REGULAR $\mathfrak{D}$-CLASSES IN SEMIGROUPS

BY

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0. **Introduction.** J. A. Green has introduced [1], in an arbitrary semigroup, certain equivalence relations which we shall denote by $\mathfrak{L}$ and $\mathfrak{R}$, their relative product $\mathfrak{D}$, and their intersection $\mathfrak{H}$. Since none of the relations is a congruence relation, the product of equivalence classes is not generally contained in an equivalence class. In §1 of this paper we obtain some information about the multiplication of such classes, restricting our attention for the most part to products of $\mathfrak{H}$-classes that lie in a $\mathfrak{D}$-class D containing an idempotent element, a restriction equivalent, as we show, to requiring that all elements of D be regular in the sense of von Neumann [4]. In §2 we use these results to obtain a theorem on matrix representations of semigroups which reduces, in the case of completely simple semigroups, to the Rees-Suschkewitsch Theorem, [5] and [6].

1. **Idempotents, inverses, and products.** Throughout the paper, S will denote an arbitrary semigroup, *i.e.*, a set closed under an associative binary operation: $a(bc)=(ab)c$ for all a, b, c in S. Green [1] has defined in S the equivalence relations $\mathfrak{L}$ and $\mathfrak{R}$ as follows:

$$a\mathfrak{L}b \text{ if and only if } Sa \cup a = Sb \cup b;$$
$$a\mathfrak{R}b \text{ if and only if } aS \cup a = bS \cup b.$$

Green showed that $\mathfrak{L}$ and $\mathfrak{R}$ are permutable and hence that their relative product $\mathfrak{D}$ is an equivalence relation: $a\mathfrak{D}b$ if and only if either (1) there exists $c\in S$ such that $a\mathfrak{L}c$ and $c\mathfrak{R}b$, or (2) there exists $d\in S$ such that $a\mathfrak{R}d$ and $d\mathfrak{L}b$. We shall denote by $\mathfrak{H}$ the intersection of $\mathfrak{R}$ and $\mathfrak{L}$: $a\mathfrak{H}b$ if and only if both $a\mathfrak{R}b$ and $a\mathfrak{L}b$.

For any $a\in S$ we shall denote by R_a, L_a, D_a, and H_a the respective $\mathfrak{R}$-, $\mathfrak{L}$-, $\mathfrak{D}$-, and $\mathfrak{H}$-equivalence classes to which a belongs. Clearly, the relations $\mathfrak{L}$ and $\mathfrak{R}$ each imply $\mathfrak{D}$, and $\mathfrak{H}$ implies both $\mathfrak{L}$ and $\mathfrak{R}$. Hence each $\mathfrak{D}$-class is both a union of $\mathfrak{L}$-classes and a union of $\mathfrak{R}$-classes, and each $\mathfrak{L}$- or $\mathfrak{R}$-class is a union of $\mathfrak{H}$-classes.

It is convenient to think of the elements of S as partitioned into a rectangular matrix of cells, each row of cells corresponding to an $\mathfrak{R}$-class and each column to an $\mathfrak{L}$-class. Each nonempty cell corresponds to an $\mathfrak{H}$-class. The permutability of $\mathfrak{R}$ and $\mathfrak{L}$ may be expressed in this way: for any a, $b\in S$, $R_a\cap L_b$ is nonempty if and only if $R_b\cap L_a$ is nonempty. We may imagine the rows and columns of the pattern to be so ordered that $\mathfrak{D}$-equivalent ones

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come together. Then the nonempty cells occur in rectangular blocks down the main diagonal of the matrix, each block constituting a $\mathcal{D}$-class. According to Green's Theorem 1, any two $\mathcal{D}$-equivalent $\mathcal{H}$-classes have the same cardinal number. Thus all the cells in a $\mathcal{D}$-block are, so to speak, filled to the same level with elements of S. An example is given at the end of the paper.

We note that $\mathcal{R}$ is a left congruence and $\mathcal{L}$ a right congruence: for all $c \in S$, $a\mathcal{R}b$ implies $ca\mathcal{R}cb$ and $a\mathcal{L}b$ implies $ac\mathcal{L}bc$. The relations $\mathcal{D}$ and $\mathcal{H}$ do not in general have either of these properties.

