ON DIFFERENTIAL GEOMETRY OF HYPERSURFACES IN THE LARGE

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1. Introduction. Let \( V^n (V^*n) \) be an orientable hypersurface of class \( C^3 \) imbedded in a Euclidean space \( E^{n+1} \) of \( n+1 \geq 3 \) dimensions with a closed boundary \( V^{n-1} (V^*n-1) \) of dimension \( n-1 \). Suppose that there is a one-to-one correspondence between the points of the two hypersurfaces \( V^n, V^n* \) such that at corresponding points the two hypersurfaces \( V^n, V^n* \) have the same normal vectors. Let \( \kappa_1, \ldots, \kappa_n \) be the \( n \) principal curvatures at a point \( P \) of the hypersurface \( V^n \), then the \( \alpha \)th mean curvature \( M_\alpha \) of the hypersurface \( V^n \) at the point \( P \) is defined by

\[
(1.1) \quad \left( \begin{array}{c} n \\ \alpha \end{array} \right) M_\alpha = \sum_{\kappa_1 \cdots \kappa_\alpha} \quad (\alpha = 1, \ldots, n),
\]

where the expression on the right side is the \( \alpha \)th elementary symmetric function of \( \kappa_1, \ldots, \kappa_n \). In particular, \( M_1 \) is the Gaussian curvature of the hypersurface \( V^n \) at the point \( P \). Let \( P^* \) be the point of the hypersurface \( V^n* \) corresponding to the point \( P \) of the hypersurface \( V^n \) under the given correspondence, \( p^* \) the oriented distance from a fixed point \( O \) in the space \( E^{n+1} \) to the tangent hyperplane of the hypersurface \( V^n* \) at the point \( P^* \), and \( dA \) the area element of the hypersurface \( V^n \) at the point \( P \). The purpose of this paper is first to derive some expressions for the integrals \( \int_{V^n} M_\alpha p^* dA \) \( (\alpha = 1, \ldots, n) \), and then to prove the following

**Theorem.** Let \( V^n (V^n*) \) be an orientable hypersurface of class \( C^3 \) imbedded in a Euclidean space \( E^{n+1} \) of \( n+1 \geq 3 \) dimensions with a positive Gaussian curvature and a closed boundary \( V^{n-1} (V^*n-1) \) of dimension \( n-1 \). Suppose that there is a one-to-one correspondence between the points of the two hypersurfaces \( V^n, V^n* \), such that at corresponding points the two hypersurfaces \( V^n, V^n* \) have the same normal vectors and equal sums of the principal radii of curvature, and such that the two boundaries \( V^{n-1}, V^*n-1 \) are congruent. Then the two hypersurfaces \( V^n, V^n* \) are congruent or symmetric.

This theorem has been obtained by T. Kubota (see [6] or [1, pp. 29–30]) for closed hypersurfaces \( V^n, V^n* \), and by the author [5] for \( n = 2 \) in a slightly different form.

2. Preliminaries. In a Euclidean space \( E^{n+1} \) of dimension \( n+1 \geq 3 \), let us
consider a fixed orthogonal frame $O\mathcal{Y}_1 \cdots \mathcal{Y}_{n+1}$ with a point $O$ as the origin. With respect to this orthogonal frame we define the vector product of $n$ vectors $A_1, \ldots, A_n$ in $E^{n+1}$ to be the vector $A_{n+1}$, denoted by $A_1 \times \cdots \times A_n$, satisfying the following conditions:

(a) the vector $A_{n+1}$ is normal to the $n$-dimensional space determined by the vectors $A_1, \ldots, A_n$,

(b) the magnitude of the vector $A_{n+1}$ is equal to the volume of the parallelepiped whose edges are the vectors $A_1, \ldots, A_n$,

(c) the two frames $OA_1 \cdots A_nA_{n+1}$ and $O\mathcal{Y}_1 \cdots \mathcal{Y}_{n+1}$ have the same orientation.

