PRIMAL IDEALS AND ISOLATED COMPONENTS IN NONCOMMUTATIVE RINGS[1]

BY

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Introduction. L. Fuchs [2] has given for Noetherian rings a theory of the representation of an ideal as an intersection of primal ideals, the theory being in many ways analogous to the classical Noether theory. An ideal Q is primal if the elements not prime to Q form an ideal, necessarily prime, called the adjoint of Q. Primary ideals are necessarily primal, but not conversely. Analogous results have been obtained by Curtis [1] for noncommutative rings with unit element, using a definition of right primal ideal which does not, however, reduce to that of Fuchs in a commutative ring. In this paper an alternative definition of right primal ideal in a general ring is given, which reduces to Fuchs' for commutative rings and to Curtis' for rings with unit element and ascending chain condition (A.C.C.) for ideals. This definition is based on a definition of "not right prime to A" which associates with any ideal A certain maximal not right prime to A ideals, analogous to Krull's maximal associated primes. These maximal not right prime to A ideals apparently are not necessarily prime unless a condition of "uniformity," which is weaker than the A.C.C., is imposed. In §3 a discussion of primal decompositions in rings without finiteness conditions is given, and in §4 the Fuchs-Curtis decomposition theorems are obtained for rings with A.C.C. for ideals. In §5 a new definition of the right associated primes of an ideal is given, and the maximal such ideals are determined. Following the methods of Murdoch [6], upper and lower right isolated B-components of an ideal A, where B is any divisor of A, are defined and their properties investigated in §6. If B is a maximal not right prime to A ideal, the isolated B-components of A are called upper and lower principal component ideals of A, and reduce to Krull's principal component ideals in commutative rings. It is shown in §7 that under certain conditions an ideal is the intersection of its upper principal components, and that any ideal in an associative ring is the intersection of its lower principal components.

1. **Notation and definitions.** We shall use R to denote an associative ring which will be noncommutative unless otherwise specified. The term ideal will always mean two-sided ideal. Proper ideals in R will be denoted by $A, B, \cdots$

Presented to the Society, June 19, 1954; received by the editors July 6, 1954.

[1] The material for this paper is taken from the author's Ph.D. thesis, written under the direction of Professor D. C. Murdoch at the University of British Columbia.

and elements of R by $a, b, x, y, \cdots$. The symbols $\sum A_\alpha$ and $A+B$ will denote ideal sums of ideals, and $\bigcup A_\alpha$ and $A \cup B$ will denote set theoretic unions. By $\{A \mid A$ has property $P\}$ and $\{x \mid x$ has property $P\}$ we shall mean respectively the set of ideals having property P and the set of elements having property P. The symbol (x) will mean the principal ideal generated by x and $(x, y, \cdots)$ the ideal generated by $x, y, \cdots$.

DEFINITION 1. $Ax^{-1} = \{y \mid yRx \subseteq A\}$ and $AB^{-1} = \{y \mid yRB \subseteq A\}$. Evidently both Ax^{-1} and AB^{-1} are ideals and both contain the ideal A.

DEFINITION 2. The element x is *not right prime* (nrp) to A if $Ax^{-1} \neq A$. Otherwise x is right prime (rp) to A.

DEFINITION 3. The ideal B is nrp to A if every b in B is nrp to A. Otherwise B is rp to A.

DEFINITION 4. The ideal A is *right primal* if $\{x \mid x$ is nrp to $A\}$ is an ideal, which is then termed the *adjoint ideal* of A. Since there are no ideals nrp to the ring R, we shall say that R is primal. As we shall consider only right primal ideals, we shall simply say "primal" instead of "right primal."

DEFINITION 5. An ideal B is *uniformly* nrp (unrp) to A if $AB^{-1} \neq A$, or equivalently if there exists some y not in A such that $yRB \subseteq A$.

DEFINITION 6. A ring is termed (right) *uniform* if for any two ideals A and B, if B is nrp to A it follows that B is unrp to A.

We note that B nrp to A in the sense of Curtis [1] implies B is unrp to A, if B is unrp to A then B is nrp to A, and if R has a unit element then B unrp to A implies B is nrp to A in the sense of Curtis.

Since the ideal sum of an ascending chain of ideals nrp to A is again nrp to A, Zorn's lemma assures the existence of ideals which are maximal in the inclusion ordered set of ideals nrp to A. Such an ideal will be termed a maximal nrp to A ideal, and thus A is primal if there is only one maximal nrp to A ideal.

If R is commutative, then x nrp to A is equivalent to x not prime to A, and thus our definition of primal agrees with that of Fuchs. If R has a unit element then A primal in the sense of Curtis implies A is primal, and if R is a uniform ring with unit element then A primal implies A is primal in the sense of Curtis. That the condition of uniformity is an essential one is shown by an example of Curtis' paper [1] of a nonuniform ring which has a primal ideal that is not primal in his sense.

2. **Primal ideals.**

DEFINITION 7. By a *prime* ideal we mean an ideal which is prime in the sense of McCoy [5], that is, P is prime if $xRy \subseteq P$ implies x or y is in P. McCoy has shown that this is equivalent to the property that if P divides the product of two ideals then P must divide at least one of them.

DEFINITION 8. A *maximal prime* of an ideal A is an ideal which is maximal in the inclusion ordered set of prime ideal divisors of A which are nrp to A.

We note that in the general case there may be no maximal primes of A

even if it happens that there are prime divisors of A which are nrp to A, since the union of an ascending chain of prime ideals is not necessarily prime.

LEMMA 1. *If B is maximal in the inclusion ordered set of ideals nrp to A and is unrp to A then B is a maximal prime of A.*

Proof. Let xRy be contained in B and y be not in B. Then since B is unrp to A there exists z not in A such that zRB is contained in A, hence $zRxRy$ is contained in A. Now B a maximal nrp to A ideal implies the existence of some $y' \in (y)+B$ such that y' is rp to A. But then $zRxRy'$ is contained in A, which implies zRx is contained in A. Thus we have $zR[(x)+B] \subseteq A$ and $(x)+B$ is nrp to A. Since B is maximal nrp to A it follows that x is in B as required.

DEFINITION 9. An ideal A is *strongly irreducible* if A cannot be expressed as an intersection, finite or infinite, of proper divisors. A is *irreducible* if A cannot be expressed as a finite intersection of proper divisors.

