THE SYMMETRIC DERIVATIVE ON THE $(k-1)$-DIMENSIONAL HYPERSPHERE

BY

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1. **Introduction.** Let x be a point on the unit $(k-1)$-dimensional hypersphere Ω in Euclidean k-space, $k \geqq 3$, and let μ be a completely additive set function of bounded variation defined on the Borel sets of Ω. Let $D(x, h)$ represent the spherical cap on Ω obtained by intersecting Ω with a sphere whose center is x and radius is $2 \sin h/2$, and let $|D(x, h)|$ be the $(k-1)$-dimensional volume of $D(x, h)$. Then μ will be said to have a symmetric derivative at x, designated by $\mu_s(x)$, if $|D(x, h)|^{-1}\mu[D(x, h)]$ tends to $\mu_s(x)$ as h tends to zero.

Let $S[d\mu] = \sum_{n=0}^{\infty} Y_n(x)$ be the Stieltjes series of surface harmonics defined by μ. We shall show in this paper that if $\mu_s(x_0)$ exists and is finite and μ satisfies the global condition $|\mu| [D(x_0', \epsilon)] = 0$ for some $\epsilon > 0$, where x_0' is the point diametrically opposite to x_0 and $|\mu|$ is the total variation of μ, then S is summable (C, δ), $\delta > (k-2)/2+1$, to $\mu_s(x_0)$. This result generalizes the well-known result for Fourier-Stieltjes series where $\delta > 1$, see [8, p. 55]. In case the global condition is not satisfied, we obtain that $S[d\mu]$ is summable (C, η) to $\mu_s(x_0)$ where $\eta > k-2$ for $k \geqq 4$ and $\eta > 3/2$ for $k=3$.

In the special case when μ is absolutely continuous and $Y_n(x_0) = O(n^{-1})$, we shall show that a necessary and sufficient condition that $S[d\mu]$ converges at x_0 to the finite value β is that $\mu_s(x_0)$ exists and equals β. This fact generalized a result previously obtained by Hardy and Littlewood [5, p. 229] for Fourier series.

2. **Definitions and notation.** λ will always designate the value $(k-2)/2$, and P_n^λ will designate the Gegenbauer (ultraspherical) polynomials defined by the equation,

$$(1 - 2r \cos \theta + r^2)^{-\lambda} = \sum_{n=0}^{\infty} r^n P_n^\lambda(\cos \theta).$$

With the help of these functions, we can associate to every additive set function defined on the Borel sets of Ω and of bounded variation there, a sequence of surface harmonics by means of the equation

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$$Y_n(x) = \frac{\Gamma(\lambda)(n+\lambda)}{2\pi^{\lambda+1}} \int_\Omega P_n^\lambda(\cos\gamma)d\mu(y)$$

where γ is the angle between x and y, that is if $x=(x_1, \cdots, x_k)$ and $y=(y_1, \cdots, y_k)$,

$$\cos\gamma = (x, y) = x_1y_1 + \cdots + x_ky_k.$$

As shown in [3, Chap. 11], $r^nY_n(x)$ gives rise to an homogeneous harmonic polynomial of degree n in Euclidean k-space. We define $S[d\mu]=\sum_{n=0}^\infty Y_n(x)$ and call this the Stieltjes-series of surface harmonics associated to μ.

By the terminology $\sum_{n=0}^\infty Y_n(x)$ is (C, α) summable to a given value, we mean the usual Cesaro summability defined for example in [8, Chap. 3].

We note that $D(x_0, h)$ defined in the introduction is the set

$$\{x, (x, x_0) \geqq \cos h\}.$$

$[\lambda]$ will mean the integral part of λ and $|\mu|(E)$ will stand for the total variation of μ in E.

