,\) and \(R_1^o = -R_1, R_2^o = -R_2.\) We have therefore shown that apart from an "inner" automorphism by an element of \(\Delta_4,\) every automorphism \(\tau\) of \(\Gamma_4\) can be described by
\[(R_1, R_2, S_0, T_0)^o = (\pm R_1, \pm R_2, \pm S_0, \pm T_0),\]

and the signs must satisfy
\[R_1^o R_2^o = R_2 R_1, \quad S_0^o T_0^o = S_0 T_0\]

Thus every automorphism \(\tau\) is given by
\[X^o = \theta(X) X X^{-1}\] for all \(X \in \Gamma_4,\)

where \(X \in \Delta_4\) and \(\theta\) is a character of \(\Gamma_4.\)

(vii) It will be shown in a future note by the author [9] that \(\Gamma_4\) has exactly one nontrivial character \(\theta,\) where \(\theta\) is the map of \(\Gamma_4\) into \(\{\pm 1\}\) induced by
\[\theta(R_1) = \theta(R_2) = \theta(S_0) = \theta(T_0) = -1.\]

This fact, together with the preceding discussion, settles the question of automorphisms of \(\Gamma_4.\) It will also be shown in the same note that \(\Gamma_{2n}, n > 2,\) has no nontrivial characters. This result will be needed in finding all automorphisms of \(\Gamma_{2n}.

5. Automorphisms of \(\Gamma_{2n}, n > 2.\) We are now ready to prove, by induction on \(n,\) the following result:

**Theorem 3.** For \(n > 2,\) every automorphism \(\tau\) of \(\Gamma_{2n}\) is given by
\[ x^r = A x A^{-1}, \]
where \( A \in \Delta_{2n} \) depends only on \( r \).

**Proof.** (i) Let \( n \geq 3 \); by the induction hypothesis and our previous results, we may assume that every automorphism \( \sigma \) of \( \Gamma_2(n-1) \) is given by
\[ X^\sigma = \theta(X) \cdot A X A^{-1}, \]
where \( A \in \Delta_{2(n-1)} \) and \( \theta \) is a character of \( \Gamma_2(n-1) \). Let \( \tau \) be an automorphism of \( \Gamma_{2n} \), and set
\[ \Psi = - I^{(2)} \cdot I^{2(n-1)}. \]
We see from Theorem 2 that after changing \( \tau \) by a suitable inner automorphism, we may take \( \Psi = \pm \Psi \). The elements of \( \Gamma_{2n} \) which commute with \( \Psi \) are of the form \( Y_1 \cdot Z_1 \), \( Y_1 \in \Gamma_2 \), \( Z_1 \in \Gamma_2(n-1) \), so that we have
\[ (Y_1 \cdot Z_1)^\tau = Y_2 \cdot Z_2. \]
Again we set
\[ (Y \cdot I)^\tau = Y^a \cdot Y^b \quad \text{for } Y \in \Gamma_2, \]
\[ (I \cdot Z)^\tau = Z^r \cdot Z^s \quad \text{for } Z \in \Gamma_2(n-1). \]
Then \( \Gamma_2^a \) and \( \Gamma_2(n-1) \) commute elementwise, and \( \Gamma_2 \) is their product. As in §4, part (ii), we deduce that one of \( \Gamma_2^a \), \( \Gamma_2(n-1) \) is \( \Gamma_2 \), and the other is contained in \( \{ \pm I \} \).

(ii) For the moment set \( \mathcal{A} = \Gamma_2^a \), \( \mathcal{B} = \Gamma_2(n-1) \). Then \( \mathcal{A} \) and \( \mathcal{B} \) commute elementwise, and their product is \( \Gamma_{2(n-1)} \). This shows that \( \mathcal{B} \) is a normal subgroup of \( \Gamma_{2(n-1)} \). We shall show that \( \mathcal{A} \subseteq \{ \pm I \} \), \( \mathcal{B} = \Gamma_2(n-1) \), and that \( \delta \) is an automorphism.

