AUTOMORPHISMS OF THE SYMPLECTIC MODULAR GROUP

BY

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1. **Introduction.** Let Ω_n denote the unimodular group consisting of all $n\times n$ integral matrices of determinant ± 1, and let $I^{(n)}$ be the identity matrix in Ω_n. We shall use 0 to denote a null matrix whose size is determined by the context, X' for the tranpose of X, and $X \dotplus Y$ for the direct sum of X and Y. We call an integral matrix *primitive* if the greatest common divisor of its maximal size minors is 1.

Define

$$\mathfrak{F} = \begin{pmatrix} 0 & I^{(n)} \\ -I^{(n)} & 0 \end{pmatrix}, \tag{1}$$

and let the symplectic group Sp_{2n} consist of all rational $2n\times 2n$ matrices $\mathfrak{M}$ satisfying

$$\mathfrak{M}\mathfrak{F}\mathfrak{M}' = \mathfrak{F}. \tag{2}$$

We define the symplectic modular group Γ_{2n} to be the group of integral matrices in Sp_{2n}. Although we shall not do so in this paper, it is sometimes more convenient to work with the factor group of Γ_{2n} over its center $\pm\mathfrak{I}$; see [1; 2; 3](1). We may also define an extended group Δ_{2n} consisting of all integral matrices $\mathfrak{M}$ for which $\mathfrak{M}\mathfrak{F}\mathfrak{M}' = \pm\mathfrak{F}$.

The automorphisms of Sp_{2n} (over any field) have previously been determined [5], as have the automorphisms of Γ_2 (see [4]). The object of this paper is to determine all automorphisms of Γ_{2n}. Let us call a homomorphism of Γ_{2n} into $\{\pm 1\}$ a *character*. Then we shall prove that every automorphism τ of Γ_{2n} is given by

$$\mathfrak{X}^\tau = \psi(\mathfrak{X})\mathfrak{A}\mathfrak{X}\mathfrak{A}^{-1} \qquad \text{for all } \mathfrak{X} \in \Gamma_{2n},$$

where ψ is a character, and $\mathfrak{A}\in\Delta_{2n}$. We may remark at this point that the mapping σ defined by

$$\mathfrak{X}^\sigma = \mathfrak{X}'^{-1} \qquad \text{for all } \mathfrak{X} \in \Gamma_{2n}$$

is obviously an automorphism. As we shall see, however, it is an inner automorphism.

Let us set

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(1) Numbers in brackets refer to the bibliography at the end of this paper.

(3) $$\mathfrak{M} = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where A, B, C, D are integral $n\times n$ matrices. Then $\mathfrak{M}\in\Gamma_{2n}$ if and only if the following conditions are satisfied:

(4) $$AB' \text{ symmetric, } CD' \text{ symmetric, } AD'-BC'=I.$$

We single out for future use certain types of elements of Γ_{2n}:

(1) Translations:

$$\mathfrak{T} = \begin{pmatrix} I & S \\ 0 & I \end{pmatrix} \text{ or } \begin{pmatrix} I & 0 \\ S & I \end{pmatrix}, \qquad S \text{ symmetric.}$$

(2) Rotations:

$$\mathfrak{R} = \begin{pmatrix} U & 0 \\ 0 & U'^{-1} \end{pmatrix}, \qquad U \in \Omega_n.$$

(3) Semi-involutions:

$$\mathfrak{S} = \begin{pmatrix} J & I-J \\ J-I & J \end{pmatrix}, \qquad J \text{ diagonal with diagonal elements 0's and 1's.}$$

Further, if $\mathfrak{M}$ given by (3) is in Γ_{2n}, then

(5) $$\mathfrak{M}^{-1} = \begin{pmatrix} D' & -B' \\ -C' & A' \end{pmatrix}.$$

Finally, if

$$\mathfrak{M}_i = \begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix} \in \Gamma_{2n_i} \qquad (i = 1, 2),$$

we define the *symplectic direct sum* $\mathfrak{M}_1 * \mathfrak{M}_2 \in \Gamma_{2(n_1+n_2)}$ by

$$\mathfrak{M}_1 * \mathfrak{M}_2 = \begin{bmatrix} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & B_2 \\ C_1 & 0 & D_1 & 0 \\ 0 & C_2 & 0 & D_2 \end{bmatrix}.$$

We may remark that as $\mathfrak{M}$ ranges over all elements of Γ_{2n}, the matrix $[-I^{(n)}\dotplus I^{(n)}]\mathfrak{M}$ ranges over all elements in $\Delta'_{2n}=\{\mathfrak{X}\in\Delta_{2n}:\mathfrak{X}\notin\Gamma_{2n}\}$. Thence $\mathfrak{M}_i\in\Delta'_{2n_i}$ $(i=1,\ 2)$ implies $\mathfrak{M}_1 * \mathfrak{M}_2\in\Delta'_{2(n_1+n_2)}$. However, $\mathfrak{M}_1\in\Gamma_{2n_1}$ and $\mathfrak{M}_2\in\Delta_{2n_2}$ implies $\mathfrak{M}_1 * \mathfrak{M}_2\notin\Delta_{2(n_1+n_2)}$.

2. **Involutions in Γ_{2n}.** It is known [4] that as x, y, and z range over all non-negative integers such that $2x+y+z=n$, the matrix

$$W(x, y, z) = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \dotplus \cdots \dotplus \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \dotplus (-I)^{(y)} \dotplus I^{(z)} \tag{6}$$

(where x 2×2 blocks occur) gives a complete set of nonconjugate involutions in Ω_n. By an $[x, y, z]$ involution in Ω_n we mean any conjugate of $W(x, y, z)$ in Ω_n. Now define

$$\mathfrak{W}(x, y, z) = W(x, y, z) \dotplus W'(x, y, z) \in \Gamma_{2n}.$$

THEOREM 1. *The matrices* $\mathfrak{W}(x, y, z)$ *with* $2x+y+z=n$ *give a complete set of nonconjugate involutions in* Γ_{2n}.