LEMMA 1. *The product LR of an $\mathcal{L}$-class L and an $\mathcal{R}$-class R is wholly contained in some one $\mathcal{D}$-class.*

Proof. Suppose $a_1, a_2 \in L$ and $b_1, b_2 \in R$. Then $a_1\mathcal{L}a_2$ and $b_1\mathcal{R}b_2$. Since $\mathcal{L}$ [$\mathcal{R}$] is a right [left] congruence, we infer that $a_1b_1\mathcal{L}a_2b_1$ and $a_2b_1\mathcal{R}a_2b_2$. Hence $a_1b_1\mathcal{D}a_2b_2$ by definition of $\mathcal{D}$.

We shall need later the following corollary to Lemma 1. Let D be a $\mathcal{D}$-class in S. Let the $\mathcal{R}$-classes and $\mathcal{L}$-classes contained in D be indexed by sets I and Λ, respectively. Thus $R_i, R_j, \cdots$ [$L_\lambda, L_\mu, \cdots$] will denote the $\mathcal{R}$- [$\mathcal{L}$-] classes of S contained in D, where $i, j, \cdots \in I$ [$\lambda, \mu, \cdots \in \Lambda$]. Denote by $H_{i\lambda}$ the $\mathcal{H}$-class $R_i \cap L_\lambda$ ($i \in I$, $\lambda \in \Lambda$).

COROLLARY. *For each $i \in I$ and $\lambda \in \Lambda$, $H_{i\lambda}^2$ is contained in some $\mathcal{D}$-class $D_{\lambda i}$ of S, and $H_{i\lambda}H_{j\mu} \subseteq D_{\lambda j}$ for all $i, j \in I$, $\lambda, \mu \in \Lambda$.*

Proof. By Lemma 1, $L_\lambda R_j$ is contained in some $\mathcal{D}$-class $D_{\lambda j}$, for each $j \in I$ and $\lambda \in \Lambda$. Hence for every $i, j \in I$ and $\lambda, \mu \in \Lambda$ we have $H_{i\lambda}H_{j\mu} \subseteq L_\lambda R_j \subseteq D_{\lambda j}$.

Following von Neumann [4], we say that an element a of S is *regular* if there exists $x \in S$ such that $axa = a$. An element a' of S will be called an *inverse* of a if $aa'a = a$ and $a'aa' = a'$. Clearly a is then also an inverse of a'. If a is regular, and $axa = a$, then, as pointed out by Thierrin [7], the element $a' = xax$ is an inverse of a. In particular, every idempotent element is regular and is an inverse (but not necessarily the only inverse) of itself. The following lemma is evident.

LEMMA 2. *If a and a' are inverse elements of a semigroup S, then $e = aa'$ and $f = a'a$ are idempotent elements such that $ea = af = a$, $a'e = fa' = a'$, and hence such that $e \in R_a \cap L_{a'}$ and $f \in R_{a'} \cap L_a$. The elements a, a', e, f all belong to the same $\mathcal{D}$-class of S.*

The next lemma is von Neumann's Lemma 6 and Green's Theorem 6 in the papers cited.

LEMMA 3 (VON NEUMANN). *The following three propositions concerning an element a of S are equivalent:*

(i) *a is regular,*

(ii) *L_a contains an idempotent element,*

(iii) *R_a contains an idempotent element.*

COROLLARY. *If a is regular, so is every element in L_a and every element in R_a.*

THEOREM 1. *If a $\mathfrak{D}$-class D in a semigroup S contains a regular element then every element of D is regular. If this be the case, every $\mathfrak{L}$-class and every $\mathfrak{R}$-class in D contains at least one idempotent element.*

Proof. Let a be a regular element of D, and let b be any element of D. By definition of $\mathfrak{D}$, there exists an element c in $L_a \cap R_b$. By the corollary to Lemma 3, c is regular since a is regular and $c \in L_a$. But then $b \in R_c$ and b is regular, by the same corollary. The second assertion of the theorem is immediate from the first and Lemma 3.

We shall call a $\mathfrak{D}$-class in S *regular* if all its members are regular, and *irregular* if none of its members is regular. By Theorem 1, every $\mathfrak{D}$-class is either regular or irregular. Henceforth we shall be concerned exclusively with regular $\mathfrak{D}$-classes.

LEMMA 4. *Any idempotent element e of S is a right identity element of L_e, a left identity element of R_e, and a two-sided identity element of H_e.*

Proof. Let $a \in L_e$. Then $a \in Se \cup e = Se$, so that $a = xe$ for some $x \in S$. Hence

$$ae = xe^2 = xe = a.$$

Similarly, $eb = b$ for every $b \in R_e$. If $c \in H_e = R_e \cap L_e$, then $ec = ce = c$.