Let $\sigma$ be a permutation on the $n$ numbers $1, \ldots, n$, then

$$A_{\sigma(1)} \times \cdots \times A_{\sigma(n)} = (\text{sgn } \sigma)A_1 \times \cdots \times A_n,$$

where $\text{sgn } \sigma$ is $+1$ or $-1$ according as the permutation $\sigma$ is even or odd. Let $i_1, \ldots, i_{n+1}$ be the unit vectors from the origin $O$ in the directions of the vectors $\mathcal{Y}_1, \ldots, \mathcal{Y}_{n+1}$ and let $A'_{\alpha}$ ($j = 1, \ldots, n+1$) be the components of the vector $A_\alpha$ ($\alpha = 1, \ldots, n$)\(^{(1)}\) with respect to the frame $O\mathcal{Y}_1 \cdots \mathcal{Y}_{n+1}$, then the scalar product of any two vectors $A_\alpha$ and $A_\beta$ and the vector product of $n$ vectors $A_1, \ldots, A_n$ are, respectively,

$$A_\alpha \cdot A_\beta = \sum_{i=1}^{n+1} A_\alpha^i A_\beta^i,$$

$$A_1 \times \cdots \times A_n = (-1)^n \begin{vmatrix} i_1 & i_2 & \cdots & i_{n+1} \\ A_1^1 & A_1^2 & \cdots & A_1^{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ A_n^1 & A_n^2 & \cdots & A_n^{n+1} \end{vmatrix}.$$

If $A'_\alpha$ are differentiable functions of $n$ variables $x^1, \ldots, x^n$, then by equation (2.3) and the differentiation of determinants

$$\frac{\partial}{\partial x^\alpha} (A_1 \times \cdots \times A_n) = \sum_{\beta=1}^{n} \left( A_1 \times \cdots \times A_{\beta-1} \times \frac{\partial A_\beta}{\partial x^\alpha} \times A_{\beta+1} \times \cdots \times A_n \right).$$

Now we consider a hypersurface $V^n$ of class $C^3$ imbedded in the space $E^{n+1}$ with a closed boundary $V^{n-1}$ of dimension $n-1$. Let $(y^1, \ldots, y^{n+1})$ be the coordinates of a point $P$ in the space $E^{n+1}$ with respect to the orthogonal frame $O\mathcal{Y}_1 \cdots \mathcal{Y}_{n+1}$. Then the hypersurface $V^n$ can be given by the parametric equations\(^{(2)}\)

\(^{(1)}\) Throughout this paper all Latin indices take the values $1$ to $n+1$ and Greek indices the values $1$ to $n$ unless stated otherwise. We shall also follow the convention that repeated indices imply summation.

\(^{(2)}\) For the remainder of this section see, for instance, [2, Chap. IV].
where $y^i$ and $f^i$ are respectively the components of the two vectors $Y$ and $F$, the parameters $x^1, \cdots, x^n$ take values in a simply connected domain $D$ of the $n$-dimensional real number space, $f^i(x^1, \cdots, x^n)$ are of the third class and the Jacobian matrix $\|\partial y^i/\partial x^n\|$ is of rank $n$ at all points of $D$. If we denote the vector $\partial Y/\partial x^\alpha$ by $Y_\alpha$ for $\alpha = 1, \cdots, n$, then the first fundamental form of the hypersurface $V^n$ at a point $P$ is

$$d\mathbf{s}^2 = g_{\alpha\beta} dx^\alpha dx^\beta,$$

where

$$g_{\alpha\beta} = Y_\alpha \cdot Y_\beta,$$

and the matrix $\|g_{\alpha\beta}\|$ is positive definite so that the determinant $g = \sqrt{g_{\alpha\beta}} > 0$.

