THE MODULI OF HYPERELLIPTIC CURVES[1]

BY

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Introduction. We can state the problem of the moduli of algebraic curves in the following way. Consider all irreducible algebraic curves, defined over a ground field k, of a fixed genus g. Can we, in some natural fashion, associate to each birational class of these curves a point of an algebraic variety N, defined over k, in such a way that for distinct birational classes we obtain distinct points? We impose the further condition that those points of N which do not represent any of the above birational classes form a proper algebraic subvariety of N, defined over k.

We shall consider this problem only for the special case of hyperelliptic curves. van der Waerden [3] has attacked the general problem. There is, however, an important gap in the proof of Theorem 7 [p. 698, loc. cit.]. We shall use the notation of van der Waerden in pointing this out.

I_ν is an irreducible algebraic system of regular plane curves of genus g [¶4, p. 698, loc. cit.]. The algebraic correspondence $\mathfrak{B}_\nu$ transforms I_ν onto itself. The difficulty arises from the fact that, as far as we know, an irreducible regular plane curve Γ of genus g may be fundamental for the correspondence $\mathfrak{B}_\nu$.

This situation has not been ruled out since Γ may correspond, under $\mathfrak{B}_\nu$, to curves in I_ν which do not belong to $\mathfrak{P}_\Gamma$. These curves could not be irreducible regular curves of genus g. Such "limiting curves" can be ignored when considering I_ν as the domain of $\mathfrak{B}_\nu$, but not when considering it as the range of this correspondence. If Γ is fundamental for $\mathfrak{B}_\nu$ then the birational class of Γ would be represented by infinitely many points on the moduli-variety M_ν.

In the special case of hyperelliptic curves the objects involved are much simpler, hence the corresponding difficulty can be handled easily.

We shall assume the basic concepts and definitions of the theory of algebraic varieties. Our terminology conforms to that of Zariski [6]. We suppose that a universal domain Ω has been fixed, once and for all. All quantities which occur will lie in Ω, and any ground field k that we choose will be such that Ω has infinite transcendence degree over it.

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1. **Hyperelliptic curves.** Let k be any algebraically closed ground field of characteristic different from 2.

DEFINITION. Let K be an algebraic function field of one variable over k whose genus g is at least 2. If K contains an element x such that $[K:k(x)]=2$ we shall say that K is a hyperelliptic function field over k. If C is any projective model of K/k we shall say that C is a hyperelliptic curve.

We recall in particular that any irreducible curve C of genus 2 is hyperelliptic. Also, any hyperelliptic curve C is birationally equivalent to a plane curve $Y^2=f(X)$, where $f(X)$ possesses no square factors and is of even degree n (here X, Y are nonhomogeneous coordinates in the plane). If C is of genus g then we have $n=2g+2$.

We now wish to consider projectively equivalent sets of points on a line. During the discussion we place no restriction on the characteristic of the universal domain. If $Q_1, Q_2, \cdots, Q_m$ are any m points on the projective line S_1 we shall denote by $\tilde{Q}=Q_1\otimes \cdots \otimes Q_m$ the symmetric product of these m points. The point $\tilde{Q}$ lies in the projective space S_m. In fact, its homogeneous coordinates are the coefficients of the binary form $f(X_0, X_1)$ of degree m whose zeros on S_1 are the points $Q_1, Q_2, \cdots, Q_m$ counted with their multiplicities. Any point in S_m represents the symmetric product of a unique set of m points on S_1.

Let T be any projective transformation, defined over the universal domain, of S_1 into itself, and let $\tilde{Q}=Q_1\otimes \cdots \otimes Q_m$. If no Q_i belongs to the singular space of T we set $\tilde{Q}^T=Q_1^T\otimes \cdots \otimes Q_m^T$, where Q_i^T is the image of Q_i under T.

DEFINITION. We shall say that two points $\tilde{Q}$, $\tilde{Q}'$ in S_m are projectively associated if there exists a nonsingular projective transformation T of S_1 onto itself for which $\tilde{Q}^T=\tilde{Q}'$. If T is defined over a field k we shall say that $\tilde{Q}$, $\tilde{Q}'$ are projectively associated over k.

Note that if T is a singular transformation we do not say that $\tilde{Q}$, $\tilde{Q}^T$ are projectively associated.

DEFINITION. Let $A=\{A_1, A_2, \cdots, A_m\}$, $B=\{B_1, B_2, \cdots, B_m\}$ be two sets, of m points each, on the line S_1 (the points in each set need not be distinct). Let $\tilde{A}=A_1\otimes \cdots \otimes A_m$, $\tilde{B}=B_1\otimes \cdots \otimes B_m$. We shall say that A and B are projectively equivalent if $\tilde{A}$ and $\tilde{B}$ are projectively associated. If $\tilde{A}$ and $\tilde{B}$ are projectively associated over a field k then we shall say that A and B are projectively equivalent over k.

Let (X_0, X_1) be a system of homogeneous coordinates on the line S_1.

DEFINITION. Two forms $F(X_0, X_1)$, $G(X_0, X_1)$ are said to be projectively equivalent (over k) if their two sets of roots, taken with proper multiplicities, represent projectively equivalent sets of points (over k) on the line S_1.

If $X=X_1/X_0$ and $F(0, 1)\neq 0$, $G(0, 1)\neq 0$ then the corresponding dehomogenized polynomials $f(X)$, $g(X)$ are said to be projectively equivalent.

We now mention one further property of hyperelliptic curves which will be useful. Again let k be any algebraically closed field of characteristic different from 2, and let C/k, C'/k be the two hyperelliptic curves $Y^2=f(X)$; $Y^2=g(X)$, where $f(X)$ and $g(X)$ possess no square factors and are of even degree. Then a necessary and sufficient condition that C and C' be birationally equivalent over k is that the polynomials $f(X)$, $g(X)$ be projectively equivalent over k.

2. **Projectively equivalent sets of points on a line.** Let k_0 be the prime field of the characteristic of our universal domain. In this section we impose no restrictions on that characteristic. Let T be a projective transformation (defined over the universal domain) of the line S_1 into itself, given by the equation

$$Y_0:Y_1 = \alpha_0X_0 + \alpha_1X_1:\alpha_2X_0 + \alpha_3X_1.$$

We shall denote by R_T the point $(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$ in projective 3-space, and shall call R_T the representative point of T. T may be singular or nonsingular. If $P(x_0, x_1)$ is any point of S_1 not in the singular space of T, we shall denote by P^T the image point of P under T.

We shall use an arrow to denote the relation of specialization. If no field is mentioned then it is to be understood that the specialization is over the prime field.

LEMMA 1. *Let $(P, R_T)\to(Q, R_{T'})$ where P, $Q\in S_1$; R_T, $R_{T'}\in S_3$. Assume that Q does not belong to the singular space of T'. Then the above specialization can be extended to the specialization $(P, R_T, P^T)\to(Q, R_{T'}, Q^{T'})$, and this extension is unique.*

Proof. The lemma becomes obvious upon writing down the algebraic relation satisfied by the coordinates of the points P, R_T, P^T.

LEMMA 2. *Let T' be a projective transformation of S_1 into itself with the singular point Q. Let k be any field and let R_T be a general point of $S_3/k(Q)$. Then if A is any point in S_1 we have $(R_T, Q^T)\to(R_{T'}, A)$, the specialization being over the field $k(Q)$.*

Proof. Let (X_0, X_1) be homogeneous coordinates in S_1, and let (Y_0, Y_1) be homogeneous coordinates in a second copy of S_1. Further, let (W_0, W_1, W_2, W_3) be homogeneous coordinates in S_3. If Q is the point (x_0', x_1') we can assume that $x_0'\neq 0$ without loss of generality. Setting $x'=x_1'/x_0'$, we find that the points R_T, Q^T satisfy the homogeneous equation

$$Y_0:Y_1 = W_0 + x'W_1:W_2 + x'W_3. \tag{1}$$

Let ψ be the irreducible algebraic correspondence defined over $k(Q)$ by the general point pair (R_T, Q^T). Then it follows from (1) that ψ is a rational transformation of the projective space S_3 onto S_1.

We quote the following well known result. If ϕ is a rational transformation of a projective space S_n onto an algebraic variety V, lying in the projective space S_v, among whose defining equations appear

$$Y_0:Y_1: \cdots :Y_v = F_0(X_0, \cdots, X_n):F_1(X_0, \cdots, X_n): \cdots :F_v(X_0, \cdots, X_n),$$

the $F_i(X)$ being forms of like degree with no common factors, and if $P(x)$ is such that $F_0(x)=F_1(x)= \cdots =F_v(x)=0$, then P is a fundamental point for the correspondence ϕ.

Apply this result to our particular case. Since Q is singular for T' the forms $W_0+x'W_1$, $W_2+x'W_3$ both vanish at $R_{T'}$. Hence $R_{T'}$ is a fundamental point for our correspondence ψ. As S_1 is of dimension 1 this implies that $R_{T'}$ corresponds, under ψ, to the entire space S_1. This completes the proof of Lemma 2.

