SOME THEOREMS ABOUT THE RIESZ FRACTIONAL INTEGRAL

BY

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I show in this paper that theorems which hold for Riemann-Liouville fractional integrals have analogues holding for the Riesz fractional integral [1]. Theorems 1, 2, and 3 are analogous to well-known results due to Hardy and Littlewood [2]. Theorem 4 is of a different character and is analogous to one recently proved by the author [3].

The Riesz fractional integral $f_\alpha(P)$ of order α is given by

$$f_\alpha(P) = K_m^{-1}\int_E r_{PQ}^{\alpha-m} f(Q)dQ, \quad \text{where} \quad K_m = \pi^{m/2}2^\alpha\Gamma(\alpha/2)[\Gamma((m-\alpha)/2)]^{-1},$$

E denotes all of Euclidean m-space, and r_{PQ} denotes the distance between P and Q.

We assume always that $f(Q)$ is L-integrable over E.

I prove the following theorems.

THEOREM 1. *If $f(P)\in\text{Lip }\beta$, $0<\beta<1$ then $f_\alpha(P)\in\text{Lip }(\alpha+\beta)$, $0<\alpha+\beta<1$.*

THEOREM 2. *If $f(P)\in L^q$, $q>1$, $1+m/q>\alpha>m/q$, then*

$$f_\alpha(P) \in \text{lip }(\alpha - m/q).$$

THEOREM 3. *If $f(P)\in L^q$ and $0<\alpha<m/q$, then*

$$f_\alpha(P) \in L^r, \quad \text{where} \quad \alpha = m(1/q - 1/r).$$

THEOREM 4. *If $f(P)\in L^q$ then*

(a) *for $0<\alpha<m$, $2<q<\infty$, $f_{\alpha/q}(P)$ is finite everywhere except possibly in a set which is of zero β-capacity for all $\beta>m-\alpha$;*

(b) *for $0<\alpha<m$, $1\leqq q\leqq 2$, $f_{\alpha/q}(P)$ is finite everywhere except possibly in a set of zero $(m-\alpha)$-capacity.*

Both (a) *and* (b) *are best possible.*

1. **Preliminaries.** If P is the point $(x_1, \cdots, x_m)$ and Q the point $(t_1, \cdots, t_m)$ we define the points $(x_1+t_1, \cdots, x_m+t_m)$ and $(x_1-t_1, \cdots, x_m-t_m)$ to be $P+Q$ and $P-Q$ respectively. The distance $|P|$ of P from the origin $0=(0, \cdots, 0)$ is given by $|P|^2=\sum_{r=1}^m x_r^2$, and $|P-Q|$ is the distance P to Q.

If, for $0\leqq\beta\leqq 1$, $f(P+H)-f(P)=O(|H|^\beta)$ uniformly in P as $|H|\to 0$, we say that $f(P)\in\text{Lip }\beta$. If, in this, O is replaced by o we say that $f(P)\in\text{lip }\beta$.

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Next, we have

$$K_m(f_\alpha(P+H)-f_\alpha(P)) = \left(\int_U + \int_{E-U}\right)(|Q-H|^{\alpha-m} - |Q|^{\alpha-m})f(Q+P)dQ,$$

where U is the unit hypersphere having the origin as center. For $|H|<1/2$ it is not difficult to establish that

$$|Q-H|^{\alpha-m} - |Q|^{\alpha-m} = O(|H|)$$

uniformly in $E-U$. The second integral is thus $O(H)$ uniformly in P, and so

$$K_m(f_\alpha(P+H)-f_\alpha(P)) = \int_U(|Q-H|^{\alpha-m} - |Q|^{\alpha-m})f(Q+P)dQ + O(|H|). \tag{2}$$

2. **Proofs of Theorems 1 and 2.** First, Theorem 1. The first term on the right-hand side of (1) of §1 may be rewritten in the form

$$\int_U(|Q-H|^{\alpha-m} - |Q|^{\alpha-m})(f(Q+P)-f(P))dQ + f(P)\left\{\int_{U'}|Q|^{\alpha-m}dQ - \int_U|Q|^{\alpha-m}dQ\right\}, \tag{1}$$

where U' is the sphere U transforms into under the transformation $Q'=Q-H$. The expression in curly brackets is dominated by $\int_S|Q|^{\alpha-m}dQ$, where $S=U'+U-U'U$.