Proof. We use induction on n. The result is trivial for $n=1$, so now let $\mathfrak{X}$ be an involution in Γ_{2n}, $n>1$. From $\mathfrak{X}^2=I^{(2n)}$ we conclude that the characteristic roots of $\mathfrak{X}$ are 1's and -1's. Let ϵ be a characteristic root of $\mathfrak{X}$; then there exists a primitive row vector $\mathfrak{x}$ such that $\mathfrak{x}\mathfrak{X}=\epsilon\mathfrak{x}$. We can then find [6] a matrix $\mathfrak{Y}\in\Gamma_{2n}$ whose first row is $\mathfrak{x}$. In that case the first row of $\mathfrak{X}_1=\mathfrak{Y}\mathfrak{X}\mathfrak{Y}^{-1}$ is $(\epsilon\ 0 \cdots 0)$. Since $\mathfrak{X}_1$ is an involution in Γ_{2n}, we obtain

$$\mathfrak{X}_1 = \begin{bmatrix} \epsilon & 0\cdots 0 & 0 & 0\cdots 0 \\ * & & 0 & \\ \vdots & A_1 & \vdots & B_1 \\ * & & 0 & \\ * & *\cdots * & \epsilon & 0\cdots 0 \\ * & & * & \\ \vdots & C_1 & \vdots & D_1 \\ * & & * & \end{bmatrix},$$

where

$$\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}$$

is itself an involution in $\Gamma_{2(n-1)}$. Continuing this procedure, we see that $\mathfrak{X}$ is conjugate in Γ_{2n} to a matrix of the form

$$\mathfrak{X}_2 = \begin{pmatrix} A & 0 \\ C & D \end{pmatrix}.$$

From the fact that $\mathfrak{X}_2$ is an involution in Γ_{2n}, we deduce at once that A is an involution in Ω_n, and $D=A'^{-1}$. However,

$$\begin{pmatrix} U & 0 \\ 0 & U'^{-1} \end{pmatrix}\begin{pmatrix} A & 0 \\ C & D \end{pmatrix}\begin{pmatrix} U^{-1} & 0 \\ 0 & U' \end{pmatrix} = \begin{pmatrix} UAU^{-1} & 0 \\ \bar{C} & U'^{-1}DU' \end{pmatrix},$$

and so by choosing $U\in\Omega_n$ properly, we find that $\mathfrak{X}$ is conjugate to $\mathfrak{X}_3$ given by

$$\mathfrak{X}_3 = \begin{pmatrix} W(x, y, z) & 0 \\ C & W'(x, y, z) \end{pmatrix}$$

with a new C. Since $\mathfrak{X}_3\in\Gamma_{2n}$ is an involution, we have

(7) $\qquad CW$ symmetric, $\qquad C$ skew-symmetric.

The proof now splits into two cases:

CASE 1. If either $y\neq 0$ or $z\neq 0$, we may set $W(x, y, z) = W_1 \dotplus (\epsilon)$, $\epsilon=\pm 1$. From (7) we find that

$$\mathfrak{X}_3 = \begin{bmatrix} W_1 & 0 & 0 & 0 \\ 0 & \epsilon & 0 & 0 \\ C_1 & -\mathfrak{x}' & W_1' & 0 \\ \mathfrak{x} & 0 & 0 & \epsilon \end{bmatrix},$$

and that

$$\mathfrak{Z} = \begin{pmatrix} W_1 & 0 \\ C_1 & W_1' \end{pmatrix}$$

is an involution in $\Gamma_{2(n-1)}$. By the induction hypothesis there exist integers x_1, y_1, z_1 with $2x_1+y_1+z_1=n-1$, such that $\mathfrak{Z}$ is conjugate to $\mathfrak{W}(x_1, y_1, z_1)$. For the moment set $P=W(x_1, y_1, z_1)$. Then in Γ_{2n}, $\mathfrak{X}_3$ is conjugate to $\mathfrak{X}_4$, where

$$\mathfrak{X}_4 = \begin{bmatrix} P & 0 & 0 & 0 \\ 0 & \epsilon & 0 & 0 \\ 0 & -\mathfrak{x}' & P' & 0 \\ \mathfrak{x} & 0 & 0 & \epsilon \end{bmatrix}$$

with a new $\mathfrak{x}$. But then

$$\mathfrak{X}_5 = \mathfrak{S}\mathfrak{X}_4\mathfrak{S}^{-1} = \begin{bmatrix} P & 0 & 0 & 0 \\ \mathfrak{x} & \epsilon & 0 & 0 \\ 0 & 0 & P' & \mathfrak{x}' \\ 0 & 0 & 0 & \epsilon \end{bmatrix} \text{ where } \mathfrak{S} = \begin{bmatrix} I^{(n-1)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & I^{(n-1)} & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}.$$

Since $\mathfrak{X}_5$ is now a direct sum $W\dotplus W'$, where W is an involution in Ω_n, the result follows upon transforming $\mathfrak{X}_5$ by a suitably chosen rotation in Γ_{2n}.

CASE 2. If both y and z are 0, we write $W(x, y, z)=L\dotplus W_1$, where

$$L = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}.$$

Then, as before, $\mathfrak{X}_3$ is conjugate to $\mathfrak{X}_4$ given by

$$\mathfrak{X}_4 = \begin{bmatrix} L & 0 & 0 & 0 \\ 0 & W_1 & 0 & 0 \\ \begin{smallmatrix} 0 & b \\ -b & 0 \end{smallmatrix} & B & L' & 0 \\ -B' & 0 & 0 & W_1' \end{bmatrix}.$$

However,

$$\mathfrak{M} = \begin{bmatrix} 0 & 0 & I^{(2)} & 0 \\ 0 & I^{(n-2)} & 0 & 0 \\ -I^{(2)} & 0 & 0 & 0 \\ 0 & 0 & 0 & I^{(n-2)} \end{bmatrix} \begin{bmatrix} I^{(n)} & 0 \\ & \\ 0 \dotplus b \dotplus 0^{(n-2)} & I^{(n)} \end{bmatrix} \in \Gamma_{2n},$$

and we have

$$\mathfrak{M}\mathfrak{X}_4\mathfrak{M}^{-1} = \begin{pmatrix} L' & B \\ 0 & W_1 \end{pmatrix} \dotplus \begin{pmatrix} L & 0 \\ B' & W_1' \end{pmatrix}.$$

The result then follows as in the previous case.