LEMMA 2. *If A is strongly irreducible then every ideal B nrp to A is unrp to A.*

Proof. Since B is nrp to A we have $A \subset Ab^{-1}$ for every $b \in B$. Hence $A \subseteq \bigcap_{b \in B} Ab^{-1}$, and since A is strongly irreducible the inclusion must be proper. Thus there exists $x \notin A$ such that $x \in Ab^{-1}$ for all $b \in B$. But then $xRB \subseteq A$ and B is unrp to A.

LEMMA 3. *Every irreducible ideal is primal. Every strongly irreducible ideal is primal with prime adjoint.*

Proof. Suppose b_1 and b_2 are nrp to an irreducible ideal A. Then $Ab_1^{-1} \supset A$ and $Ab_2^{-1} \supset A$, hence $A \subseteq Ab_1^{-1} \cap Ab_2^{-1}$, and since A is irreducible the inclusion must be proper. Hence there exists $x \notin A$ such that $x \in Ab_1^{-1} \cap Ab_2^{-1}$. But then $xR(b_1-b_2) \subseteq A$ and b_1-b_2 is nrp to A. Thus the set of elements nrp to A form an ideal and A is primal. If A is strongly irreducible then, as we have just seen, A is primal. Then by Lemma 2 the adjoint P of A is unrp to A, hence by Lemma 1, P is a prime ideal.

The following theorem follows at once from Theorem 1.4 of Curtis [1], and Lemma 3.

THEOREM 1. *Every ideal is the intersection of its strongly irreducible primal divisors.*

DEFINITION 10. If P is a prime ideal divisor of A, the (right) *upper isolated P-component* of A, $U(A, P)$, is the intersection of all ideals which contain A and are such that every element not in P is right prime to them. The upper isolated R-component of A is defined to be A.

This definition has been shown by Murdoch [6] to be equivalent to his definition, except for the case of the upper isolated R-component of A, which in his definition is the ring R. As may be readily verified, however, if the (right) lower isolated R-component of A is also defined to be A, then all results in Murdoch's paper [6] remain valid, the only changes being simplifications in certain theorems where particular cases no longer have to be considered.

LEMMA 4. *If A is primal with prime adjoint P, then $A = U(A, P)$, the upper isolated P-component of A.*

Proof. By definition $U(A, P)$ is the intersection of all ideals B such that $B \supseteq A$ and if $x \notin P$ then x is rp to B. But A is itself such an ideal and the result follows at once.

THEOREM 2. *Any ideal A is the intersection of its upper isolated P_α-components, where the P_α are the adjoints of the strongly irreducible primal divisors A_α of A.*

Proof. By Theorem 1 we have that A is the intersection of the A_α, and by Lemma 3 each P_α is prime. Thus the theorem is meaningful as stated.

By Lemma 4 we have $A_\alpha = U(A_\alpha, P_\alpha)$ and Murdoch has shown in [6] that if $P_\alpha \supseteq A_\alpha \supseteq A$, then $U(A_\alpha, P_\alpha) \supseteq U(A, P_\alpha) \supseteq A$. We thus obtain $A = \bigcap A_\alpha = \bigcap U(A_\alpha, P_\alpha) \supseteq \bigcap U(A, P_\alpha) \supseteq A$, and the equality follows.

3. Representations by primal ideals.

DEFINITION 11. A representation

$$A = A_1 \cap A_2 \cap \cdots \cap A_n \tag{1}$$

of an ideal A as the intersection of ideal divisors of A will be called *irredundant* if no A_i contains the intersection of the remaining ones, and *reduced* if no A_i can be replaced by a proper divisor.

LEMMA 5. *If* (1) *is a reduced representation of A by primal ideals A_i with prime adjoints P_i, then an ideal B is nrp to A if and only if B is contained in one of the P_i.*

Proof. (i) If B is nrp to A, for any $b \in B$ we may find $x_b \notin A$ such that $x_b R b \subseteq A \subseteq A_i$ for all i. But $x_b \notin A$ implies $x_b \notin A_i$ for some i, hence b is nrp to A_i and $b \in P_i$ since A_i is primal. Since b is arbitrary in B, we conclude that $B \subseteq P_1 \cup P_2 \cup \cdots \cup P_n$. We may suppose the indexing to be such that $B \subseteq P_1 \cup P_2 \cup \cdots \cup P_k$ but B is not contained in the union of any proper subset of $P_1, P_2, \cdots, P_k$. Then we may choose $p_i \in B \cap P_i$ such that $p_i \notin P_j$ for $j \neq i$, for each $i = 1, 2, \cdots, k$.

If $k = 2$, then $p_1 + p_2$ is in B, hence in either P_1 or P_2, either of which is contrary to the choice of p_1 and p_2. Thus $k \neq 2$.

If $k > 2$, then $p_1 R p_2 \not\subseteq P_k$ since neither of p_1, p_2 is in P_k and P_k is prime. Choose r_1 so that $p_1 r_1 p_2 = p_1' p_2 \notin P_k$. Then $p_1' p_2 R p_3 \not\subseteq P_k$ and we may choose r_2 so that $p_1' p_2 r_2 p_3 = p_1' p_2' p_3 \notin P_k$. Continuing in this way we obtain $p = p_1' p_2'$

$\cdots p_{k-1}\notin P_k$. But $b=p+p_k\in B$, and b must be in some P_i for $i\leqq k$. If $i<k$ then $p_k\in P_i$ contrary to assumption, and if $i=k$ then $p\in P_k$ contrary to assumption. Hence $k\not> 2$ and we conclude that $k=1$, and thus $B\subseteq P_1$.

(ii) For $p_1\in P_1$, let y be such that $y\notin A_1$ and $yRp_1\subseteq A_1$. Then $A_1'=(y)+A_1\supset A_1$. Since (1) is reduced, there must exist $y'\in A_1'\cap A_2\cap\cdots\cap A_n$ and such that $y'\notin A$. But $y'Rp_1\subseteq(y)Rp_1+A_1Rp_1\subseteq A_1$, hence $y'Rp_1\subseteq A$ and p_1 is nrp to A. But since p_1 is arbitrary in P_1 we conclude that P_1 is nrp to A, and similarly the other P_i are nrp to A.