Let $N$ be the unit normal vector of the hypersurface $V^n$ at a point $P$, and $N_\alpha$ the vector $\partial N/\partial x^\alpha$, then

$$N_\alpha = - b_{\alpha\beta} g_{\beta\gamma} Y_\gamma,$$

where

$$b_{\alpha\beta} = b_{\beta\alpha} = - Y_\alpha \cdot N_\beta$$

are the coefficients of the second fundamental form of the hypersurface $V^n$ at the point $P$, and $g^{\beta\gamma}$ denotes the cofactor of $g_{\beta\gamma}$ in $g$ divided by $g$ so that

$$g^{\alpha\gamma} g_{\beta\gamma} = \delta^\alpha_\gamma,$$

$\delta^\alpha_\gamma$ being the Kronecker deltas. The $n$ principal curvatures $\kappa_1, \cdots, \kappa_n$ of the hypersurface $V^n$ at the point $P$ are the roots of the determinant equation

$$\begin{vmatrix} b_{\alpha\beta} - \kappa g_{\alpha\beta} \end{vmatrix} = 0.$$

From equations (1.1) and (2.12) follow immediately

$$M_n = b/g, \quad nM_1 = b_{\alpha\beta} g^{\alpha\beta}, \quad nM_{n-1} = g_{\alpha\beta} B_{\alpha\beta}/g,$$

where $b = |b_{\alpha\beta}| \neq 0$ and $B_{\alpha\beta}$ is the cofactor of $b_{\alpha\beta}$ in $b$.

The third fundamental form of the hypersurface $V^n$ at the point $P$ is

$$dN \cdot dN = f_{\alpha\beta} dx^\alpha dx^\beta,$$

where we have placed

$$f_{\alpha\beta} = N_\alpha \cdot N_\beta.$$
From equations (2.8), (2.9), and (2.11), it follows immediately that

\[ f_{\alpha\beta} = b_{\alpha\rho} b_{\beta\sigma} g^{\rho\sigma}, \]

and therefore that

\[ g^{\alpha\beta} = f_{\alpha\rho} b^{\alpha\sigma} b^{\beta\sigma}, \]

where \( b^{\alpha\sigma} = B^{\alpha\sigma}/b \). It is easily seen that the principal radii of curvature \( r_\alpha (\alpha = 1, \ldots, n) \) of the hypersurface \( V^n \) at the point \( P \) are the roots of the determinant equation

\[ \left| b_{\alpha\beta} - rf_{\alpha\beta} \right| = 0, \]

from which we obtain

\[ r_1 \cdots r_n = b/f, \quad \sum_{\alpha=1}^{n} r_\alpha = b_{\alpha\beta} f^{\alpha\beta}, \quad \sum r_1 \cdots r_{n-1} = f_{\alpha\beta} B^{\alpha\beta}/f, \]

where \( f^{\alpha\beta} \) denotes the cofactor of \( f_{\alpha\beta} \) in \( f = \left| f_{\alpha\beta} \right| \) divided by \( f \). From equations (2.13) and (2.19) it follows immediately that

\[ f = M_n^2 g > 0. \]

The area element of the hypersurface \( V^n \) at the point \( P \) is given by

\[ dA = g^{1/2} dx^1 \cdots dx^n. \]

Now we choose the direction of the unit normal vector \( N \) in such a way that the two frames \( P Y_1 \cdots Y_n N \) and \( O \mathcal{Y}_1 \cdots \mathcal{Y}_{n+1} \) have the same orientation. Then from equations (2.3) and (2.21) it follows that

\[ g^{1/2} N = Y_1 \times \cdots \times Y_n, \]

\[ \left| Y_1, \cdots, Y_n, N \right| = g^{1/2}. \]

Let \( u^1, \ldots, u^{n-1} \) be the local coordinates of a point \( P \) on the boundary \( V^{n-1} \), then the first fundamental form of the boundary \( V^{n-1} \) at the point \( P \) is

\[ ds^2 = a_{\lambda\mu} du^\lambda du^\mu \] (\( \lambda, \mu = 1, \ldots, n - 1 \)),

where

\[ a_{\lambda\mu} = \frac{g_{\alpha\beta}}{\partial u^\lambda / \partial u^\mu}, \]

and the matrix \( \left| a_{\lambda\mu} \right| \) is positive definite so that the determinant \( a = \left| a_{\lambda\mu} \right| > 0 \). The coefficients of the second fundamental form of the boundary \( V^{n-1} \) corresponding to the unit normal vector \( N \) of the hypersurface \( V^n \) at the point \( P \) are