Now $mS<\pi^{m/2}[\Gamma((m+2)/2)]^{-1}\{(1+|H|)^m-1\}=O(|H|)$ and $|Q|^{\alpha-m}<2^{m-\alpha}$ in S for $|H|<1/2$. Consequently, the second term in (2) is $O(|H|)$.

To deal with the first term we note that it is of order $\int_U||Q-H|^{\alpha-m}-|Q|^{\alpha-m}||Q|^\beta dQ$ and apply a uniform dilatation transformation of ratio $1:|H|$ and then a rotation which takes the transform of H into the point $1=(1, 0, \cdots, 0)$. The first term is then seen to be less than

$$|H|^{\alpha+\beta}\int_E||Q-1|^{\alpha-m} - |Q|^{\alpha-m}||Q|^\beta dQ = O(|H|^{\alpha+\beta}),$$

since it is again a simple matter to establish that the integral is finite. This proves Theorem 1.

Next, Theorem 2. Let $S(r)$ denote the hypersphere of radius r centered at the origin and write

$$A(\delta) = S(\delta) - S(|H|), \qquad B(\delta) = U - S(\delta),$$

where δ will presently be defined. Split the right-hand side of (1) into integrals

I_1 over $S(|H|)$, I_2 over $A(\delta)$, and I_3 over $B(\delta)$. Then, firstly

$$
\begin{aligned}
|I_1| &\leqq \left\{\int_{S(|H|)} \left| |Q-H|^{\alpha-m} - |Q|^{\alpha-m}\right|^{q'} dQ\right\}^{1/q'} \\
&\quad \cdot \left\{\int_{S(|H|)} |f(Q+P)|^{q} dQ\right\}^{1/q} \\
&= |H|^{\alpha-m/q}\left\{\int_{U} \left| |Q-1|^{\alpha-m} - |Q|^{\alpha-m}\right|^{q'} dQ\right\}^{1/q'} o(1)
\end{aligned}
$$

as $|H|\to 0$: we use the same transformation on the integral as before. Thus $I=o(|H|^{\alpha-m/q})$. Further

$$
\begin{aligned}
|I_2| &\leqq \left\{\int_{A(\delta)} \left| |Q-H|^{\alpha-m} - |Q|^{\alpha-m}\right|^{q'} dQ\right\}^{1/q'} \\
&\quad \cdot \left\{\int_{A(\delta)} |f(Q+P)|^{q} dQ\right\}^{1/q}.
\end{aligned}
$$

It is again easy to show that, for $|H|<\delta/3$,

$$\left| |Q-H|^{\alpha-m} - |Q|^{\alpha-m}\right| \leqq C|H|\,|Q|^{\alpha-m-1}$$

and thus

$$|I_2| \leqq C|H|\left\{\int_{A(\delta)} |Q|^{(\alpha-m-1)q'} dQ\right\}^{1/q'} \left\{\int_{A(\delta)} |f(Q+P)|^{q} dQ\right\}^{1/q}.$$

Further

$$\int_{A(\delta)} |Q|^{(\alpha-m-1)q'} dQ < |H|^{(\alpha-1)q'-m(q'-1)} \int_{E-U} |Q|^{(\alpha-m-1)q'} dQ,$$

so that

$$|I_2| \leqq C|H|^{\alpha-m/q}\left\{\int_{A(\delta)} |f(Q+P)|^{q} dQ\right\}^{1/q}.$$

Given any $\epsilon>0$, we can choose δ so that $\int_{A(\delta)}|f(Q+P)|^{q}dQ$ is less than $(\epsilon/C)^q$, and so

$$|I_2| < \epsilon |H|^{\alpha-m/q}.$$

Finally

$$|I_3| \leqq \left\{\int_{B(\delta)} \left| |Q-H|^{\alpha-m} - |Q|^{\alpha-m}\right|^{q'} dQ\right\}^{1/q'} \left\{\int_{E} |f(Q+P)|^{q} dQ\right\}^{1/q}.$$

For fixed δ, $|Q-H|^{\alpha-m}-|Q|^{\alpha-m}=O(|H|)$ uniformly in $B(\delta)$, and so $I_3=O(|H|)$.

Thus, finally, $K_m(f_\alpha(P+H)-f_\alpha(P))=o(|H|^{\alpha-m/q})$, giving the required result.

