A NEW CLASS OF CONTINUED FRACTION EXPANSIONS FOR THE RATIOS OF HYPERGEOMETRIC FUNCTIONS

BY

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1. **Introduction.** In this paper a new class of continued fraction expansions of the type

$$1 + \frac{a_1 z}{b_1 z + 1} + \frac{a_2 z}{b_2 z + 1} + \frac{a_3 z}{b_3 z + 1} + \cdots \tag{1.1}$$

for the ratios of two hypergeometric functions is described in detail.

The hypergeometric function $F(a, b, c, z)$, where a and b are any complex constants, and *c is a complex constant different from zero or a negative integer*, is the well-known infinite series with radius of convergence equal to one,

$$\begin{aligned} F(a, b, c, z) = 1 &+ \frac{abz}{c} + \frac{a(a+1)b(b+1)z^2}{2!c(c+1)} \\ &+ \frac{a(a+1)(a+2)b(b+1)(b+2)z^3}{3!c(c+1)(c+2)} + \cdots. \end{aligned} \tag{1.2}$$

If a or b is zero or a negative integer, (1.2) reduces to a polynomial. The well-known continued fraction of Gauss [2](1) for the ratio of two hypergeometric functions is

$$\begin{aligned} \frac{F(a, b+1, c+1, z)}{F(a, b, c, z)} = \frac{1}{1} - \frac{\dfrac{a(c-b)z}{c(c+1)}}{1} - \frac{\dfrac{(b+1)(c-a+1)z}{(c+1)(c+2)}}{1} \\ - \frac{\dfrac{(a+1)(c-b+1)z}{(c+2)(c+3)}}{1} - \frac{\dfrac{(b+2)(c-a+2)z}{(c+3)(c+4)}}{1} - \cdots. \end{aligned} \tag{1.3}$$

It converges throughout the z-plane exterior to the cut along the real axis from 1 to $+\infty$ except possibly at certain isolated points, is equal to the function $F(a, b+1, c+1, z):F(a, b, c, z)$ in the neighborhood of the origin, and furnishes the analytic continuation of the function in the interior of the cut plane.

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(1) Numbers in brackets refer to the bibliography at the end of the paper.

Here, *new* continued fraction expansions of the form (1.1) are derived for the functions $F(a, b, c, z):F(a, b+1, c+1, z)$ (formula (2.1)), $F(a, b, c, z):F(a+1, b, c+1, z)$ (formula (2.1′)), $F(a, b, c, z):F(a+1, b, c, z)$ (formula (2.3)), $F(a, b, c, z):F(a, b+1, c, z)$ (formula (2.3′)), $F(a, b, c, z):F(a, b, c+1, z)$ (formulas (2.7) and (2.7′)), and $F(a, b, c, z):F(a+1, b+1, c+1, z)=(ab/c)\cdot F(a, b, c, z):F'(a, b, c, z)$ (formulas (2.9) and (2.9′).

In §3, the regions of convergence for these continued fractions are studied, and it is found that they converge over the entire z-plane exterior to a cut along the unit circle, except possibly at certain isolated points which are poles of the functions represented. They are equal to the ratio of the two hypergeometric functions which generate the expansions for $|z|<1$, and they converge to entirely different functions for $|z|>1$. In §4, certain equal continued fractions are studied. They are equal as consequences of the convergence found in §3. In §5 and §6, special continued fraction expansions and specialized functions obtained from (2.1) are considered.

2. **New expansions of the type (1.1) for the ratios of two hypergeometric functions.** This type occurs, for example, as the even part of certain continued fractions studied by the author [1, p. 931]. The reciprocal of the continued fractions of Gauss, that is, the ratio of two certain hypergeometric functions, can be expressed in this form, namely(²),

$$\frac{F(a, b, c, z)}{F(a, b+1, c+1, z)} \sim 1 - \cfrac{\frac{a(c-b)z}{c(c+1)}}{\frac{(a-b)z}{c+1} + 1 - \cfrac{\frac{(a+1)(c+1-b)z}{(c+1)(c+2)}}{\frac{(a+1-b)z}{c+2} + 1 - \cfrac{\frac{(a+2)(c+2-b)z}{(c+2)(c+3)}}{\frac{(a+2-b)z}{c+3} + 1 - \cdots}}}. \tag{2.1}$$

The continued fraction expansion for $F(a, b+1, c+1, z):F(a, b, c, z)$ is then

$$\frac{1}{1} - \frac{\frac{a(c-b)z}{c(c+1)}}{\frac{(a-b)z}{c+1} + 1} - \frac{\frac{(a+1)(c+1-b)z}{(c+1)(c+2)}}{\frac{(a+1-b)z}{c+2} + 1} - \frac{\frac{(a+2)(c+2-b)z}{(c+2)(c+3)}}{\frac{(a+2-b)z}{c+3} + 1} - \cdots.$$

The expansion (2.1) is obtained by the successive substitution of the Gauss identities [2, p. 133]

(²) The symbol $\sim$ denotes a formal expansion. If at any time a partial numerator vanishes, the continued fraction breaks off with the preceding term. In this case, the symbol $\sim$ can be replaced by the $=$ sign.

$$(2.2)\quad \begin{aligned} &\text{(i) } F(a, b+1, c+1, z) - F(a, b, c, z) \\ &\qquad\qquad = \frac{a(c-b)z}{c(c+1)} F(a+1, b+1, c+2, z), \\ &\text{(ii) } F(a, b+1, c, z) - F(a+1, b, c, z) \\ &\qquad\qquad = \frac{(a-b)z}{c} F(a+1, b+1, c+1, z). \end{aligned}$$

On interchanging a and b, since the hypergeometric function is symmetric in a and b, one obtains

$$(2.1') \quad \frac{F(a, b, c, z)}{F(a+1, b, c+1, z)} \sim 1 - \cfrac{\cfrac{b(c-a)z}{c(c+1)}}{\cfrac{(b-a)z}{c+1} + 1 - \cfrac{\cfrac{(b+1)(c+1-a)z}{(c+1)(c+2)}}{\cfrac{(b+1-a)z}{c+2} + 1 - \cfrac{\cfrac{(b+2)(c+2-a)z}{(c+2)(c+3)}}{\cfrac{(b+2-a)z}{c+3} + 1 - \cdots}}}.$$

Similarly, one obtains the expansion

$$(2.3) \quad \frac{F(a, b, c, z)}{F(a+1, b, c, z)} \sim 1 - \cfrac{\cfrac{bz}{c}}{\cfrac{(b-a)z}{c} + 1 - \cfrac{\cfrac{(b+1)(c-a)z}{c(c+1)}}{\cfrac{(b+1-a)z}{c+1} + 1 - \cfrac{\cfrac{(b+2)(c+1-a)z}{(c+1)(c+2)}}{\cfrac{(b+2-a)z}{c+2} + 1 - \cdots}}},$$

when one substitutes successively the identities of Gauss [2, p. 133],

$$(2.4)\quad \begin{aligned} &\text{(i)} \quad F(a+1, b, c, z) - F(a, b, c, z) = \frac{bz}{c} F(a+1, b+1, c+1, z), \\ &\text{(ii)} \; F(a+1, b, c, z) - F(a, b+1, c, z) = \frac{(b-a)z}{c} F(a+1, b+1, c+1, z), \\ &\text{(iii)} \; F(a+1, b, c+1, z) - F(a, b, c, z) = \frac{b(c-a)z}{c(c+1)} F(a+1, b+1, c+2, z). \end{aligned}$$