\[
\Omega_{\lambda\mu} = \sum_{i=1}^{n+1} N_i \left( \frac{\partial^2 y^i}{\partial u^\nu \partial u^\mu} - \left\{ \frac{\nu}{\lambda \mu} \right\}_a \cdot \frac{\partial y^i}{\partial u^\nu} \right) \quad (\lambda, \mu, \nu = 1, \ldots, n - 1),
\]

where

\[
\left\{ \frac{\nu}{\lambda \mu} \right\}_a
\]

is a Christoffel symbol of the second kind formed with respect to the \(a\)'s and \(u\)'s. Similarly, for the hypersurface \(V^n\) we have

\[
b_{\alpha\beta} N = \frac{\partial^2 Y}{\partial x^\alpha \partial x^\beta} - \left\{ \frac{\gamma}{\alpha \beta} \right\}_{\gamma'} Y_{\gamma'},
\]

where

\[
\left\{ \frac{\gamma}{\alpha \beta} \right\}_{\gamma'}
\]

is a Christoffel symbol of the second kind formed with respect to the \(g\)'s and \(x\)'s. From equations (2.26) and (2.27) it is easily seen that

\[
\Omega_{\lambda\mu} = b_{\alpha\beta} \frac{\partial x^\alpha}{\partial u^\lambda} \frac{\partial x^\beta}{\partial u^\mu}.
\]

3. Some integral formulas. Suppose that there is a one-to-one correspondence between the points of two hypersurfaces \(V^n, V^{*n}\) of class \(C^3\) imbedded in a space \(E^{n+1}\) with positive Gaussian curvatures and closed boundaries \(V^{n-1}, V^{*n-1}\) of dimension \(n-1\) respectively such that the two hypersurfaces \(V^n, V^{*n}\) have the same normal vectors at corresponding points. Without loss of generality we may assume that the corresponding points of the two hypersurfaces \(V^n, V^{*n}\) have the same local coordinates \(x^1, \ldots, x^n\). Then §2 can be applied to the hypersurface \(V^n\), and for the corresponding quantities for the hypersurface \(V^{*n}\) we shall use the same symbols with a star.

At first, we observe that the vector \(Y_1 \times \cdots \times Y_{a-1} \times N \times Y_{a+1} \times \cdots \times Y_n\) is perpendicular to the normal vector \(N\) and that the \(n\) vectors \(Y_1, \ldots, Y_n\) are linearly independent in the tangent hyperplane of the hypersurface \(V^n\) at the point \(P\). Therefore the vector \(Y_1 \times \cdots \times Y_{a-1} \times N \times Y_{a+1} \times \cdots \times Y_n\) can be written in the form

\[
Y_1 \times \cdots \times Y_{a-1} \times N \times Y_{a+1} \times \cdots \times Y_n = a^{\alpha \beta} N_\beta.
\]

Taking the scalar products of the both sides of equations (3.1) with the vector \(Y_\gamma\) and making use of equations (2.2), (2.3), (2.10), (2.23), we obtain

\[
a^{\alpha \beta} \delta_{\beta \gamma} = g^{1/2} a^{\alpha \gamma} \quad (\alpha, \gamma = 1, \ldots, n).
\]

Solving equations (3.2) for \(a^{\alpha \beta}\) for each fixed \(\alpha\) and substituting the results in
The given text is a segment from a mathematical journal article by C. C. Hsiung. The page contains a series of mathematical equations and proofs, starting with the equation:

\[ (3.3) \quad Y_1 \times \cdots \times Y_{\alpha-1} \times N \times Y_{\alpha+1} \times \cdots \times Y_n = g^{1/2} b^{\alpha \beta} N_\beta. \]