3. **Proof of Theorem** 3. We first prove a many-dimensional generalization of a theorem due to Hardy and Littlewood [2, Theorem 3].

LEMMA. *If $f(P)\in L^q$, $g(Q)\in L^r$, $1/q+1/r>1$, $q>1$, $r>1$ and $\mu=2-1/q-1/r$ then*

$$\int_E\int_E |Q-P|^{-m\mu}f(P)g(Q)dPdQ \leqq KM_q(f)M_r(g), \tag{1}$$

where $M_q(f)=\{\int_E|f(P)|^q dP\}^{1/q}$ and $M_r(g)$ is similarly defined.

I prove here the case $m=3$, which is sufficiently typical.

Since an arithmetic mean is greater than the corresponding geometric mean we have

$$\begin{aligned}|P-Q|^2 &= (x_1-t_1)^2+(x_2-t_2)^2+(x_3-t_3)^2\\ &\geqq 3|x_1-t_1|^{2/3}|x_2-t_2|^{2/3}|x_3-t_3|^{2/3}\end{aligned}$$

and so

$$|P-Q|^{-3\mu} \leqq C|x_1-t_1|^{-\mu}|x_2-t_2|^{-\mu}|x_3-t_3|^{-\mu}.$$

Consequently the left-hand side of (1) is not greater than a constant multiple of

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{f(x_1,x_2,x_3)g(t_1,t_2,t_3)}{|x_1-t_1|^{\mu}|x_2-t_2|^{\mu}|x_3-t_3|^{\mu}} \cdot dt_3dx_3dt_2dx_2dt_1dx_1. \tag{2}$$

By the Hardy-Littlewood theorem mentioned, which is the case $m=1$ of the lemma,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|x_3-t_3|^{-\mu}f(x_1,x_2,x_3)g(t_1,t_2,t_3)dt_3dx_3$$

is dominated by $CF(x_1,x_2)G(t_1,t_2)$, where $F(x_1,x_2)=\{\int_{-\infty}^{\infty}|f(x_1,x_2,x_3)|^q dx_3\}^{1/q}$ and $G(t_1,t_2)$ is defined analogously.

Hence [2] is dominated by

$$C_1\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|x_1-t_1|^{-\mu}|x_2-t_2|^{-\mu}F(x_1,x_2)G(t_1,t_2)dt_2dx_2dt_1dx_1.$$

Applying the case $m=1$ of the lemma again to the inner two integrals we find that (1) is dominated by

$$C_1C_2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|x_1-t_1|^{-\mu}F(x_1)G(t_1)dt_1dx_1,$$

where $F(x_1)=\left\{\int_{-\infty}^{\infty}|F(x_1, x_2)|^q dx_2\right\}^{1/q}$ and $G(t_1)$ is defined analogously.

A final application of the lemma with $m=1$ shows that (1) is dominated by $C_1C_2C_3M_q(F)M_r(G)$. Since $M_q(F)$ equals $\left\{\int_E|f(P)|^q dP\right\}^{1/q}$ and a similar result holds for $M_r(G)$, we have the required result.

To prove Theorem 3 it is sufficient to prove that, for every $g(P)$ such that $M_{r'}(g)\leqq 1$,

$$\int_E f_\alpha(P)g(P)dP \leqq KM_q(f).$$

The left-hand side of this is equal to

$$K_m^{-1}\int_E\int_E|P-Q|^{\alpha-m}f(Q)g(P)dQdP \tag{3}$$

and, since $\alpha-m=m(1/q-1/r)-m=-m(2-1/q-1/r')$, the lemma applies and shows (3) to be, in modulus, not greater than $K'M_q(f)M_{r'}(g)\leqq K'M_q(f)$, thus proving the theorem.

4. **Preliminaries about Theorem** 4. We say, with Frostman [4, p. 26], that a non-negative additive set function $\mu(S)$ defined for all Borel sets in E is a distribution if $\mu(E)=1$. Further, if $S\subset E$ and $\mu(S)=1$ we say that the distribution is concentrated on S.

Let S be a given set. Suppose that there is a distribution concentrated on S such that

$$V_\beta=\sup_{P\in E}\int_E|Q-P|^{-\beta}d\mu(Q)$$

is finite. Then we say that S is of positive β-capacity. Otherwise S is said to be of zero β-capacity. Clearly, if S is of positive β-capacity, it is of positive γ-capacity for all $\gamma<\beta$. Further, if it is of zero β-capacity, it is of zero γ-capacity for all $\gamma>\beta$.