LEMMA 3. *Let $\tilde{A}=A_1\otimes \cdots \otimes A_m$, where $A_1, \cdots, A_m$ are distinct points of S_1, and let T be a nonsingular projective transformation of S_1 onto itself. Then if $\tilde{Q}$, $\tilde{B}\in S_m$, $R_{T'}\in S_3$ and $(\tilde{Q}, \tilde{Q}^T, R_T)\to(\tilde{A}, \tilde{B}, R_{T'})$ either*

(a) *T' is nonsingular and $\tilde{B}=\tilde{A}^{T'}$ or*

(b) *T' is singular and $\tilde{B}$ contains the same point repeated at least $m-1$ times.*

Proof. Let $\tilde{Q}=Q_1\otimes \cdots \otimes Q_m$, $\tilde{B}=B_1\otimes \cdots \otimes B_m$. For a certain ordering of the points $A_1, \cdots, A_m, B_1, \cdots, B_m$ we have $(Q_1, \cdots, Q_m, Q_1^T, \cdots, Q_m^T, R_T)\to(A_1, \cdots, A_m, B_1, \cdots, B_m, R_{T'})$. At most one A_i can be a singular point of T'. We see from Lemma 1 that for any value of i such that A_i is not such a point we have $B_i=A_i^{T'}$. If T' is nonsingular this holds for all $i=1, 2, \cdots, m$; hence $\tilde{B}=\tilde{A}^{T'}$. On the other hand, if T' is singular, for at least $m-1$ values of i we have $B_i=A_i^{T'}=$ the image point of S_1 under T'. This completes the proof of our lemma.

LEMMA 4. *Let $\tilde{A}$, T be defined as in Lemma 3, and let $\tilde{Q}\in S_m$ be such that $(\tilde{Q}, \tilde{Q}^T)\to(\tilde{A}, \tilde{B})$ for some $\tilde{B}\in S_m$. Then there exists a nonsingular projective transformation T^* of S_1 such that $(\tilde{A}, \tilde{A}^{T^*})\to(\tilde{A}, \tilde{B})$.*

Proof. Extend the given specialization to

$$(\tilde{Q}, \tilde{Q}^T, R_T) \to (\tilde{A}, \tilde{B}, R_{T'}). \tag{2}$$

If T' is nonsingular it follows from Lemma 3 that $\tilde{B}=\tilde{A}^{T'}$. Hence in this case we may choose $T^*=T'$.

Next suppose that T' is singular. Then we may choose T^* so that R_{T^*} is a general point of $S_3/k_0(\tilde{Q}, \tilde{A})$. Then $(\tilde{Q}, R_{T^*})\to(\tilde{Q}, R_T)$. Extend this to a specialization of $\tilde{Q}^{T^*}$. As T is nonsingular it follows from Lemma 3 that the only such extension is $(\tilde{Q}, \tilde{Q}^{T^*}, R_{T^*})\to(\tilde{Q}, \tilde{Q}^T, R_T)$. Note the specialization (2) above. By the transitivity of specializations we have therefore $(\tilde{Q}, \tilde{Q}^{T^*}, R_{T^*})\to(\tilde{A}, \tilde{B}, R_{T'})$. If none of the A_i are singular for T' we must have $\tilde{B}=\tilde{A}^{T'}$. Since R_{T^*} is a general point of $S_3/k_0(\tilde{A})$ we finally obtain $(\tilde{A}, \tilde{A}^{T^*})\to(\tilde{A}, \tilde{B})$, the desired result.

The remaining case to consider is that in which some A_i is singular for T'. This can happen for only one i. For the sake of definiteness we shall suppose that A_1 is the singular point of T'. We now use Lemma 2. Choose the field k of that lemma to be $k_0(A_1, A_2, \cdots, A_m)$. Since S_3 is absolutely irreducible and $k_0(A_1, A_2, \cdots, A_m)$ is a finite algebraic extension of $k_0(\tilde{A})$, R_{T^*} is also a general point of S_3 over this larger field. Hence, according to Lemma 2, we have $(A_1^{T^*}, R_{T^*})\rightarrow(A, R_{T'})$ for any point A in S_1', the specialization being over the field $k_0(A_1, \cdots, A_m)$. In particular, choose $A=B_1$. We then have $(A_1, \cdots, A_m, A_1^{T^*}, R_{T^*})\rightarrow(A_1, \cdots, A_m, B_1, R_{T'})$. Extend this to a specialization of $A_2^{T^*}, \cdots, A_m^{T^*}$. The only such extension is $(A_1, \cdots, A_m, A_1^{T^*}, A_2^{T^*}, \cdots, A_m^{T^*}, R_{T^*})\rightarrow(A_1, \cdots, A_m, B_1, A_2^{T'}, \cdots, A_m^{T'}, R_{T'})$. By the usual argument $B_i=A_i^{T'}$, $i=2, 3, \cdots, m$. Hence $(\tilde{A}, \tilde{A}^{T^*})\rightarrow(\tilde{A}, \tilde{B})$ as we desired.

LEMMA 5. *Let $\tilde{Q}=Q_1\otimes \cdots \otimes Q_m$, where the Q_i are distinct and $m\geqq 3$. Let $V\langle\tilde{Q}\rangle$ be the smallest algebraic variety in S_m which contains every point in S_m projectively associated to $\tilde{Q}$. Then $V\langle\tilde{Q}\rangle$ is absolutely irreducible, is of dimension 3, and is defined over the field $k_0(\tilde{Q})$. The points $\tilde{A}$ of $V\langle\tilde{Q}\rangle$ which are not projectively associated to $\tilde{Q}$ form a proper algebraic subvariety of $V\langle\tilde{Q}\rangle$, this subvariety also being defined over the field $k_0(\tilde{Q})$.*

Proof. Choose R_T to be a general point of $S_3/k_0(\tilde{Q})$. Consider the irreducible algebraic correspondence ψ defined over $k_0(\tilde{Q})$ by the general point pair $(R_T, \tilde{Q}^T)$. This correspondence transforms S_3 onto some subvariety H of S_m, H being defined and irreducible over $k_0(\tilde{Q})$.

Applying the principle of counting constants to the correspondence ψ we find dim $H=3$ (it is here that we must have $m\geqq 3$). Obviously $V\langle\tilde{Q}\rangle\subset H$. We shall show that these two varieties are in fact equal. To do this, we first show that H is absolutely irreducible. Let k be the algebraic closure of the field $k_0(\tilde{Q})$. Any general point of an irreducible component of H/k is also a general point of H/k_0, hence can be written $\tilde{Q}^{T'}$ for some T', $R_{T'}\in S_3$. R_T is not only a general point of $S_3/k_0(\tilde{Q})$ but also of S_3/k. Hence $R_T\rightarrow R_{T'}$, over the field k. Extending this specialization to one of $\tilde{Q}^T$, we find at once $(R_T, \tilde{Q}^T)\rightarrow(R_{T'}, \tilde{Q}^{T'})$, over the field k. Hence $\tilde{Q}^T$ is also a general point of H/k. Since k is algebraically closed we deduce that H is absolutely irreducible.

We are now in a position to prove that $V\langle\tilde{Q}\rangle=H$. For let k' be any field of definition of $V\langle\tilde{Q}\rangle$ such that $k'\supset k_0(\tilde{Q})$. Choose $\tilde{B}$ to be a general point of H/k'. Then $\tilde{B}$ is projectively associated to $\tilde{Q}$, hence $\tilde{B}\in V\langle\tilde{Q}\rangle$. Therefore $H\subset V\langle\tilde{Q}\rangle$, hence these two varieties are identical.

All of Lemma 5 has now been proved, except for the last statement. But it is easy to see that a point $\tilde{A}$ of $V\langle\tilde{Q}\rangle$ is not projectively associated to $\tilde{Q}$ if and only if it corresponds under ψ to a singular projective transformation of S_1. Since the set of all such singular transformations is a subvariety of S_3, defined over k_0, the proof is complete.

The symbol $V\langle\tilde{Q}\rangle$ will be used only when it is understood that $\tilde{Q}$ is the symmetric product of distinct points on S_1.

Choose $P_1, P_2, \cdots, P_m$ to be m algebraically independent general points of S_1/k_0 and set $\tilde{P}=P_1\otimes \cdots \otimes P_m$. Let T be such that R_T is a general point of $S_3/k_0(\tilde{P})$. Consider Ψ, the irreducible algebraic correspondence defined over k_0 by the general point pair $(\tilde{P}, \tilde{P}^T)$. Assume $m\geqq 3$. By its definition, $\Psi(\tilde{P}) = V\langle\tilde{P}\rangle$. Hence dim $\Psi(\tilde{P})=3$, and so for any point $\tilde{Q}\in S_m$, dim $\Psi(\tilde{Q})\geqq 3$.

Now let $\tilde{Q}$ be the symmetric product of m distinct points on S_1. Then it easily follows from Lemma 4 that $\Psi(\tilde{Q})\subset V\langle\tilde{Q}\rangle$. But $V\langle\tilde{Q}\rangle$ is of dimension 3, and is absolutely irreducible. Hence $\Psi(\tilde{Q}) = V\langle\tilde{Q}\rangle$. Thus, if $m\geqq 3$, Ψ/k_0 is an irreducible algebraic correspondence which corresponds to any $\tilde{Q}$ representing a set of m distinct points on S_1 to $V\langle\tilde{Q}\rangle$.

3. **The algebraic system** M. For the definitions of the terms (e.g. incidence correspondence, involution) used in the theory of algebraic systems and for the basic properties of such systems the reader is referred to Zariski [6].

Let $\tilde{P}$ be a general point of S_m/k_0, and let $V\langle\tilde{P}\rangle$ be the absolutely irreducible 3-dimensional variety of Lemma 5. Then $V\langle\tilde{P}\rangle$ defines a unique absolutely prime 3-cycle Z. We shall use $|Z|$ to stand for the set of points in the cycle Z.