We have thus shown that any involution $\mathfrak{X} \in \Gamma_{2n}$ is conjugate to some $\mathfrak{W}(x, y, z)$. On the other hand, if $\mathfrak{W}(x, y, z)$ and $\mathfrak{W}(x_0, y_0, z_0)$ were conjugate in Γ_{2n}, they would certainly be conjugate in Ω_{2n}. This implies [4] that $x = x_0$, $y = y_0$, and $z = z_0$.

The conjugates of $\mathfrak{W}(x, y, z)$ in Γ_{2n} will be called (x, y, z) involutions.

3. **Characterization of the $\pm(0, 1, n-1)$ involutions.** In Sp_{2n}, every involution is conjugate to one of the form $I^{(2p)} \dotplus -I^{(2q)}$, with $p+q=n$. Any involution in the class of $I^{(2p)} \dotplus -I^{(2q)}$ is said to have *signature* $\{p, q\}$ (see [5]). One easily proves that any (x, y, z) involution in Γ_{2n} has signature $\{x+z, x+y\}$, and that the negative of an (x, y, z) involution is of type (x, z, y).

It is known that an abelian set of involutions of signature $\{p, q\}$ in Sp_{2n} cannot contain more than $C_{n,p}$ elements (see [5, Theorem 2; 7, §19]). We shall use this fact in proving the following basic result:

THEOREM 2. *Under any automorphism of* Γ_{2n}, *the image of a* $(0, 1, n-1)$ *involution is either a* $(0, 1, n-1)$ *involution or a* $(0, n-1, 1)$ *involution.*

Proof. (i) An abelian set of involutions in Γ_{2n}, each of type (x, y, z), we shall call an (x, y, z) set. Let $f(x, y, z)$ be the number of elements in an (x, y, z) set of largest size. The above-quoted result shows that

$$f(x, y, z) \leqq C_{n, x+z},$$

so for $(x, y, z) = \pm(0, 0, n)$, $\pm(0, 1, n-1)$, $\pm(1, 0, n-2)$ we have $f(x, y, z) \leqq n$.

We now show that $f(x, y, z) > n$ except for the 6 cases given above.

From an abelian set $\mathfrak{K}$ of $[x, y, z]$ involutions in Ω_n, one obtains an (x, y, z) set in Γ_{2n} by taking the set of matrices $U \dotplus U'^{-1}$, $U \in \mathfrak{K}$. We know, however, that there exist abelian sets of $[x, y, z]$ involutions in Ω_n containing more than n elements, except for the 6 cases listed above (see [8, §§12 and 13]).

(ii) The $\pm(0, 0, n)$ involutions in Γ_{2n} are $\pm I^{(2n)}$, so that certainly a $(0, 1, n-1)$ involution cannot be mapped onto a $\pm(0, 0, n)$ involution by an automorphism of Γ_{2n}. It remains to prove that the image cannot be of type $\pm(1, 0, n-2)$. To begin with, a simple calculation shows that two rotations $U \dotplus U'^{-1}$ and $V \dotplus V'^{-1}$ are conjugate in Γ_{2n} if and only if U and V are conjugate in Ω_n. For $n > 2$, there are at least two nonconjugate $[1, 0, n-2]$ sets in Ω_n, each containing n elements; on the other hand, there is a unique (up to conjugacy) abelian set of n $[0, 1, n-1]$ involutions in Ω_n (see [8, §12]). Hence for $n > 2$, the image of a $(0, 1, n-1)$ involution in Γ_{2n} must be of type $\pm(0, 1, n-1)$.

(iii) The case $n = 1$ is trivial, and so we are left with $n = 2$. Now we have

$$I^{(2)*} - I^{(2)} = \left(I^{(2)*}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)^2,$$

so any $(0, 1, 1)$ involution in Γ_4 is the square of some element of Γ_4. We show that the $(1, 0, 0)$ involutions in Γ_4 are not squares. For suppose that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^2 = \begin{pmatrix} L & 0 \\ 0 & L' \end{pmatrix}, \text{ where } \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_4 \text{ and } L = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}.$$

From (5) we then have

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} L & 0 \\ 0 & L' \end{pmatrix}\begin{pmatrix} D' & -B' \\ -C' & A' \end{pmatrix}.$$

This implies that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{bmatrix} a & 0 & 0 & 0 \\ (a-d)/2 & d & 0 & b \\ c & -2c & a & (a+d)/2 \\ -2c & 4c & 0 & -d \end{bmatrix}.$$

Using $AD' - BC' = I$, we find that

$$-d^2 - 4bc = 1,$$

whence $d^2 \equiv -1 \pmod 4$, since a, b, c, d are integers. This is impossible, and so the theorem is proved.

4. Automorphisms of Γ_4. As is usually the case with determination of automorphisms of a group of matrices, the lower the dimension the more difficult the proof. We begin by stating in (i) some earlier results (see [4]) which will be needed.

(i) The group Δ_2 coincides with Ω_2, and Γ_2 is the subgroup Ω_2^+ consisting of all elements of Ω_2 with determinant $+1$. For the remainder of this paper we let

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{8}$$

Then S and T generate Γ_2, and in any relation $S^{m_1}T^{n_1}S^{m_2}T^{n_2} \cdots = I$ the sum $m_1+n_1+m_2+n_2+ \cdots$ is always even. Hence the elements $X \in \Gamma_2$ can be classified as even or odd according to the parity of the sum of the exponents when X is expressed as a product of powers of S and T. The only nontrivial character of Γ_2 is defined by

$$\epsilon(X) = \begin{cases} 1, & X \text{ even,} \\ -1, & X \text{ odd.} \end{cases}$$

Then every automorphism τ of Γ_2 is given by

$$X^\tau = \lambda(X) A X A^{-1} \qquad \text{for all } X \in \Gamma_2,$$

where λ is a character, and $A \in \Omega_2$.