LEMMA. *For $1<q<2$, and for every $\epsilon>0$ for which $q-\epsilon>1$, we have*

$$\int_S\left\{\int_E|Q-P|^{(\alpha/q')-m}d\mu(Q)\right\}^{q-\epsilon}dP\leqq A(\alpha,\epsilon,m,q,S)V_{m-\alpha}^{(q-\epsilon)/(q-\epsilon)'}, \tag{1}$$

where $A(\alpha, \epsilon, m, q, S)$ is a constant depending only on the parameters shown and S is a bounded set.

For $2\leqq q\leqq\infty$ we have

$$\int_E\left\{\int_E|Q-P|^{(\alpha/q')-m}d\mu(Q)\right\}^{q}dP\leqq A(\alpha,m)V_{m-\alpha}^{q-1}, \tag{2}$$

where $A(\alpha, m)$ is a constant depending only on the parameters shown.

We have

$$\left\{\int_E |Q-P|^{(\alpha/q')-m}d\mu(Q)\right\}^{q-\epsilon} = \left\{\int_E |Q-P|^{-\alpha/q}|Q-P|^{\alpha-m}d\mu(Q)\right\}^{q-\epsilon}$$

$$\leqq \left\{\int_E |Q-P|^{-\alpha(q-\epsilon)/q}|Q-P|^{\alpha-m}d\mu(Q)\right\}$$

$$\cdot \left\{\int_E |Q-P|^{\alpha-m}d\mu(Q)\right\}^{(q-\epsilon)/(q-\epsilon)'}$$

by Hölder's inequality. The second factor is not greater than $V^{(q-\epsilon)/(q-\epsilon)'}$ while the first is $\int_E |Q-P|^{\alpha\epsilon/q-m}d\mu(Q)$. The left-hand side of (1) is therefore not greater than

$$V_{m-\alpha}^{(q-\epsilon)/(q-\epsilon)'}\int_S dP\int_E |Q-P|^{\alpha\epsilon/q-m}d\mu(Q).$$

We invert the order of integration and note that

$$\int_S |Q-P|^{\alpha\epsilon/q-m}dP = A(\alpha, \epsilon, m, q, S), \text{ say.}$$

Furthermore $\int_E d\mu(Q)=1$. (1) now follows.

To prove (2) I first show the result true for $q=2$ and then that this implies its truth for $q>2$. For this latter part of the proof I am indebted to Professor J. E. Littlewood.

We have first, on inverting the order of integration,

$$\tag{3} \begin{aligned} &\int_E\left\{\int_E |Q-P|^{(\alpha-m)/2}d\mu(Q)\right\}^2 dP \\ &\quad = \int_E\int_E\int_E |Q-P|^{(\alpha-m)/2}|R-P|^{(\alpha-m)/2}dP\,d\mu(Q)\,d\mu(R). \end{aligned}$$

To deal with the inner integral we dilate E uniformly, taking Q as the center of dilatation, in the ratio $1:|Q-R|$ and then rotate the dilated space so that the transform of $Q-R$ goes into the point 1. The inner integral then becomes

$$|Q-R|^{\alpha-m}\int_E |U|^{(\alpha-m)/2}|U+1|^{(\alpha-m)/2}dU = B(\alpha, m)|Q-R|^{\alpha-m}.$$

Consequently, the right-hand side of (3) is dominated by

$$B(\alpha, m)\int_E\int_E |Q-R|^{\alpha-m}d\mu(Q) \leqq B(\alpha, m)V_{m-\alpha}\mu(E).$$

Since $\mu(E)=1$ this gives the result for $q=2$.

For $q>2$, we have

$$\int_E\left\{\int_E |Q-P|^{\alpha/q'-m}d\mu(Q)\right\}^q dP$$

$$=\int_E\left\{\int_E |Q-P|^{((q-2)/q)(\alpha-m)}|Q-P|^{(\alpha-2m)/q}d\mu(Q)\right\}^q dP$$

and this, by Hölder's inequality, does not exceed

$$J=\int_E\left\{\int_E |Q-P|^{\alpha-m}d\mu(Q)\right\}^{q-2}\left\{\int_E |Q-P|^{(\alpha-m)/2}d\mu(Q)\right\}^2 dP.$$

The first curly bracket does not exceed $V_{m-\alpha}^{q-2}$ (by the definition of $V_{m-\alpha}$). So

$$J\leqq V_{m-\alpha}^{q-2}\int_E\left\{\int_E |Q-P|^{(\alpha-m)/2}d\mu(Q)\right\}^2 dP$$

and this, by the result for $q=2$, does not exceed $V_{m-\alpha}^{q-2}B(\alpha, m)V_{m-\alpha}$. This gives the result for $q>2$.