3. **Statement of main results.** We shall prove the following theorem for Stieltjes-series of surface harmonics:

THEOREM 1. *Let μ be a completely additive set function defined on the Borel sets of Ω and of bounded variation on Ω, and let $S[d\mu]=\sum_{n=0}^\infty Y_n(x)$. Suppose $\mu_s(x_0)$ exists and is finite. Then $S[d\mu]$ is (C, η), $\eta>\max(3/2, k-2)$, summable to $\mu_s(x_0)$. If, furthermore, μ satisfies the condition that $|\mu|[D(x_0', \epsilon)]=0$ for some $\epsilon>0$, where x_0' is diametrically opposite to x, then $S[d\mu]$ is (C, δ) summable to $\mu_s(x_0)$, $\delta>\lambda+1$.*

Concerning the convergence of series of surface harmonics, we shall prove the following theorem:

THEOREM 2. *Let f be an integrable function on Ω and define $\mu(E)$, for E a Borel set on Ω, by $\mu(E)=\int_E f(x)d\Omega(x)$ where $d\Omega(x)$ is the $(k-1)$-dimensional volume element on Ω. Let $S[f]=\sum_{n=0}^\infty Y_n(x)$, and suppose that $Y_n(x_0)=O(n^{-1})$. Then a necessary and sufficient condition that $S[f]$ converges at x_0 to β is that $\mu_s(x_0)=\beta$.*

REMARK 1. By [1, Theorem 2], it is easily seen that the condition $|\mu|[D(x_0', \epsilon)]=0$ for some $\epsilon>0$ in Theorem 1 can be replaced by the condition that in $D(x_0', \epsilon)$, μ is absolutely continuous with $\mu(E)=\int_E f(y)d\Omega(y)$ and that

$$\int_{D(x_0',\epsilon)} \frac{|f(y)|}{[1-(x_0', y)]^{\lambda/2}} d\Omega(y) < \infty.$$

4. **Fundamental lemmas.** Before proceeding with the proof of these theorems we shall prove some lemmas. By $S_n^{\alpha,\lambda}(\cos\theta)$, we shall designate the sum

$$S_n^{\alpha,\lambda}(\cos\theta) = \sum_{j=0}^{n} (j+\lambda)P_j^{\lambda}(\cos\theta)A_{n-j}^{\alpha}$$

where $\sum_{n=0}^{\infty} A_n x^n = (1-x)^{-(\alpha+1)}$ with $\alpha > 1$. By $[g(\cos\theta)]'$, we shall mean $dg(\cos\theta)/d\theta$.

LEMMA 1. *$|[S_n^{\alpha,\lambda}(\cos\theta)]'| \leqq K(\epsilon)n^{\lambda+1}\theta^{-(\alpha+\lambda+1)}$ for $n^{-1} \leqq \theta \leqq \pi - \epsilon$ and $[\lambda]+2 > \alpha > [\lambda]+1$, where $K(\epsilon)$ is a constant depending on ϵ but not on n.*

We first observe that

$$(1)\qquad \sum_{n=0}^{\infty} (n+\lambda)[P_n^{\lambda}(\cos\theta)]'r^n = -\frac{2\lambda(\lambda+1)r(1-r^2)\sin\theta}{(1-2r\cos\theta+r^2)^{\lambda+2}}.$$

Let us suppose that $\lambda > 1$. Then since

$$(2)\qquad \frac{1}{(1-r)^{\alpha+1}} \frac{(1-r^2)r\sin\theta}{(1-2r\cos\theta+r^2)^{\lambda+2}}$$
$$= \frac{1}{(1-r)^{\alpha}} \frac{(1-r^2)r\sin\theta}{(1-2r\cos\theta+r^2)^{\lambda+1}} \quad \frac{1}{(1-r)} \frac{1}{(1-2r\cos\theta+r^2)}$$

we obtain from (1) that

$$(3)\qquad [S_n^{\alpha,\lambda}(\cos\theta)]' = K_1 \sum_{j=0}^{n} [S_j^{\alpha-1,\lambda-1}(\cos\theta)]'T_{n-j}(\cos\theta)$$

where $T_n(\cos\theta) = \sum_{j=0}^{n} P_n^1(\cos\theta)$ and K_1 is a constant. But

$$P_j^1(\cos\theta) = \sin(j+1)\theta/\sin\theta;$$

therefore,

$$(4)\qquad T_n(\cos\theta) = \frac{\cos\theta/2 - \cos(2n+3)\theta/2}{2\sin\theta\sin\theta/2}.$$