For each involution \( W \in \Gamma_2(n-1) \) we have \( (W^2) = I^2 = I \). Suppose that \( W^2 = \pm I \) for every involution \( W \in \Gamma_{2(n-1)} \); since the involutions in \( \Gamma_{2(n-1)} \) generate all of \( \Gamma_{2(n-1)} \) (this follows readily from [3]), this would mean that \( \mathcal{B} \subseteq \{ \pm I \} \), and so \( \beta \) would map \( \Gamma_2 \) homomorphically onto \( \Gamma_2(n-1) \). We may then show that \( \beta \) is an isomorphism; for, suppose that \( Y^\beta = I \), \( Y \neq I \). Then
\[ (Y \cdot I)^\tau = Y^a \cdot I. \]
Since \( \mathcal{B} \subseteq \{ \pm I \} \), certainly \( \Gamma_{2(n-1)} \) is not contained in \( \{ \pm I \} \), and so \( \Gamma_2^a \subseteq \{ \pm I \} \), that is, \( a \) is a character. Therefore \( Y^a = \pm I \). But \( Y^a = I \) is impossible, since then \( (Y \cdot I)^\tau = I^{(2n)} \) and \( Y = I \). On the other hand, \( Y^a = - I \) is impossible, since in that case \( (Y \cdot I)^\tau = \Psi \), so \( (Y \cdot I) = \pm \Psi \). Therefore we would have \( Y = - I \), and this gives a contradiction because \( - I = S^2 \), and \( a \) a character, together imply \( - I^a = I \). Therefore \( \beta \) is an isomorphism. However, this is itself impossible because \( \Gamma_2 \) has no involutions other than \( \pm I \), whereas \( \Gamma_{2(n-1)} \) has such involutions for \( n > 2 \).

We conclude from the above that there is at least one involution \( W \in \Gamma_{2(n-1)} \).
for which \( W^g \neq \pm I \). However, \( B \) is a normal subgroup of \( \Gamma_{2(n-1)} \), and \( W^g \in B \). Therefore \( B \) contains all of the conjugates of \( W^g \) in \( \Gamma_{2(n-1)} \). It is not difficult to see that if \( W^g \neq \pm I \), the only elements of \( \Gamma_{2(n-1)} \) which commute elementwise with all conjugates of \( W^g \) are \( \pm I \). Hence \( A \subseteq \{ \pm I \} \), and \( B = \Gamma_{2(n-1)} \). Consequently

\[
(Y \cdot Z)^r = \theta(Z) Y^\alpha \cdot \lambda(Y) Z^g,
\]

where \( \theta \) and \( \lambda \) are characters, \( \alpha \) is a homomorphism of \( \Gamma_2 \) onto itself, and \( \delta \) a homomorphism of \( \Gamma_{2(n-1)} \) onto itself. We deduce readily that \( \alpha \) and \( \delta \) are automorphisms, whence incidentally \( \Psi^r = \Psi \).

By the discussion at the beginning of the proof, we know that there exist matrices \( C \in \Omega_2 \), \( D \in \Delta_{2(n-1)} \), and characters \( \mu \), \( \nu \) such that

\[
Y^\alpha = \mu(Y) CYC^{-1}, \quad Z^g = \nu(Z) DZD^{-1}.
\]

If \( C \cdot D \in \Delta_{2n} \), define \( \tau_1 \) by

\[
\Psi^{\tau_1} = (C \cdot D)^{-1} \Psi(C \cdot D),
\]

so that

\[
(Y \cdot Z)^{\tau_1} = \theta(Z) \mu(Y) Y \cdot \lambda(Y) \nu(Z) Z.
\]

However, possibly \( C \cdot D \in \Delta_{2n} \). In that case, if \( K = (-1) \pm (1) \), then \( CK \cdot D \in \Delta_{2n} \), and we define \( \tau_2 \) by

\[
\Psi^{\tau_2} = (CK \cdot D)^{-1} \Psi(CK \cdot D).
\]

Thus, changing notation, we may assume that

\[
(Y \cdot Z)^r = \theta(Z) \mu(Y) HYH^{-1} \cdot \lambda(Y) \nu(Z) Z,
\]

for any \( Y \in \Gamma_2 \), \( Z \in \Gamma_{2(n-1)} \), where \( \theta \), \( \mu \), \( \lambda \), \( \nu \) are characters, and where either \( H = I(2) \) or \( H = K \).