COROLLARY 1. *No $\mathfrak{H}$-class contains more than one idempotent element.*

COROLLARY 2. *If a and b belong to a regular $\mathfrak{D}$-class then $a\mathfrak{L}b$ [$a\mathfrak{R}b$] if and only if $Sa = Sb$ [$aS = bS$].*

Proof. Evidently it will suffice to show that $Sa \cup a = Sa$, i.e., $a \in Sa$. By Lemma 3, R_a contains an idempotent element e, and, by Lemma 4, $ea = a$, whence $a \in Sa$.

THEOREM 2. *Let e and f be $\mathfrak{D}$-equivalent idempotent elements of a semigroup S. Then each element a of $R_e \cap L_f$ has a unique inverse a' in $R_f \cap L_e$. Eight of the sixteen products of the elements e, f, a, a' with one another are given in the following table:*

	e	f	a	a'
e	e		a	
f		f		a'
a		a		e
a'	a'		f	

Proof. Since $e\mathfrak{D}f$, the $\mathfrak{H}$-classes $R_e \cap L_f$ and $R_f \cap L_e$ are not empty. Let $a \in R_e \cap L_f$. From $a\mathfrak{R}e$ and $a\mathfrak{L}f$ we have $ea = a = af$, $e = ax$, $f = ya$ for some $x, y \in S$. (Note Corollary 2 to Lemma 4.) Let $a' = fxe$. Then

$$
\begin{aligned}
fa' &= a'e = a', \\
aa' &= afxe = axe = e^2 = e, \\
a'a &= fa'a = yaa'a = yea = ya = f.
\end{aligned}
$$

Since $aa'a=ea=a$ and $a'aa'=fa'=a'$, a and a' are mutually inverse. From $fa'=a'$ and $a'a=f$ we have $a'\mathcal{R}f$. From $a'e=a'$ and $aa'=e$ we have $a'\mathcal{L}e$. Hence $a'\in R_f\cap L_e$.

To show the uniqueness of a', let b be any inverse of a in $R_f\cap L_e$. From $aba=a$ and $bab=b$ we have

$$
\begin{aligned}
ab &\in R_a \cap L_b = R_e \cap L_e = H_e, \\
ba &\in R_b \cap L_a = R_f \cap L_f = H_f.
\end{aligned}
$$

Since ab and ba are idempotent, it follows from Corollary 1 to Lemma 4 that $ab=e$ and $ba=f$. Hence

$$b = bab = be = baa' = fa' = a'.$$

The following corollary locates, so to speak, all the inverses of a regular element a of a semigroup.

COROLLARY 1. *If a be a regular element of a semigroup then*

(i) *every inverse of a lies in D_a;*

(ii) *an $\mathcal{H}$-class H_b contains an inverse of a if and only if both of the $\mathcal{H}$-classes $R_a\cap L_b$ and $R_b\cap L_a$ contain idempotent elements;*

(iii) *no $\mathcal{H}$-class contains more than one inverse of a.*

Proof. That every inverse of a lies in D_a follows from Lemma 2. If H_b contains an inverse a' of a, then, again by Lemma 2, $R_a\cap L_b$ $(=R_a\cap L_{a'})$ and $R_b\cap L_a$ $(=R_{a'}\cap L_a)$ contain the idempotents aa' and $a'a$, respectively. Conversely, if $e\in R_a\cap L_b$, $f\in R_b\cap L_a$, $e^2=e$, and $f^2=f$, then $a\in R_a\cap L_a=R_e\cap L_f$, and by the theorem a has an inverse a' in $R_f\cap L_e$ $(=R_b\cap L_b=H_b)$. The uniqueness of a' follows from the theorem.

From Corollary 1 we see at once that there is a one-to-one correspondence between the set of all inverses of a and the set of all pairs (e, f) of idempotent elements with $e\in R_a$, $f\in L_a$. The next corollary is an evident consequence of this fact.

COROLLARY 2. *In order that each element of a regular semigroup have a unique inverse it is necessary and sufficient that each $\mathcal{R}$-class and each $\mathcal{L}$-class contain exactly one idempotent element.*

COROLLARY 3 (VAGNER). *If S be a regular semigroup, and if the idempotent elements of S commute with one another, then each element of S has a unique inverse.*

Proof. Let e and f be idempotents such that $e\mathcal{R}f$. By Lemma 4, $ef=f$ and $fe=e$. By hypothesis, $ef=fe$; hence $e=f$. Since S is regular, each $\mathcal{R}$-class contains at least one idempotent, hence exactly one. Similarly, each $\mathcal{L}$-class contains exactly one idempotent. The result then follows from Corollary 2.