On interchanging a and b, one also obtains the expansion

$$(2.3') \quad \frac{F(a, b, c, z)}{F(a, b+1, c, z)} \sim 1 - \cfrac{\cfrac{az}{c}}{\cfrac{(a-b)z}{c} + 1 - \cfrac{\cfrac{(a+1)(c-b)z}{c(c+1)}}{\cfrac{(a+1-b)z}{c+1} + 1 - \cfrac{\cfrac{(a+2)(c+1-b)z}{(c+1)(c+2)}}{\cfrac{(a+2-b)z}{c+2} + 1 - \cdots}}}.$$

In the formation of the expansions (2.3) and (2.3′), the following relations are found between (2.1′) and (2.3), and between (2.1) and (2.3′), respectively:

$$(2.5) \quad \frac{F(a, b-1, c, z)}{F(a+1, b-1, c, z)} = 1 - \cfrac{\cfrac{(b-1)z}{c}}{\cfrac{(b-1-a)z}{c} + \cfrac{F(a, b, c, z)}{F(a+1, b, c+1, z)}},$$

$$(2.5') \quad \frac{F(a-1, b, c, z)}{F(a-1, b+1, c, z)} = 1 - \cfrac{\cfrac{(a-1)z}{c}}{\cfrac{(a-1-b)z}{c} + \cfrac{F(a, b, c, z)}{F(a, b+1, c+1, z)}}.$$

By the Gauss identity

$$(2.6) \quad F(a, b, c+1, z) - F(a, b, c, z) = \frac{-abz}{c(c+1)} F(a+1, b+1, c+2, z),$$

and (2.2)(i), (2.4)(i) and (ii), the expansion

$$(2.7) \quad \frac{F(a, b, c, z)}{F(a, b, c+1, z)} \sim 1 + \cfrac{\cfrac{abz}{c(c+1)}}{\cfrac{-bz}{c+1} + 1 - \cfrac{\cfrac{(a+1)(c+1-b)z}{(c+1)(c+2)}}{\cfrac{(a+1-b)z}{c+2} + 1 - \cfrac{\cfrac{(a+2)(c+2-b)z}{(c+2)(c+3)}}{\cfrac{(a+2-b)z}{c+3} + 1 - \cdots}}},$$

is generated, or, by the interchange of a and b,

$$\frac{F(a, b, c, z)}{F(a, b, c+1, z)} \sim 1 + \cfrac{\frac{abz}{c(c+1)}}{\frac{-az}{c+1} + 1 - \cfrac{\frac{(b+1)(c+1-a)z}{(c+1)(c+2)}}{\frac{(b+1-a)z}{c+2} + 1 - \cfrac{\frac{(b+2)(c+2-a)z}{(c+2)(c+3)}}{\frac{(b+2-a)z}{c+3} + 1 - \cdots}}}. \tag{2.7'}$$

Here one also has the following relations between (2.1) and (2.7), and between (2.1′) and (2.7′), respectively:

$$\frac{F(a-1, b, c-1, z)}{F(a-1, b, c, z)} = 1 + \cfrac{\frac{(a-1)bz}{c(c-1)}}{\frac{-bz}{c} + \frac{F(a, b, c, z)}{F(a, b+1, c+1, z)}}, \tag{2.8}$$

$$\frac{F(a, b-1, c-1, z)}{F(a, b-1, c, z)} = 1 + \cfrac{\frac{a(b-1)z}{c(c-1)}}{\frac{-az}{c} + \frac{F(a, b, c, z)}{F(a+1, b, c+1, z)}}. \tag{2.8'}$$

Finally, by the Gauss identities (2.4)(i) (with a and b interchanged), (2.4)(iii), and (2.2)(ii), one obtains the continued fraction expansion

$$\frac{ab}{c}\cdot\frac{F(a, b, c, z)}{F'(a, b, c, z)} = \frac{F(a, b, c, z)}{F(a+1, b+1, c+1, z)} \sim \frac{-az}{c} + 1 - \cfrac{\frac{(b+1)(c-a)z}{c(c+1)}}{\frac{(b+1-a)z}{c+1} + 1 - \cfrac{\frac{(b+2)(c+1-a)z}{(c+1)(c+2)}}{\frac{(b+2-a)z}{c+2} + 1 - \cfrac{\frac{(b+3)(c+2-a)z}{(c+2)(c+3)}}{\frac{(b+3-a)z}{c+3} + 1 - \cdots}}}, \tag{2.9}$$

or, on the interchange of a and b,

$$(2.9')\qquad \frac{F(a, b, c, z)}{F(a+1, b+1, c+1, z)} \sim \frac{-bz}{c} + 1 - \cfrac{\frac{(a+1)(c-b)z}{c(c+1)}}{\frac{(a+1-b)z}{c+1} + 1 - \cfrac{\frac{(a+2)(c+1-b)z}{(c+1)(c+2)}}{\frac{(a+2-b)z}{c+2} + 1 - \cdots}}.$$

Here also the following identities hold between (2.1′) and (2.9), and (2.1) and (2.9′), respectively:

$$(2.10)\qquad \frac{F(a, b-1, c, z)}{F(a+1, b, c+1, z)} = \frac{-az}{c} + \frac{F(a, b, c, z)}{F(a+1, b, c+1, z)},$$

$$(2.10')\qquad \frac{F(a-1, b, c, z)}{F(a, b+1, c+1, z)} = \frac{-bz}{c} + \frac{F(a, b, c, z)}{F(a, b+1, c+1, z)}.$$

Similar relations exist between the other expansions in this section.

The question of the convergence of the expansions in this section is now considered.

3. **The convergence of the continued fractions in §2**(3). Expansion (2.1) is first considered. It is a "limit-periodic" continued fraction (cf. Perron [3, pp. 280 ff.]) of the form (1.1) in which $\lim_{p\to\infty} a_p z = -z$, $\lim_{p\to\infty} (b_p z+1) = z+1$. The roots of the auxiliary equation $x^2 = \lim_{p\to\infty} (b_p z+1)x + \lim_{p\to\infty} a_p z$, or $x^2-(z+1)x+z=0$, are 1 and z. They are of unequal modulus if $|z| \neq 1$. Then, by Theorems 41 and 42 of Perron [3, p. 286], (2.1) converges (at least in the "wider sense") if $|z| \neq 1$, and there exist positive numbers M and n such that, for $v \geqq n$, the continued fraction

$$(3.1)\qquad b_v z + 1 + \frac{a_{v+1}z}{b_{v+1}z + 1} + \frac{a_{v+2}z}{b_{v+2}z + 1} + \cdots$$

converges uniformly for $|z| < 1$ and $M > |z| > 1$. Now, by a proof entirely analogous to that of Perron [3, p. 342] for Stieltjes-type continued fractions, since (3.1) converges uniformly in $|z| < 1$, (2.1) is equal to the corresponding power series within the unit circle.