So if we assume that $[S_n^{\alpha-1,\lambda-1}(\cos\theta)]'$ satisfies the conclusion of the lemma, we see from (3) and (4) that

$$|[S_n^{\alpha,\lambda}(\cos\theta)]'| \leqq K(\epsilon)\theta^{-(\alpha+\lambda-1)-2}\sum_{j=0}^{n} j^{\lambda}.$$

Therefore in order to prove the lemma we need prove it only in the special cases $\lambda = 1/2$ and $\lambda = 1$.

To do this we introduce

$$S_n^{\alpha,0}(\cos\theta) = A_n^{\alpha}/2 + \sum_{j=1}^{n} (\cos j\theta)A_{n-j}^{\alpha}$$

and observe by [8, p. 56] that

$$(5)\qquad \left| [S_n^{\alpha,0}(\cos\theta)]' \right| \leqq K(\epsilon)n\theta^{-(\alpha+1)} \qquad \text{for } n^{-1} \leqq \theta \leqq \pi - \epsilon,\ 1 < \alpha < 2.$$

For the case $\lambda = 1/2$, we rewrite (2) in the form

$$\frac{1}{(1-r)^{\alpha+1}} \frac{(1-r^2)r\sin\theta}{(1-2r\cos\theta+r^2)^{5/2}} = \frac{1}{(1-r)^{\alpha+1}} \frac{(1-r^2)r\sin\theta}{(1-2r\cos\theta+r^2)^2} \frac{1}{(1-2r\cos\theta+r^2)^{1/2}}$$

and obtain that

$$(6)\qquad [S_n^{\alpha,1/2}(\cos\theta)]' = K_2 \sum_{j=0}^{n} [S_j^{\alpha,0}(\cos\theta)]' P_{n-j}^{1/2}(\cos\theta)$$

where K_2 is a constant. From [7, p. 160], $|P_n^{1/2}(\cos\theta)| \leqq (n\sin\theta)^{-1/2}$. So we conclude from (5) and (6) that

$$\left| [S_n^{\alpha,1/2}(\cos\theta)]' \right| \leqq K(\epsilon)n^{3/2}\theta^{-(\alpha+3/2)} \qquad \text{for } n^{-1} \leqq \theta \leqq \pi - \epsilon \text{ and } 1 < \alpha < 2$$

which proves the lemma in the special case $\lambda = 1/2$.

For the case $\lambda = 1$, let us assume that $2 < \alpha < 3$. Then by (2), we obtain that

$$[S_n^{\alpha,1}(\cos\theta)]' = K_3 \sum_{j=0}^{n} [S_j^{\alpha-1,0}(\cos\theta)]' T_{n-j}(\cos\theta)$$

and (4) and (5) give us that

$$\left| [S_n^{\alpha,1}(\cos\theta)]' \right| \leqq K(\epsilon)n^2\theta^{-(\alpha+2)} \qquad \text{for } n^{-1} \leqq \theta \leqq \pi - \epsilon,$$

and the proof to the lemma is complete.

LEMMA 2. *$\left| [S_n^{\alpha,\lambda}(\cos\theta)]' \right| \leqq Kn^{\alpha+2\lambda+2}$ for $0 \leqq \theta \leqq n^{-1}$ and $\alpha \geqq 0$, where K is a constant independent of n.*

To prove this lemma we write

$$\frac{1}{(1-r)^{\alpha-1}} \frac{(1-r^2)r\sin\theta}{(1-2r\cos\theta+r^2)^{\lambda+2}} = \frac{(1-r^2)r\sin\theta}{(1-r)^{\alpha+1}(1-2r\cos\theta+r^2)^{\lambda+3/2}} \frac{1}{(1-2r\cos\theta+r^2)^{1/2}}$$

and obtain that

$$(7)\qquad [S^{\alpha,\lambda}(\cos\theta)]' = K_1 \sum_{i=0}^{n} [S_j^{\alpha,\lambda-1/2}(\cos\theta)]' P_{n-j}^{1/2}(\cos\theta),$$

$S_n^{\alpha,0}(\cos\theta)$ being defined as in Lemma 1.