(ii) Now let τ be any automorphism of Γ_4. After changing τ by a suitable inner automorphism, we may assume that $\mathfrak{P}^\tau = \pm \mathfrak{P}$, where

$$\mathfrak{P} = I^{(2)} * -I^{(2)}.$$

Since $\mathfrak{P}$ and $-\mathfrak{P}$ are conjugate in Γ_4, assume in fact that $\mathfrak{P}^\tau = \mathfrak{P}$. Then any element of Γ_4 which commutes with $\mathfrak{P}$ maps into another such element, so that

$$(Y_1 * Z_1)^\tau = Y_2 * Z_2,$$

where Y_1, Y_2, Z_1, $Z_2 \in \Gamma_2$. Let us set

$$(Y * I)^\tau = Y^\alpha * Y^\beta \qquad \text{for } Y \in \Gamma_2,$$

$$(I * Z)^\tau = Z^\gamma * Z^\delta \qquad \text{for } Z \in \Gamma_2.$$

Then α, β, γ, δ are all homomorphisms of Γ_2 into itself, since

$$(Y_1 * Z_1)(Y_2 * Z_2) = Y_1 Y_2 * Z_1 Z_2.$$

Further, since $Y * I$ and $I * Z$ commute, so do Y^α and Z^γ for all pairs of elements Y, $Z \in \Gamma_2$; also, every element of Γ_2 is a product $Y^\alpha Z^\gamma$ for some such pair. Since $S \in \Gamma_2$, there exists an element $X \in \Gamma_2$ such that $SX^{-1} \in \Gamma_2^\alpha$ and $X \in \Gamma_2^\gamma$. But then X commutes with SX^{-1}, whence $X = \pm I$ or $\pm S$. Therefore either $S \in \Gamma_2^\alpha$ or $S \in \Gamma_2^\gamma$.

Suppose now that $S \in \Gamma_2^\alpha$; since every element of Γ_2^γ commutes with S, we see that $\Gamma_2^\gamma \subset \{\pm I, \pm S\}$. However, $S \in \Gamma_2^\gamma$ would imply the finiteness of Γ_2^α, whence $\Gamma_2 = \Gamma_2^\alpha \Gamma_2^\gamma$ could not be true. Therefore $\Gamma_2^\gamma \subset \{\pm I\}$, and then certainly $\Gamma_2^\alpha = \Gamma_2$. Similarly, one of Γ_2^β, Γ_2^δ is Γ_2, and the other is included in $\{\pm I\}$.

Now we use the fact that $(-\mathfrak{P})^\tau = -\mathfrak{P}$, that is

$$(-I * I)^\tau = -I * I.$$

Therefore $(-I)^\alpha = -I$; but if $\Gamma_2^\alpha \subset \{\pm I\}$, the fact that $-I = S^2$ would imply $(-I)^\alpha = I$. Hence $\Gamma_2^\alpha = \Gamma_2$, $\Gamma_2^\gamma \subset \{\pm I\}$, and therefore $\Gamma_2^\beta \subset \{\pm I\}$, $\Gamma_2^\delta = \Gamma_2$.

Next we prove that α is an automorphism; we need merely prove that $Y^\alpha = I$ implies $Y = I$. But if $Y^\alpha = I$, then $(Y * I)^\tau = I * \pm I$. Since $(I * I)^\tau = I * I$ and $(I * -I)^\tau = I * -I$, this implies that $Y = I$. By the same reasoning, δ is also an automorphism.

(iii) Now define

$$Y_1 \circ Y_2 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \circ \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{bmatrix} 0 & a_1 & 0 & b_1 \\ a_2 & 0 & b_2 & 0 \\ 0 & c_1 & 0 & d_1 \\ c_2 & 0 & d_2 & 0 \end{bmatrix}.$$

Then $Y_1 \circ Y_2 \in \Gamma_4$ if and only if $Y_1, Y_2 \in \Gamma_2$. The elements of Γ_4 which anticommute with $\mathfrak{P}$ are of the form $Y_1 \circ Y_2$, and we have

$$(A * B)(C \circ D) = AC \circ BD,$$
$$(A \circ B)(C * D) = AD \circ BC,$$
$$(A \circ B)(C \circ D) = AD * BC.$$

Suppose now that $(I \circ I)^\tau = U \circ V$. Since $(I \circ I)^2 = I * I$, we have $(U \circ V)^2 = UV * VU = I * I$, so $V = U^{-1}$. But now let

$$\mathfrak{X}^\sigma = (U^{-1} * I)\mathfrak{X}^\tau(U * I).$$

Then $\mathfrak{P}^\sigma = \mathfrak{P}$, σ and τ differ by an inner automorphism, and $(I \circ I)^\sigma = I \circ I$. Changing notation, we henceforth assume $\mathfrak{P}^\tau = \mathfrak{P}$ and $(I \circ I)^\tau = I \circ I$. From

$$(I \circ I)(Y * Z)(I \circ I) = Z * Y$$

we deduce

$$(I \circ I)(Z^\gamma Y^\alpha * Y^\beta Z^\delta)(I \circ I) = Y^\gamma Z^\alpha * Z^\beta Y^\delta.$$

Therefore

$$Z^\gamma Y^\alpha = Z^\beta Y^\delta$$

for all $Y, Z \in \Gamma_2$. Hence $\beta = \gamma$, $\alpha = \delta$. We have thus shown that for any Y, $Z \in \Gamma_2$ we have

$$(Y * Z)^{\tau} = \lambda(Z)Y^{\alpha} * \lambda(Y)Z^{\alpha},$$

where λ is a character, and α is an automorphism of Γ_2.