Since one knows the function to which (2.1) converges within $|z| < 1$, the question of the function to which (2.1) converges for $|z| > 1$ will next be investigated. By an equivalence transformation, (2.1) can be written in the form

(3) In any of the expansions in §2, for certain values of a, b, or c, a partial numerator of a continued fraction may be zero. In this case, the continued fraction is finite, and its value can be computed therefrom. Furthermore, it is understood throughout that in each case those values of a, b, or c are excluded which make indeterminate the function to which the continued fraction in question converges.

$$c\cdot\frac{F(a,b,c,z)}{F(a,b+1,c+1,z)}\sim c-\frac{a(c-b)z}{(a-b)z+c+1}-\frac{(a+1)(c+1-b)z}{(a+1-b)z+c+2}-\frac{(a+2)(c+2-b)z}{(a+2-b)z+c+3}-\cdots,$$

or, if one adds $(a-1-b)z$ to both sides and applies (2.2)(ii) with a replaced by $a-1$,

$$c\cdot\frac{F(a-1,b+1,c,z)}{F(a,b+1,c+1,z)}\sim(a-1-b)z+c-\frac{a(c-b)z}{(a-b)z+c+1}-\frac{(a+1)(c+1-b)z}{(a+1-b)z+c+2}-\cdots, \tag{3.2}$$

where the equality sign holds if $|z|<1$. To the continued fraction (3.2), Theorem 5, p. 488 of [3] concerning

$$D+\frac{A+B+C}{D+E}+\frac{A+2B+4C}{D+2E}+\frac{A+3B+9C}{D+3E}+\cdots \tag{3.3}$$

is now applied. Continued fraction (3.3) is of the form (3.2), where $A=-(a-1)(c-1-b)z$, $B=-(a-b+c-2)z$, $C=-z$, $D=(a-1-b)z+c$, $E=z+1$. Then $(E^2+4C)^{1/2}=\pm(1-z)$, where the $+$ or $-$ sign must be chosen so that $R(1+z)/(\pm(1-z))>0$. According to Theorem 5, the continued fraction (3.3) is equal to the quotient $(E^2+4C)^{1/2}\cdot F_1(\alpha,\beta,\gamma,x):F_1(\alpha+1,\beta+1,\gamma+1,x)$, where $F_1(\alpha,\beta,\gamma,x)=F(\alpha,\beta,\gamma,x)/\Gamma(\gamma)$. Consequently, (3.2) is equal to $\pm(1-z)\gamma F(\alpha,\beta,\gamma,x):F(\alpha+1,\beta+1,\gamma+1,x)$, where α, β are the roots of the equation $C\rho^2-B\rho+A=0$, or $\rho^2-(a-b+c-2)\rho+(a-1)(c-1-b)=0$, so that $\alpha=a-1$, $\beta=c-1-b$. Also

$$\gamma=\frac{B+C}{2C}\left(1-\frac{E}{(E^2+4C)^{1/2}}\right)+\frac{D}{(E^2+4C)^{1/2}}$$
$$=\frac{a-b+c-1}{2}\left(1\mp\frac{1+z}{1-z}\right)+\frac{(a-1-b)z+c}{\pm(1-z)},$$
$$x=\frac{1-\dfrac{E}{(E^2+4C)^{1/2}}}{2}=\frac{1\pm\dfrac{1+z}{1-z}}{2}.$$

There are two cases here due to the two signs of the radical: (i) For $R((1+z)/(1-z))>0$, or, for the same condition, $|z|<1$, $\gamma=c$, and $x=z/(z-1)$. (ii) For $R((1+z)/(1-z))<0$, or $|z|>1$, $\gamma=a-1-b$, $x=1/(1-z)$.

In case (i), for $|z|<1$, (3.2) is equal to

$$(3.4) \qquad (1-z)\cdot cF\left(a-1, c-1-b, c, \frac{z}{z-1}\right) : F\left(a, c-b, c+1, \frac{z}{z-1}\right).$$

Since (cf. [5, pp. 265, 267]),

$$F\left(a-1, c-1-b, c, \frac{z}{z-1}\right) = (1-z)^{a-1}F(a-1, b+1, c, z),$$

$$F\left(a, c-b, c+1, \frac{z}{z-1}\right) = (1-z)^{a}F(a, b+1, c+1, z),$$

(3.4) becomes $c\cdot F(a-1, b+1, c, z) : F(a, b+1, c+1, z)$, which is a check on the value of (3.2) for $|z|<1$.

In case (ii), for $|z|>1$, (3.2) is equal to

$$(3.5) \qquad \begin{aligned} &-(1-z)(a-1-b)F\left(a-1, c-1-b, a-1-b, \frac{1}{1-z}\right) \\ &\qquad\qquad : F\left(a, c-b, a-b, \frac{z}{1-z}\right). \end{aligned}$$

Since [5, pp. 265, 267]

$$\begin{aligned} (1-z)^{-a+1}F&\left(a-1, c-1-b, a-1-b, \frac{1}{1-z}\right) \\ &= (-z)^{-a+1}F\left(a-1, a-c, a-1-b, \frac{1}{z}\right), \end{aligned}$$

$$(1-z)^{-a}F\left(a, c-b, a-b, \frac{1}{1-z}\right) = (-z)^{-a}F\left(a, a-c, a-b, \frac{1}{z}\right),$$

(3.5) becomes

$$(a-1-b)zF\left(a-1, a-c, a-1-b, \frac{1}{z}\right) : F\left(a, a-c, a-b, \frac{1}{z}\right),$$

which is the value of (3.2) for $|z|>1$. Hence the value of (2.1) for $|z|>1$ is

$$\frac{(a-1-b)z}{c}\left[F\left(a-1, a-c, a-1-b, \frac{1}{z}\right) : F\left(a, a-c, a-b, \frac{1}{z}\right) - 1\right],$$

or

$$b(a-c)F\left(a, a+1-c, a+1-b, \frac{1}{z}\right) : c(a-b)F\left(a, a-c, a-b, \frac{1}{z}\right)$$

by (2.4)(iii). On interchanging a and b, one likewise obtains the values of (2.1′).

The following theorem has thus been proved.

THEOREM 3.1[4]. *The continued fractions* (2.1) *and* (2.1′) *converge throughout the cut z-plane where the cut is along the entire circumference of the unit circle, except possibly at certain isolated points which are poles of the function represented. For* $|z|<1$, *the continued fraction* (2.1) *is equal to the function* $F(a, b, c, z):F(a, b+1, c+1, z)$; *for* $|z|>1$, (2.1) *is equal to*

$$b(a-c)F\left(a, a+1-c, a+1-b, \frac{1}{z}\right):c(a-b)F\left(a, a-c, a-b, \frac{1}{z}\right).$$

For $|z|<1$, *the continued fraction* (2.1′) *is equal to* $F(a, b, c, z):F(a+1, b, c+1, z)$; *for* $|z|>1$, (2.1′) *is equal to*

$$a(b-c)F\left(b, b+1-c, b+1-a, \frac{1}{z}\right):\ c(b-a)F\left(b, b-c, b-a, \frac{1}{z}\right).$$

By a method entirely analogous to that of the proof of Theorem 3.1, the following theorem concerning the convergence of the remaining expansions in §2 can be proved. Or, since these continued fractions are either of the type (2.1) or (2.1′), the results of Theorem 3.1 can be used.

For this purpose, (2.3) is transformed into

$$\begin{aligned} (3.6)\quad &(c-1)\cdot\frac{F(a+1, b-1, c-1, z)}{F(a+1, b, c, z)} \\ &= (c-a-1)\cdot\frac{F(a, b, c, z)}{F(a+1, b, c, z)} + a + (b-a-1)z \\ &\sim c-1+(b-1-a)z \frac{-b(c-1-a)z}{(b-a)z+c-}\ \frac{(b+1)(c-a)z}{(b+1-a)z+c+1-}\cdots. \end{aligned}$$

The equality in (3.6) is shown by means of the Gauss identity [2, p. 130]

$$(3.7)\quad (c-1-a)F(a, b, c, z) + aF(a+1, b, c, z) - (c-1)F(a, b, c-1, z) = 0,$$

and (2.4)(ii). From (3.6) the values of (2.3) are found. For $|z|>1$, (2.3) has the value

$$\left[(b-1-a)z\left\{F\left(b-1, b+1-c, b-1-a, \frac{1}{z}\right)\right.\right.$$
$$\left.\left.:F\left(b, b+1-c, b-a, \frac{1}{z}\right)-1\right\}-a\right]\Big/(c-1-a).$$

This is equal to $aF(b, b+1-c, b+1-a, 1/z):(a-b)F(b, b+1-c, b-a, 1/z)$ by

[4] The method of proof of this theorem as well as the function to which (2.1) converges for $|z|>1$ was indicated to the author by Professor Oskar Perron.