Making use of equations (2.4), (2.9), (2.13), (2.22) and the relation

\[ Y_1 \times \cdots \times Y_{\beta-1} \times Y_{\beta+1} \times \cdots \times Y_{\alpha-1} \times N \times Y_{\alpha+1} \times \cdots \times Y_n \]

\[ = - Y_1 \times \cdots \times Y_{\beta-1} \times N \times Y_{\beta+1} \times \cdots \times Y_{\alpha-1} \times Y_{\alpha+1} \times \cdots \times Y_n \]

\[(\alpha > \beta; \alpha, \beta = 1, \ldots, n), \]

it is easily seen that

\[ \sum_{\alpha=1}^{n} \frac{\partial}{\partial x^\alpha} (Y_1 \times \cdots \times Y_{\alpha-1} \times N \times Y_{\alpha+1} \times \cdots \times Y_n) \]

\[ = \sum_{\alpha=1}^{n} Y_1 \times \cdots \times Y_{\alpha-1} \times N_{\alpha} \times Y_{\alpha+1} \times \cdots \times Y_n = - n g^{1/2} M_1 N. \]

Thus, from equations (3.3) and (3.4),

\[ (3.5) \quad n g^{1/2} M_1 N = - \frac{\partial}{\partial x^\alpha} (g^{1/2} b^{\alpha \beta} N_\beta). \]

Taking the scalar products of the both sides of equation (3.5) with the position vector \( Y^* \) of the corresponding point \( P^* \) of the point \( P \), we obtain in consequence of the relation \( b^2 = fg \), obtained from equations (2.13) and (2.19),

\[ (3.6) \quad n M_1 g^{1/2} p^* = \frac{1}{f^{1/2}} \sum_{\alpha=1}^{n} B^{\alpha \beta} N_\beta \cdot Y^* - \sum_{\alpha=1}^{n} \frac{\partial}{\partial x^\alpha} \left( \frac{1}{f^{1/2}} B^{\alpha \beta} Y^* \cdot N_\beta \right), \]

where we have placed

\[ (3.7) \quad p^* = Y^* \cdot N. \]

Integrating equation (3.6) with respect to \( x^1, \cdots, x^n \) over the hypersurface \( V^n \) and applying the general Green's theorem (cf., for instance, [3, pp. 75–76]) to the second term on the right side of equation (3.6), we then obtain

\[ \int_{V^n} n M_1 p^* dA = \int_{V^n} \frac{1}{f^{1/2}} \sum_{\alpha=1}^{n} B^{\alpha \beta} N_\beta \cdot Y^* dx^1 \cdots dx^n \]

\[ - \int_{V^n} \frac{1}{f^{1/2}} \sum_{\alpha=1}^{n} (-1)^{\alpha-1} B^{\alpha \beta} Y^* \cdot N_\beta dx^1 \cdots dx^{\alpha-1} dx^{\alpha+1} \cdots dx^n. \]

In order to use the formula (3.8) to derive an analogous expression for the integral \( \int_{V^n} M_{\alpha} p^* dA \) for a general \( \alpha (\alpha = 1, \ldots, n) \), in the space \( E^{n+1} \) we first consider a hypersurface \( \overline{V^n} \) parallel to the hypersurface \( V^n \) so that the two hypersurfaces \( V^n, \overline{V^n} \) have the same normals. It is evident that the
vector equation of the hypersurface $\mathcal{V}^n$ can be written in the form

\begin{equation}
(3.9) \quad \mathcal{V} = Y - tN,
\end{equation}

where $t$ is a real parameter. From equations (3.9), $N \cdot N = 1$ and $N \cdot \mathcal{V}_a = N \cdot \partial \mathcal{V} / \partial x^a = 0$, it follows immediately that $\partial t / \partial x^a = 0$ and therefore that $t$ is constant. Making use of equations (2.8), (2.10), (2.15) and their analogous ones for the hypersurface $\mathcal{V}^n$, we obtain the coefficients of the first and the second fundamental forms of the hypersurface $\mathcal{V}^n$:

\begin{align}
(3.10) & \quad \tilde{g}_{a\beta} = g_{a\beta} + 2b_{a\beta}^t + f_{a\beta}^t, \\
(3.11) & \quad b_{a\beta} = b_{a\beta} + f_{a\beta}^t,
\end{align}

from which it follows easily by an elementary calculation that

\begin{align}
(3.12) & \quad b = b\Delta, \\
(3.13) & \quad \bar{g} = g\Delta^2, \\
(3.14) & \quad | \tilde{r} \delta_{a\beta} - \tilde{g}_{a\beta} | = | (\bar{r} - t) b_{a\beta} - g_{a\beta} | \Delta,
\end{align}