Corollary 3 was proved by V. V. Vagner [8]. The converse was shown by A. E. Liber [2]: if each element of a semigroup S has a unique inverse in S, then the idempotent elements of S commute. This was also proved by Munn and Penrose [3].

THEOREM 3. *Let a and b be elements of a semigroup S. Then $ab\in R_a\cap L_b$ if and only if $R_b\cap L_a$ contains an idempotent element; if this be the case then*

$$aH_b = H_ab = H_aH_b = H_{ab} = R_a \cap L_b.$$

Proof. Assume first that $ab\in R_a\cap L_b$. Since $ab\in R_a$, either $ab=a$ or $abx=a$ for some $x\in S$. Since $ab\in L_b$, either $ab=b$ or $yab=b$ for some $y\in S$. Let us suppose $abx=a$ and $yab=b$, and set $e=ya$. Then

$$e = ya = y(abx) = (yab)x = bx,$$

whence

$$e^2 = (ya)(bx) = y(abx) = e.$$

From $e=ya$ and $a=a(bx)=ae$ we have $e\mathcal{L}a$. From $e=bx$ and $b=(ya)b=eb$ we have $e\mathcal{R}b$. Thus $e\in R_b\cap L_a$. The cases $ab=a$ or $ab=b$ or both are handled exactly as above by suppressing the x or the y or both.

Conversely, suppose that $R_b\cap L_a$ contains an idempotent element e. Let $a_1\in H_a$, $b_1\in H_b$. We proceed to show that $a_1b_1\in R_a\cap L_b$, from which we conclude that

$$H_aH_b \subseteq R_a \cap L_b.$$

From $e\mathcal{R}b\mathcal{R}b_1$ we have, by Lemma 4, $eb_1=b_1$. But from $a_1\mathcal{L}a\mathcal{L}e$ we have $a_1b_1\mathcal{L}eb_1$. Hence $a_1b_1\mathcal{L}b_1\mathcal{L}b$. Similarly, from $e\mathcal{L}a\mathcal{L}a_1$ we have $a_1e=a_1$, and hence $b_1\mathcal{R}e$ implies $a_1b_1\mathcal{R}a_1\mathcal{R}a$. From $a_1b_1\mathcal{L}b$ and $a_1b_1\mathcal{R}a$ we conclude that $a_1b_1 \in R_a\cap L_b$.

Finally, it will suffice to show that

$$R_a \cap L_b \subseteq H_ab,$$

assuming that $R_b\cap L_a$ contains an idempotent e. (The proof that $R_a\cap L_b\subseteq aH_b$ is similar.) Let $c\in R_a\cap L_b$. Since $e\mathcal{R}b$ we have $eb=b$ and $bx=e$ for some $x\in S$. (Note Corollary 2 to Lemma 4.) Since $b\mathcal{L}c$ we have $yb=c$ and $zc=b$ for some $y, z\in S$. Let $a_1=ye$. Then $a_1=ye=ybx=cx$, whence $e=bx=zcx=za_1$, so that $a_1\mathcal{L}e\mathcal{L}a$. And $a_1b=yeb=yb=c$, whence $a_1\mathcal{R}c\mathcal{R}a$. Therefore $a_1\mathcal{H}a$, and $c=a_1b \in H_ab$.

COROLLARY 1. *If e is an idempotent element of S and $a\mathcal{L}e$ $[b\mathcal{R}e]$ then $H_aH_e=H_a$ $[H_eH_b=H_b]$.*

Proof. Assume $a\mathcal{L}e$ and take $b=e$ in the theorem. Then $e\in H_e=R_e\cap L_e$

$=R_e \cap L_a$, and from the theorem we conclude that $H_a H_e = H_{ae} = H_a$. Proof of the dual is similar.

The next two corollaries constitute Green's Theorem 7 and corollary thereto.

COROLLARY 2 (GREEN). *If e is an idempotent element of S then H_e is a group.*

Proof. Let $a \in H_e$. By the theorem, taking $b = a$,

$$aH_e = H_e a = R_a \cap L_a = H_e.$$

But any subset T of a semigroup S with the property that $aT = Ta = T$ for every $a \in T$ is a subgroup of S.

REMARK. Clearly H_e is a maximal subgroup of S; the maximal subgroups of S are precisely those $\mathcal{H}$-classes that contain idempotents.