(2.4) (iii) and (3.7), or by (2.5). On interchanging a and b in (2.3), one obtains the values of (2.3′).

The convergence of expansions (2.7) and (2.7′) is obtained from Theorem 3.1 and the use of relations (2.8) and (2.8′). The value of (2.7) for $|z|>1$ is thus

$$1 + \cfrac{ab/c}{-a + (a-b)\cdot F\left(a, a-c, a-b, \frac{1}{z}\right):F\left(a+1, a-c, a+1-b, \frac{1}{z}\right)}.$$

By the identity (3.7) and the Gauss identity [2, p. 130]

$$(b-a)F(a, b, c, z) + aF(a+1, b, c, z) - bF(a, b+1, c, z) = 0,$$

this quantity is transformed into $(c-a)F(a, a+1-c, a+1-b, 1/z):cF(a, a-c, a+1-b, 1/z)$. The values of (2.7′) are obtained from (2.7) by the interchange of a and b.

The convergence of the continued fractions (2.9) and (2.9′) is determined from Theorem 3.1 and the relations (2.10) and (2.10′). For example, for $|z|>1$, (2.9) is equal to

$$\frac{-bz}{c} + \frac{(b-a)z}{c}F\left(b, b+1-c, b-a, \frac{1}{z}\right):F\left(b+1, b+1-c, b+1-a, \frac{1}{z}\right)$$

or

$$-azF\left(b, b+1-c, b+1-a, \frac{1}{z}\right):cF\left(b+1, b+1-c, b+1-a, \frac{1}{z}\right)$$

by (3.7).

THEOREM 3.2. *The continued fractions* (2.3), (2.3′), (2.7), (2.7′), (2.9), *and* (2.9′) *converge throughout the cut z-plane where the cut is along the entire circumference of the unit circle, except possibly at certain isolated points which are poles of the function represented. For* $|z|<1$, *the continued fraction* (2.3) *is equal to the function* $F(a, b, c, z):F(a+1, b, c, z)$; *for* $|z|>1$, (2.3) *is equal to*

$$a\cdot F\left(b, b+1-c, b+1-a, \frac{1}{z}\right):(a-b)\cdot F\left(b, b+1-c, b-a, \frac{1}{z}\right).$$

For $|z|<1$, *the continued fraction* (2.3′) *is equal to* $F(a, b, c, z):F(a, b+1, c, z)$; *for* $|z|>1$, (2.3′) *is equal to*

$$b\cdot F\left(a, a+1-c, a+1-b, \frac{1}{z}\right):(b-a)\cdot F\left(a, a+1-c, a-b, \frac{1}{z}\right).$$

For $|z|<1$, *the continued fraction* (2.7) *is equal to the function* $F(a, b, c, z)$ $:F(a, b, c+1, z)$; *for* $|z|>1$, (2.7) *is equal to*

$$(c-a)\cdot F\left(a, a+1-c, a+1-b, \frac{1}{z}\right) : c\, F\left(a, a-c, a+1-b, \frac{1}{z}\right).$$

For $|z|<1$, *the continued fraction* (2.7′) *is equal to the function* $F(a, b, c, z) : F(a, b, c+1, z)$; *for* $|z|>1$, (2.7′) *is equal to*

$$(c-b)\cdot F\left(b, b+1-c, b+1-a, \frac{1}{z}\right) : c\cdot F\left(b, b-c, b+1-a, \frac{1}{z}\right).$$

For $|z|<1$, *the continued fraction* (2.9) *is equal to the function* $F(a, b, c, z) : F(a+1, b+1, c+1, z)$; *for* $|z|>1$, (2.9) *is equal to*

$$-az\cdot F\left(b, b+1-c, b+1-a, \frac{1}{z}\right) : c\cdot F\left(b+1, b+1-c, b+1-a, \frac{1}{z}\right).$$

For $|z|<1$, *the continued fraction* (2.9′) *is equal to the function* $F(a, b, c, z) : F(a+1, b+1, c+1, z)$; *for* $|z|>1$, (2.9′) *is equal to*

$$-bz\cdot F\left(a, a+1-c, a+1-b, \frac{1}{z}\right) : c\cdot F\left(a+1, a+1-c, a+1-b, \frac{1}{z}\right).$$

4. **Equal continued fractions**[5]. Since expansions (2.7) and (2.7′) both converge to the generating function $F(a, b, c, z) : F(a, b, c+1, z)$ if $|z|<1$, the following theorem on equal infinite continued fractions holds. The same result can be obtained from (2.9) and (2.9′).

THEOREM 4.1. *For values of z such that* $|z|<1$,

$$\frac{\dfrac{abz}{c(c+1)}}{\dfrac{-bz}{c+1}+1-\dfrac{\dfrac{(a+1)(c+1-b)z}{(c+1)(c+2)}}{\dfrac{(a+1-b)z}{c+2}+1-\dfrac{\dfrac{(a+2)(c+2-b)z}{(c+2)(c+3)}}{\dfrac{(a+2-b)z}{c+3}+1-\cdots}}}$$

(4.1)

$$=\frac{\dfrac{abz}{c(c+1)}}{\dfrac{-az}{c+1}+1-\dfrac{\dfrac{(b+1)(c+1-a)z}{(c+1)(c+2)}}{\dfrac{(b+1-a)z}{c+2}+1-\dfrac{\dfrac{(b+2)(c+2-a)z}{(c+2)(c+3)}}{\dfrac{(b+2-a)z}{c+3}+1-\cdots}}}.$$

From the above equal continued fractions, a number of special cases are obtained.

For $a-c=n=1, 2, \cdots$, in (4.1), the following finite continued fraction

[5] It is understood throughout this section concerning equal continued fractions that (i) if a partial numerator is zero, the continued fraction breaks off with the preceding term; (ii) both continued fractions can diverge (if they are ∞), or, if one of the continued fractions is indeterminate, the other diverges unless it is finite, in which case it is also indeterminate.

is found equal to an infinite one,

$$\frac{\dfrac{(c+n)bz}{c(c+1)}}{\dfrac{-bz}{c+1}+1}-\frac{\dfrac{(c+n+1)(c+1-b)z}{(c+1)(c+2)}}{\dfrac{(c+n-1-b)z}{c+2}+1}-\frac{\dfrac{(c+n+2)(c+2-b)z}{(c+2)(c+3)}}{\dfrac{(c+n+2-b)z}{c+3}+1}-\cdots$$

$$(4.2)\qquad =\frac{\dfrac{(c+n)bz}{c(c+1)}}{\dfrac{-(c+n)z}{c+1}+1}-\frac{\dfrac{(b+1)(-n+1)z}{(c+1)(c+2)}}{\dfrac{(b+1-c-n)z}{c+2}+1}$$

$$-\frac{\dfrac{(b+2)(-n+2)z}{(c+2)(c+3)}}{\dfrac{(b+2-c-n)z}{c+3}+1}-\cdots-\frac{\dfrac{-(b+n-1)z}{(c+n-1)(c+n)}}{\dfrac{-(b-1-c)z}{c+n}+1},\qquad |z|<1.$$