where $\tilde{g} = | \tilde{g}_{a\beta} |$, $b = | b_{a\beta} |$ and $\bar{r}_a$ being the principal curvatures of the hypersurface $\mathcal{V}^n$. In consequence of equations (3.12), (3.13), (3.14) and (2.12), (2.13), (2.21) together with their analogues for the hypersurface $\mathcal{V}^n$, we have

\begin{align}
(3.15) & \quad M_{n} d\bar{A} = M_{n} dA, \\
(3.16) & \quad \tilde{r}_a = r_a + t,
\end{align}

where $d\bar{A}$ is the area element of the hypersurface $\mathcal{V}^n$.

Similarly, let $\mathcal{V}^* n$ be a hypersurface in the space $E^{n+1}$ parallel to the hypersurface $\mathcal{V}^n$ and having the vector equation

\begin{equation}
(3.19) \quad \mathcal{V}^* = Y^* - tN,
\end{equation}

where $t$ is the same arbitrary constant as in equation (3.9). For this one-to-one correspondence between the points of the two hypersurfaces $\mathcal{V}^n$, $\mathcal{V}^* n$, equation (3.8) can be written as, by means of equations (1.1) and (3.16),

\begin{equation}
(3.20) \quad \int_{\mathcal{V}^n} \tilde{g}^* \left( \sum_{\alpha=1}^{n} \tilde{r}_1 \tilde{r}_2 \cdots \tilde{r}_{n-1} \right) M_{n} d\bar{A}
\end{equation}

\begin{equation}
\begin{aligned}
&= \int_{\mathcal{V}^n} \frac{1}{f^{1/2}} \sum_{\alpha=1}^{n} \bar{B}^{a\beta} N_\beta \cdot \mathcal{V}^* d\mathcal{V}^1 \cdots d\mathcal{V}^n \\
&\quad - \int_{\mathcal{V}^* n} \frac{1}{f^{1/2}} \sum_{\alpha=1}^{n} (-1)^{\alpha-1} \bar{B}^{a\beta} N_\beta \cdot \mathcal{V}^* d\mathcal{V}^1 \cdots d\mathcal{V}^{\alpha-1} d\mathcal{V}^{\alpha+1} \cdots d\mathcal{V}^n,
\end{aligned}
\end{equation}
where $\bar{p}^* = \bar{Y}^*. N = p^*-t$, $\bar{B}^{\alpha \beta}$ is the cofactor of $b_{\alpha \beta}$ in $b$, $\bar{V}_{\alpha}^* = \partial \bar{Y}^* / \partial x^\alpha$, and $\bar{V}_n^{n-1}$ is the boundary of the hypersurface $\bar{V}^n$. Substitution, in equation (3.20), of equations (3.17), (3.18), (2.15) and the analogue of equation (2.10) for the hypersurface $V^{*n}$ yields immediately

$$
\int_{\bar{V}^n} (p^* - t) \sum_{\alpha=1}^n (n - \alpha + 1) (\sum_{r_1} r_1 \cdots r_{n-1}) l^{n-\alpha} M_n dA
$$

(3.21)

$$
= - \int_{\bar{V}^n} \frac{1}{f_{1/2}} \sum_{\alpha=1}^n \bar{B}^{\alpha \beta} (b_{\beta \alpha}^* + t f_{\alpha \beta}) dx^1 \cdots dx^n
$$

+ \int_{\bar{V}^{n-1}} \frac{1}{f_{1/2}} \sum_{\alpha=1}^n (-1)^{\alpha-1} \bar{V}^*. N_\beta \bar{B}^{\alpha \beta} dx^1 \cdots dx^{\alpha-1} dx^{\alpha+1} \cdots dx^n,
$$

which is an identity in $t$. Hence, by equating the coefficients of $t^0, t, \cdots, t^{n-1}$ on the both sides of equation (3.21) and using equation (1.1), we can obtain $n$ formulas, one of which is equation (3.8). These $n$ formulas have been obtained by the author [4] for the case where the two hypersurfaces $V^n, V^{*n}$ coincide.