(iv) From the discussion in part (i) of this section, we know that there exists a character μ and an element $A \in \Delta_2$ such that $X^{\alpha} = \mu(X)AXA^{-1}$ for all $X \in \Gamma_2$. We remark next that if $\mathfrak{B} \in \Delta_{2n}$, the map ϕ defined by $\mathfrak{X}^{\phi} = \mathfrak{B}\mathfrak{X}\mathfrak{B}^{-1}$ for each $\mathfrak{X} \in \Gamma_{2n}$ is clearly an automorphism of Γ_{2n}. In particular, let us define an automorphism σ of Γ_4 by

$$\mathfrak{X}^{\sigma} = (A^{-1} * A^{-1})\mathfrak{X}^{\tau}(A * A) \qquad \text{for all } \mathfrak{X} \in \Gamma_4.$$

Calling this new automorphism τ instead of σ, we then know that

$$(Y * Z)^{\tau} = \lambda(Z)\mu(Y)Y * \lambda(Y)\mu(Z)Z$$

for each pair $Y, Z \in \Gamma_2$, and further that

$$(I \circ I)^{\tau} = (A^{-1} * A^{-1})(I \circ I)(A * A) = I \circ I.$$

Thence we have

$$(Y \circ Z)^{\tau} = (Y * Z)^{\tau}(I \circ I)^{\tau} = \lambda(Z)\mu(Y)Y \circ \lambda(Y)\mu(Z)Z.$$

(v) We apply the above results to the 4 generators of Γ_4, which are given by (see [3])

$$\mathfrak{R}_1 = I \circ I, \qquad \mathfrak{R}_2 = T \dotplus T'^{-1}, \qquad \mathfrak{S}_0 = S * I, \qquad \mathfrak{T}_0 = T * I$$

(where S and T are defined by (8)). We have at once

$$\mathfrak{R}_1^{\tau} = \mathfrak{R}_1, \qquad \mathfrak{S}_0^{\tau} = \pm S * \pm I, \qquad \mathfrak{T}_0 = \pm T * \pm I, \qquad \mathfrak{S}_0^{\tau}\mathfrak{T}_0^{\tau} = \mathfrak{S}_0\mathfrak{T}_0,$$

(the last equation holding because $\mathfrak{S}_0\mathfrak{T}_0$ is a square).

We use now (and again later) an argument due to Hua [5] to find the possible images $\mathfrak{R}_2^{\tau}$. Observe that

$$\begin{bmatrix} I^{(2)} & \begin{matrix} 2n & 0 \\ 0 & 0 \end{matrix} \\ 0 & I^{(2)} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} I^{(2)} & \begin{matrix} 0 & 0 \\ 0 & 2m \end{matrix} \\ 0 & I^{(2)} \end{bmatrix}$$

are elements of Γ_4 which are invariant under τ; their product is also invariant. Hence the group of all elements of Γ_4 which commute element-wise with the set of matrices of the form

$$\begin{bmatrix} I^{(2)} & \begin{matrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{matrix} \\ 0 & I^{(2)} \end{bmatrix}, \qquad \lambda_1, \lambda_2 \text{ even integers},$$

is mapped onto itself by τ. This group is readily found to consist of all elements of Γ_4 of the form

$$\begin{pmatrix} E & B \\ 0 & E \end{pmatrix}, \text{ where } E = \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \text{ and } EB' = BE.$$

The squares of these elements are the matrices of Γ_4 given by

$$\begin{pmatrix} I & M \\ 0 & I \end{pmatrix}, \tag{9}$$

where M is symmetric and all elements of M are even. Hence

$$\begin{pmatrix} I & M \\ 0 & I \end{pmatrix}^\tau = \begin{pmatrix} I & M_1 \\ 0 & I \end{pmatrix}$$

for even symmetric M, and M_1 is also even and symmetric.

Next observe that

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}\begin{pmatrix} I & M \\ 0 & I \end{pmatrix}\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} = \begin{pmatrix} I & 0 \\ -M & I \end{pmatrix}.$$

Since

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}^\tau = (S_*S)^\tau = \pm S_* \pm S,$$

we see that for even symmetric N we have

$$\begin{pmatrix} I & 0 \\ N & I \end{pmatrix}^\tau = \begin{pmatrix} I & 0 \\ N_1 & I \end{pmatrix}, \tag{10}$$

with N_1 even and symmetric.

Now let Σ be the group of matrices of the form (9) with M even and symmetric, and let Σ' be the group of matrices given by (10) with even symmetric N. Then τ maps both Σ and Σ' onto themselves, and so any element commuting with both Σ and Σ' maps into another such element. However, these elements are precisely the rotations in Γ_4. Hence for each $U \in \Omega_2$ we have

$$\begin{pmatrix} U & 0 \\ 0 & U'^{-1} \end{pmatrix}^\tau = \begin{pmatrix} U^\sigma & 0 \\ 0 & (U^\sigma)'^{-1} \end{pmatrix}.$$

The map $U \to U^\sigma$ is an automorphism σ of Ω_2, and we already know from $\mathfrak{P}^\tau = \mathfrak{P}$ and $\mathfrak{R}_1^\tau = \mathfrak{R}_1$ that $S^\sigma = S$. Consequently (see [4]) there are only 4 possibilities for T^σ, given by

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix}.$$

(vi) We next apply τ to both sides of the equation

$$(S*I)\begin{pmatrix} T^2 & 0 \\ 0 & T'^{-2} \end{pmatrix}(S*I)^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ -2 & 0 & 0 & 1 \end{bmatrix}$$

and use equation (10). This shows that

$$(T^2)^\sigma = \begin{pmatrix} 1 & \pm 2 \\ 0 & 1 \end{pmatrix},$$

and so either

$$T^\sigma = T \text{ or } T^\sigma = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} = T_1 \text{ (say)}.$$

Now we show that $\mathfrak{S}_0^\tau = \pm\mathfrak{S}_0$, $\mathfrak{T}_0^\tau = \pm\mathfrak{T}_0$. For, $\mathfrak{R}_2$ and $\mathfrak{T}_0$ commute; hence so do $\mathfrak{R}_2^\tau$ and $\mathfrak{T}_0^\tau$. However, $\mathfrak{R}_2^\tau = \mathfrak{R}_2$ or $\mathfrak{R}_2^\tau = T_1 \dotplus T_1'^{-1}$, and it is easily verified that $\pm(T * -I)$ does not commute with either of these two possible images of $\mathfrak{R}_2$. Therefore $\mathfrak{T}_0^\tau = \pm(T*I)$, whence $\mathfrak{S}_0^\tau = \pm(S*I)$.