Let M be the irreducible algebraic system of 3-cycles defined over k_0 by taking Z to be its general cycle. Recall that Ψ was the algebraic correspondence whose general point pair over k_0 was $(\tilde{P}, \tilde{P}^T)$, R_T being a general point of $S_3/k_0(\tilde{P})$. Hence $|Z| =\Psi(\tilde{P})$ and therefore for any point $\tilde{B}\in S_m$, $\Psi(\tilde{B})$ is the set-theoretic union of all cycles in M which contain $\tilde{B}$. If $\tilde{B}$ is the symmetric product of distinct points we already know that $\Psi(\tilde{B}) = V\langle\tilde{B}\rangle$. Hence, in that case, there is only one cycle Z' of M which passes through $\tilde{B}$, and $|Z'| = V\langle\tilde{B}\rangle$. This holds in particular for any general point $\tilde{B}$ of S_m/k_0. Thus the algebraic system M is an involution. This also shows that if a cycle Z' of M contains a point $\tilde{B}$ which is the symmetric product of distinct points on S_1 then every point in $|Z'|$ is either projectively associated to $\tilde{B}$ or else contains a single point repeated at least $m-1$ times.

We shall use the same letter M to denote the representative variety of the algebraic system. Let χ^{-1} be the incidence correspondence of M. Since the cycle Z is absolutely prime, χ^{-1} is irreducible over the field k_0. Let $(\tilde{P}', Z)$ be a general pair of χ^{-1}/k_0. $\tilde{P}$ is a general point of S_m/k_0, hence $\tilde{P}\rightarrow\tilde{P}'$. This can be extended to $(\tilde{P}, Z)\rightarrow(\tilde{P}', Z')$ for some cycle Z'. Since M is an involution, $Z=Z'$. Therefore $(\tilde{P}, Z)\rightarrow(\tilde{P}', Z)$. Since $\tilde{P}$ belongs to the point set $|Z|$, it follows that $(\tilde{P}, Z)$ is a general pair of χ^{-1}/k_0. We might have noted earlier that χ^{-1} transforms M onto S_m (i.e. S_m is the carrier of the algebraic system M). Namely, $\tilde{P}$ belongs to the carrier of M, and $\tilde{P}$ is a general point of S_m/k_0.

We are now in a position to show that the algebraic system M is absolutely irreducible. Let Z' be a general cycle of some irreducible component of

$M/\bar{k}_0$, where $\bar{k}_0$ is the algebraic closure of k_0. Choose a point $\tilde{P}' \in |Z'|$ which is the symmetric product of distinct points on the line S_1.

Since $\tilde{P}$ is a general point of $S_m/\bar{k}_0$, we have $\tilde{P} \rightarrow \tilde{P}'$, over the field $\bar{k}_0$. This can be extended to the specialization $(\tilde{P}, Z) \rightarrow (\tilde{P}', \overline{Z}')$ which holds, over the field $\bar{k}_0$, for some cycle $\overline{Z}'$ in M. Hence $\tilde{P}' \in |\overline{Z}'|$, and so $\overline{Z}' = Z'$. We conclude that $Z \rightarrow Z'$ over $\bar{k}_0$, and thus M is absolutely irreducible.

Since M is an involution of 3-cycles whose carrier is S_m it follows at once, from the principle of counting constants, that dim $M = m-3$.

LEMMA 6. *Let W be the subset of M consisting of all cycles Z' in M which do not have the property that $|Z'| = V\langle \tilde{Q} \rangle$ for some point $\tilde{Q}$ in S_m. Then W is a proper algebraic subvariety of M, defined over the field k_0.*

Proof. $|Z'| = V\langle \tilde{Q} \rangle$ for some $\tilde{Q}$ in S_m if and only if $|Z'|$ contains a point $\tilde{Q}$ which is the symmetric product of m distinct points on S_1. The set of all points $\tilde{Q}$ which do not have this property form a subvariety H of S_m, the variety H being defined over the field k_0. W consists of those cycles Z' in M such that $\chi^{-1}(Z') = |Z'| \subset H$. It is a general fact that the set of all cycles of a given dimension and order which lie on a fixed variety H, defined over a field k, form an algebraic system, defined over k. Thus, in our case W is an algebraic subvariety of M, defined over k_0. W is a proper subvariety since the general point of M/k_0 does not belong to W.

We shall next investigate the algebraic system M more closely by intersecting its cycles with a certain $(m-3)$-dimensional linear space. Fix 3 distinct points $\overline{A}_1$, $\overline{A}_2$, $\overline{A}_3$ on the line S_1 whose coordinates lie in k_0. For convenience (also because in the case of characteristic 2 there are no other rational points on S_1) we shall take them to be the points (1, 0), (0, 1), (1, 1). Consider in S_m the set of points $\tilde{Q}$ which represent 0-dimensional cycles to which $\overline{A}_1$, $\overline{A}_2$, $\overline{A}_3$ belong. This point set is a linear space L'_{m-3}, defined over k_0.

LEMMA 7. *Let Z be any general cycle of the algebraic system M/k_0. Then $|Z| \cap L'_{m-3}$ contains a finite number of points, any two of which are conjugate points over the field $k_0(Z)$. Further, any point $\tilde{A} \in |Z| \cap L'_{m-3}$ is a general point of L'_{m-3}/k_0.*

Proof. Let Φ be the irreducible algebraic correspondence defined over k_0 by the general point pair $(Z, \tilde{A})$, where $\tilde{A}$ is some point in $|Z| \cap L'_{m-3}$. By the principle of counting constants, applied to Φ, we find dim M+dim $\Phi(Z)$ = tr.d. $k_0(\tilde{A})/k_0$+dim $\Phi^{-1}(\tilde{A})$.

Since Z is a general cycle of M/k_0, $|Z| = V\langle \tilde{P} \rangle$, where $\tilde{P}$ is a general point of S_m/k_0. $\tilde{A}$ belongs to L'_{m-3}, hence the set of m points on S_1 which $\tilde{A}$ represents includes the 3 distinct points $\overline{A}_1$, $\overline{A}_2$, $\overline{A}_3$. Therefore $\tilde{A}$, which belongs to $V\langle \tilde{P} \rangle$, does not contain any point of multiplicity $m-1$ and thus $\tilde{A}$ is projectively associated to $\tilde{P}$. It follows that Z is the only cycle in M which contains $\tilde{A}$. Hence dim $\Phi^{-1}(\tilde{A}) = 0$.

The above remarks show that if $\tilde{A}'$ is any other point in $|Z|\cap L'_{m-3}$ then $\tilde{A}'$ is projectively associated to $\tilde{A}$. Any such $\tilde{A}'$ contains (as a 0-dimensional cycle) the points $\overline{A}_1$, $\overline{A}_2$, $\overline{A}_3$ of S_1. Hence there are only a finite number of such points $\tilde{A}'$.

Therefore $\dim \Phi(Z)=0$. Taking into account the fact that $\dim M=m-3$, we can conclude that tr.d. $k_0(\tilde{A})/k_0=m-3$. Thus $\tilde{A}$ is a general point of L'_{m-3}/k_0. If $\tilde{A}'$ is any point in $|Z|\cap L'_{m-3}$ then $\tilde{A}\rightarrow\tilde{A}'$. Extend this specialization to one of Z. Since only one cycle of M passes through $\tilde{A}'$ it follows by the usual argument that $(\tilde{A}, Z)\rightarrow(\tilde{A}', Z)$. Reversing the procedure enables us to conclude that $\tilde{A}$ and $\tilde{A}'$ are conjugate points over the field $k_0(Z)$. Therefore Φ is independent of the particular point chosen from $|Z|\cap L'_{m-3}$.

LEMMA 8. *Let $\tilde{Q}\in S_m$ be the symmetric product of m distinct points on S_1 ($m\geqq 3$). Then the function field of $V\langle\tilde{Q}\rangle$ is separably generated over the ground field $k_0(\tilde{Q})$.*

Proof. Let R_T be a general point of $S_3/k_0(\tilde{Q})$. Then $\tilde{Q}^T$ is a general point of $V\langle\tilde{Q}\rangle$ over the ground field $k_0(\tilde{Q})$. Hence $k_0(\tilde{Q}, \tilde{Q}^T)$ is the function field of $V\langle\tilde{Q}\rangle$ over $k_0(\tilde{Q})$. Clearly, $k_0(\tilde{Q})\subset k_0(\tilde{Q}, \tilde{Q}^T)\subset k_0(\tilde{Q}, R_T)$. Since any subfield of a separably generated field is also separably generated we conclude that $k_0(\tilde{Q}, \tilde{Q}^T)$ is separably generated over $k_0(\tilde{Q})$.

LEMMA 9. *Let Z be a general cycle of M/k_0 and let $\tilde{A}\in|Z|\cap L'_{m-3}$. Then $k_0(\tilde{A})$ is a finite separable algebraic extension of $k_0(Z)$.*

Proof. The field $k_0(Z)$ is the smallest field of definition of the algebraic variety $|Z|$ over which the function field of $|Z|$ is separably generated (Zariski [6]). Since $|Z|=V\langle\tilde{A}\rangle$ it follows from Lemma 8 that the function field of $|Z|$ is separably generated over $k_0(\tilde{A})$. Hence $k_0(Z)$ is a subfield of $k_0(\tilde{A})$.