If we assume that the conclusion to the lemma holds for $S_n^{\alpha,\lambda-1/2}(\cos\theta)$, we can use the fact that $|P_n^{1/2}(\cos\theta)|\leqq 1$ and obtain from (7) that

$$|[S_n^{\alpha,\lambda}(\cos\theta)]'| \leqq K_2\sum_{j=0}^{n} j^{\alpha+2\lambda+1} \leqq Kn^{\alpha+2\lambda+2}.$$

So to prove the lemma, we need only show that the conclusion of the lemma holds for $S_n^{\alpha,0}(\cos\theta)$. But

$$|[S_n^{\alpha,0}(\cos\theta)]'| = \left|\sum_{j=0}^{n} j\sin j\theta A_{n-j}^{\alpha}\right| \leqq Kn^{\alpha+2},$$

and the proof is complete.

We now state a lemma of Kogbetliantz [6, p. 139].

LEMMA 3. *For* $-1<\alpha\leqq k-1$ *and for* $0\leqq\theta\leqq\pi$

$$|S_n^{\alpha,\lambda}(\cos\theta)| \leqq K(n+1)^{k-2}(\sin\theta/2)^{-(\alpha+1)}.$$

We next define the expression $B_n^\lambda(h)$ by

$$B_n^\lambda(h) = \int_0^h P_n^\lambda(\cos\theta)(\sin\theta)^{2\lambda}d\theta \Big/ P_n^\lambda(1)\int_0^h(\sin\theta)^{2\lambda}d\theta \tag{8}$$

and prove the following lemmas:

LEMMA 4. $|B_n^\lambda(h)|\leqq K(hn)^{-\lambda}$ *for* $0\leqq h\leqq\pi/2$ *where* K *is a constant independent of* n.

LEMMA 5. $|B_{n+1}^\lambda(h)-B_n^\lambda(h)|\leqq Kh^2n$ *for* $0\leqq h\leqq\pi$ *where* K *is a positive constant independent of* n.

To prove Lemma 4 we have by [7, p. 80, pp. 166 and 167] that

$$\begin{aligned} |P_n^\lambda(\cos\theta)| &\leqq K_1\theta^{-\lambda}n^{\lambda-1} \qquad \text{for } 0\leqq\theta\leqq\pi/2,\\ P_n^\lambda(1) &= \binom{n+2\lambda-1}{n},\\ |P_n^\lambda(\cos\theta)| &\leqq P_n^\lambda(1). \end{aligned} \tag{9}$$

We conclude that the left side of the inequality in Lemma 4 is majorized by a constant multiple of

$$n^{\lambda-1}\int_0^h\theta^{-\lambda}\theta^{2\lambda}d\theta \Big/ n^{2\lambda-1}\int_0^h\theta^{2\lambda}d\theta,$$

which gives the right side of the inequality, and the lemma is proved.

To prove Lemma 5, we let $g(n)$ equal the square of the normalizing coefficient of $P_n^\lambda(r)$, that is [3, p. 174]

$$g(n) = \int_{-1}^{1} [P_n^\lambda(r)]^2(1-r^2)^{\lambda-1/2}dr = \frac{\pi^{1/2}\Gamma(\lambda+1/2)}{(n+\lambda)\Gamma(\lambda)} P_n^\lambda(1) \tag{10}$$

and obtain from the Christoffel-Darboux formula [3, p. 159] that

$$\frac{P_n^\lambda(\cos\theta)P_{n+1}^\lambda(1) - P_{n+1}^\lambda(\cos\theta)P_n^\lambda(1)}{2\sin^2\theta/2} = 2g(n)\frac{n+\lambda}{n+1}\sum_{j=0}^{n}[g(j)]^{-1}P_j^\lambda(\cos\theta)P_j^\lambda(1). \tag{11}$$