Next suppose that $\mathfrak{R}_2^\tau = T_1 \dotplus T_1'^{-1}$; then define τ_1 by $\mathfrak{X}^{\tau_1} = \mathfrak{P}\mathfrak{X}^\tau\mathfrak{P}^{-1}$. Then $\mathfrak{S}_0^{\tau_1} = \mathfrak{S}_0$, $\mathfrak{T}_0^{\tau_1} = \mathfrak{T}_0$, and $\mathfrak{R}_1^{\tau_1} = -\mathfrak{R}_1$, $\mathfrak{R}_2^{\tau_1} = -\mathfrak{R}_2$. We have therefore shown that apart from an "inner" automorphism by an element of Δ_4, every automorphism τ of Γ_4 can be described by

$$(\mathfrak{R}_1, \mathfrak{R}_2, \mathfrak{S}_0, \mathfrak{T}_0)^\tau = (\pm\mathfrak{R}_1, \pm\mathfrak{R}_2, \pm\mathfrak{S}_0, \pm\mathfrak{T}_0),$$

and the signs must satisfy

$$\mathfrak{R}_1^\tau\mathfrak{R}_2^\tau = \mathfrak{R}_1\mathfrak{R}_2, \qquad \mathfrak{S}_0^\tau\mathfrak{T}_0^\tau = \mathfrak{S}_0\mathfrak{T}_0.$$

Thus every automorphism τ is given by

$$\mathfrak{X}^\tau = \theta(\mathfrak{X})\mathfrak{A}\mathfrak{X}\mathfrak{A}^{-1} \qquad \text{for all } \mathfrak{X} \in \Gamma_4,$$

where $\mathfrak{A} \in \Delta_4$ and θ is a character of Γ_4.

(vii) It will be shown in a future note by the author [9] that Γ_4 has exactly one nontrivial character θ, where θ is the map of Γ_4 into $\{\pm 1\}$ induced by

$$\theta(\mathfrak{R}_1) = \theta(\mathfrak{R}_2) = \theta(\mathfrak{S}_0) = \theta(\mathfrak{T}_0) = -1.$$

This fact, together with the preceding discussion, settles the question of automorphisms of Γ_4. It will also be shown in the same note that Γ_{2n}, $n>2$, has no nontrivial characters. This result will be needed in finding all automorphisms of Γ_{2n}.

5. **Automorphisms of Γ_{2n}, $n>2$.** We are now ready to prove, by induction on n, the following result:

THEOREM 3. *For $n>2$, every automorphism τ of Γ_{2n} is given by*

$$\mathfrak{X}^{\tau} = \mathfrak{A}\mathfrak{X}\mathfrak{A}^{-1},$$

where $\mathfrak{A} \in \Delta_{2n}$ *depends only on* τ.

Proof. (i) Let $n \geqq 3$; by the induction hypothesis and our previous results, we may assume that every automorphism σ of $\Gamma_{2(n-1)}$ is given by

$$X^{\sigma} = \theta(X) \cdot AXA^{-1},$$

where $A \in \Delta_{2(n-1)}$ and θ is a character of $\Gamma_{2(n-1)}$. Let τ be an automorphism of Γ_{2n}, and set

$$\mathfrak{P} = -I^{(2)} * I^{2(n-1)}.$$

We see from Theorem 2 that after changing τ by a suitable inner automorphism, we may take $\mathfrak{P}^{\tau} = \pm\mathfrak{P}$. The elements of Γ_{2n} which commute with $\mathfrak{P}$ are of the form $Y_1 * Z_1$, $Y_1 \in \Gamma_2$, $Z_1 \in \Gamma_{2(n-1)}$, so that we have

$$(Y_1 * Z_1)^{\tau} = Y_2 * Z_2.$$

Again we set

$$(Y * I)^{\tau} = Y^{\alpha} * Y^{\beta} \qquad \text{for } Y \in \Gamma_2,$$

$$(I * Z)^{\tau} = Z^{\gamma} * Z^{\delta} \qquad \text{for } Z \in \Gamma_{2(n-1)}.$$

Then Γ_2^{α} and $\Gamma_{2(n-1)}^{\gamma}$ commute elementwise, and Γ_2 is their product. As in §4, part (ii), we deduce that one of Γ_2^{α}, $\Gamma_{2(n-1)}^{\gamma}$ is Γ_2, and the other is contained in $\{\pm I\}$.

(ii) For the moment set $\mathcal{A} = \Gamma_2^{\beta}$, $\mathcal{B} = \Gamma_{2(n-1)}^{\delta}$. Then $\mathcal{A}$ and $\mathcal{B}$ commute elementwise, and their product is $\Gamma_{2(n-1)}$. This shows that $\mathcal{B}$ is a normal subgroup of $\Gamma_{2(n-1)}$. We shall show that $\mathcal{A} \subset \{\pm I\}$, $\mathcal{B} = \Gamma_{2(n-1)}$, and that δ is an automorphism.