We next remark that $\tilde{A}$ is an absolutely simple point of $|Z|$. One method of proving this is as follows. For a suitable choice of T, $\tilde{A}^T$ is a general point of $|Z|/k_0(Z)$. Since the function field of this latter variety is separably generated we conclude that $\tilde{A}^T$ is an absolutely simple point of $|Z|$. After a suitable transcendental extension of the ground field $k_0(Z)$, the transformation T induces a birational transformation of $|Z|$ onto itself which maps $\tilde{A}$ into $\tilde{A}^T$ and is regular at $\tilde{A}$. Therefore $\tilde{A}$ also is an absolutely simple point of $|Z|$.

The local field of the point $\tilde{A}$ on the variety $|Z|/k_0(Z)$ is $k_0(\tilde{A})$ (since $k_0(Z)\subset k_0(\tilde{A})$), hence we conclude that $k_0(\tilde{A})$ is separably generated over $k_0(Z)$. But we already know that $k_0(\tilde{A})$ is a finite algebraic extension of $k_0(Z)$. We can now add the adjective separable to this description.

For a further study of the correspondence Φ introduced previously we find it necessary to consider the associated forms, in the sense of Chow, of the cycles in M. Let U^i be the set of indeterminates $(U_0^i, U_1^i, \cdots, U_m^i)$,

$i=1, 2, 3, 4$; and let t be the order of the cycles in M. Then the associated form of any cycle in M is a multiply homogeneous form $G(U^1, U^2, U^3, U^4)$, homogeneous of degree t in each set of variables U^i.

Let $F(U^1, U^2, U^3, U^4)$ be the associated form of the general cycle Z of M/k_0. Choose $u^i=(u_0^i, u_1^i, \cdots, u_m^i)$, $i=1, 2, 3$, to be three algebraically independent general points of $S_m/k_0(Z)$. Then, from the properties of associated forms it follows that the form $F(u^1, u^2, u^3, U)$ in the $m+1$ indeterminates $U_0, U_1, \cdots, U_m$ factors into linear factors over a suitable algebraic extension of the field $k_0(Z, u^1, u^2, u^3)$,

$$F(\overset{1}{u}, \overset{2}{u}, \overset{3}{u}, U) = \prod_{j=1}^{t} (y_0^j U_0 + y_1^j U_1 + \cdots + y_m^j U_m).$$

Let $C_j=(y_0^j, y_1^j, \cdots, y_m^j)$, $j=1, 2, \cdots, t$. The points $C_1, C_2, \cdots, C_t$ are distinct, as the cycle Z is absolutely prime.

Choose π_i to be the hyperplane in S_m defined by $u_0^i X_0+u_1^i X_1+\cdots+u_m^i X_m=0$, and let $L_{m-3}=\pi_1\cap\pi_2\cap\pi_3$. Then L_{m-3} is a general linear $(m-3)$-space over the field $k_0(Z)$, and the points $C_1, C_2, \cdots, C_t$ are the intersections of L_{m-3} with $|Z|$. These t points determine a 0-cycle of order t whose associated form is $F(u^1, u^2, u^3, U)$. We call this 0-cycle the intersection cycle of L_{m-3} with the cycle Z and denote it by $Z\cap L_{m-3}$.

Now let $\bar{\pi}_i(\bar{u}_0^i, \bar{u}_1^i, \cdots, \bar{u}_m^i)$, $i=1, 2, 3$, be three hyperplanes, defined over k_0, such that $\bar{\pi}_1\cap\bar{\pi}_2\cap\bar{\pi}_3$ is the L'_{m-3} we have been considering. L'_{m-3} meets $|Z|$ in only a finite number of points, hence $F(\bar{u}^1, \bar{u}^2, \bar{u}^3, U)$ is not identically zero. Thus this form, as a form in $U_0, U_1, \cdots, U_m$, is a specialization over k_0 of $F(u^1, u^2, u^3, U)$. Hence it is the associated form of a certain 0-cycle which we call $Z\cap L'_{m-3}$, the intersection cycle of L'_{m-3} with the cycle Z. It can be easily seen that $|Z\cap L'_{m-3}|$ consists exactly of the points in $|Z|\cap L'_{m-3}$. As far as we know there may not be t distinct points in this point set, although $Z\cap L'_{m-3}$ is a cycle of order t.

Now let W be the subvariety of M defined in Lemma 6. If Z' is any cycle in $M-W$ then $|Z'|=V\langle\tilde{Q}\rangle$ for some $\tilde{Q}$ in S_m. In this case the point set $|Z'|\cap L'_{m-3}$ is finite, the proof being exactly the same as that given previously for the general cycle Z of M. Hence if $F'(U^1, U^2, U^3, U^4)$ is the associated form of the cycle Z' then $F'(\bar{u}^1, \bar{u}^2, \bar{u}^3, U)$ is not identically zero. Thus the specialization $Z\rightarrow Z'$ can be extended in one and only one way to the cycle $Z\cap L'_{m-3}$. Write this specialization $(Z, Z\cap L'_{m-3})\rightarrow(Z', Z'\cap L'_{m-3})$.

We shall call the 0-cycle $Z'\cap L'_{m-3}$, defined in this way, the intersection cycle of L'_{m-3} with the cycle Z'. The associated form of $Z'\cap L'_{m-3}$ is $F'(\bar{u}^1, \bar{u}^2, \bar{u}^3, U)$. The point sets $|Z'|\cap L'_{m-3}$ and $|Z'\cap L'_{m-3}|$ are identical.

Let Φ be the algebraic correspondence defined in the course of the proof of Lemma 7. It can be shown, by an argument used many times before, that Φ is absolutely irreducible.

LEMMA 10. *If the cycle Z' belongs to $M-W$ then $\Phi(Z')=|Z'\cap L'_{m-3}|$.*

Proof. Let $(Z, \tilde{A})$ be a general pair of Φ/k_0. Then if $\tilde{B}\in\Phi(Z')$ we have $(Z, \tilde{A})\rightarrow(Z', \tilde{B})$. From $\tilde{A}\in|Z|$ it follows that $\tilde{B}\in|Z'|$. Also $\tilde{A}\in L'_{m-3}$, a variety defined over k_0. Hence $\tilde{B}\in|Z'\cap L'_{m-3}|$.

Conversely, if $\tilde{B}\in|Z'\cap L'_{m-3}|$ we first consider the specialization $(Z, Z\cap L'_{m-3})\rightarrow(Z', Z'\cap L'_{m-3})$. Since $\tilde{B}\in|Z'\cap L'_{m-3}|$ there exists a point $\tilde{A}'$ such that $(Z, Z\cap L'_{m-3}, \tilde{A}')\rightarrow(Z', Z'\cap L'_{m-3}, \tilde{B})$ and $\tilde{A}'\in|Z\cap L'_{m-3}|$. Thus $(Z, \tilde{A}')\rightarrow(Z', \tilde{B})$. By Lemma 7 $\tilde{A}$ and $\tilde{A}'$ are conjugate points over $k_0(Z)$. Hence $\tilde{B}\in\Phi(Z')$. The proof of Lemma 10 is now complete.

Now let k be any ground field in our universal domain. Then the variety M and the algebraic correspondence Φ are defined and irreducible over k. If Z is a general cycle of M/k it is also such of M/k_0. If $(Z, \tilde{A})$ is a general pair of Φ/k then $\tilde{A}\in|Z\cap L'_{m-3}|$. Hence by Lemma 9 the field $k_0(\tilde{A})$ is a finite separable algebraic extension of the field $k_0(Z)$. But then also the field $k(\tilde{A})$ is a finite separable algebraic extension of the field $k(Z)$. Also a slight change in the proof of Lemma 7 shows that if $\tilde{A}$, $\tilde{A}'$ are any two points in $|Z\cap L'_{m-3}|$ then $\tilde{A}$ and $\tilde{A}'$ are conjugate points over the field $k(Z)$ as well as over $k_0(Z)$.

Let Z be a general cycle of M/k. Consider a derived normal model $\overline{M}/k$ of M/k with general point $\overline{P}$ such that $(Z, \overline{P})$ is a general pair, over k, of the birational transformation Ψ between M and $\overline{M}$ (Zariski [4]). Neither Ψ nor Ψ^{-1} have any fundamental points, while Ψ^{-1} is single valued on the whole of $\overline{M}$. Let $\overline{W}=\Psi(W)$, W being defined as in Lemma 6.

Choose any $\tilde{A}\in|Z\cap L'_{m-3}|$, and let ϕ be the irreducible algebraic correspondence defined over k by the general point pair $(\overline{P}, \tilde{A})$. ϕ is independent of the particular point $\tilde{A}$ chosen in $|Z\cap L'_{m-3}|$. Since $k(\overline{P})=k(Z)\subset k(\tilde{A})$, ϕ^{-1} is rational.

Lemma 11. *If $\overline{Q}\in\overline{M}-\overline{W}$ then $\phi(\overline{Q})=|Z'\cap L'_{m-3}|$, where $Z'=\Psi^{-1}(\overline{Q})$.*

Proof. $(\overline{P}, Z)\rightarrow(\overline{Q}, Z')$, over the field k. We can extend this to the specialization $(\overline{P}, Z, Z\cap L'_{m-3})\rightarrow(\overline{Q}, Z', Z'\cap L'_{m-3})$, over the field k. From this point on the proof is identical to that of Lemma 10.

We now leave these considerations temporarily in order to prove the following general theorem about involutions. The field k is still any ground field in our universal domain Ω.