4. **Proof of the theorem.** In order to prove the theorem stated in the introduction, we may assume, for simplicity, that the local coordinate $x^1, \cdots, x^n$ of the two hypersurfaces $V^n, V^{*n}$ be so chosen that

$$ f_{\alpha \beta} = 0 \text{ for } \alpha \neq \beta. $$

Then from equation (3.11) it follows that

$$
\bar{B}^{\alpha \beta} = \frac{f}{f_{\alpha \alpha f_{\beta \beta}}} \left[ -b_{\beta \alpha} t^{n-2} + \sum_{\gamma=1, \gamma \neq \alpha, \gamma \neq \beta}^n (b_{\beta \gamma} b_{\gamma \alpha} - b_{\gamma \gamma} b_{\alpha \beta}) t^{n-3} / f_{\gamma \gamma} \right] + \cdots,
$$

(4.1)

for $\beta \neq \alpha$,

$$
\bar{B}^{\alpha \alpha} = \frac{f}{f_{\alpha \alpha}} \left[ l^{n-1} + \sum_{\gamma=1, \gamma \neq \alpha}^n \frac{b_{\gamma \gamma}}{f_{\gamma \gamma}} l^{n-2} + \frac{1}{2} \sum_{\beta, \gamma=1, \gamma \neq \alpha; \beta \neq \gamma}^n \frac{b_{\beta \beta} b_{\gamma \gamma} - b_{\beta \gamma}^2}{f_{\beta \beta} f_{\gamma \gamma}} l^{n-3} \right] + \cdots,
$$

(4.2)

where the unwritten terms are of degrees $< n - 3$ in $t$. Moreover, an elementary calculation from equation (2.18) leads to

$$
\sum_{\alpha, \beta=1; \alpha \neq \beta}^{n} b_{\alpha \alpha} b_{\beta \beta} - b_{\alpha \beta}^2 / f_{\alpha \alpha f_{\beta \beta}}.
$$

(4.3)

Thus, by equating the coefficients of $t^{n-2}$ on the both sides of equation (3.21) and using equations (4.1), (4.2), (4.3), we obtain
Replacing the hypersurface $V^n$ by the hypersurface $V^*n$ in equation (4.4) gives

\[
(n - 1) \int_{V^n} \rho^* \left( \sum_{\alpha=1}^{n} r^*_{\alpha} \right) M^* dA^* = \int_{V^n} \sum_{\alpha, \beta=1}^{n} \frac{f_{1/2}}{f_{aaf_{\beta\beta}}} (b^*_{\beta\alpha} b^*_{\alpha\alpha} - b_{\alpha\beta} b^*_{\beta\alpha}) dx^1 \cdots dx^n
\]

\[\quad + \int_{V^{*n}} \sum_{\alpha, \beta=1}^{n} (-1)^{\alpha-1} \frac{f_{1/2}}{f_{aaf_{\beta\beta}}} V^* \cdot (N_{\alpha\beta} b^*_{\beta\alpha} - N_{\beta\alpha} b^*_{\alpha\beta}) dx^1 \cdots dx^{\alpha-1} dx^{\alpha+1} \cdots dx^n.\]

(4.5)

Since under the given one-to-one correspondence between the points of the two hypersurfaces $V^n$, $V^*n$, the two boundaries $V^{n-1}$, $V^{*n-1}$ are congruent, we may assume that the corresponding points of the boundaries $V^{n-1}$, $V^{*n-1}$ have the same local coordinates $u^1, \ldots, u^{n-1}$. Then the second fundamental forms of the boundaries $V^{n-1}$, $V^{*n-1}$ corresponding to the common unit normal vector $N$ of the hypersurfaces $V^n$, $V^*n$ at the corresponding points of the boundaries $V^{n-1}$, $V^{*n-1}$ are equal (see, for instance, [2, p. 192]), and therefore from equation (2.28) and its analogue for the hypersurface $V^*n$ it follows that $b_{\alpha\beta} = b^*_{\alpha\beta}$ at corresponding points of the two boundaries $V^{n-1}$, $V^{*n-1}$. Thus the second integrals on the right side of equations (4.4), (4.5) are equal. On the other hand, by the assumption of the theorem we have $\sum_{\alpha=1}^{n} r_{\alpha} = \sum_{\alpha=1}^{n} r^*_{\alpha}$, and from equations (2.13), (2.19) and the analogous ones for the hypersurface $V^*n$ it is seen at once that $M_{n^2}^{1/2} = M_{n^*2}^{1/2}$. Hence subtracting equation (4.4) from equation (4.5) yields