From (8) and (9), we see that the left side of the inequality in Lemma 5 is majorized by a constant multiple of

$$\max_{0\leq\theta\leq h}\frac{|P_{n+1}^\lambda(1)P_n^\lambda(\cos\theta) - P_n^\lambda(1)P_{n+1}^\lambda(\cos\theta)|}{n^{4\lambda-2}}.$$

But by (9), (10), and (11) this expression in turn is majorized by a constant multiple of

$$\frac{h^2}{n^{4\lambda-2}}n^{2\lambda-2}\sum_{j=0}^{n} jj^{2\lambda-1} = h^2O(n),$$

which is the right side of the inequality in Lemma 5, and the proof of the lemma is complete.

5. **Proof of Theorem 1.** Let us first suppose that $\mu(E)=0$ for E a Borel set contained in $D(x_0', \epsilon)$ where ϵ is a positive number between 0 and $\pi/2$. Then with $\delta>\lambda+1$ but less than $[\lambda]+2$, we have

$$S_n^\delta(x) = \sum_{j=0}^{n} Y_j(x)A_{n-j}^\delta = \frac{\Gamma(\lambda)}{2\pi^{\lambda+1}}\int_{\Omega-D(x_0',\epsilon)} S_n^{\delta,\lambda}(\cos\gamma)d\mu(y) \tag{12}$$

with $\cos\gamma=(x, y)$ and $S_n^{\delta,\lambda}(\cos\gamma)$ defined in the beginning of §4.

Next, we introduce the continuous function $f(x)$ which has the following properties:

(i) $f(x)=0$ for x in $D(x_0', \epsilon)$,

(ii) $f(x_0)=\mu_s(x_0)$,

(iii) $\int_\Omega f(x)d\Omega(x)=\mu[D(x_0, \pi-\epsilon)]$,

and set $\mu_1(E)=\int_E f(x)d\Omega(x)$ for E a Borel set on Ω. By [1, Theorem 2], $R_n^\delta(x_0)/A_n^\delta\to\mu_s(x_0)$ for $\delta>\lambda$ when R_n is defined by the last expression in (12) with μ replaced by μ_1. Consequently to prove the second part of this theorem it is sufficient to show that

$$(13)\qquad \frac{2\pi^{\lambda+1}}{\Gamma(\lambda)}\left[R_n^{\delta}(x_0) - S_n^{\delta}(x_0)\right] = \int_{\Omega - D(x_0', \epsilon)} S_n^{\delta,\lambda}[(x_0, y)]d\mu_2(y) = o(n^{\delta})$$

with $[\lambda]+2>\delta>\lambda+1$ and $\mu_2=\mu_1-\mu$.

To do this, we define a one-dimensional completely additive set function σ of bounded variation on $[0, \pi]$ in the following manner:

To every Borel set Z on $[0, \pi]$ associate the set Z^{k-1} on Ω where $Z^{k-1} = \{x,\ x\in\Omega \text{ and } (x, x_0)=\cos\theta \text{ where } \theta\in Z\}$. Then σ is defined by $\sigma(Z) = \mu_2(Z^{k-1})$. So in particular if $Z=[0, h]$, $\sigma(Z)=\mu_2[D(x_0, h)]$.

With this definition we obtain that

$$(14)\qquad \begin{aligned} \int_{\Omega - D(x_0', \epsilon)} S_n^{\delta,\lambda}[(x_0, y)]d\mu_2(y) &= \int_0^{\pi-\epsilon} S_n^{\delta,\lambda}(\cos\theta)d\sigma(\theta) \\ &= -\int_0^{\pi-\epsilon} \mu_2[D(x_0, \theta)][S_n^{\delta,\lambda}(\cos\theta)]'d\theta \end{aligned}$$

since both $\mu_2[D(x_0, \pi-\epsilon)]$ and $\mu_2[D(x_0, 0)]$ are zero.