For each involution $W \in \Gamma_{2(n-1)}$ we have $(W^{\delta})^2 = I^{\delta} = I$. Suppose that $W^{\delta} = \pm I$ for every involution $W \in \Gamma_{2(n-1)}$; since the involutions in $\Gamma_{2(n-1)}$ generate all of $\Gamma_{2(n-1)}$ (this follows readily from [3]), this would mean that $\mathcal{B} \subset \{\pm I\}$, and so β would map Γ_2 homomorphically *onto* $\Gamma_{2(n-1)}$. We may then show that β is an isomorphism; for, suppose that $Y^{\beta} = I$, $Y \neq I$. Then

$$(Y * I)^{\tau} = Y^{\alpha} * I.$$

Since $\mathcal{B} \subset \{\pm I\}$, certainly $\Gamma_{2(n-1)}^{\gamma}$ is not contained in $\{\pm I\}$, and so $\Gamma_2^{\alpha} \subset \{\pm I\}$, that is, α is a character. Therefore $Y^{\alpha} = \pm I$. But $Y^{\alpha} = I$ is impossible, since then $(Y * I)^{\tau} = I^{(2n)}$ and $Y = I$. On the other hand, $Y^{\alpha} = -I$ is impossible, since in that case $(Y * I)^{\tau} = \mathfrak{P}$, so $(Y * I) = \pm\mathfrak{P}$. Therefore we would have $Y = -I$, and this gives a contradiction because $-I = S^2$, and α a character, together imply $(-I)^{\alpha} = I$. Therefore β is an isomorphism. However, this is itself impossible because Γ_2 has no involutions other than $\pm I$, whereas $\Gamma_{2(n-1)}$ has such involutions for $n > 2$.

We conclude from the above that there is at least one involution $W \in \Gamma_{2(n-1)}$

for which $W^\delta \neq \pm I$. However, $\mathcal{B}$ is a normal subgroup of $\Gamma_{2(n-1)}$, and $W^\delta \in \mathcal{B}$. Therefore $\mathcal{B}$ contains all of the conjugates of W^δ in $\Gamma_{2(n-1)}$. It is not difficult to see that if $W^\delta \neq \pm I$, the only elements of $\Gamma_{2(n-1)}$ which commute elementwise with all conjugates of W^δ are $\pm I$. Hence $\mathcal{A} \subset \{\pm I\}$, and $\mathcal{B} = \Gamma_{2(n-1)}$. Consequently

$$(Y * Z)^\tau = \theta(Z) Y^\alpha * \lambda(Y) Z^\delta,$$

where θ and λ are characters, α is a homomorphism of Γ_2 onto itself, and δ a homomorphism of $\Gamma_{2(n-1)}$ onto itself. We deduce readily that α and δ are automorphisms, whence incidentally $\mathfrak{P}^\tau = \mathfrak{P}$.

By the discussion at the beginning of the proof, we know that there exist matrices $C \in \Omega_2$, $D \notin \Delta_{2(n-1)}$, and characters μ, ν such that

$$Y^\alpha = \mu(Y) C Y C^{-1}, \qquad Z^\delta = \nu(Z) D Z D^{-1}.$$

If $C * D \in \Delta_{2n}$, define τ_1 by

$$\mathfrak{X}^{\tau_1} = (C * D)^{-1} \mathfrak{X}^\tau (C * D),$$

so that

$$(Y * Z)^{\tau_1} = \theta(Z)\mu(Y) Y * \lambda(Y)\nu(Z) Z.$$

However, possibly $C * D \in \Delta_{2n}$. In that case, if $K = (-1) \dotplus (1)$, then $CK * D \in \Delta_{2n}$, and we define τ_2 by

$$\mathfrak{X}^{\tau_2} = (CK * D)^{-1} \mathfrak{X}^\tau (CK * D).$$

Thus, changing notation, we may assume that

$$(Y * Z)^\tau = \theta(Z)\mu(Y) H Y H^{-1} * \lambda(Y)\nu(Z) Z, \tag{11}$$

for any $Y \in \Gamma_2$, $Z \in \Gamma_{2(n-1)}$, where θ, μ, λ, ν are characters, and where either $H = I^{(2)}$ or $H = K$.

(iii) Suppose now that $Y \in \Gamma_2$, $Z \in \Gamma_{2(n-1)}$ are given by

$$Y = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad Z = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Then define $Y *^i Z$ to be the $2n \times 2n$ matrix $\mathfrak{M}$ obtained by placing the elements of Y at the intersections of the ith and $(n+i)$th rows and columns, filling in the remaining places in those rows and columns with 0's, and letting the matrix obtained from $\mathfrak{M}$ by deleting the ith and $(n+i)$th rows and columns be identical with Z. Then $Y *^i Z$ is a generalization of the previously defined symplectic direct sum, and in fact $Y *^1 Z = Y * Z$.

Now set

$$\mathfrak{P}_i = -I^{(2)} *^i I^{2(n-1)} = I^{(2)} * Q_i, \text{ say.}$$

Then Q_i is a square in $\Gamma_{2(n-1)}$ (since $-I = S^2$), and so from (11) we have

$$\mathfrak{P}_i^\tau = I * Q_i = \mathfrak{P}_i.$$

As before it then follows for $Y \in \Gamma_2$, $Z \in \Gamma_{2(n-1)}$ that

$$(Y *^i Z)^\tau = (F_i(Z) f_i(Y) A_i Y A_i^{-1}) *^i (g_i(Y) G_i(Z) B_i Z B_i^{-1}), \tag{12}$$

where $A_i \in \Omega_2$, $B_i \in \Delta_{2(n-1)}$, and F_i, f_i, g_i, G_i are characters.

(iv) Next let X and $Y \in \Gamma_2$, $Z \in \Gamma_{2(n-2)}$. Applying τ to both sides of the equation

$$X * (Y * Z) = Y *^2 (X * Z)$$

and using (12), we obtain

$$\begin{aligned} &[F_1(Y*Z) f_1(X) A_1 X A_1^{-1}] * [g_1(X) G_1(Y*Z) B_1(Y*Z) B_1^{-1}] \\ &\qquad = [F_2(X*Z) f_2(Y) A_2 Y A_2^{-1}] *^2 [g_2(Y) G_2(X*Z) B_2(X*Z) B_2^{-1}]. \end{aligned} \tag{13}$$

In particular for $X = -I$, $Y = I$, $Z = I$ this yields

$$B_2(-I * I) B_2^{-1} = -I * I,$$

so that

$$B_2 = \pm A_1 * C_2$$

and further

$$B_1 = \pm A_2 * \pm C_2.$$

We use these expressions for B_1 and B_2 in (13) and obtain

$$\begin{aligned} F_1(Y*Z) f_1(X) &= g_2(Y) G_2(X*Z), \\ F_2(X*Z) f_2(Y) &= g_1(X) G_1(Y*Z), \\ g_1(X) G_1(Y*Z) &= g_2(Y) G_2(X*Z). \end{aligned}$$

These imply that $f_1 = g_1$ and $f_2 = g_2$.