5. **Proof of Theorem** 4. Let

$$S_n(P)=\int_E |Q-P|^{\alpha/q-m}[f(Q)]_n dQ,$$

where

$$[f(Q)]_n=\begin{cases}|f(Q)| & \text{for} \quad |f(Q)|\leqq n\\ n & \text{for} \quad |f(Q)|>n.\end{cases}$$

$S_n(P)$ is always defined and finite, and to prove the theorem it is sufficient to show that $S_n(P)$ is bounded everywhere except possibly in a set of zero β-capacity, where $\beta=m-\alpha$ for $1\leqq q\leqq 2$ and $\beta>m-\alpha$ for $q>2$.

Assume, then, that $S_n(P)$ is unbounded in a set M of positive β-capacity. It is then unbounded in a bounded set S of positive β-capacity. Then, first, there is a distribution concentrated on S such that $\int_E|Q-P|^{-\beta}d\mu(Q)$ is bounded for all P. Secondly, there is a function $n(P)\leqq n$, taking only integer values such that $\int_S S_{n(P)}(P)d\mu(P)$ exists and is unbounded as $n\to\infty$. This is an adaptation of a known result used by Salem and Zygmund [5, embodied in the proof of Theorem II], but a proof is perhaps not unwelcome.

Let $\overline{S}_n(P)=\sup S_m(P)$ for $0\leqq m\leqq n$. Then for all $P\in S$, $\{\overline{S}_n(P)\}^{-1}\to 0$ as $n\to\infty$. By Egoroff's theorem on uniform convergence it follows that there is a set $S'\subset S$ such that $\mu(S-S')$ is as small as we please, and in which $\{\overline{S}_n(P)\}^{-1}\to 0$ uniformly. It follows that, given any large number G, there is an integer $n_0=n_0(G)$ such that, for all $P\in S'$, $\overline{S}_n(P)>G$ for all $n>n_0(G)$. Choose $n(P)$ such that $S_{n(P)}(P)=\overline{S}_n(P)$. Then

$$\int_S S_{n(P)}(P)d\mu(P) > G\mu(S') \quad \text{for} \quad n > n_0,$$

and so

$$\int_S S_{n(P)}(P)d\mu(P) \to +\infty \qquad \text{as } n \to \infty.$$

I show this last to be impossible. We have

$$\left|\int_S S_{n(P)}(P)d\mu(P)\right| = \left|\int_S\int_E |Q-P|^{\alpha/q-m}[f(Q)]_{n(P)}dQd\mu(P)\right|$$
$$\leqq \int_E |f(Q)| \int_S |Q-P|^{\alpha/q-m}d\mu(P)dQ$$

and this does not exceed $M_q(f)M_{q'}[\int_S|Q-P|^{\alpha/q-m}d\mu(P)]$. Now $M_q(f)<+\infty$ by hypothesis, and we have only to show that

$$M_{q'}\left[\int_E |Q-P|^{\alpha/q-m}d\mu(P)\right] \tag{1}$$

is bounded.

If $1 \leqq q \leqq 2$ then $q' \geqq 2$ and (2) of the lemma of §4 immediately gives (1). If $q>2$ we write $\beta = m-\gamma$. Since $\gamma<\alpha$ there is an $r<q$ such that $\alpha/q=\gamma/r$. We may suppose β so near $m-\alpha$ that $2<r<q$ since the result, if true for a given β, is true for a larger β. We may now rewrite (1) in the form

$$M_{r'-\epsilon}\left[\int_E |P-Q|^{\gamma/r-m}d\mu(Q)\right],$$

which, since $r'>2$, is shown to be bounded by invoking (1) of the lemma.