THEOREM 3. *If* (1) *is a reduced representation of A by primal ideals with prime adjoints P_i, then the maximal nrp to A ideals are the maximal primes of A and are in fact the maximal elements of the inclusion ordered set $P_1, P_2, \cdots, P_n$.*

Proof. Suppose P is maximal nrp to A, i.e., P is nrp to A and if $Q\supset P$ then Q is rp to A. By Lemma 5, P is contained in some P_i. But P_i is nrp to A for all i by the other half of Lemma 5, hence the maximality of P implies that $P=P_i$ for some P_i which is maximal in the set $P_1, P_2, \cdots, P_n$. Thus P is prime and hence a maximal prime of A. If conversely P_j is maximal in $P_1, P_2, \cdots, P_n$, then P_j must be a maximal nrp to A ideal, for if not there exists $Q\supset P_j$ and such that Q is nrp to A. But then by Lemma 5 we would have Q contained in some P_i and $P_j\subset Q\subseteq P_i$ contrary to the maximality of P_j.

LEMMA 6. *Let* (1) *be a reduced representation of A by primal ideals with prime adjoints P_i. Then A is primal if and only if one P_j divides all the others, in which case P_j is the adjoint of A.*

Proof. (i) Let $P_j\supseteq P_i$ for all i, so that $P_j=P_1\cup P_2\cup\cdots\cup P_n$. Then by Lemma 5, B nrp to A implies that $B\subseteq P_j$ and since by Lemma 5 again P_j is nrp to A, it follows that A is primal with adjoint P_j.

(ii) Let A be primal with adjoint P. Then since P is nrp to A, $P\subseteq P_j$ for some j by Lemma 5. Also by Lemma 5, P_i is nrp to A, hence $P_i\subseteq P$ for all i, or $P_i\subseteq P\subseteq P_j$ for all i. Then $P=P_j$ and the lemma is proved.

DEFINITION 12. If (1) is an irredundant representation of A by primal ideals A_i, and is such that $A_i\cap A_j$ is not primal if $i\neq j$, it will be called a *short representation* of A by primal ideals.

THEOREM 4. *Let* (1) *be a reduced representation of A by primal ideals with prime adjoints P_i. Then A has a short representation by primal ideals whose adjoints are the maximal primes of A.*

Proof. Since (1) is reduced we may assume that the representation is irredundant, since the intersection of any subset of $A_1, A_2, \cdots, A_n$ is also reduced. Let the indexing be such that $P_1, P_2, \cdots, P_r$ are the maximal elements of the set $P_1, P_2, \cdots, P_n$. Let A_1' denote the intersection of those

A_i such that $P_i \subseteq P_1$, and let A_j' denote the intersection of those A_i such that $P_i \subseteq P_j$ but $P_i \not\subseteq P_k$ if $k<j$, $j=2, 3, \cdots, r$. Each of $A_1', A_2', \cdots, A_r'$ satisfies the conditions of Lemma 6 and hence they are all primal with prime adjoints $P_1, P_2, \cdots, P_r$. Now for $j \neq k$, the intersection of the A_i forming A_j' and A_k' is a reduced intersection of primal ideals not all of whose adjoints are contained in any one adjoint, hence by Lemma 6, $A_j' \cap A_k'$ is not primal and the representation $A = A_1' \cap A_2' \cap \cdots \cap A_r'$ is short. By Theorem 3, $P_1, P_2, \cdots, P_r$ are the maximal primes of A, and the theorem is proved.

COROLLARY. *If* (1) *is a representation of A by strongly irreducible ideals, then A has a short representation by primal ideals whose adjoints are the maximal primes of A.*

Proof. By Lemma 3 each A_i is primal with prime adjoint. We may assume the representation is irredundant. Since the A_i are irreducible, Lemma II of E. Noether's paper [4], which remains valid in general rings, assures that the representation is reduced. The result now follows from the theorem.

THEOREM 5. *In any short reduced representation of A by primal ideals with prime adjoints, the adjoints and the number of primal components are uniquely determined.*

Proof. Let $A = A_1 \cap A_2 \cap \cdots \cap A_n$, where P_i is the adjoint of A_i and $A = A_1' \cap A_2' \cap \cdots \cap A_m'$, where P_j' is the adjoint of A_j' be any two such representations of A. Since both representations are short, no P_i properly contains another P_j and no P_i' properly contains another P_j'. Then by Theorem 3 both $P_1, P_2, \cdots, P_n$ and $P_1', P_2', \cdots, P_m'$ are the set of maximal primes of A, hence $m=n$ and the P_i are the P_j' in some order.

By Lemma 1, every primal ideal in a uniform ring has a prime adjoint. Thus for uniform rings the results of this section are valid without the stipulation that the primal ideals in question have prime adjoints.

For later use we include here the following lemma.

LEMMA 7. *Let* (1) *be a reduced representation of A by primal ideals such that the adjoint Q_i of A_i is unrp to A_i. Then an ideal is nrp to A if and only if it is contained in some Q_i, and every ideal nrp to A is unrp to A.*

Proof. Since Q_1 is unrp to A_1, there exists $x \notin A_1$ such that $xRQ_1 \subseteq A_1$. Let $A_1' = (x) + A_1$, and we have $A_1' \cap A_2 \cap \cdots \cap A_n \supset A$ since (1) is reduced and $A_1' \supset A_1$. Then there exists $y \notin A$ such that $y \in A_1' \cap A_2 \cap \cdots \cap A_n$. Now $yRQ_1 \subseteq (x)RQ_1 + A_1RQ_1 \subseteq A_1$, and it follows that $yRQ_1 \subseteq A$. Thus Q_1 is unrp to A, and similarly each Q_i is unrp to A.

By Lemma 1 the Q_i are all prime, and thus it follows from Lemma 5 that an ideal B is nrp to A if and only if B is contained in some Q_i. But since each Q_i is unrp to A this implies that every ideal nrp to A is unrp to A.

4. **Ascending chain condition.**

Throughout this section we shall assume that the ring R satisfies the ascending chain condition for ideals.

THEOREM 6. *If A is an irreducible ideal and B is nrp to A, then B is unrp to A.*

Proof. The ascending chain condition implies that B has a finite basis, or $B=(b_1, b_2, \cdots, b_n)$. Now $(Ab_1^{-1}\cap Ab_2^{-1}\cap \cdots \cap Ab_n^{-1})RB = AB^{-1}RB\subseteq A$. For each i, $Ab_i^{-1}\supset A$, hence $Ab_1^{-1}\cap Ab_2^{-1}\cap \cdots \cap Ab_n^{-1}\supseteq A$, and equality is impossible trivially if $n=1$, and if $n>1$ since A is irreducible. Thus there exists $x\in Ab_1^{-1}\cap Ab_2^{-1}\cap \cdots \cap Ab_n^{-1}$ such that $x\notin A$. For this x we have $xRB\subseteq A$, and hence B is unrp to A.