(iii) Suppose now that \( Y \in \Gamma_2 \), \( Z \in \Gamma_{2(n-1)} \) are given by

\[
Y = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad Z = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.
\]

Then define \( Y \cdot Z \) to be the \( 2n \times 2n \) matrix \( \mathbb{M} \) obtained by placing the elements of \( Y \) at the intersections of the \( i \)th and \( (n+i) \)th rows and columns, filling in the remaining places in those rows and columns with 0's, and letting the matrix obtained from \( \mathbb{M} \) by deleting the \( i \)th and \( (n+i) \)th rows and columns be identical with \( Z \). Then \( Y \cdot Z \) is a generalization of the previously defined symplectic direct sum, and in fact \( Y \cdot Z = Y \cdot Z \).

Now set

\[
\Psi_i = - I(2) \cdot i I_{2(n-1)} = I(2) \cdot Q_i, \text{ say}.
\]

Then \( Q_i \) is a square in \( \Gamma_{2(n-1)} \) (since \( -I = S^2 \)), and so from (11) we have
\( \Psi_i = I \cdot Q_i = \Psi_i. \)

As before it then follows for \( Y \in \Gamma_2, \ Z \in \Gamma_{2(n-1)} \) that

\[
(Y^i Z)^\tau = (F_i(Z)f_i(Y)A_i(YA_i^{-1})^i(g_i(Y)G_i(Z)B_iZB_i^{-1}),
\]

where \( A_i \in \Omega_2, \ B_i \in \Delta_{2(n-1)} \), and \( F_i, f_i, g_i, G_i \) are characters.

(iv) Next let \( X \) and \( Y \in \Gamma_2, \ Z \in \Gamma_{2(n-2)} \). Applying \( \tau \) to both sides of the equation

\[
X \cdot (Y \cdot Z) = Y^2(X \cdot Z)
\]

and using (12), we obtain

\[
[F_1(Y \cdot Z)f_1(X)A_1(XA_1^{-1}) \cdot [g_1(X)G_1(Y \cdot Z)B_1(Y \cdot Z)B_1^{-1}] = [F_2(X \cdot Z)f_2(Y)A_2(YA_2^{-1}) \cdot [g_2(Y)G_2(X \cdot Z)B_2(X \cdot Z)B_2^{-1}].
\]

In particular for \( X = -I, \ Y = I, \ Z = I \) this yields

\[
B_2(-I \cdot I)B_2^{-1} = -I \cdot I,
\]

so that

\[
B_2 = \pm A_1 \cdot C_2
\]

and further

\[
B_1 = \pm A_2 \cdot \pm C_2.
\]

We use these expressions for \( B_1 \) and \( B_2 \) in (13) and obtain

\[
F_1(Y \cdot Z)f_1(X) = g_2(Y)G_2(X \cdot Z),
F_2(X \cdot Z)f_2(Y) = g_1(X)G_1(Y \cdot Z),
g_1(X)G_1(Y \cdot Z) = g_2(Y)G_2(X \cdot Z).
\]

These imply that \( f_1 = g_1 \) and \( f_2 = g_2 \).

Continuing in this way we see that each \( B_i \) decomposes completely, and in fact if

\[
\mathcal{D} = A_1 \cdot A_2 \cdot \cdots \cdot A_n,
\]

then \( B_i \) is obtained from \( \mathcal{D} \) by deleting \( A_i \) and possibly changing signs of some of the remaining \( A \)'s. Furthermore, if any \( A_i \in \Delta_4' \), then every \( A_i \in \Delta_4' \), since each \( B_i \in \Delta_{2(n-1)} \). Therefore \( \mathcal{D} \in \Delta_{2n} \). After a further inner automorphism of \( \Gamma_{2n} \) by a factor of \( \mathcal{D}^{-1} \), we may assume hereafter that

\[
(Y^i Z)^\tau = f_i(Y)[F_i(Z)Y^i G_i(Z)B_iZB_i^{-1}]
\]

for \( Y \in \Gamma_2, \ Z \in \Gamma_{2(n-1)} \), where \( f_i, F_i \) and \( G_i \) are characters and each \( B_i \) is of the form \((\pm I) \cdot \cdots \cdot (\pm I)\), and in fact we may take \( B_1 = I \).