Now by construction $\mu_2[D(x_0, \theta)]=o[|D(x_0, \theta)|]=o(\theta^{2\lambda+1})$ as $\theta\to 0$. Therefore by Lemma 2,

$$(15)\qquad \begin{aligned} \int_0^{n^{-1}} \mu_2[D(x_0, \theta)][S_n^{\delta,\lambda}(\cos\theta)]'d\theta &= o\left(\int_0^{1/n} \theta^{2\lambda+1} n^{\delta+2\lambda+2} d\theta\right) \\ &= o(n^{\delta}), \end{aligned}$$

and by Lemma 1,

$$(16)\qquad \begin{aligned} &\left|\int_{n^{-1}}^{h} \mu_2[D(x_0, \theta)][S_n^{\delta,\lambda}(\cos\theta)]'d\theta\right| \\ &\qquad \leqq \psi_1(h)n^{\lambda+1}\int_{n^{-1}}^{h} \theta^{2\lambda+1}\theta^{-(\delta+\lambda+1)}d\theta \leqq \psi_2(h)n^{\delta}, \end{aligned}$$

where $\psi_i(h)\,(i=1, 2)$ tends to zero with h.

From Lemma 1, we also obtain that

$$(17)\qquad \int_h^{\pi-\epsilon} \mu_2[D(x_0, \theta)][S_n^{\delta,\lambda}(\cos\theta)]'d\theta = O(n^{\lambda+1-\delta}n^{\delta}) = o(n^{\delta}).$$

We therefore conclude from (13), (14), (15), (16), and (17) that

$$\limsup_{n\to\infty} \left| R_n^{\delta}(x_0) - S_n^{\delta}(x_0)\right| n^{-\delta} = 0.$$

Since $R_n^{\delta}(x_0)/A_n^{\delta}\to\mu_s(x_0)$, the second part of Theorem 1 is proved.

To prove the first part of the theorem, we suppose that $k-2<\eta<k-1$ for $k=4$ and $3/2<\eta<2$ for $k=3$ and set $\mu=\mu_3+\mu_4$, where for Borel sets E,

$\mu_3(E)=0$ for $E\subset\Omega-D(x_0, 3\pi/4)$ and $\mu_4(E)=0$ for $E\subset D(x_0, 3\pi/4)$. Then

$$\frac{S_n^\eta(x_0)}{A_n^\eta} = \frac{\Gamma(\lambda)}{2A_n^\eta\pi^{\lambda+1}}\int_\Omega S_n^{\eta,\lambda}(\cos\gamma)\left[d\mu_3(y) + d\mu_4(y)\right] = I_n^\eta + II_n^\eta.$$

By the part of the theorem proved above, $I_n^\eta\to\mu_s(x_0)$ for $\eta>\lambda+1$. By Lemma 3 with $k-2<\eta<k-1$,

$$\left| II_n^\eta\right| \leqq \frac{K(n+1)^{k-2}\int_{\Omega-D(x_0,3\pi/4)}\left| d\mu_4(y)\right|}{\left(\sin\frac{3\pi}{8}\right)^{\eta+1} n^\eta}.$$

We conclude that II_n^η tends to zero as $n\to\infty$, and the proof of Theorem 1 is complete.

REMARK 2. To show that η cannot be taken equal to $k-2$ in the above proof, choose μ_4 to be the mass distribution with mass 1 at x_0' and zero elsewhere. Then

$$II_n^{k-2} = \frac{\Gamma(\lambda)}{2A_n^{k-2}\pi^{\lambda+1}}\sum_{j=0}^{n}(j+\lambda)(-1)^j P_j^\lambda(1)A_{n-j}^{k-2}.$$

By [8, p. 43] for II_n^{k-2} to tend to zero, $(n+\lambda)P_n(1)$ would have to be $o(n^{k-2})$. But $(n+\lambda)P_n^\lambda(1)n^{-(k-2)}\to 1/(k-3)!$ as $n\to\infty$. So in Theorem 1, η must be chosen greater than $k-2$.

6. **Proof of Theorem 2.** The sufficiency condition of Theorem 2 follows immediately from Theorem 1 and the usual Tauberian theorem for Cesàro summability [4, p. 121].