THEOREM 7. *Every ideal A has a short reduced representation by primal ideals whose adjoints are the maximal primes of A and are unrp to A.*

Proof. The A.C.C. implies that A has a finite representation, which we may assume to be irredundant, by irreducible ideals $A_1, A_2, \cdots, A_n$. By Lemma II of E. Noether's paper [4] which, together with Lemma IV of the same paper, remains valid in the noncommutative case, the representation is reduced. Then by Lemma 3 each A_i is primal, hence by Theorem 6 the adjoint P_i of A_i is unrp to A_i. Then each P_i is prime by Lemma 1, and by Lemma 7 is unrp to A. Suppose $P_1, P_2, \cdots, P_m$ are the maximal elements of the inclusion ordered set $P_1, P_2, \cdots, P_n$. Let A_j' be the intersection of those A_i whose adjoints P_i are divisible by P_j but not by P_k for $k<j$, $j=1, 2, \cdots, m$. By Lemma 6 each A_j' is primal. If we replace the A_i composing A_1' by A_1', E. Noether's Lemma IV assures that the representation remains reduced, and similarly for $A_2', A_3', \cdots, A_m'$. By Lemma 6 the resulting representation, $A=A_1'\cap A_2'\cap \cdots \cap A_m'$, is short and by Theorem 3 the maximal primes of A are exactly $P_1, P_2, \cdots, P_m$.

COROLLARY 1. *Every ideal A has a finite set of maximal primes, which are the maximal nrp to A ideals.*

Proof. The result is an immediate consequence of Theorem 7 and Theorem 3.

COROLLARY 2. *If A is primal the adjoint of A is a prime ideal.*

THEOREM 8. *Let $A=A_1'\cap A_2'\cap \cdots \cap A_m'$ be the short reduced representation of Theorem 7, and $A_i=\{\cap B_\alpha \mid B_\alpha\supseteq A,\ B_\alpha$ primal with adjoint $Q_\alpha\subseteq P_i\}$. Then A_i is primal with adjoint P_i and is the minimal primal divisor of A whose adjoint is contained in P_i. Also, $A=A_1\cap A_2\cap \cdots \cap A_m$ is a short representation of A.*

Proof. (i) There exists $x_i\notin A_i'$ such that $x_iRP_i\subseteq A_i'$. Then there exists $x_i'\notin A$ such that $x_i'\in A_1'\cap \cdots \cap A_{i-1}'\cap (x_i, A_i')\cap A_{i+1}'\cap \cdots \cap A_m'$ since

$A=A_1'\cap A_2'\cap\cdots\cap A_m'$ is reduced. Thus $x_i'\notin A_i'$, $x_i'\notin A_i$. But

$$x_i'RP_i\subseteq A_1'\cap\cdots\cap A_{i-1}'\cap[(x_i)RP_i+A_i'RP_i]\cap A_{i+1}'\cap\cdots\cap A_m'=A.$$

Hence $x_i'RP_i\subseteq A_i$ while $x_i'\notin A_i$ and P_i is nrp to A_i. If y is nrp to A_i, then $xRy\subseteq A_i$ for some $x\notin A_i$. Then $x\notin B$ for some $B\supseteq A$, B primal with adjoint $Q\subseteq P_i$, and $xRy\subseteq B$ implies $y\in Q$, hence $y\in P_i$. Thus P_i is the adjoint of A_i, and our first assertion is proved.

(ii) Clearly $A=A_1\cap A_2\cap\cdots\cap A_m$. If $m>1$, suppose $A=A_2\cap\cdots\cap A_m$. Now there exists $x\notin A$ such that $xRP_1\subseteq A$, hence P_1 is nrp to one of A_2, A_3, $\cdots$, A_m. Thus P_1 is divisible by one of $P_2,\cdots,P_m$, a contradiction, and the representation must be irredundant. If the representation is not short, suppose $A_1\cap A_2$ is primal with adjoint P. Then $xRP\subseteq A_1\cap A_2$ for some $x\notin A_1\cap A_2$. Say $x\notin A_1$. Then P is nrp to A_1, hence $P\subseteq P_1$ and we have $A_1\cap A_2\supseteq A_1$ by definition of A_1. But this implies $A_1=A_1\cap A_2$ and the representation is redundant, which is impossible.

THEOREM 9. *Every ideal is the intersection of its upper isolated components* $U(A,P_1)$, $U(A,P_2)$, $\cdots$, $U(A,P_n)$ *where* $P_1,P_2,\cdots,P_n$ *are the maximal primes of* A.

Proof. By Theorem 7, A has a representation $A=A_1\cap A_2\cap\cdots\cap A_n$ where A_i is primal with adjoint P_i. As in the proof of Theorem 2, we have $A\subseteq U(A,P_i)\subseteq U(A_i,P_i)=A_i$. Hence it follows that $A\subseteq\bigcap_{i=1}^n U(A,P_i)\subseteq\bigcap_{i=1}^n A_i=A$, and the theorem is proved.

THEOREM 10. *Any ring satisfying the A.C.C. for ideals is a uniform ring.*

Proof. If A is an ideal in a ring with A.C.C., let B be any ideal nrp to A. Then B is contained in a maximal nrp to A ideal P which by Theorem 7, Corollary 1 is a maximal prime of A, and by Theorem 7 is unrp to A. Thus there exists $x\notin A$ such that $xRP\subseteq A$, hence $xRB\subseteq xRP\subseteq A$ and B is unrp to A.

That a uniform ring need not satisfy the A.C.C. can be shown by the following example.

Let F be a field with a valuation Φ such that the value group of F is the rational numbers. Let R be the ring of all $f\in F$ for which $\Phi(f)\geqq 1$. It may be readily shown that if A is a proper ideal in R and B is any ideal of R we can always find $x\in R$ such that $x\notin A$ while $xB\subseteq A$, hence $xRB\subseteq A$ and B is unrp to A. But the A.C.C. clearly does not hold in R. We may remark that this is an example of a ring in which every proper ideal is primal with adjoint R.