\[
(4.6) \int_{V^n} \sum_{\alpha, \beta=1}^{n} \frac{f_{1/2}}{f_{aaf_{\beta\beta}}} \left[ (b_{\beta\alpha} b^*_{\alpha\alpha} - b_{\alpha\beta} b^*_{\beta\alpha}) - (b^*_{\alpha\beta} b^*_{\beta\alpha} - b_{\alpha\beta} b^*_{\beta\alpha}) \right] dx^1 \cdots dx^n = 0.
\]

Adding together equation (4.6) and the analogous one by interchanging the two hypersurfaces $V^n$, $V^*n$, we obtain

\[
(4.7) \int_{V^n} \sum_{\alpha, \beta=1}^{n} \frac{f_{1/2}}{f_{aaf_{\beta\beta}}} \left[ (b_{\alpha\alpha} - b^*_{\alpha\alpha}) (b_{\beta\beta} - b^*_{\beta\beta}) - (b_{\alpha\beta} - b^*_{\alpha\beta})^2 \right] dx^1 \cdots dx^n = 0.
\]
From the assumption $\sum_{\alpha=1}^{n} r_{\alpha} = \sum_{\alpha=1}^{n} r_{\alpha}^{*}$, equation (2.19) and the analogous one for the hypersurface $V^{*n}$, we have

\[
(4.8) \quad \sum_{\alpha=1}^{n} \frac{1}{f_{\alpha\alpha}} (b_{aa} - b_{aa}^{*}) = 0,
\]

and therefore

\[
\sum_{\alpha, \beta=1; \alpha \neq \beta}^{n} \frac{1}{f_{\alpha\alpha}f_{\beta\beta}} (b_{aa} - b_{aa}^{*})(b_{\beta\beta} - b_{\beta\beta}^{*})
= \sum_{\alpha, \beta=1; \alpha \neq \beta}^{n} \frac{1}{f_{\alpha\alpha}f_{\beta\beta}} (b_{aa} - b_{aa}^{*})(b_{\beta\beta} - b_{\beta\beta}^{*}) - \sum_{\alpha=1}^{n} \frac{1}{f_{\alpha\alpha}^{2}} (b_{aa} - b_{aa}^{*})^{2}
= - \sum_{\alpha=1}^{n} \frac{1}{f_{\alpha\alpha}^{2}} (b_{aa} - b_{aa}^{*})^{2}.
\]

Thus equation (4.7) is reduced to

\[
(4.9) \quad \int_{V^{n}} \left[ \sum_{\alpha=1}^{n} \frac{1}{f_{\alpha\alpha}^{2}} (b_{aa} - b_{aa}^{*})^{2} + \sum_{\alpha, \beta=1; \alpha \neq \beta}^{n} \frac{1}{f_{\alpha\alpha}f_{\beta\beta}} (b_{ab} - b_{ab}^{*})^{2} \right] f^{1/2} dx^{1} \cdots dx^{n} = 0.
\]

It is obvious that the integrand of equation (4.9) is non-negative, and therefore equation (4.9) holds when and only when

\[
(4.10) \quad b_{ab} = b_{ab}^{*} \quad (\alpha, \beta = 1, \cdots, n),
\]

from which, equations (2.11), (2.17), and the analogous ones for the hypersurface $V^{*n}$ we obtain that $g_{ab} = g_{ab}^{*} \quad (\alpha, \beta = 1, \cdots, n)$. Hence the proof of the theorem is complete.

**References**


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