Theorem 1. *Let N be an involution, defined and irreducible over the field k, and let V be the carrier of N. If Z' is any cycle of N containing a point Q such that:*

(1) *Z' is the only cycle of N passing through Q',*

(2) *V/k is analytically irreducible at Q',*

then the representative variety N/k of the involution N is analytically irreducible at the representative point of the cycle Z'.

We recall that a variety V/k is said to be analytically irreducible at a point Q' if the completion of the local ring of the point Q' on V/k is an integral domain. V/k is analytically irreducible at Q' if and only if the points which correspond to Q' on a derived normal model of V/k are isomorphic over k (Zariski [5]). Now let π/k be an irreducible algebraic correspondence of V onto an algebraic variety H. We assume that π is quasi-rational (i.e. it is single valued at any general point of V/k), and that Q' is a point of V at which V is analytically irreducible and which is not a fundamental point of π. Then the points on H which correspond, under π, to Q' are isomorphic over k. This result follows easily from the special case in which H is a derived normal model of V/k.

We now proceed with the proof of Theorem 1. Let χ^{-1} be the incidence correspondence of N onto V. Since N is an involution χ^{-1} is irreducible over k. From the involutorial character of N we see that χ/k is a quasi-rational transformation of V onto N. Let (Q, Z) be a general point pair of χ/k. Consider a derived normal model $\overline{N}/k$ of N/k with general point $\overline{P}$ chosen so that $(Z, \overline{P})$ is a general pair, over k, of the birational transformation Ψ between N and $\overline{N}$. Let π be the irreducible algebraic correspondence of V onto $\overline{N}$ defined over k by the general point pair $(Q, \overline{P})$. Then $\pi=\chi\cdot\Psi$, that is, π is the set-theoretic product of χ and Ψ (Zariski [5, p. 85]). From this we deduce that π is quasi-rational. Now let Z', Q' be as in the hypothesis of Theorem 1. Let $\Psi(Z')=\{\overline{P}_1', \overline{P}_2', \cdots, \overline{P}_s'\}$. Since $\chi(Q')=Z'$, while $\pi=\chi\cdot\Psi$ we find $\pi(Q') =\{\overline{P}_1', \overline{P}_2', \cdots, \overline{P}_s'\}$.

But V is analytically irreducible at Q' and π is quasi-rational. Since we have just shown that π is finitely valued at Q' we conclude that the points $\overline{P}_1', \overline{P}_2', \cdots, \overline{P}_s'$ are isomorphic over k. Thus N/k is analytically irreducible at Q'.

We now return to the particular involution M in S_m.

THEOREM 2. *M/k is analytically irreducible at all points not in the subvariety W.*

Proof. $Z'\in M-W$ if and only if $|Z'|=V\langle\tilde{Q}\rangle$ for some point $\tilde{Q}\in S_m$. If $|Z'|=V\langle\tilde{Q}\rangle$ then Z' is the only cycle in M through $\tilde{Q}$. Since S_m is the carrier of the algebraic system M the assumptions of Theorem 1 concerning analytical irreducibility are certainly satisfied. Hence the desired conclusion follows.

We need one further result concerning the intersections of L_{m-3}' with the cycles of M.

LEMMA 12. *Let Z' be any cycle in $M-W$, and let $\tilde{Q}\in S_m$ be such that $|Z'|=V\langle\tilde{Q}\rangle$. Then if there are v projective transformations T' of S_1 onto itself such that $\tilde{Q}^{T'}=\tilde{Q}$ the point set $|Z'\cap L_{m-3}'|$ contains s/v distinct points, where s is the number of permutations of m objects taken 3 at a time.*

Proof. There are exactly s distinct projective transformations T' of S_1 onto itself such that $\tilde{Q}^{T'}\in|Z'\cap L_{m-3}'|$. Since v of these leave $\tilde{Q}$ invariant we obtain the stated result.

4. **The moduli-variety for hyperelliptic curves.** If V is any algebraic variety defined over the field k we shall denote by V^0 the set of points in V algebraic over the field k. We shall use the term algebraic variety to refer to V^0, as well as to V.

In this section we shall show how the algebraic system M leads in a natural fashion to the construction of a normal variety $\overline{M}^0$, of dimension $2g-1$, whose points except for the points of a proper algebraic subvariety $\overline{W}^0$ are in one-one correspondence with the birational classes of hyperelliptic curves of genus g. We shall further prove that $\overline{M}^0$ has certain properties which characterize it up to a regular birational transformation (see Theorem 3). Thus we shall show that the use of the representation of hyperelliptic curves of genus g by $2g+2$ points on the line leads to an essentially uniquely determined moduli-variety.

Let k be any algebraically closed field of characteristic different from 2. If g is any integer greater than 1 set $m=2g+2$. In §1 we have shown that one can associate, in a natural manner, to any hyperelliptic function field K/k of genus g various sets of m distinct points on the line S_1. Each point appearing in such a set is rational over k. Two different sets of points correspond to the same function field if and only if they are projectively equivalent over the field k. We are going to use this representation of function fields to give an algebro-geometric structure to the set of all hyperelliptic function fields K/k of genus g.

Let π/k be an irreducible algebraic correspondence of S_m^0 onto an algebraic variety N^0 such that:

(a) *If $\tilde{Q}\in S_m^0$ is the symmetric product of m distinct points on S_1 then π is single valued at $\tilde{Q}$.*

(b) *If $\tilde{Q}$, $\tilde{Q}'$ both satisfy the hypothesis of* (a) *then $\pi(\tilde{Q})=\pi(\tilde{Q}')$ if and only if $\tilde{Q}$, $\tilde{Q}'$ are projectively associated over the field k.*

Then to every hyperelliptic function field K/k of genus g we can associate by means of π a unique point of the algebraic variety N^0.

Since $m=2g+2>3$ we can consider the algebraic system M of §3. M is defined and irreducible over k. Let χ^{-1} be the incidence correspondence of M. Then χ is an algebraic correspondence of S_m onto M. χ is single valued at those points of S_m which represent m distinct points on S_1. Two such points of S_m correspond, under χ, to the same point on M if and only if they represent projectively equivalent sets of points on S_1.

Clearly a point $\tilde{Q}$ in S_m lies in S_m^0 if and only if $\tilde{Q}=Q_1\otimes Q_2\otimes \cdots \otimes Q_m$, where each $Q_i\in S_1^0$, $i=1, 2, \cdots, m$. Further, if $\tilde{Q}$, $\tilde{Q}'\in S_m^0$ are projectively associated then they are projectively associated over the field k. Also, if a cycle Z' of M contains a point $\tilde{Q}\in S_m^0$ which is the symmetric product of distinct points of S_1 then it follows from Lemma 5 that $k(Z')=k$, hence $Z'\in M^0$. Putting these remarks together we find that the restriction of the algebraic correspondence χ to S_m^0 satisfies the conditions (a), (b) imposed on

π above. This shows the existence of such correspondences π. M^0 plays the role of N^0 when π is chosen to be χ.

Every point of M, except for the proper algebraic subvariety W/k, corresponds under χ to a point $\tilde{Q}$ of S_m such that $\tilde{Q}$ is the symmetric product of distinct points on S_1. Hence the points of M^0 which interest us are exactly the points in M^0-W^0. It follows from Theorem 2 that M^0/k is analytically irreducible at all such points (M^0/k is said to be analytically irreducible at a point $Q\in M^0$ if M/k is analytically irreducible at Q). If $\overline{M}/k$ is a derived normal model of M/k, with ψ the birational transformation of M onto $\overline{M}$, then ψ is single valued at all points of M^0-W^0 (since k is algebraically closed). Let $\bar{\chi}$ be the set-theoretic product $\chi\cdot\psi$. Then if we replace M^0 by $\overline{M}^0$, χ by $\bar{\chi}$ our conditions (a), (b) are still satisfied. Hence correspondences π exist such that the image variety N^0 is normal.

Let π satisfy conditions (a), (b). Then π determines uniquely an irreducible algebraic correspondence, defined over k, of S_m onto N. The restriction of this new correspondence to algebraic points (over k) is π. We denote this extended correspondence by the same letter π. Since π is single valued at almost every algebraic point of S_m/k it is single valued at almost every point of S_m/k. In particular, it is single valued at every general point $\tilde{P}$ of S_m/k.

Lemma 13. *Let $(\tilde{P}, P)$ be a general point pair of π/k, and let T be any nonsingular projective transformation of S_1 onto itself which is defined over k. Then the points $\tilde{P}^T$, P are corresponding points under π.*

Proof. Let H be the subvariety of S_m which consists of all points $\tilde{B}$ in S_m such that $\tilde{B}$ is not the symmetric product of m distinct points on S_1. Let $\tilde{Q}$ be any point in $S_m^0-H^0$, and let $Q=\pi(\tilde{Q})$. If T meets the conditions of Lemma 13 then $\tilde{Q}$, $\tilde{Q}^{T^{-1}}$ are projectively associated over k, hence $\tilde{Q}=\pi(\tilde{Q}^{T^{-1}})$. Therefore $(\tilde{P}, P)\rightarrow(\tilde{Q}^{T^{-1}}, Q)$, over the field k. Extend this to a specialization of $\tilde{P}^T$. Since T is defined over k this extension must yield $(\tilde{P}, P, \tilde{P}^T)\rightarrow(\tilde{Q}^{T^{-1}}, Q, \tilde{Q})$, over the field k. Hence, in particular, $(\tilde{P}^T, P)\rightarrow(\tilde{Q}, Q)$, over the field k.