Murdoch [6] has defined an ideal Q to be right primary if every element not in the McCoy radical [5] of Q is rp to Q.

LEMMA 8. *If* Q *is right primary with radical* P, *then* Q *is primal with prime adjoint* P.

Proof. By a result of Murdoch [6] P is nrp to Q. By definition all elements not in P are rp to Q, hence the set of all elements nrp to Q is exactly P and Q is primal with adjoint P. By Corollary 2, Theorem 7, P is prime.

THEOREM 11. *Let* $A = Q_1 \cap Q_2 \cap \cdots \cap Q_n$ *be an irredundant representation of* A *by right primary ideals with radicals* $P_1, P_2, \cdots, P_n$. *Then the maximal elements of the set* $P_1, P_2, \cdots, P_n$ *are the maximal primes of* A.

Proof. Murdoch has shown in [6] that an element x is rp to A if and only if x is in the complement of every P_i. Hence each P_i is nrp to A and if B is nrp to A then $B \subseteq P_1 \cup P_2 \cup \cdots \cup P_n$. By Lemma 8 each P_i is prime. We now repeat the argument used in part (i) of the proof of Lemma 5, and obtain $B \subseteq P_i$ for some i. The argument now proceeds exactly as in the proof of Theorem 3.

5. **Maximal primes and associated primes.** If a minimal prime ideal of A is defined to be an ideal which is minimal in the inclusion ordered set of prime divisors of A, then the intersection of the minimal primes of A has been shown in [5] to be the McCoy radical of A.

For commutative rings Fuchs has characterized in [3] the intersection of the maximal primes of an ideal. In the case of ideals which possess reduced representations by primal ideals with prime adjoints it is possible to extend Fuchs' result to noncommutative rings. As we have seen, this condition is satisfied for any ideal in a ring with A.C.C.

DEFINITION 13. If A is any ideal in a general ring, the *adjoint ideal* of A is defined to be the set of all x such that (x, y) is nrp to A whenever y is nrp to A. That this set does form an ideal is easily shown. In virtue of the fact that if R is commutative then x nrp to A is equivalent to x not prime to A, we see that the above definition is the same as Fuchs' [3] in the commutative case.

We note that A is trivially contained in the adjoint of A. In the event that A is primal, then the adjoint Q of A defined previously coincides with the adjoint Q' of Definition 13.

THEOREM 12. *The adjoint ideal* Q' *of* A *is the intersection of the maximal nrp to* A *divisors of* A.

Proof. Suppose $x \in Q'$ and B is any ideal nrp to A. Then $(x)+B$ is nrp to A since $y \in (x)+B$ implies $y \in (x, b)$ for some $b \in B$ and thus y is nrp to A by the definition of Q'. Hence Q' is contained in every maximal nrp to A ideal. Conversely, let x be in every maximal nrp to A ideal and y be nrp to A. Then y is in some maximal nrp to A ideal B, $(x, y) \subseteq B$ and (x, y) is nrp to A so that $x \in Q'$ and the theorem is proved.

THEOREM 13. *If* A *has a reduced representation by primal ideals with prime adjoints, then the adjoint of* A *is the intersection of the maximal primes of* A.

Proof. By Theorem 3 the maximal nrp to A ideals are precisely the maximal primes of A, namely the maximal elements of the set of adjoint primes of the primal ideals in the given representation of A. The result now follows at once from Theorem 12.

McCoy [5] has noted that any prime divisor of A contains a minimal prime of A. Thus it follows that the adjoint of an ideal possessing a reduced representation by primal ideals with prime adjoints always contains the McCoy radical of A.

We turn now to a consideration of the prime ideals "associated" with a given ideal. Such ideals have been defined by Krull [4] for noncommutative rings. Since our point of view is considerably different from that of Krull, however, we shall give a new definition which is derived from the method of Murdoch [6]. He has shown that if an ideal A in a ring with A.C.C. has a short representation by right primary ideals with radicals $P_1, P_2, \cdots, P_n$, then P_i is nrp to A, and a prime P which divides A is nrp to $U(A, P)$ if and only if P is one of $P_1, P_2, \cdots, P_n$.

DEFINITION 14. A prime ideal P containing A is a (right) *associated prime* of A if P is nrp to A and also nrp to $U(A, P)$.

Thus in a ring with A.C.C., if an ideal A has a short representation by right primary ideals, their radicals are exactly the associated primes of A.

LEMMA 9. *Let A be an ideal with a short reduced representation by primal ideals $A_1, A_2, \cdots, A_n$ whose adjoints $P_1, P_2, \cdots, P_n$ are such that P_i is unrp to A_i. Then P_i is unrp to $U(A, P_i)$ for $i=1, 2, \cdots, n$.*

Proof. By Lemma 1, each P_i is a prime ideal. By Lemma 7 each P_i is unrp to A. Hence there exists $x_i \notin A$ such that $x_iRP_i \subseteq A$ and thus $x_iRP_i \subseteq U(A, P_j)$ for every j. Now $x_i \notin A$ implies $x_i \notin A_j$ for some j. Then $x_iRP_i \subseteq A_j$ implies $P_i \subseteq P_j$ but since P_i and P_j are both maximal this implies $P_i = P_j$ and $i = j$. From $A_i = U(A_i, P_i) \supseteq U(A, P_i)$ it follows that $x_i \notin U(A, P_i)$ and P_i is unrp to $U(A, P_i)$.

THEOREM 14. *In a uniform ring, if an ideal A has a short reduced representation by primal ideals with adjoints $P_1, P_2, \cdots, P_n$, then a prime ideal P is a maximal prime of A if and only if P is a maximal element in the inclusion ordered set of associated primes of A.*

Proof. By Lemma 9 each P_i is an associated prime of A, and by Theorem 3 the P_i are the maximal nrp to A ideals, hence the maximal primes of A. Clearly every associated prime of A is contained in a maximal nrp to A ideal, and hence in some P_i. Thus if P is a maximal associated prime of A it must be one of the P_i, hence a maximal prime of A. Conversely, every maximal prime of A is one of the P_i and hence a maximal associated prime of A.

COROLLARY. *If A is an ideal in a ring with the A.C.C. for ideals, then the maximal primes of A are the maximal associated primes of A.*

Proof. By Theorem 7 every ideal A has a short reduced representation by primal ideals, and by Theorem 10, R is a uniform ring. The result now follows at once from Theorem 14.