To prove the necessity, we set $F_h(x)=\left|D(x, h)\right|^{-1}\int_{D(x,h)}f(z)d\Omega(z)$ and obtain that

$$\begin{aligned}(18)\qquad &\int_\Omega P_n^\lambda[(x, y)]F_h(y)d\Omega(y)\\ &= \left| D(x, h)\right|^{-1}\int_\Omega f(z)\left[\int_\Omega P_n^\lambda[(x, y)]X_{D(y,h)}(z)d\Omega(y)\right]d\Omega(z)\\ &= \left| D(x, h)\right|^{-1}\int_\Omega f(z)\left[\int_{D(z,h)} P_n^\lambda[(x, y)]d\Omega(y)\right]d\Omega(z)\end{aligned}$$

where $X_E(x)$ stands for the characteristic function of the set E.

Then by [3, p. 243],

$$(19)\qquad P_n^\lambda[(x, y)] = P_n^\lambda(1)\left|\Omega\right|\left[h(n)\right]^{-1}\sum_{j=1}^{h(n)} S_n^j(x)S_n^j(y)$$

where $h(n)$ is the maximum number of linearly independent surface harmonics of degree n on Ω, and $S_n^j(x), j=1, \cdots, h(n)$, are a set of linear independent, orthonormal surface harmonics on Ω.

On the other hand, by [3, p. 240] with $z=(1, O, \cdots, o)$ and x given by spherical coordinates in terms of z, $S_n^j(x)$ can be chosen as a constant multiple of

$$(20) \qquad Y(m_q; \theta_q, \pm \phi) = e^{\pm i m_{2\lambda}\phi} \prod_{q=0}^{2\lambda-1} (\sin \theta_{q+1})^{m_{q+1}} P_{m_q - m_{q+1}}^{m_{q+1}+\lambda-1/2q}(\cos \theta_{q+1})$$

where $n=m_0 \geqq m_1 \geqq \cdots \geqq m_{2\lambda} \geqq 0$. [For the spherical coordinate notation see [3, p. 233] with $p=2\lambda$.] Let us call $S_n^1(x)$ the function obtained in (20) when the sequence $(n, 0, \cdots, 0)$ is used. Then $S_n^1(x)$ is the function $P_n^\lambda[(x, z)]$, normalized.

We shall now show that,

$$(21) \qquad \int_{D(z,h)} S_n^j(y) d\Omega(y) = 0, \qquad \text{for } j \neq 1.$$

For there must then exist some $q \neq 0$ such that $m_q \neq 0$. Let q_1 be the last such q. Recalling that

$$d\Omega(y) = (\sin \theta_1)^{2\lambda} (\sin \theta_2)^{2\lambda-1} \cdots (\sin \theta_{2\lambda}) d\theta_1 \cdots d\theta_{2\lambda} d\phi$$

where $0 \leqq \theta_q \leqq \pi$ $(q=1, \cdots, 2\lambda)$, and $0 \leqq \phi \leqq 2\pi$, we see immediately that (21) holds in case $q_1 = 2\lambda$. Let us suppose then that $q_1 \neq 2\lambda$. Then

$$(\sin \theta_{q_1+1})^{m_{q_1+1}} P_{m_{q_1}-m_{q_1+1}}^{m_{q_1+1}+\lambda-1/2q_1}(\cos \theta_{q_1+1}) = P_{m_{q_1}}^{\lambda-1/2q_1}(\cos \theta_{q_1+1}).$$

But by [2, p. 177]

$$\int_0^\pi P_n^{\lambda-1/2q}(\cos \theta)(\sin \theta)^{2\lambda-q} d\theta = 0 \qquad \text{for } n \neq 0$$

and consequently (21) holds.