For $n=1$ in (4.2), $b-c\neq 1, 2, \cdots$,

$$(4.3)\qquad \frac{\dfrac{bz}{c}}{\dfrac{-bz}{c+1}+1}-\frac{\dfrac{(c+1-b)z}{c+1}}{\dfrac{(c+2-b)z}{c+2}+1}-\frac{\dfrac{(c+2-b)z}{c+2}}{\dfrac{(c+3-b)z}{c+3}+1}-\cdots$$

$$=\frac{bz}{c(-z+1)},\qquad |z|<1;$$

for $c=b+1$, $b\neq -1, -2, \cdots$, in (4.3),

$$\frac{\dfrac{bz}{b+1}}{\dfrac{-bz}{b+2}+1}-\frac{\dfrac{2z}{b+2}}{\dfrac{3z}{b+3}+1}-\frac{\dfrac{3z}{b+3}}{\dfrac{4z}{b+4}+1}-\cdots=\frac{bz}{(b+1)(-z+1)},\qquad |z|<1;$$

and, for $b-c=k=1, 2, \cdots$, in (4.3),

$$\frac{\dfrac{(c+k)z}{c}}{\dfrac{-(c+k)z}{c+1}+1}-\frac{\dfrac{(-k+1)z}{c+1}}{\dfrac{(-k+2)z}{c+2}+1}-\frac{\dfrac{(-k+2)z}{c+2}}{\dfrac{(-k+3)z}{c+3}+1}-\cdots-\frac{\dfrac{-z}{c+k-1}}{1}$$

$$=\frac{(c+k)z}{c(-z+1)},\qquad |z|<1,$$

which is an equation between two rational functions.

The following theorem deals with another equality between two infinite continued fractions. It is obtained from (2.9) and from

$$\frac{F(a,b,c,z)}{F'(a,b,c,z)} = \frac{c}{ab} - \frac{(a+b+1)z}{ab} + \frac{\dfrac{(a+1)(b+1)z(1-z)}{ab(c+1)}}{1 - \dfrac{(a+b+3)z}{c+1}} + \frac{\dfrac{(a+2)(b+2)z(1-z)}{(c+1)(c+2)}}{1 - \dfrac{(a+b+5)z}{c+2}} + \cdots, \tag{4.4}$$

which is valid for

$$z \neq 0, \qquad R(z) < 1/2, \qquad a, b \neq -1, -2, \cdots.$$

Relation (4.4) can be obtained directly from the Gauss identity [2, p. 133]

$$ab\,F(a,b,c,z) - [c - (a+b+1)z]F'(a,b,c,z) - z(1-z)F''(a,b,c,z) = 0,$$

or from Perron [3, p. 486].

THEOREM 4.2. *For* $z\neq 0$, $R(z)<1/2$, $|z|<1$, $a, b\neq 0, -1, -2, \cdots$,

$$\frac{c}{ab} - \frac{(a+b+1)z}{ab} + \frac{\dfrac{(a+1)(b+1)z(1-z)}{ab(c+1)}}{1 - \dfrac{(a+b+3)z}{c+1}} + \frac{\dfrac{(a+2)(b+2)z(1-z)}{(c+1)(c+2)}}{1 - \dfrac{(a+b+5)z}{c+2}} + \cdots$$

$$= \frac{c}{ab} - \frac{z}{b} - \frac{\dfrac{(b+1)(c-a)z}{ab(c+1)}}{\dfrac{(b+1-a)z}{c+1} + 1} - \frac{\dfrac{(b+2)(c-a+1)z}{(c+1)(c+2)}}{\dfrac{(b+2-a)z}{c+2} + 1} - \cdots. \tag{4.5}$$

From (4.5) *for the values of z allowed in Theorem* 4.2, one can obtain for

$$a - c = n = 0, 1, 2, \cdots$$

the following relation between a finite and an infinite continued fraction:

$$\frac{c}{b(c+n)} - \frac{(c+n+b+1)z}{b(c+n)} + \cfrac{\frac{(c+n+1)(b+1)z(1-z)}{b(c+n)(c+1)}}{1 - \frac{(c+n+b+3)z}{c+1}} + \cfrac{\frac{(c+n+2)(b+2)z(1-z)}{(c+1)(c+2)}}{1 - \frac{(c+n+b+5)z}{c+2}} + \cdots$$

(4.6)

$$= \frac{c}{b(c+n)} - \frac{z}{b} - \cfrac{\frac{(b+1)(-n)z}{(c+n)b(c+1)}}{\frac{(b+1-c-n)z}{c+1} + 1} - \cfrac{\frac{(b+2)(-n+1)z}{(c+1)(c+2)}}{\frac{(b+2-c-n)z}{c+2} + 1} - \cdots$$

$$- \cfrac{\frac{-(b+n)z}{(c+n-1)(c+n)}}{\frac{(b-c)z}{c+n} + 1}, \quad R(z) < \frac{1}{2}.$$

If $n=0$, (4.6) becomes

$$\frac{1}{b} - \frac{(c+b+1)z}{bc} + \cfrac{\frac{(b+1)z(1-z)}{bc}}{1 - \frac{(c+b+3)z}{c+1}} + \cfrac{\frac{(b+2)z(1-z)}{c+1}}{1 - \frac{(c+b+5)z}{c+2}} + \cdots$$

(4.7)

$$= \frac{1-z}{b}, \quad R(z) < \frac{1}{2}.$$

If $c=b+1$ in (4.7), one obtains the periodic continued fraction (cf. [3, p. 491]),

$$\frac{1}{b}\left[1 - 2z + \frac{z(1-z)}{1-2z} + \frac{z(1-z)}{1-2z} + \cdots\right]$$

$$= \frac{1-z}{b}.$$

This is the same value of this periodic continued fraction found by Theorem 38, p. 276 of [3]. If $b=-k$, a negative integer, from (4.7) it follows that

$$\frac{1}{-k} - \frac{(c-k+1)z}{-kc} + \frac{\dfrac{(-k+1)z(1-z)}{-kc}}{1 - \dfrac{(c-k+3)z}{c+1}} + \frac{\dfrac{(-k+2)z(1-z)}{c+1}}{1 - \dfrac{(c-k+5)z}{c+2}} + \cdots + \frac{\dfrac{-z}{c+k-2}}{1-z} = \frac{1-z}{-k},$$

a relation between two rational functions.

It is remarked that if (2.9) is written in the form

$$\frac{F'(a,b,c,z)}{F(a,b,c,z)} = \frac{ab}{c-az} - \frac{(b+1)(c-a)z}{(c+1)+(b+1-a)z} - \frac{(b+2)(c+1-a)z}{(c+2)+(b+2-a)z} - \frac{(b+3)(c+2-a)z}{(c+3)+(b+3-a)z} - \cdots,$$

valid if $|z|<1$, and if both sides of the equality are multiplied by z, each partial numerator and denominator multiplied by $1/(1-z)$, and the substitution $z/(1-z)=\xi$ is made, then this continued fraction becomes

$$\frac{ab\xi}{c+(c-a)\xi} - \frac{(b+1)(c-a)\xi(1+\xi)}{(c+1)+(b+c+2-a)\xi} - \frac{(b+2)(c+1-a)\xi(1+\xi)}{(c+2)+(b+c+4-a)\xi} - \cdots,$$

valid if $R(\xi)>-1/2$. Consequently, by connecting this with equation (22), p. 490 of [3], one obtains the following equality between infinite continued fractions, *valid for* $|\xi|<1$, $R(\xi)>-1/2$:

$$\begin{aligned}&\frac{ab\xi}{c+(c-a)\xi} - \frac{(b+1)(c-a)\xi(1+\xi)}{(c+1)+(b+c+2-a)\xi} - \frac{(b+2)(c+1-a)\xi(1+\xi)}{(c+2)+(b+c+4-a)\xi} - \cdots \\ &= \frac{ab\xi}{c-(a+b+1-c)\xi} + \frac{(a+1)(b+1)\xi}{(c+1)-(a+b+2-c)\xi} + \frac{(a+2)(b+2)\xi}{(c+2)-(a+b+3-c)\xi} + \cdots \\ &= (a+b-c)\xi + \frac{(c-a)(c-b)\xi}{c-(c+1-a-b)\xi} + \frac{(c-a+1)(c-b+1)\xi}{(c+1)-(c+2-a-b)\xi} + \cdots.\end{aligned}$$

Since the reciprocal of the continued fraction (1.3) converges to the same value as the expansion (2.1) for $|z|<1$, the following theorem holds.