We note that by the corollary the representations of an ideal in a ring with A.C.C. obtained in Theorems 8 and 9 are identical. For if P is a maximal prime of A, then $U(A, P)$ is primal with adjoint P and by definition is contained in every primal divisor of A whose adjoint is contained in P, hence is the minimal such divisor of A.

As we noted in §1, if R is a uniform ring with unit element then the definition of B nrp to A is equivalent to that of Curtis [1]. Hence the respective definitions of the maximal primes of A are also equivalent in such a ring. For a ring with unit element satisfying the A.C.C. for ideals, Curtis defined the (right) isolated B-component ideal of A to be the ideal $I(A, B)=AB^{-q}$ for $q\geqq 0$ and such that $AB^{-q}=AB^{-q-1}$, and a (right) associated prime ideal of A to be a prime ideal P such that $I(A, P)\supset A$ and $[I(A, P)]^{-1}A\subseteq P$ where $[I(A, P)]^{-1}A=\{\sum C \mid I(A, P)C\subseteq A\}$. He then proved that the maximal primes of A are the maximal elements of the inclusion ordered set of (right) associated prime ideals of A. Thus we see that for a ring with unit element and A.C.C. for ideals, the two definitions of associated prime ideals of an ideal A both lead to the same set of maximal associated prime ideals.

6. **Isolated components of an ideal.** McCoy has defined in [5] an m-system to be a set M of elements of R with the property that if x and y are in M then there exists r in R such that xry is in M. The null set is also considered an m-system. Thus an ideal is prime if and only if its complement is an m-system. Murdoch has defined [6] a right M-n-system to be a set N containing an m-system M and with the property that for given m in M and n in N there exists r in R such that nrm is in N. If M is the null set then the only M-n-system is M itself. He then defined the (right) upper isolated M-component of an ideal A not intersecting M to be the set of elements x such that every right M-n-system containing x also contains an element of A. We shall adopt these definitions with the exception that if M is the null set then any set in R is a right M-n-system, a change which results in our Definition 10 of the (right) isolated P-component of A where P is a prime divisor of A. We now define the (right) upper isolated B-component, $U(A, B)$, where B is any ideal divisor of A. We shall call a set M *entirely rp* (erp) to A if every element in M is rp to A.

DEFINITION 15. If $B\supseteq A$, then the (right) *upper isolated B-component* of A, $U(A, B)$, is the intersection of all ideal divisors of A to which the set M of elements rp to B is erp.

DEFINITION 16. Let B be a (proper) ideal in R and M be the set of elements in R which are rp to B. If M is non-null a set N containing M is a (right) *B-ν-system* if for every $m\in M$ and $n\in N$ there exists some $r\in R$ such that $nrm\in N$. If M is null (R is nrp to B) then any set is a B-ν-system. Similarly, any set is considered to be an R-ν-system.

We note that $C(B)$, the complement in R of B, is a B-ν-system. For if M is non-null then $m \in M$ and $xRm \subseteq B$ implies $x \in B$, hence $m \in M$ and $x \in C(B)$ implies the existence of some $r \in R$ such that $xrm \in C(B)$, while if M is null then $C(B)$ is a B-ν-system by definition.

LEMMA 10. *Let N be any B-ν-system disjoint from A. Then A is contained in a maximal ideal Q disjoint from N and M is erp to Q.*

Proof. Since the union of any ascending chain of ideals containing A and disjoint from N is again an ideal containing A and disjoint from N the existence of Q follows at once from Zorn's lemma. If $y \notin Q$ then, by the maximality of Q, there exists $n \in N$ such that $n \in (y)+Q$. Then n has the form $n = iy + ry + yr' + \sum_{ij} r_i y r_j + q$, for r, r', r_i, r_j all in R and q in Q. Now if m is rp to B there exists $x \in R$ such that $nxm \in N$, or $nxm = iyxm + ryxm + yr'xm + \sum_{ij} r_i y r_j xm + qxm$ is in N. But if $yRm \subseteq Q$ then $nxm \in Q \cap N$, contradicting Q disjoint from N. Hence $yRm \subseteq Q$ implies $y \in Q$ and m is rp to Q as required.

THEOREM 15. *The complement in R of $U(A, B)$ is the maximal B-ν-system disjoint from A, and $U(A, B)$ itself is the set consisting of all x such that every B-ν-system containing x intersects A.*

Proof. Let N be the complement of $U(A, B)$. Then $x \in N$ implies that $x \notin Q$ for some ideal $Q \supseteq A$ and such that M is erp to Q. Hence for $m \in M$ there exists $r \in R$ such that $xrm \in N$. Thus N is a B-ν-system, and is trivially disjoint from A. Suppose N' is any B-ν-system which is disjoint from A. Then by Lemma 10 we have $A \subseteq Q'$ for Q' an ideal disjoint from N' and such that M is erp to Q'. But then $U(A, B) \subseteq Q'$ by definition, hence $N \supseteq C(Q') \supseteq N'$. Thus N is the maximal B-ν-system disjoint from A. The second assertion follows at once from the first, and the theorem is proved.

COROLLARY. *If $B \supseteq A \supseteq A'$, then $U(A, B) \supseteq U(A', B)$.*

Proof. Every ideal J containing A to which M is erp also contains A', and hence the intersection of all such ideals containing A contains the intersection of all such ideals containing A'.

We may remark that the last property in Theorem 15 could have been used to define $U(A, B)$, and the others derived therefrom. Such a method would have been more conventional, the proof proceeding by way of the standard three lemmas (cf. e.g. McCoy [5] or Murdoch [6]), our versions of which would have read as follows: (i) if $R \supseteq B \supseteq A$ then there exists a unique maximal B-ν-system disjoint from A, (ii) same as Lemma 10, and (iii) a set Q is a minimal ideal dividing A such that M is erp to Q if and only if $C(Q)$ is a maximal B-ν-system disjoint from A.

In the event P is a prime ideal divisor of A, then the ideal $U(A, P)$ is the upper isolated P-component of A of Definition 12, as in this case the set of elements rp to P is just the complement of P.