We thus obtain from (18), (19), (20), and (21) that

$$(22) \qquad \begin{aligned} &\int_\Omega P_n^\lambda[(x, y)] F_h(y) d\Omega(y) \\ &\qquad = \zeta(n) \int_{D(w,h)} P_n^\lambda[(y, w)] d\Omega(y) \int_\Omega f(z) P_n^\lambda[(x, z)] d\Omega(z) \end{aligned}$$

where

$$\begin{aligned} \zeta(n) &= \frac{P_n^\lambda(1) \, |\Omega|}{h(n) \, |D(w, h)|} \left\{ \int_\Omega (P_n^\lambda[(x, w)])^2 d\Omega(x) \right\}^{-1} \\ &= \frac{1}{D(w, h) P_n^\lambda(1)} \qquad \text{by [3, p. 236, formula 29].} \end{aligned}$$

Consequently with $S[f]=\sum_{n=0}^{\infty} Y_n(x)$ and $B_n^\lambda(h)$ as in Lemmas 4 and 5, we conclude from (22) that

$$S[F_h] = \sum_{n=0}^{\infty} Y_n(x)B_n^\lambda(h)$$

and, furthermore, from the continuity of $F_h(x)$, from the fact that $Y_n(x) = O(n^{-1})$, and from Lemma 4, that

$$F_h(x_0) = \sum_{n=0}^{q[h^{-1}]} Y_n(x_0)B_n^\lambda(h) + \sum_{n=q[h^{-1}]+1}^{\infty} Y_n(x_0)B_n^\lambda(h) \tag{23}$$

where q is a fixed, large positive integer.

Since there clearly is no loss of generality in assuming that

$$R_n = \sum_{j=0}^{n} Y_n(x_0) \to 0,$$

we shall make this assumption and shall show this implies that $F_h(x_0)\to 0$.

By Lemma 4 the second sum on the right in (23) is majorized by a constant multiple of

$$h^{-\lambda} \sum_{n=q[h^{-1}]}^{\infty} n^{-1(+\lambda)} = O(q^{-\lambda}). \tag{24}$$

Using Abel summation by parts on the first sum on the right side in (23), we obtain that this sum is equal to

$$\sum_{n=0}^{q[h^{-1}]-1} R_n[B_n^\lambda(h) - B_{n+1}^\lambda(h)] + R_{q[h^{-1}]}B_{q[h^{-1}]}^\lambda(h). \tag{25}$$

By Lemma 4 the second term in (25) tends to zero as $h\to 0$. By Lemma 5 the first term in (25) is majorized by

$$h^2 \sum_{n=0}^{q[h^{-1}]} o(n)$$

which tends to zero with h. We consequently conclude from (23) and (24) that

$$\limsup_{h\to 0} |F_h(x_0)| = O(q^{-\lambda}).$$

But this fact implies that $\lim_{h\to 0} F_h(x_0)=0$, and the proof of Theorem 2 is complete.

In closing we point out that with no change in the proof of Theorem 2 the condition that μ be absolutely continuous can be replaced with one requiring only that μ $[O(x, h)]$ be continuous at x_0 for h small.

Bibliography

1. K. K. Chen, *On the Cesàro-summability of the Laplace's series of hyperspherical functions*, The Science Reports of the Tôhoku Imperial University vol. 17 (1928) pp. 1073–1089.

2. A. Erdélyi, W. Magnus, F. Oberhettinger, F. G. Tricomi, *Higher transcendental functions*, vol. 1, New York, 1953.

3. ———, *Higher transcendental functions*, vol. 2, New York, 1953.

4. G. H. Hardy, *Divergent series*, Oxford, 1949.

5. G. H. Hardy and J. E. Littlewood, *Abel's theorem and its converse*, Proc. London Math. Soc. vol. 18 (1920) pp. 205–235.

6. E. Kogbetliantz, *Recherches sur la sommabilité des séries ultraspheriques par la méthod des moyennes arithmetiques*, Jour. de Math. vol. 3 (1924) pp. 107–187.

7. G. Szegö, *Orthogonal polynomials*, New York, 1939.

8. A. Zygmund, *Trigonometrical series*, Warsaw, 1935.

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Page 329, lines 20–21. For "those with superscript "^{Q}" partial recursive, uniformly in Q);" read "those with superscript "^{Q}" partial recursive uniformly in Q);".