6. **Theorem 4 is best possible.** We show this by constructing a function $f(P)\in L^q$ and a set M of positive β-capacity (where $\beta=m-\alpha$ when $1\leqq q\leqq 2$, and β is any number greater than $m-\alpha$ when $q>2$) at every point of which $f_{\alpha/q}(P)$ is infinite. It will avoid unnecessary complication and fully illustrate the general procedure if this is done for the simplest case $m=2$.

M is constructed as follows. Let $\{\xi_n\}$ be any sequence such that $0<\xi_n<1/2$. Let M_0 be the unit square $0\leqq x_1\leqq 1$, $0\leqq x_2\leqq 1$. From M_0 remove the rectangle $\xi_1<x_1<1-\xi_1$, $0\leqq x_2\leqq 1$ thus leaving the set M_1. From the left-hand rectangle in M_1 remove the rectangle $\xi_1\xi_2<x_1<\xi_1(1-\xi_2)$, $0\leqq x_2\leqq 1$, and make a similar symmetric removal from the right-hand rectangle of M_1, thus leaving a set consisting of 4 closed rectangles of length 1 and breadth $\xi_1\xi_2$. If we continue in this manner we are left, after the nth removal, with a set M_n consisting of 2^n closed rectangles each of length 1 and breadth $\xi_1\xi_2 \cdots \xi_n$. Consequently

$$mM_n = 2^n \xi_1 \xi_2 \cdots \xi_n.$$

It is known [5, p. 40] that the projection S of $M = \lim M_n$ on the x-axis will be of positive β-capacity if and only if

$$\sum_{n=1}^{\infty} 2^{-n}(\xi_1\xi_2 \cdots \xi_n)^{-\beta} < \infty. \tag{1}$$

If S is of positive β-capacity there is a distribution ν concentrated on S such that $\int_0^1 |x_1 - t|^{-\beta} d\nu(t)$ is bounded for all x_1. Let μ be an additive set function defined over E by

$$\mu(X) = \iint_X d\nu(x_1)dx_2.$$

Then

$$\int_M |P - Q|^{-\beta-1} d\mu(Q) = \int_0^1 \int_0^1 [(x_1 - t_1)^2 + (x_2 - t_2)^2]^{-(\beta+1)/2} dt_2 d\nu(t_1).$$

In the inner integral make the substitution $x_2 - t_2 = (x_1 - t_1)u$. It is then dominated by

$$|x_1 - t_1|^{-\beta} \int_{-\infty}^{\infty} (1 + u^2)^{-(\beta+1)/2} du = A(\beta) |x_1 - t_1|^{-\beta}.$$

Consequently, since μ is a distribution concentrated on M,

$$\int_M |P - Q|^{-\beta-1} d\mu(Q) = \int_E |P - Q|^{-\beta-1} d\mu(Q) \leqq A(\beta) \int_0^1 |x_1 - t_1|^{-\beta} d\nu(t_1),$$

which is bounded. *Thus M is of positive $(\beta+1)$-capacity if S is of positive β-capacity.*

Define $\{f_n(P)\}$ over M_0 by

$$\begin{aligned} f_0(P) &= 0 \text{ in } M_0, \\ f_n(P) &= (\xi_1\xi_2 \cdots \xi_n)^{-\alpha/q} n^{-1} \text{ in } M_n, \\ f_n(P) &= f_{n-1}(P) \text{ in } M_0 - M_n. \end{aligned}$$

Since $\{f_n(P)\}$ is, eventually, an increasing sequence of measurable functions the function $f(P)$ given by

$$f(P) = \lim_{n\to\infty} f_n(P) \text{ in } M_0,$$

$$f(P) = 0 \text{ in } E - M_0$$

exists and is measurable over E.

It is easily seen that, for $n = 1, 2, \cdots,$

$$f(P) = 0 \text{ in } M_0 - M_1,$$
$$f(P) = (\xi_1\xi_2 \cdots \xi_n)^{-\alpha/q}n^{-1} \text{ on } M_n - M_{n+1}$$

so that

$$\int_E |f(P)|^q dP = \int_{M_0} |f(P)|^q dP = \sum_{n=1}^{\infty} \int_{M_n - M_{n+1}} |f(P)|^q dP$$

$$(2) \qquad = \sum_{n=1}^{\infty} (\xi_1\xi_2 \cdots \xi_n)^{-\alpha} n^{-q}(mM_n - mM_{n+1})$$

$$= \sum_{n=1}^{\infty} (1 - 2\xi_{n+1})2^n(\xi_1\xi_2 \cdots \xi_n)^{1-\alpha}n^{-q}.$$

For $q>2$, we may choose $\delta>0$ so that $2(1+\delta)<q$, and then put

$$2\xi_n^{1-\alpha} = 1 + (1 + \delta)n^{-1}.$$

Then $2^{-n}(\xi_1\xi_2 \cdots \xi_n)^{\alpha-1} \sim Cn^{-1-\delta}$ so that (1) with $\beta=1-\alpha$ is satisfied, showing S to be of positive $(1-\alpha)$-capacity, and hence that M is of positive $(2-\alpha)$-capacity.