Murdoch has defined in [6] the right lower isolated component of an ideal A, relative to an m-system M disjoint from A, to be the set of all elements x of R such that $xRm \subseteq A$ for some m of M. Thus if P is a prime divisor of A the right lower isolated component of A relative to P is the set $\{x \mid xRm \subseteq A$ for some $m \notin P\}$. We now extend this concept to the case of B any ideal divisor of A, and establish relations between the upper and lower isolated B-components of A corresponding to those obtained by Murdoch.

Definition 17. If B is an ideal containing A, let M be the set of all m in R which are rp to B. If M is non-null the (right) *lower isolated B-component* of A is the ideal $L(A, B) = \{ \sum Am^{-1} \mid m \in M\}$. If M is null then $L(A, B)$ is A, and $L(A, R)$ is also A.

We note that if P is a prime ideal different from R this definition agrees with that of Murdoch, since then the set M is an m-system.

Lemma 11. *If* $R \supseteq B \supseteq A$ *then* $U(A, B) \supseteq L(A, B) \supseteq A$.

Proof. By definition $L(A, R) = A = U(A, R)$, and if M is null then $L(A, B) = A = U(A, B)$ also by definition.

Suppose M is non-null and $R \neq B$. That $L(A, B) \supseteq A$ follows at once from the fact that $Am^{-1} \supseteq A$ for any $m \in M$. If $x \in L(A, B)$ then $x = \sum_{i=1}^{n} x_i$ for $x_i \in Am_i^{-1}$, $m_i \in M$. Now $x_i \in Am_i^{-1}$ implies $x_iRm_i \subseteq A$ and every B-ν-system containing x_i certainly contains an element of A, hence $x_i \in U(A, B)$ for $i = 1, 2, \cdots, n$. Thus $x = \sum_{i=1}^{n} x_i$ is in $U(A, B)$ and $L(A, B) \subseteq U(A, B)$.

Theorem 16.

$$\begin{aligned} &\text{(a)} \quad U(U(A, B), B) = U(A, B), \\ &\text{(b)} \quad L(U(A, B), B) = U(A, B), \\ &\text{(c)} \quad U(L(A, B), B) = U(A, B). \end{aligned}$$

Proof. (a) The complement in R of $U(A, B)$ is a B-ν-system N not intersecting $U(A, B)$ by Theorem 15. Hence it is certainly the maximal such system, and it follows from the same theorem that $C(N) = U(A, B)$ is the upper isolated B-component of $U(A, B)$.

(b) By (a) just proved and Lemma 11 we obtain

$$L(U(A, B), B) \subseteq U(U(A, B), B) = U(A, B) \subseteq L(U(A, B), B)$$

and the desired equality follows.

(c) Since $L(A, B) \subseteq U(A, B)$ we may apply the corollary to Theorem 15 to obtain $U(L(A, B), B) \subseteq U(U(A, B), B)$. Since $A \subseteq L(A, B)$ we apply the same corollary to obtain $U(A, B) \subseteq U(L(A, B), B)$. Combining these results with part (a) we have $U(L(A, B), B) \subseteq U(U(A, B), B) = U(A, B) \subseteq U(L(A, B), B)$ and the equality follows.

Definition 18. For all ordinal numbers α we define $L^{\alpha}(A, B)$ by induction as follows: $L^{1}(A, B) = L(A, B)$. If α is not a limit ordinal then $L^{\alpha}(A, B)$

$=L(L^{\alpha-1}(A, B), B)$, and if α is a limit ordinal then $L^{\alpha}(A, B)$ is the union of all $L^{\beta}(A, B)$ for $\beta<\alpha$.

Evidently if $\beta<\alpha$ then $L^{\beta}(A, B)\subseteq L^{\alpha}(A, B)$.

THEOREM 17. *For all ordinal numbers* α, $U(A, B)\supseteq L^{\alpha}(A, B)$.

Proof. For $\alpha=1$ the result is known by Lemma 11. We assume the result for all ordinals less than α and proceed by induction.

(i) If α is not a limit ordinal and so has an immediate predecessor $\alpha-1$ then we have

$$\begin{aligned} L^{\alpha}(A, B) &= L(L^{\alpha-1}(A, B), B) && \text{by definition,} \\ &\subseteq U(L^{\alpha-1}(A, B), B) && \text{by Lemma 11,} \\ &\subseteq U(U(A, B), B) && \text{by Theorem 15, Corollary,} \\ &= U(A, B) && \text{by Theorem 16(a).} \end{aligned}$$

(ii) If α is a limit ordinal, then by the definition of $L^{\alpha}(A, B)$ we have that $x\in L^{\alpha}(A, B)$ implies $x\in L^{\beta}(A, B)$ for some $\beta<\alpha$. But then by the inductive assumption $x\in U(A, B)$ and hence $L^{\alpha}(A, B)\subseteq U(A, B)$.

THEOREM 18. *For any ordinal number* α, $L^{\alpha}(A, B)=L^{\alpha+1}(A, B)$ *if and only if* $L^{\alpha}(A, B)=U(A, B)$.

Proof. (i) If $L^{\alpha}(A,B)=U(A,B)$ then by Theorem 16(b) we have $L^{\alpha+1}(A,B) =L(U(A, B), B)=U(A, B)$.

(ii) Suppose $L^{\alpha}(A, B)=L^{\alpha+1}(A, B)$ for some α, and $xRm\subseteq L^{\alpha}(A, B)$ for some m rp to B. (If M is null, the result is trivial, as then all the component ideals are just A.) Then $x\in L^{\alpha}(A, B)m^{-1}\subseteq L^{\alpha+1}(A, B)=L^{\alpha}(A, B)$ and m is rp to $L^{\alpha}(A, B)$ if m is rp to B. But $U(A, B)$ is the minimal ideal containing A to which every m rp to B is rp, hence $U(A, B)\subseteq L^{\alpha}(A, B)$. Since $L^{\alpha}(A, B) \subseteq U(A, B)$ for all α, the result follows.

COROLLARY 1. *There exists an ordinal* α, *finite or transfinite, such that* $L^{\alpha}(A, B)=U(A, B)$.

Proof. Under inclusion the $L^{\sigma}(A, B)$ form a bounded, well ordered set such that the union of any subset is again an $L^{\sigma}(A, B)$. Hence there exists a maximal element $L^{\alpha}(A, B)$ which must contain all $L^{\sigma}(A, B)$, thus $L^{\alpha}(A, B) =L^{\alpha+1}(A, B)=U(A, B)$.

COROLLARY 2. *If the A.C.C. for ideals holds in* R/A *then* $L^{n}(A, B)=U(A, B)$ *for some finite* n.