Let $G(\pi)$ be the graph of the algebraic correspondence π. We must show that the point $\tilde{P}^T\times P$ belongs to $G(\pi)$. The algebraic variety $(H\times N)\cap G(\pi)$ is a proper algebraic sub-variety of $G(\pi)$. We have just shown above that any algebraic point $\tilde{Q}\times Q$ of $G(\pi)/k$ which does not lie in $H\times N$ is a specialization of $\tilde{P}^T\times P$. Hence $\tilde{P}^T\times P$ can be specialized, over k, to almost every algebraic point in $G(\pi)/k$. But $\tilde{P}^T\times P$ has the same dimension over k as $\tilde{P}\times P$, which is a general point of $G(\pi)/k$. Hence $\tilde{P}^T\times P$ belongs to $G(\pi)$, and in fact is a general point of $G(\pi)/k$. Lemma 13 has thus been proved.

Lemma 14. *If $(\tilde{P}, P)$ is a general point pair of π/k then $V\langle\tilde{P}\rangle\subset\pi^{-1}(P)$.*

Proof. Let V be the smallest algebraic variety (over any field of definition) which contains all points $\tilde{P}^T$ such that T satisfies the conditions of Lemma 13. Let k' be a field of definition of V. Choose the projective transformation

T' so that $R_{T'}$ is a general point of $S_3/k'(\tilde{P})$. We consider the irreducible algebraic correspondence ψ/k' defined by the general point pair $(R_{T'}, \tilde{P}^{T'})$. Then ψ transforms S_3 onto $V\langle\tilde{P}\rangle$. We clearly have $V\subset V\langle\tilde{P}\rangle$. Consider the algebraic variety $\psi^{-1}[V]\subset S_3$. (By $\psi^{-1}[V]$ we mean the set-theoretic union of the subvarieties of S_3 which correspond, under ψ^{-1}, to the irreducible components of V/k'.) Then $R_T\in\psi^{-1}[V]$ for any T which is nonsingular and is defined over k. But S_3 is the smallest algebraic variety which contains all such points R_T. Hence $\psi^{-1}[V]=S_3$, and thus $V=V\langle\tilde{P}\rangle$. Applying Lemma 13 we obtain the desired conclusion $V\langle\tilde{P}\rangle\subset\pi^{-1}(P)$.

The variety $\pi^{-1}(P)/k(P)$ is irreducible. We next show that this variety is of dimension 3. Let $\tilde{Q}$ be a point in $S_m^0-H^0$, and let $Q=\pi(\tilde{Q})$. Then it is an immediate consequence of our condition (b) on π that $V\langle\tilde{Q}\rangle$ is an irreducible component of $\pi^{-1}(Q)$ over k. The dimension of any irreducible component of $\pi^{-1}(Q)$ is greater than or equal to the dimension of $\pi^{-1}(P)$. (This follows from a well known extension of the principle of counting constants.) Since $\dim V\langle\tilde{Q}\rangle=3$ it follows that $\dim \pi^{-1}(P)\leqq 3$. However, we have shown above that $\pi^{-1}(P)$ contains the 3-dimensional variety $V\langle\tilde{P}\rangle$. Hence $\dim \pi^{-1}(P)=3$. We thus conclude that $V\langle\tilde{P}\rangle$ is an absolutely irreducible component of $\pi^{-1}(P)$. Let V' be any other absolutely irreducible component of $\pi^{-1}(P)$, and let $\tilde{P}'$ be a general point of V' over the algebraic closure of $k(P)$. Then it follows from the argument just concluded that $V'=V\langle\tilde{P}'\rangle$. Hence any absolutely irreducible component of $\pi^{-1}(P)$ is of the form $V\langle\tilde{P}'\rangle$, where $\tilde{P}'$ is a general point of $\pi^{-1}(P)/k(P)$.

We are now in a position to show that $\pi^{-1}(P)$ is absolutely irreducible. Assume that this is false. Let $V\langle\tilde{P}_1\rangle$, $V\langle\tilde{P}_2\rangle$ be two distinct absolutely irreducible components of $\pi^{-1}(P)$. Consider the cycles Z_1, Z_2 in M such that $|Z_i|=V\langle\tilde{P}_i\rangle$, $i=1, 2$. We can specialize the pair of cycles (Z_1, Z_2), over the field k, to a pair of cycles (Z_1', Z_2') subject to the conditions $Z_1'\neq Z_2'$, $k(Z_1')=k(Z_2')=k$, and $Z_1'\notin W$, $Z_2'\notin W$. Here W is the subvariety of M which is described in Lemma 6. Next choose points $\tilde{Q}_i'\in|Z_i'|$, $i=1, 2$, such that each $\tilde{Q}_i'$ is the symmetric product of distinct points of S_1^0. Then $\tilde{Q}_1'$, $\tilde{Q}_2'$ are not projectively associated. Points $\tilde{A}_1$, $\tilde{A}_2$ can be chosen in $|Z_1|$, $|Z_2|$ respectively such that $(Z_1, Z_2, \tilde{A}_1, \tilde{A}_2)$ specializes, over the field k, to $(Z_1', Z_2', \tilde{Q}_1', \tilde{Q}_2')$. This specialization can be extended to include one of P. Let us suppose that P is specialized to a point Q in this extension. We have $|Z_1|\cup|Z_2|\subset\pi^{-1}(P)$, hence $\tilde{A}_i\in\pi^{-1}(P)$, $i=1, 2$. Therefore $\tilde{Q}_i'\in\pi^{-1}(Q)$, $i=1, 2$. This contradicts condition (b) which has been imposed on π, since $\tilde{Q}_1'$, $\tilde{Q}_2'$ belong to S_m^0 and are not projectively associated, although each of them is the symmetric product of distinct points on S_1. Hence we have reached a contradiction, and so $\pi^{-1}(P)$ is absolutely irreducible. Thus if $(\tilde{P}, P)$ is a general point pair of π/k we have shown that $\pi^{-1}(P)=V\langle\tilde{P}\rangle$.

Since $V\langle\tilde{P}\rangle$ is absolutely irreducible it follows that the field $k(P)$ is a quasi-maximally algebraic subfield of $k(\tilde{P}, P)$. On the other hand, since π is single valued the field $k(\tilde{P}, P)$ is a purely inseparable extension of $k(\tilde{P})$. In case our universal domain Ω is of characteristic 0 we could therefore conclude that $k(P)$ is a maximally algebraic subfield of $k(\tilde{P})$. This is not true if Ω is not of characteristic 0.

Let us return to the correspondence χ of S_m onto M. We have already shown that χ meets conditions (a) and (b). Let $(\tilde{P}, Z)$ be a general point pair of χ/k. We wish to show that $k(Z)$ is a maximally algebraic subfield of $k(\tilde{P})$, even in the case of nonzero characteristic. We see this as follows. Since $\tilde{P} \in |Z|$ it is a consequence of Lemma 8 that the function field of $|Z|/k_0(\tilde{P})$ is separably generated. But $k_0(Z)$ is the smallest field of definition for $|Z|$ over which the function field of $|Z|$ is separably generated. Hence $k_0(Z) \subset k_0(\tilde{P})$, and therefore $k(Z) \subset k(\tilde{P})$. Also $k(\tilde{P})$ is separably generated over $k(Z)$. But $k(Z)$ is quasi-maximally algebraic in $k(\tilde{P})$ (since $|Z|$ is absolutely irreducible). Hence $k(Z)$ is a maximally algebraic subfield of $k(\tilde{P})$.

We now impose an additional condition on our algebraic correspondence π:

(c) *$k(P)$ is a maximally algebraic subfield of $k(\tilde{P})$, where $(\tilde{P}, P)$ is a general point pair of π/k.*

χ is an example of a correspondence which satisfies conditions (a), (b) and (c). The only purpose of (c) is to get rid of purely inseparable extensions in the case of characteristic p.

Let π/k be any irreducible algebraic correspondence, of S_m onto N, which satisfies (a), (b), (c). Then we have $\pi^{-1}(P) = V\langle\tilde{P}\rangle$, where $(\tilde{P}, P)$ is a general point pair of π/k. On the other hand there is a general cycle Z of M/k such that $|Z| = V\langle\tilde{P}\rangle$. Consider the irreducible algebraic correspondence ψ/k defined over k by the general pair (P, Z). The domain of the correspondence ψ is N, while its range is M. Since $|Z| = \pi^{-1}(P)$, for any point P' in N the set-theoretic union of the cycles in $\psi(P')$ equals the variety $\pi^{-1}(P')$. Hence ψ is single-valued at any general point of N/k, and thus $k(P, Z)$ is a purely inseparable extension of $k(P)$. We next show that ψ^{-1} is single-valued at Z. For if $(P, Z) \rightarrow (Q, Z)$, over the field k, then there exists a point $\tilde{A} \in |Z|$ such that $(\tilde{A}, P, Z) \rightarrow (\tilde{P}, Q, Z)$, over the field k. Since $\tilde{A} \in |Z|$ we have $\pi(\tilde{A}) = P$. From the fact that $(\tilde{P}, Q)$ is a specialization of $(\tilde{A}, P)$ over the field k we deduce that $\pi(\tilde{P}) = Q$. But we already know that $\pi(\tilde{P}) = P$. Therefore $P = Q$, and so P is the only point of N which corresponds to Z under ψ^{-1}. Hence we also have shown that $k(P, Z)$ is a purely inseparable extension of $k(Z)$.