THEOREM 4.3. *For values of z such that $|z|<1$,*

$$\frac{\dfrac{a(c-b)z}{c(c+1)}}{1}-\frac{\dfrac{(b+1)(c+1-a)z}{(c+1)(c+2)}}{1}-\frac{\dfrac{(a+1)(c+1-b)z}{(c+2)(c+3)}}{1}$$

$$-\frac{\dfrac{(b+2)(c+2-a)z}{(c+3)(c+4)}}{1}-\cdots \tag{4.8}$$

$$=\frac{\dfrac{a(c-b)z}{c(c+1)}}{\dfrac{(a-b)z}{c+1}+1}-\frac{\dfrac{(a+1)(c+1-b)z}{(c+1)(c+2)}}{\dfrac{(a+1-b)z}{c+2}+1}-\frac{\dfrac{(a+2)(c+2-b)z}{(c+2)(c+3)}}{\dfrac{(a+2-b)z}{c+3}+1}-\cdots.$$

As a special case, for $a-c=n=1, 2, \cdots$, in (4.8), the following relation exists between a finite and an infinite continued fraction:

$$\frac{1}{1}-\frac{\dfrac{(b+1)(-n+1)z}{(c+1)(c+2)}}{1}-\frac{\dfrac{(c+n+1)(c+1-b)z}{(c+2)(c+3)}}{1}-\frac{\dfrac{(b+2)(-n+2)z}{(c+3)(c+4)}}{1}-\cdots$$

$$-\frac{\dfrac{(c+2n-2)(c-b+n-2)z}{(c+2n-4)(c+2n-3)}}{1}-\frac{\dfrac{-(b+n-1)z}{(c+2n-3)(c+2n-2)}}{1} \tag{4.9}$$

$$=\frac{1}{\dfrac{(c+n-b)z}{c+1}+1}-\frac{\dfrac{(c+n+1)(c+1-b)z}{(c+1)(c+2)}}{\dfrac{(c+n+1-b)z}{c+2}+1}$$

$$-\frac{\dfrac{(c+n+2)(c+2-b)z}{(c+2)(c+3)}}{\dfrac{(c+n+2-b)z}{c+3}+1}-\cdots, \qquad |z|<1.$$

For $n=1$, $b-c\neq 1, 2, \cdots$, in (4.9),

$$1=\frac{1}{\dfrac{(c+1-b)z}{c+1}+1}-\frac{\dfrac{(c+1-b)z}{c+1}}{\dfrac{(c+2-b)z}{c+2}+1}-\frac{\dfrac{(c+2-b)z}{c+2}}{\dfrac{(c+3-b)z}{c+3}+1}-\cdots, \tag{4.10}$$

$$|z|<1,$$

which is the same result as obtained in (4.3). For $c=b+1, b\neq -2, -3, \cdots$, in (4.10), and for $b-c=k=1, 2, \cdots$, in (4.10), one also obtains the same results as for the special cases of (4.3).

5. **Special continued fraction expansions obtainable from** (2.1). From (2.1) and Theorem 3.1 one obtains as a special case (cf., for example, [3, pp. 349–352]),

$$1 - \cfrac{\frac{1-n^2}{1\cdot 3}z^2}{\frac{z^2}{3}+1} - \cfrac{\frac{9-n^2}{3\cdot 5}z^2}{\frac{3}{5}z^2+1} - \cfrac{\frac{25-n^2}{5\cdot 7}z^2}{\frac{5}{7}z^2+1} - \cdots$$

$$(5.1)\quad = \begin{cases} \dfrac{F\left(\frac{1-n}{2}, -\frac{n}{2}, \frac{1}{2}, z^2\right)}{F\left(\frac{1-n}{2}, \frac{2-n}{2}, \frac{3}{2}, z^2\right)} = \dfrac{nz[(1+z)^n+(1-z)^n]}{(1+z)^n-(1-z)^n}, & |z| < 1; \\[2ex] \dfrac{n^2F\left(\frac{1-n}{2}, \frac{2-n}{2}, \frac{3}{2}, \frac{1}{z^2}\right)}{F\left(\frac{1-n}{2}, -\frac{n}{2}, \frac{1}{2}, \frac{1}{z^2}\right)} = \dfrac{nz[(z+1)^n-(z-1)^n]}{(z+1)^n+(z-1)^n}, & |z| > 1. \end{cases}$$

If one replaces z by $1/z$, this expansion becomes

$$z - \cfrac{\frac{1-n^2}{1\cdot 3}z}{\frac{1}{3}+z^2} - \cfrac{\frac{9-n^2}{3\cdot 5}z^2}{\frac{3}{5}+z^2} - \cfrac{\frac{25-n^2}{5\cdot 7}z^2}{\frac{5}{7}+z^2} - \cdots$$

$$= \begin{cases} \dfrac{n[(z+1)^n+(z-1)^n]}{(z+1)^n-(z-1)^n}, & |z| > 1; \\[2ex] \dfrac{n[(1+z)^n-(1-z)^n]}{(1+z)^n+(1-z)^n}, & |z| < 1. \end{cases}$$

By subtracting n and taking the reciprocal, one has

$$1 + \cfrac{2n}{z-n} - \cfrac{\frac{1-n^2}{1\cdot 3}z}{\frac{1}{3}+z^2} - \cfrac{\frac{9-n^2}{3\cdot 5}z^2}{\frac{3}{5}+z^2} - \cfrac{\frac{25-n^2}{5\cdot 7}z^2}{\frac{5}{7}+z^2} - \cdots$$

$$= \begin{cases} ((z+1)/(z-1))^n, & |z| > 1; \\ -((1+z)/(1-z))^n, & |z| < 1. \end{cases}$$

Furthermore,

$$1 + \frac{2n}{z - n -}\;\frac{\frac{1^2 + n^2}{1\cdot 3}z}{\frac{1}{3} - z^2 +}\;\frac{\frac{3^2 + n^2}{3\cdot 5}z^2}{\frac{3}{5} - z^2 +}\;\frac{\frac{5^2 + n^2}{5\cdot 7}z^2}{\frac{5}{7} - z^2 + \cdots}$$

$$= \begin{cases} \left(\dfrac{iz + 1}{iz - 1}\right)^{in} = \exp\,(2n \arctan 1/z), & |z| > 1; \\ -\left(\dfrac{1 + iz}{1 - iz}\right)^{in} = -\exp\,(-2n \arctan z), & |z| < 1. \end{cases}$$