In the event R is a commutative ring, then the set of elements rp to an ideal B is the set of elements prime to B, which forms a multiplicative system. Thus both $U(A, B)$ and $L(A, B)$ as we have defined them are the same as the components defined by Murdoch in [6], which he has shown to be both equivalent to Krull's isolated component in the commutative case.

7. **Principal components of an ideal.** In Theorem 2 we saw that any ideal A is the intersection of its upper isolated P_α-components, where the P_α are the prime adjoints of the strongly irreducible primal divisors of A. If the ring R satisfies the A.C.C. for ideals, Theorem 9 shows that A is the finite intersection of its upper isolated components corresponding to the maximal primes of A. Since for any divisor B of A we have $A \subseteq L(A, B) \subseteq U(A, B)$, Theorems 2 and 9 remain valid if we replace "upper" by "lower." Moreover, we can then obtain a result for general rings similar to the modified version of Theorem 9.

DEFINITION 19. If $R \supseteq Q$ and Q is a maximal nrp to A ideal, then $L(A, Q)$ is a (right) *lower principal component* of A, and $U(A, Q)$ is a (right) *upper principal component* of A.

We note that if R is commutative, these definitions are equivalent to that of Krull's principal component.

THEOREM 19. *In any ring R, any ideal A is the intersection of its lower principal components.*

Proof. Let $\{Q_\alpha\}$ be the set of maximal nrp to A ideals.

(i) If $R = Q_\alpha$ for some α, then $L(A, Q_\alpha) = A$ by definition, or if R is nrp to Q_α for some α, then $L(A, Q_\alpha) = A$ by definition, and since $A \subseteq \bigcap L(A, Q_\alpha)$ the result follows trivially in either of these cases.

(ii) Suppose $R \neq Q_\alpha$ and R is rp to Q_α for all α, and suppose $x \in \bigcap L(A, Q_\alpha)$. Then if Q is any Q_α, we have $x = \sum_{i=1}^{n} x_i$ where there exists m_i rp to Q such that $x_i R m_i \subseteq A$, $i = 1, 2, \cdots, n$. Let $B = \{y \mid xRy \subseteq A\}$. Clearly B is an ideal. Now we have $xR(m_1Rm_2 \cdots m_{n-1}Rm_n) = (\sum_{i=1}^{n} x_i)R(m_1Rm_2 \cdots m_{n-1}Rm_n) \subseteq A$, hence $m_1Rm_2 \cdots m_{n-1}Rm_n \subseteq B$.

Now if $B \subseteq Q$, then $(m_1Rm_2 \cdots m_{n-1})Rm_n \subseteq Q$ and m_n rp to Q implies $(m_1Rm_2 \cdots m_{n-2})Rm_{n-1} \subseteq Q$. Continuing this process, we find $m_1 \in Q$, a contradiction. Hence $B \not\subseteq Q_\alpha$ for any α, and B must be rp to A. But then there exists $b \in B$ such that $yRb \subseteq A$ implies $y \in A$. Now $xRb \subseteq A$ by definition of B, hence $x \in A$, and we have shown $\bigcap L(A, Q_\alpha) \subseteq A$. But $A \subseteq \bigcap L(A, Q_\alpha)$ trivially and the result follows.

It appears unlikely that a similar result holds in general for the upper principal components of an ideal. However, under some circumstances such a result can be obtained.

THEOREM 20. *If the adjoint P_α of a strongly irreducible divisor A_α of A is* rp *to A, then $U(A, P_\alpha)$ is redundant in $A = \bigcap U(A, P_\alpha)$.*

Proof. We recall that by Theorem 1, $A = \bigcap A_\alpha$, where $\{A_\alpha\}$ is the set of strongly irreducible divisors of A, and by Theorem 2, $A = \bigcap U(A, P_\alpha)$, where

P_α is the adjoint of A_α. Now if $U(A, P_\alpha)$ is irredundant, then A_α is also. Hence $A_\alpha \not\supseteq \bigcap_{\beta\neq\alpha} A_\beta$ and $A_\alpha + \bigcap_{\beta\neq\alpha} A_\beta \supset A_\alpha$. Since, by Lemma 2, P_α is unrp to A_α, there exists $y \notin A_\alpha$ such that $yRP_\alpha \subseteq A_\alpha$. Let $A_\alpha' = (y) + A_\alpha \supset A_\alpha$ and we have $A_\alpha' \cap [A_\alpha + \bigcap_{\beta\neq\alpha} A_\beta] = A_\alpha + A_\alpha' \cap (\bigcap_{\beta\neq\alpha} A_\beta) \supset A_\alpha$ by Dedekind's law and the fact that A_α is irreducible. Hence there exists $x \notin A_\alpha$ such that $x \in A_\alpha + (y)$ and $x \in A_\beta$ for $\beta \neq \alpha$. Now $x = a_\alpha + y'$ for some $a_\alpha \in A_\alpha$ and $y' \in (y)$, hence $xRP_\alpha \subseteq a_\alpha RP_\alpha + y'RP_\alpha \subseteq A_\alpha$. But $xRP_\alpha \subseteq A_\beta$ for all $\beta \neq \alpha$, hence $xRP_\alpha \subseteq A$ and P_α is unrp to A. Thus if P_α is rp to A, then $U(A, P_\alpha)$ must be redundant.

THEOREM 21. *If in $A = \bigcap U(A, P_\alpha)$, where the P_α are the adjoints of the strongly irreducible divisors of A, all the redundant components can be eliminated, then A is the intersection of its upper principal components.*

Proof. If all the redundant components can be eliminated (certainly any finite number can), then by Theorem 20, the remaining ones have P_α unrp to A. Now each of these P_α is contained in a maximal nrp to A ideal Q_α. If $Q_\alpha = R$, then $U(A, Q_\alpha) = A \subseteq U(A, P_\alpha)$ by definition. If $Q_\alpha \neq R$ let $M = \{x \mid x \text{ rp to } Q_\alpha\}$. Then $M \subseteq C(P_\alpha)$, and if $C(P_\alpha)$ is erp to an ideal $B \supseteq A$, so is M. Hence $U(A, P_\alpha) \supseteq U(A, Q_\alpha) \supseteq A$ in this case also. Thus $A = \bigcap U(A, Q_\alpha)$, the intersection of the upper principal components of A.

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