(v) Define
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\[
U_1 = \begin{bmatrix}
0 & 0 \cdots & 0 & 1 \\
1 & 0 \cdots & 0 & 0 \\
0 & 1 \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 \cdots & 1 & 0
\end{bmatrix}, \quad U_2 = T + I^{(n-2)},
\]

where \( T \) is given by (8). Then the generators of \( \Gamma_2 \) are (see [3]):

\[
R_1 = U_1 + U_1^{-1}, \quad R_2 = U_2 + U_2^{-1}, \quad \mathcal{S}_0 = T \cdot I, \quad \mathcal{G}_0 = S \cdot I.
\]

From (14) we find at once that

\[
\mathcal{L}_0' = \pm \mathcal{L}_0, \quad \mathcal{G}_0' = \pm \mathcal{G}_0, \quad \text{and} \quad \mathcal{G}_0 \mathcal{S}_0' = \mathcal{G}_0 \mathcal{S}_0.
\]

Next, the rotations of \( \Gamma_2 \) map onto rotations under \( \tau \), since the rotations are generated by the elements \( Y^i Z \), \( i = 1, \cdots, n \), where \( Y \) and \( Z \) have the forms

\[
Y = \begin{pmatrix}
a & 0 \\
0 & d
\end{pmatrix}, \quad Z = \begin{pmatrix}
A & 0 \\
0 & D
\end{pmatrix},
\]

and the image of any such \( Y^i Z \) is of the same kind. Therefore \( \tau \) induces an automorphism \( \sigma \) on the group \( \Omega_n \), where

\[
\begin{pmatrix}
V & 0 \\
0 & V^{-1}
\end{pmatrix}^\tau = \begin{pmatrix}
V^\sigma & 0 \\
0 & (V^\sigma)^{-1}
\end{pmatrix}.
\]

We then know [4] that there exists \( H \in \Omega_n \) such that

\[
V^\sigma = HV^\omega H^{-1}
\]

for all \( V \in \Omega_n \), where either \( V^\omega = V \) for all \( V \) or \( V^\omega = V^{-1} \) for all \( V \).

We know furthermore that \( \tau \) maps every rotation \( \mathcal{P}_i \) onto itself, from which we see that \( H \) is diagonal, with diagonal elements \( \pm 1 \)'s. Replace \( \tau \) by \( \tau_1 \) defined by

\[
\mathcal{X}_1 = (H + H) \mathcal{X} (H + H)
\]

and change notation. We again have \( \mathcal{L}_0' = \pm \mathcal{L}_0, \mathcal{G}_0' = \pm \mathcal{G}_0, \) and \( \mathcal{G}_0 \mathcal{S}_0' = \mathcal{G}_0 \mathcal{S}_0 \), but now \( V^\sigma = V^\omega \) for each \( V \in \Omega_n \). The argument given in §4, parts (iii) and (iv) shows that \( \mathcal{R}_2' = T'^{-1} + T \) is impossible, so \( V^\sigma = V \) for all \( V \in \Omega_n \). Therefore \( \tau \) is given by

\[
(R_1, R_2, L_0, G_0)^\tau = (R_1, R_2, \pm L_0, \pm G_0).
\]

However, as we have already mentioned, \( \Gamma_2 \) has no nontrivial character for \( n \geq 3 \). Hence \( L_0' = L_0, \mathcal{G}_0' = \mathcal{G}_0 \). This completes the proof of the theorem.
6. We remark finally that if $M \in \Gamma_{2n}$ is given by (3), then

$$M^{-1} = \begin{pmatrix} D & -C \\ -B & A \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix},$$

so the automorphism $\sigma: M^\sigma = M^{-1}$ is inner.

Furthermore, any element of $\Delta_{2n}$ can be written as the product of an element of $\Gamma_{2n}$ and $-I^{(n)} + I^{(n)}$, so every automorphism of $\Gamma_{2n}$ can be obtained by using inner automorphisms by elements in $\Gamma_{2n}$, coupled with the automorphism

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \rightarrow \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} A & -B \\ -C & D \end{pmatrix}.$$

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