The situation now is as follows. Both of the fields $k(Z)$, $k(P)$ are maximally algebraic subfields of $k(\tilde{P})$, while every element in $k(Z)$ is algebraic over $k(P)$, and conversely. Therefore the fields $k(Z)$ and $k(P)$ are equal, and thus the varieties M/k and N/k are birationally equivalent. The correspondence ψ is a birational transformation of M/k onto N/k. We wish to investigate this correspondence ψ more closely. In order to do so we impose one more condition on the range N of π. Namely, we assume that the variety N/k is normal (we really only need assume that N/k is normal at points Q' of the type defined immediately below).

Let Q' be any point in N such that $Q'=\pi(\tilde{Q}')$, where $\tilde{Q}'$ is the symmetric product of m distinct points on S_1^0. We wish to consider the total transform $\psi(Q')$ of Q' on M under ψ. We have already remarked that the set-theoretic union of the cycles in $\psi(Q')$ equals $\pi^{-1}(Q')$. Let Z' be the unique cycle in M such that $|Z'| = V\langle\tilde{Q}'\rangle$. Since $V\langle\tilde{Q}'\rangle \subset \pi^{-1}(Q')$ we see that $Z' \in \psi(Q')$.

In order to avoid difficulties in terminology instead of thinking of ψ as a correspondence between the cycles of the algebraic system M and the points of the variety N we now consider the domain of ψ to be the representative points of the cycles of M. We now shall show that the representative point of the cycle Z' is an irreducible component of the variety $\psi(Q')/k$. For if Z'' is any other cycle whose representative point lies in $\psi(Q')$ then $|Z''|$ contains no point $\tilde{A}$ which is the symmetric product of distinct points on S_1. This follows from condition (b) on π, plus the fact that $|Z''| \subset \pi^{-1}(Q')$. Since Z'' contains no such point $\tilde{A}$, obviously Z' is not a specialization of Z'', over the field k. Hence we draw our desired conclusion. We shall use this result to show that Q' is not a fundamental point for the birational transformation ψ.

Notice that ψ^{-1} is single valued at the representative point of Z'. For if $(P, Z)\rightarrow(P', Z')$, over the field k, then there exists a point $\tilde{A} \in |Z|$ such that $(\tilde{A}, P, Z)\rightarrow(\tilde{Q}', P', Z')$, over the field k. Since $P \in \pi(\tilde{A})$ we see that $P' \in \pi(\tilde{Q}')$. But π is single valued at $\tilde{Q}'$, therefore $P'=Q'$. Hence ψ^{-1} is single valued at the representative point of Z', and therefore this latter point is not a fundamental point for ψ^{-1}. We have an irreducible component of $\psi(Q')$ which is of the same dimension as Q' (namely zero) and which is not a fundamental variety for ψ^{-1}. Further, Q' is a normal point of N/k. Hence it follows from Zariski's "Main Theorem" on birational transformations that Q' is not a fundamental point for ψ (Zariski [4, Corollary, p. 527]). Since Q' is a normal point of N/k, we see that ψ is semi-regular at Q', and therefore the representative point of Z' is the only point which corresponds to Q' under ψ. Recalling again that ψ^{-1} is single valued at this point, it follows that ψ^{-1} is a local normalization of the variety M/k at the representative point of Z'.

Consider our derived normal model $\overline{M}/k$ of M/k with general point $\overline{P}$ chosen so that the point pair $(Z, \overline{P})/k$ determines a birational transformation of M onto $\overline{M}$ which serves as a local normalization of M at every point of M. Let $\overline{W}$ be the total transform of W on $\overline{M}$ under this birational transformation. Then if we let $\bar{\psi}'$ be the birational transformation between $\overline{M}$ and N defined over k by the general point pair $(\overline{P}, P)$ this birational transformation $\bar{\psi}'$ is regular at all points $\overline{P}' \in \overline{M}^0$ such that $\overline{P}' \notin \overline{W}$. Hence the variety N is essentially uniquely determined. We also see that the set of points of N^0 which correspond under π^{-1} to points $\tilde{Q} \in S_m^0$ which are the symmetric product of distinct points of S_1 fill out the complement of an algebraic subvariety V^0 of N^0.

In order to summarize the results of this section in the form of a theorem we first make the following definition. H^0 will denote the subvariety of S_m^0

which consists of all points $\tilde{Q}'$ which are not the symmetric product of distinct points on S_1.

DEFINITION. *Let N^0/k be an algebraic variety which is the range of an irreducible algebraic correspondence π/k, of S_m^0 onto N^0, with the following properties:*

(1) *π is single valued at any point $\tilde{Q}' \in S_m^0 - H^0$ (H^0 is defined above),*

(2) *If $\tilde{Q}'$, $\tilde{Q}'' \in S_m^0 - H^0$ then $\pi(\tilde{Q}') = \pi(\tilde{Q}'')$ if and only if $\tilde{Q}'$, $\tilde{Q}''$ are projectively associated over k,*

(3) *If $(\tilde{P}, P)$ is a general point pair of π/k then $k(P)$ is a maximally algebraic subfield of $k(\tilde{P})$,*

(4) *N^0/k is normal at every point in the total transform $\pi(S_m^0 - H^0)$ of $S_m^0 - H^0$.*

If $m = 2g+2$ we then shall call N^0/k a moduli-variety for the hyperelliptic function fields, over k, of genus g. Any point $P' \in \pi(S_m^0 - H^0)$ will be said to represent the function field determined by any point $\tilde{Q}' \in S_m^0 - H^0$ such that $\pi(\tilde{Q}') = P'$.

We recall that for fields of characteristic zero condition (3) above is superfluous in the presence of conditions (1) and (2).

THEOREM 3. *Let g be any integer >1, and set $m = 2g+2$. Let $\tilde{P}$ be a general point of S_m/k, and let Z be the absolutely prime cycle defined by the variety $V\langle\tilde{P}\rangle$. Consider the algebraic system M defined over k by taking its general cycle to be Z. Then $\overline{M}^0/k$ is a moduli-variety for the hyperelliptic function fields of genus g over k, $\overline{M}$ being a derived normal model of M/k. Every point of $\overline{M}^0$ which does not lie in the proper algebraic subvariety $\overline{W}^0$ of $\overline{M}^0$ represents one of these function fields, and no point of $\overline{M}^0$ which represents such a function field lies in $\overline{W}^0$. Further if N^0/k is any moduli-variety for these function fields then the varieties N^0/k and $\overline{M}^0/k$ are birationally equivalent. The set of points of N^0 which represent function fields is the complement of an algebraic subvariety V^0 of N^0, and there exists a birational transformation ψ of N^0 onto $\overline{M}^0$ such that $\psi(V^0) = \overline{W}^0$, $\psi^{-1}(\overline{W}^0) = V^0$, ψ is regular at all points in $N^0 - V^0$, and points $P' \in N^0 - V^0$, $\overline{P}' \in \overline{M}^0 - \overline{W}^0$ correspond under ψ if and only if they represent the same function field.*

5. **Ramification in finitely valued algebraic correspondences.** In this section k will be any algebraically closed field. Let V/k, V'/k be two irreducible varieties, with general points P, P' and function fields $K = k(P)$, $K' = k(P')$. We assume that K' is a finite separable algebraic extension of K, of degree n. Then the irreducible algebraic correspondence ϕ defined over k by (P, P') as general point pair is a finitely (namely n) valued transformation of V onto V', while ϕ^{-1} is rational.

Let $Q \in V$, $Q' \in V'$ be corresponding nonfundamental points, under ϕ. Assume that Q is a normal point of V and that Q' is a normal point of V'. Let U/k, U'/k be the varieties whose respective general points are Q and Q'.

Then we shall say that Q' is ramified over Q if the subvariety U' of V' is ramified over the subvariety U of V, and is unramified in the contrary case. For the definition of ramification of subvarieties see Abhyankar [1, Definition 1 and also 2, §2].

In the above notation, let $Q_1', Q_2', \cdots, Q_t'$ be the points of V' corresponding to Q. Assume that each of the Q_i' is a nonfundamental, normal point. Then we say that Q is a branch point (of the given transformation) if at least one of the Q_i' is ramified over Q. In the contrary case Q is a nonbranch point. We shall need the following lemma.

Lemma 15. *Let ϕ be an irreducible n-valued algebraic correspondence of V/k onto V'/k such that ϕ^{-1} is rational. Let Q be a normal point of V which is not fundamental for ϕ. Then if $\phi\{Q\}=\{Q_1', \cdots, Q_t'\}$, where each Q_i' is normal and nonfundamental, a necessary and sufficient condition for Q to be a nonbranch point for ϕ is that ϕ be n-valued at Q (i.e., $t=n$).*

This lemma follows from the remarks preceeding Lemma 3 of Abhyankar [1, p. 577].

Lemma 16. *Let ϕ, Q, V, V' be as in Lemma 15. Let $Q'\in V'$ correspond to Q under ϕ, and assume that Q' is normal and nonfundamental. Further, assume that Q and Q' are rational over k, and that Q' is unramified over Q. Then if o_Q^*, $o_{Q'}^*$ are the completions of the local rings of Q, Q' on V, V' respectively, we have $o_Q^*=o_{Q'}^*$ (after the proper identifications have been made).*

This well known result is implicitly contained in Abhyankar [1; 2].