Further, (2) is clearly finite, so that $f \in L^q$ over E.

Let $P(x_1, x_2)$ be any point of M. Let

$$M_n(P) = M_n \cdot S[t_2; x_2 - \epsilon_n \leqq t_2 \leqq x_2 + \epsilon_n], \qquad \text{where } \epsilon_n = \xi_1\xi_2 \cdots \xi_n/2;$$

$$M_n^*(P) = (M_n - M_{n+1})S[t_2; x_2 - \delta_n \leqq t_2 \leqq x_2 + \delta_n],$$
$$\text{where } \delta_n = \xi_1\xi_2 \cdots \xi_n(1 - 2\xi_{n+1})/2.$$

$M_n(P)$ then consists of 2^n squares each of side $\xi_1\xi_2 \cdots \xi_n$, while $M_n^*(P) \subset M_n(P)$ and consists of 2^n squares each of side $\xi_1\xi_2 \cdots \xi_n(1-2\xi_{n+1})$. No square in $M_n^*(P)$ contains P, but one of the squares, I_n (say), is contained in that one of the squares, J_n (say), of $M_n(P)$ which itself contains P. Furthermore, the I_n $(n=1, 2, \cdots)$ are disjoint.

Now $|Q-P| < 2^{1/2}\xi_1 \cdots \xi_n$ for Q in J_n, and so certainly for Q in I_n, and thus

$$K_2 f_{\alpha/q}(P) = \int_{M_0} |Q - P|^{\alpha/q-2} f(Q)dQ = \sum_{n=1}^{\infty} \int_{M_n - M_{n+1}} \geqq \sum_{n=1}^{\infty} \int_{I_n}.$$

This last is not less than

$$\sum_{n=1}^{\infty} (2^{1/2}\xi_1 \cdots \xi_n)^{\alpha/q-2}(\xi_1 \cdots \xi_n)^{-\alpha/q}n^{-1}(\xi_1 \cdots \xi_n)^2(1 - 2\xi_{n+1})^2$$

$$= 2^{\alpha/2q-1} \sum_{n=1}^{\infty} (1 - 2\xi_{n+1})^2 n^{-1} = +\infty.$$

Consequently, $f_{\alpha/q}(P)$ is infinite at every point of M, giving the required example in the case of $q>2$, thus showing part (a) of Theorem 4 best possible.

For the case $q\leqq 2$, let β be any positive number less than $1-\alpha$ and let ξ be such that $2\xi^{(1-\alpha+\beta)/2}=1$. Consider the set M with $\xi_n=\xi$ for all n. Since $2\xi^{\beta}>1$, M is of positive $(\beta+1)$-capacity. Defining $f(P)$ as before, we use exactly the same argument to show that $f_{\alpha/q}(P)=+\infty$ at every point of M. Furthermore, since $2\xi^{1-\alpha}<1$, (2) is bounded, so that $f\in L^q$.

This shows part (b) of Theorem 4 best possible.

7. **The lemma of §4 is best possible.** Consider, e.g., (2) of the lemma. Suppose this is not the case, i.e. that there is an $\epsilon>0$ for which, in general,

$$M_{q+\epsilon}\left[\int_E |Q-P|^{\alpha/q'-m}d\mu(Q)\right]<\infty.$$

If, then, $f(P)\in L^{(q+\epsilon)'}$ we may say that

$$\left|\int_E S_{n(P)}(P)d\mu(P)\right|\leqq M_{(q+\epsilon)'}(f)M_{q+\epsilon}\left[\int_E |Q-P|^{\alpha/q'-m}d\mu(Q)\right]$$

which is bounded. This would imply that (b) of Theorem 4 is not best possible. Since it is best possible we have shown (2) best possible. A similar argument using (a) would show (1) best possible.

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