Continuing in this way we see that each B_i decomposes completely, and in fact if

$$\mathfrak{D} = A_1 * A_2 * \cdots * A_n,$$

then B_i is obtained from $\mathfrak{D}$ by deleting A_i and possibly changing signs of some of the remaining A's. Furthermore, if any $A_i \in \Delta_2'$, then every $A_i \in \Delta_2'$, since each $B_i \in \Delta_{2(n-1)}$. Therefore $\mathfrak{D} \in \Delta_{2n}$. After a further inner automorphism of Γ_{2n} by a factor of $\mathfrak{D}^{-1}$, we may assume hereafter that

$$(Y *^i Z)^\tau = f_i(Y) [F_i(Z) Y *^i G_i(Z) B_i Z B_i^{-1}] \tag{14}$$

for $Y \in \Gamma_2$, $Z \in \Gamma_{2(n-1)}$, where f_i, F_i and G_i are characters and each B_i is of the form $(\pm I) * \cdots * (\pm I)$, and in fact we may take $B_1 = I$.

(v) Define

$$U_1 = \begin{bmatrix} 0 & 0\cdots 0 & 1 \\ 1 & 0\cdots 0 & 0 \\ 0 & 1\cdots 0 & 0 \\ \cdot & \cdots\cdot\cdot & \cdot \\ \cdot & \cdots\cdot\cdot & \cdot \\ 0 & 0\cdots 1 & 0 \end{bmatrix}, \qquad U_2 = T \dot{+} I^{(n-2)},$$

where T is given by (8). Then the generators of Γ_{2n} are (see [3]):

$$\mathfrak{R}_1 = U_1 \dot{+} U_1'^{-1}, \qquad \mathfrak{R}_2 = U_2 \dot{+} U_2'^{-1}, \qquad \mathfrak{T}_0 = T * I, \qquad \mathfrak{S}_0 = S * I.$$

From (14) we find at once that

$$\mathfrak{T}_0^\tau = \pm \mathfrak{T}_0, \qquad \mathfrak{S}_0^\tau = \pm \mathfrak{S}_0, \quad \text{and} \quad \mathfrak{S}_0^\tau \mathfrak{T}_0^\tau = \mathfrak{S}_0 \mathfrak{T}_0.$$

Next, the rotations of Γ_{2n} map onto rotations under τ, since the rotations are generated by the elements $Y *^i Z$, $i = 1, \cdots, n$, where Y and Z have the forms

$$Y = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}, \qquad Z = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix},$$

and the image of any such $Y *^i Z$ is of the same kind. Therefore τ induces an automorphism σ on the group Ω_n, where

$$\begin{pmatrix} V & 0 \\ 0 & V'^{-1} \end{pmatrix}^\tau = \begin{pmatrix} V^\sigma & 0 \\ 0 & (V^\sigma)'^{-1} \end{pmatrix}.$$

We then know [4] that there exists $H \in \Omega_n$ such that

$$V^\sigma = H V^\omega H^{-1} \qquad \text{for all } V \in \Omega_n,$$

where either $V^\omega = V$ for all V or $V^\omega = V'^{-1}$ for all V.

We know furthermore that τ maps every rotation $\mathfrak{P}_i$ onto itself, from which we see that H is diagonal, with diagonal elements ± 1's. Replace τ by τ_1 defined by

$$\mathfrak{X}^{\tau_1} = (H \dot{+} H)\mathfrak{X}^\tau (H \dot{+} H)$$

and change notation. We again have $\mathfrak{T}_0^\tau = \pm \mathfrak{T}_0$, $\mathfrak{S}_0^\tau = \pm \mathfrak{S}_0$, and $\mathfrak{S}_0^\tau \mathfrak{T}_0^\tau = \mathfrak{S}_0 \mathfrak{T}_0$, but now $V^\sigma = V^\omega$ for each $V \in \Omega_n$. The argument given in §4, parts (iii) and (iv) shows that $\mathfrak{R}_2^\tau = T'^{-1} \dot{+} T$ is impossible, so $V^\sigma = V$ for all $V \in \Omega_n$. Therefore τ is given by

$$(\mathfrak{R}_1, \mathfrak{R}_2, \mathfrak{T}_0, \mathfrak{S}_0)^\tau = (\mathfrak{R}_1, \mathfrak{R}_2, \pm \mathfrak{T}_0, \pm \mathfrak{S}_0).$$

However, as we have already mentioned, Γ_{2n} has no nontrivial character for $n \geqq 3$. Hence $\mathfrak{T}_0^\tau = \mathfrak{T}_0$, $\mathfrak{S}_0^\tau = \mathfrak{S}_0$. This completes the proof of the theorem.

6. We remark finally that if $\mathfrak{M}\in\Gamma_{2n}$ is given by (3), then

$$\mathfrak{M}'^{-1} = \begin{pmatrix} D & -C \\ -B & A \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix},$$

so the automorphism σ: $\mathfrak{M}^\sigma = \mathfrak{M}'^{-1}$ is inner.

Furthermore, any element of Δ_{2n} can be written as the product of an element of Γ_{2n} and $-I^{(n)} \dotplus I^{(n)}$, so every automorphism of Γ_{2n} can be obtained by using inner automorphisms by elements in Γ_{2n}, coupled with the automorphism

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \to \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} A & -B \\ -C & D \end{pmatrix}.$$

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