6. **Singular points on the moduli-variety for hyperelliptic curves of genus** g. We now return to the considerations of §4. The field k is restricted to be any algebraically closed field of characteristic different from 2. We wish to discuss the question of singular points on the moduli-variety for hyperelliptic curves of genus g. Naturally we are only interested in those points of the moduli-variety which represent hyperelliptic function fields of genus g. According to Theorem 3 we can restrict our attention to the variety $\overline{M}^0$, constructed for the value $m=2g+2$.

To discuss the singularity of points of $\overline{M}^0$ we look at the variety $\overline{M}/k$. We find it necessary to consider the irreducible algebraic correspondence ϕ/k defined immediately before the statement of Lemma 11. The domain of the correspondence ϕ is $\overline{M}$, while its range is the linear space L_{m-3}' of §4. If $(\overline{P}, \tilde{A})$ is a general point pair of ϕ/k, then by Lemma 9 the field $k(\tilde{A})$ is a separable algebraic extension of $k(\overline{P})$. Hence ϕ^{-1} is a rational correspondence of the type discussed in §5. The varieties $\overline{N}$, L_{m-3}' play the roles of V, V' respectively. Note that since L_{m-3}'/k is a linear variety it is certainly normal at each of its points.

Recall the birational transformation Ψ, of M/k onto $\overline{M}/k$, which serves as a local normalization of M/k at each of its points, and again let $\overline{W}=\Psi(W)$,

W being defined in Lemma 6. We also recall that dim $M/k=\dim \overline{M}/k=m-3=2g-1$.

We shall say that a point $\tilde{Q}'\in S_m$ possesses a nonidentical self-projective transformation if there exists a nonsingular projective transformation T of S_1 onto itself such that $\tilde{Q}'^T=\tilde{Q}'$, T not the identity.

LEMMA 17. *Let $\overline{Q}'$ be any point of $\overline{M}$, not lying in $\overline{W}$, such that $\Psi^{-1}(\overline{Q}')=Z'$, and $|Z'|=V\langle\tilde{Q}'\rangle$, where $\tilde{Q}'$ possesses a nonidentical self-projective transformation. If there does not exist a point $\overline{Q}''$ of $\overline{M}$ such that:*

(1) *tr.d. $k(\overline{Q}'')/k=m-4$,*

(2) *$\Psi^{-1}(\overline{Q}'')=Z''$, $|Z''|=V\langle\tilde{Q}''\rangle$, where $\tilde{Q}''$ possesses a nonidentical self-projective transformation,*

(3) *$\overline{Q}''\to\overline{Q}'$, over the field k,*

then $\overline{Q}'$ is a singular point of $\overline{M}/k$.

Lemma 15 and Lemma 12 imply that $\overline{Q}'$ is a branch point. The nonexistence of branch points satisfying conditions (1), (2), (3) implies, in view of Lemmas 12 and 15, that there is no $m-4$ dimensional irreducible subvariety of $\overline{M}$ which passes through $\overline{Q}'$ and which is a branch subvariety. Therefore by Theorem 1 of [1] $\overline{Q}'$ is singular.

LEMMA 18. *Let $\overline{Q}'$ be any point of $\overline{M}$, not lying in $\overline{W}$, such that $\Psi^{-1}(\overline{Q}')=Z'$, $|Z'|=V\langle\tilde{Q}'\rangle$ where $\tilde{Q}'$ does not possess any nonidentical self-projective transformations. Then $\tilde{Q}'$ is a simple point of $\overline{M}/k$.*

Proof. By Lemma 15, $\tilde{Q}'$ is a nonbranch point of ϕ. Hence we can find a nonbranch point $\tilde{Q}''$ of $\overline{M}$, rational over k and not lying on $\overline{W}$, such that $\tilde{Q}'\to\tilde{Q}''$, over the field k (Abhyankar [1, Lemma 3]). Let $\tilde{A}''$ be a point of L'_{m-3} corresponding to $\overline{Q}''$. By Lemma 16 we have $o^*_{Q''}=o^*_{A''}$. Therefore $\overline{Q}''$ is a simple point and hence so is $\overline{Q}'$.

LEMMA 19. *Let Z' be any cycle in M, not lying in W, such that $|Z'|=V\langle\tilde{Q}'\rangle$ where $\tilde{Q}'$ possesses a nonidentical self-projective transformation. Then tr.d. $k(Z')/k\leqq m-2$.*

Proof. We need examine only the dimension of the points $\tilde{A}/k\in S_m$ which possess nonidentical self-projectivities and represent sets of m distinct points on the line. Any such point $\tilde{A}$ is left invariant by a nonsingular projective transformation T of finite order $h\geqq 2$. Let $A=\tilde{A}_1\otimes A_2\otimes\cdots\otimes A_m$, $A_i\in S_1$, $i=1, 2, \cdots, m$. T has at most 2 fixed points on S_1, hence at most 2 of the A_i are fixed points for T. For any i such that A_i is not a fixed point for T, h is the smallest integer such that $A_i^{T^h}=A_i$. Hence for any i, $1\leqq i\leqq m$, such that A_i is not a fixed point of T, $\tilde{A}$ contains the h distinct points $A_i, A_i^T, \cdots, A_i^{T^{h-1}}$. In this way the points $A_1, \cdots, A_m$ are divided into subclasses by T, each subclass being of length h, except for at most 2 subclasses each of length 1.

Since T is of finite order the point R_T/k is of dimension at most 2. The fixed points of T are algebraic over the field $k(R_T)$. Hence we find

$$\text{tr.d. } k(\tilde{A})/k = \text{tr.d. } k(A_1 \times A_2 \times \cdots \times A_m)/k \leqq 2 + \frac{m}{h} \leqq 2 + \frac{m}{2} \cdot$$

If $\tilde{A} \in S_m$ possesses a nonidentical self-projectivity so does any point in S_m projectively associated to $\tilde{A}$. Hence the maximum possible value of tr.d. $k(Z')/k$ is obtained by subtracting 3 from the maximum found above for $\tilde{A}$. Hence tr.d. $k(Z')/k \leqq m/2-1$.

By considering involutions on S_1 it is easy to see that it is always possible to attain this maximum.

We now apply our results to the moduli-variety $\overline{M}^0$ of hyperelliptic curves of genus g. Every hyperelliptic function field possesses at least one nonidentical automorphism over k. Such a field possesses further k-automorphisms if and only if it is represented on $\overline{M}^0$ by a point $\overline{Q}'$ such that

$$\Psi^{-1}(\overline{Q}') = Z', \quad |Z'| = V\langle \tilde{Q}' \rangle,$$

where $\tilde{Q}'$ possesses a nonidentical self-projective transformation. Hence it follows from Lemma 18 that if $\overline{Q}' \in \overline{M}^0$ represents a hyperelliptic function field K/k which possesses the minimum number of k-automorphisms then $\overline{Q}'$ is a simple point of $\overline{M}^0$. On the other hand, if K/k possesses further k-automorphisms, and if $g>2$, then it follows from Lemmas 17 and 19 that $\overline{Q}'$ is a singular point of $\overline{M}^0$. We see this in the following manner. Since $m=2g+2$, it follows from Lemma 19 that if $\overline{Q}' \in \overline{M}$ is such that $\Psi^{-1}(\overline{Q}')=Z'$, $|Z'| = V\langle \tilde{Q}' \rangle$, where $\tilde{Q}'$ possesses a nonidentical self-projective transformation, then the point $\overline{Q}'/k$ is of dimension at most g. The integer $m-4$ which occurs in Lemma 17 equals $2g-2$. If $g>2$, then $g<2g-2$. Hence there exist no points $\overline{Q}''$ of $\overline{M}$ which meet conditions (1), (2) of Lemma 17. Thus $\overline{Q}'$ is a singular point of $\overline{M}/k$. We have proved the following theorem.

THEOREM 4. *Let $\overline{M}^0$ be the moduli-variety for hyperelliptic function fields K/k of genus g. If $\overline{Q}' \in \overline{M}^0$ represents a function field K/k which possesses only* 1 *nonidentical k-automorphism then $\overline{Q}'$ is a simple point of $\overline{M}^0/k$. If $g>2$, and $\overline{Q}'$ represents a function field K/k with further k-automorphisms then $\overline{Q}'$ is a singular point of $\overline{M}^0/k$.*

If $g=2$, and the universal domain is not of characteristic 5, we can exhibit a singular point on $\overline{M}^0$. It is the point $\overline{Q}$ which represents the function field $K=k(x, y)$, where (x, y) satisfies the equation $y^2=x^6-x$. It is easy to see that $\overline{Q}$ is an isolated point of the algebraic subvariety of $\overline{M}^0$ which represents function fields possessing more than the minimum number of k-automorphisms. This follows from the fact that the 6-tuple of roots of $x^6-x=0$ is left invariant by a cyclic group of projectivities of order 5. This group consists of the five projectivities $x \rightarrow \eta^i x$; $1 \leqq i \leqq 5$, where η is a primitive 5th root of unity. It can be shown by direct calculation that no other projective transformation leaves $x^6 - x$ invariant. Since this group has no proper subgroups it follows that $\bar{Q}$ is not a specialization of any other point $\bar{Q}'' \in \bar{M}$ which arises from a function field which has more than the minimum number of k-automorphisms. It therefore follows from Lemma 17 that $\bar{Q}$ is a singular point of $\bar{M}^0/k$. The methods used here cannot settle the question of whether there exist any other singular points on $\bar{M}^0$ (for $g=2$) which represent function fields.

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