Finally,

$$1 - \frac{\frac{2}{3}z^2}{\frac{2}{3}z^2 + 1 -}\;\frac{\frac{4}{5}z^2}{\frac{4}{5}z^2 + 1 -}\;\frac{\frac{6}{7}z^2}{\frac{6}{7}z^2 + 1 - \cdots} = \frac{F\left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, z^2\right)}{F\left(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, z^2\right)}$$

$$= \frac{z(1 - z^2)^{1/2}}{\arcsin z}, \qquad |z| < 1.$$

Now, consider (2.1) with $b = 0$ and c replaced by $c-1$. Then (2.1) becomes

$$(5.2)\qquad \frac{1}{F(a, 1, c, z)} = 1 - \frac{\frac{az}{c}}{\frac{az}{c} + 1 -}\;\frac{\frac{(a+1)z}{c+1}}{\frac{(a+1)z}{c+1} + 1 -}\;\frac{\frac{(a+2)z}{c+2}}{\frac{(a+2)z}{c+2} + 1 - \cdots},$$

$$|z| < 1\text{(6)}.$$

As special cases of (5.2), one obtains the following expansions (cf., for example, [2, p. 127] or [3, pp. 348–352]):

$$\text{(i)}\qquad (1 + z)^{-n} = \frac{1}{F(-n, 1, 1, -z)} = 1 + \frac{-nz}{nz + 1}$$

$$+ \frac{\frac{(-n+1)z}{2}}{\frac{-(-n+1)z}{2} + 1 +}\;\frac{\frac{(-n+2)z}{3}}{\frac{-(-n+2)z}{3} + 1 + \cdots}, \qquad |z| < 1;$$

(6) This is precisely the continued fraction one obtains by using the algorithm given by Euler [1a, p. 370] for obtaining this continued fraction from the corresponding power series for $1:F(a, 1, c, z)$ (cf. [3, p. 210]).

$$\text{(ii)}\quad \frac{z}{\log(1+z)} = \frac{1}{F(1,1,2,-z)} = 1 + \cfrac{\frac{z}{2}}{\frac{-z}{2}+1+\cfrac{\frac{2z}{3}}{\frac{-2z}{3}+1}}$$

$$+\cfrac{\frac{3z}{4}}{\frac{-3z}{4}+1+\cdots}, \qquad |z| < 1;$$

$$\text{(iii)}\quad \frac{2z}{\log\dfrac{1+z}{1-z}} = \frac{1}{F\left(\frac{1}{2}, 1, \frac{3}{2}, z^2\right)} = 1 - \cfrac{\frac{1}{3}z^2}{\frac{1}{3}z^2+1-\cfrac{\frac{3}{5}z^2}{\frac{3}{5}z^2+1}}$$

$$-\cfrac{\frac{5}{7}z^2}{\frac{5}{7}z^2+1-\cdots}, \qquad |z| < 1;$$

$$\text{(iv)}\quad \frac{\sin z\cos z}{z} = \frac{1}{F\left(1, 1, \frac{3}{2}, \sin z^2\right)} = 1 - \cfrac{\frac{2}{3}\sin z^2}{\frac{2}{3}\sin z^2+1-\cfrac{\frac{4}{5}\sin z^2}{\frac{4}{5}\sin z^2+1}}$$

$$-\cfrac{\frac{6}{7}\sin z^2}{\frac{6}{7}\sin z^2+1-\cdots}, \qquad |\sin z^2| < 1;$$

$$\text{(v)}\quad \frac{\tan z}{z} = \frac{1}{F\left(\frac{1}{2}, 1, \frac{3}{2}, -\tan z^2\right)} = 1 + \cfrac{\frac{1}{3}\tan z^2}{-\frac{1}{3}\tan z^2+1}$$

$$+\cfrac{\frac{3}{5}\tan z^2}{\frac{-3}{5}\tan z^2+1+\cfrac{\frac{5}{7}\tan z^2}{\frac{-5}{7}\tan z^2+1+\cdots}}, \qquad |\tan z^2| < 1;$$

(vi) $$\frac{z}{\arctan z} = \frac{1}{F\left(\frac{1}{2}, 1, \frac{3}{2}, -z^2\right)} = 1 + \frac{\frac{1}{3}z^2}{\frac{-1}{3}z^2 + 1} + \frac{\frac{3}{5}z^2}{\frac{-3}{5}z^2 + 1} + \cdots, \qquad |z| < 1;$$

(vii) $$\int_0^z \frac{dt}{1+t^n} = zF\left(\frac{1}{n}, 1, 1 + \frac{1}{n}, -z^n\right) = \frac{z}{1} + \frac{\frac{z^{n+1}}{n+1}}{\frac{-z^n}{n+1} + 1} + \frac{\frac{n+1}{2n+1}z^n}{\frac{-n+1}{2n+1}z^n + 1} + \frac{\frac{2n+1}{3n+1}z^n}{\frac{-2n+1}{3n+1}z^n + 1} + \cdots, \qquad |z^n| < 1.$$

6. **Continued fraction expansions for the ratios of specialized hypergeometric functions.** On replacing z by z/b in (2.1) and then letting b tend to ∞, one obtains

$$\frac{\phi(a, c, z)}{\phi(a, c+1, z)} \sim 1 + \frac{\frac{az}{c(c+1)}}{\frac{-z}{c+1} + 1} + \frac{\frac{(a+1)z}{(c+1)(c+2)}}{\frac{-z}{c+2} + 1} + \frac{\frac{(a+2)z}{(c+2)(c+3)}}{\frac{-z}{c+3} + 1} + \cdots, \tag{6.1}$$

where

$$\phi(a, c, z) = 1 + \frac{a}{c}z + \frac{a(a+1)}{2!c(c+1)}z^2 + \cdots, \tag{6.2}$$

a and c not simultaneously $0, -1, -2, \cdots$.

On multiplying both sides of (6.1) by c and adding $-z$, one can transform (6.1) into

$$\begin{aligned} -z + \frac{c\cdot\phi(a, c, z)}{\phi(a, c+1, z)} &= c\cdot\frac{\phi(a-1, c, z)}{\phi(a, c+1, z)} \\ &\sim c - z + \frac{az}{-z + (c+1)} + \frac{(a+1)z}{-z + (c+2)} + \frac{(a+2)z}{-z + (c+3)} + \cdots. \end{aligned} \tag{6.3}$$

Expansion (6.3) is exactly expansion (8), p. 477 of [3] with $\beta=a-1$, $\gamma=c$. Hence the convergence of (6.3) as stated in the following theorem is proved in precisely the same manner as that of expansion (8) in [3].

THEOREM 6.1. *The continued fraction* (6.3) *converges uniformly to the meromorphic function* $c\cdot\phi(a-1, c, z):\phi(a, c+1, z)$ *over every closed bounded region which contains none of the poles of the function, and where* $z\neq 0$.

On replacing z by z/a in (2.1) and then letting a tend to ∞, one obtains expansion (6.4). The following theorem can be proved by means of the Worpitzky theorem for small values of $|z|$ and then for every bounded closed region by Theorems 41 and 42, p. 268 of [3].

THEOREM 6.2. *The continued fraction*

$$\frac{\phi(b, c, z)}{\phi(b+1, c+1, z)} = 1 - \cfrac{\frac{(c-b)z}{c(c+1)}}{\frac{z}{c+1}+1 - \cfrac{\frac{(c+1-b)z}{(c+1)(c+2)}}{\frac{z}{c+2}+1 - \cfrac{\frac{(c+2-b)z}{(c+2)(c+3)}}{\frac{z}{c+3}+1 - \cdots}}} \tag{6.4}$$

converges uniformly to the meromorphic function $\phi(b, c, z):\phi(b+1, c+1, z)$ *over every closed bounded region which contains none of the poles of the function, and where* $z\neq 0$.

In particular, for $b=0$, with c replaced by $c-1$, (6.4) becomes

$$\frac{1}{\phi(1, c, z)} = 1 - \cfrac{\frac{z}{c}}{\frac{z}{c}+1 - \cfrac{\frac{z}{c+1}}{\frac{z}{c+1}+1 - \cfrac{\frac{z}{c+2}}{\frac{z}{c+2}+1 - \cdots}}}. \tag{6.5}$$

If $c=1$, and z is replaced by $-z$, this formula becomes

$$e^z = 1 + \cfrac{z}{-z+1 + \cfrac{\frac{z}{2}}{\frac{-z}{2}+1 + \cfrac{\frac{z}{3}}{\frac{-z}{3}+1 + \cdots}}}.$$

On comparing (6.1) and (6.4), with $a=c-b$, since the equality sign now can replace the $\sim$ sign in (6.1) by Theorem 6.1, one finds that

$$\phi(b, c, z)/\phi(b+1, c+1, z) = \phi(c-b, c, -z)/\phi(c-b, c+1, -z)\text{(7)}.$$

Now, if z/a replaces z in (6.1) and a tends to ∞, (6.1) becomes

$$\text{(6.6)}\qquad \frac{\psi(c, z)}{\psi(c+1, z)} = 1 + \frac{\dfrac{z}{c(c+1)}}{1} + \frac{\dfrac{z}{(c+1)(c+2)}}{1} + \frac{\dfrac{z}{(c+2)(c+3)}}{1} + \cdots,$$

where

$$\psi(c, z) = 1 + \frac{z}{c} + \frac{z^2}{2!c(c+1)} + \cdots.$$

Expansion (6.6) is exactly the same as one obtains by specializing the Gauss continued fraction (cf., for example, [3, pp. 313, 353]), whence one can write the Lambert continued fraction for $(e^z - e^{-z})/(e^z + e^{-z})$, tan z, and other functions.

If one replaces z by $-cz$ and lets c tend to ∞ in (2.1), then

$$\text{(6.7)}\qquad \begin{aligned} \frac{\Omega(a, b, z)}{\Omega(a, b+1, z)} \sim 1 &+ \frac{az}{-(a-b)z+1} + \frac{(a+1)z}{-(a+1-b)z+1} \\ &+ \frac{(a+2)z}{-(a+2-b)z+1} + \cdots, \end{aligned}$$

where

$$\text{(6.8)}\qquad \begin{aligned} \Omega(a, b, z) = 1 &- \frac{abz}{1!} + \frac{a(a+1)b(b+1)z^2}{2!} \\ &- \frac{a(a+1)(a+2)b(b+1)(b+2)z^2}{3!} + \cdots. \end{aligned}$$

Here (6.7) is the ratio of two divergent series if $z \neq 0$.

When one applies the Pringsheim convergence criteria (cf. [3. p. 262])(8), one finds that (6.7) converges uniformly in the region for which

$$\left| \frac{az}{(b-a)z+1} \right| \leqq C,$$

$$\left| \frac{(a+p-1)z}{[-(a-b+p-1)z+1][-(a-b+p-2)z+1]} \right| \leqq \frac{1}{4}, \qquad z \neq 0,$$

where C is a positive (arbitrarily large) number and $p \geqq 2$, that is, it converges uniformly exterior to a circle about the origin. By Theorem 2, p. 478 of [3], (6.7) can be identified with the continued fraction

(7) This formula is in [3, p. 313]. It is, however, proved there by means of Gauss continued fractions.

(8) This procedure was suggested by Professor Perron.

$$\text{(6.9)}\qquad \frac{C+A+B}{C+D+}\frac{A+2B}{C+2D+}\frac{A+3B}{C+3D+}\cdots, \qquad B\neq 0, D\neq 0,$$

where $A=(a-1)z$, $B=z$, $C=(-a+b+1)z+1$, $D=-z$. The value to which (6.9) converges, by Theorem 2, is

$$D\cdot\phi_1\left(\frac{A}{B}, \frac{B+CD}{D^2}, \frac{B}{D^2}\right):\phi_1\left(\frac{A}{B}+1, \frac{B+CD}{D^2}+1, \frac{B}{D^2}\right),$$

where $\phi_1(a, c, z)=\phi(a, c, z)/\Gamma(c)$ and $\phi(a, c, z)$ is the series defined in (6.2). Hence the value to which (6.7) converges exterior to the origin is

$$\begin{aligned}(a-b-1)z &- z\cdot\phi_1\left(a-1, a-b-1, \frac{1}{z}\right):\phi_1\left(a, a-b, \frac{1}{z}\right)\\ &= (a-b-1)z\left[1-\phi\left(a-1, a-b-1, \frac{1}{z}\right):\phi\left(a, a-b, \frac{1}{z}\right)\right]\\ &= \frac{b}{b-a}\left[\phi\left(a, a-b+1, \frac{1}{z}\right):\phi\left(a, a-b, \frac{1}{z}\right)\right],\end{aligned}$$

since the ϕ-function (6.2) satisfies the identity

$$\phi(a, c, z) = \phi(a+1, c+1, z) + \frac{a-c}{c(c+1)}z\cdot\phi(a+1, c+2, z).$$

The value of (6.7) can also be expressed in terms of the series equivalents of the ϕ-function, as stated in the following theorem.

THEOREM 6.3. *The continued fraction* (6.7) *converges uniformly in the region exterior to a circle about the origin to the value*

$$\begin{aligned}\frac{b}{b-a}\left[\phi\left(a, a-b+1, \frac{1}{z}\right):\phi\left(a, a-b, \frac{1}{z}\right)\right] &= (a-b-1)z\\ \text{(6.10)}\qquad - z\cdot&\left[\frac{1}{\Gamma(a-b-1)}+\sum_{p=1}^{\infty}\frac{(a-1)a(a+1)\cdots(a+p-2)}{z^p\cdot p!\cdot\Gamma(a-b+p-1)}\right]\\ &:\left[\frac{1}{\Gamma(a-b)}+\sum_{p=1}^{\infty}\frac{a(a+1)\cdots(a+p-1)}{z^p\cdot p!\cdot\Gamma(a-b+p)}\right],\end{aligned}$$

provided a and $-b$ *do not both belong to the sequence* $0, -1, -2, \cdots$.

In particular, for $b=0$, (6.7) becomes

$$\text{(6.11)}\qquad \frac{1}{\Omega(a, 1, z)}\sim 1+\frac{az}{-az+1+}\frac{(a+1)z}{-(a+1)z+1+}\frac{(a+2)z}{-(a+2)z+1+}\cdots \equiv 0, \qquad z\neq 0,$$

by Theorem 6.3.

For $a=1$, (6.7) becomes

$$\text{(6.12)}\quad \frac{\Omega(1, b, z)}{\Omega(1, b+1, z)} \sim 1 + \frac{z}{(b-1)z+1} + \frac{2z}{(b-2)z+1} + \frac{3z}{(b-3)z+1} + \cdots,$$

and has the value

$$b\cdot\phi\left(1, 2-b, \frac{1}{z}\right):(b-1)\cdot\phi\left(1, 1-b, \frac{1}{z}\right)$$

$$= -bz - \frac{z}{\Gamma(-b)}:\left[\frac{1}{\Gamma(1-b)} + \sum_{p=1}^{\infty} \frac{1}{z^p\cdot\Gamma(-b+p+1)}\right], \qquad z \neq 0.$$

The results of Theorem 6.3, and, in fact, many of the other results shown here, can be expressed in a number of other forms (cf. [3, Chapter 11]). For example, since

$$\phi(\beta, \gamma, z) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}e^{tz}dt, \qquad \Gamma(\beta+1) = \beta\cdot\Gamma(\beta),$$

in terms of integrals the continued fraction (6.7) ***exterior to the origin has the value***

$$\text{(6.13)}\qquad \int_0^1 t^{a-1}(1-t)^{-b}e^{tz}dt \Big/ \int_0^1 t^{a-1}(1-t)^{-b-1}e^{tz}dt.$$

Specialized expansions similar to those considered above can also be derived from the continued fractions (2.3), (2.3′), (2.7), (2.7′), (2.9), and (2.9′) by means of the relations (2.5), (2.5′), (2.8), (2.8′), (2.10), and (2.10′), which relate these expansions to (2.1) and (2.1′).

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