COROLLARY 3 (GREEN). *If a, b, and ab all belong to the same $\mathcal{H}$-class H, then H is a group.*

Proof. By hypothesis,

$$ab \in H = R_a \cap L_b = R_b \cap L_a,$$

whence, by the theorem, H contains an idempotent, and, by Corollary 2, H is a group.

COROLLARY 4. *If a and a' are mutually inverse elements of S, then $aa' = a'a$ if and only if a and a' belong to the same $\mathcal{H}$-class H. If this be the case, H is a group, and a and a' are inverses therein in the sense of group theory, i.e., $aa' = a'a = e$, where e is the identity element of H.*

Proof. If $aa' = a'a$ then, in the notation of Lemma 2, $e = f$. Since, by that lemma, $e \in R_a \cap L_{a'}$ and $f \in R_{a'} \cap L_a$, we conclude that $R_a = R_{a'}$ and $L_{a'} = L_a$, whence $H_a = H_{a'} = H_e$. Conversely, suppose $H_a = H_{a'}$ and let $e = aa'$, $f = a'a$. By Lemma 2, e and f are idempotent and

$$e \in R_a \cap L_{a'} = R_a \cap L_a = H_a;$$

similarly, $f \in H_a$. From Corollary 1 to Lemma 4 we conclude that $e = f$, *i.e.*, $aa' = a'a$. We then have $H_a = H_{a'} = H_e$; H_e is a group by Green's theorem (Corollary 2), and $aa' = a'a = e$.

THEOREM 4. *Any inverse of an idempotent element is the product of two idempotent elements, but need not itself be idempotent.*

Proof. Let $c^2 = c$ and let c' be an inverse of c. By Lemma 2, e $(=cc')$ and f $(=c'c)$ are idempotents belonging respectively to $R_c \cap L_{c'}$ and $R_{c'} \cap L_c$. Since c is idempotent and $e \mathcal{R} c$, Lemma 4 assures us that $ce = e$ as well as $ec = c$; similarly, $fc = f$ as well as $cf = c$. Hence

$$c\cdot fe\cdot c = cf\cdot ec = cc = c, \qquad fe\cdot c\cdot fe = f\cdot ecf\cdot e = fce = fe.$$

Thus fe is an inverse ot c. Since c is idempotent and $c\in R_e\cap L_f$, setting $a=f$ and $b=e$ in Theorem 3 allows us to conclude that $fe\in R_f\cap L_e$. But $c'\in R_f\cap L_e$; and, by Theorem 2, c can have at most one inverse in $R_f\cap L_e$. Hence $c'=fe$.

As an example to show that c' need not be idempotent, let S be the 2×2 matrix semigroup (see §2) over the group with zero $G^0=\{e, 0\}$, with structure constants $p_{11}=p_{12}=p_{22}=e$, $p_{21}=0$. Let $c=(e; 2, 1)$ and $c'=(e; 1, 2)$. Then c and c' are inverses, $c^2=c$, but $c'^2=0$.

We conclude this section of the paper with a theorem which asserts, among other things, that in a regular $\mathfrak{D}$-class D the maximal subgroups ($=\mathfrak{H}$-classes containing idempotents) are all isomorphic with one another, and that, by an appropriate redefinition of multiplication, the $\mathfrak{H}$-classes in D that do not contain idempotents can be turned into groups isomorphic with the maximal subgroups in D. The conclusion asserted in (i) of the next theorem is just that of Lemma 2.63 in Rees [5], although our hypotheses are somewhat weaker; with the aid of our Theorem 1, Rees' proof can be carried over almost verbatim. We shall also spare the reader the tedious but straightforward proofs of (ii)–(iv).

THEOREM 5. *Let e and f be $\mathfrak{D}$-equivalent idempotent elements of a semigroup S, let a be an arbitrary (but fixed) element of $R_e\cap L_f$, and for each $x\in R_e\cap L_f$ let x' denote the inverse of x in $R_f\cap L_e$.*

(i) *The groups H_e and H_f are isomorphic; in fact, the mappings $x\rightarrow a'xa$ and $y\rightarrow aya'$ are mutually inverse isomorphisms of H_e onto H_f and H_f onto H_e, respectively.*

(ii) *For any $x, y\in R_e\cap L_f$, define $x\circ y$ to be $xa'y$; and for any $u, v\in R_f\cap L_e$, define $u\circ v$ to be uav. With respect to the new operations, the sets $R_e\cap L_f$ and $R_f\cap L_e$ form groups A and A', respectively, and the correspondence $x\leftrightarrow x'$ is an anti-isomorphism between A and A'.*

(iii) *Let $x, y\in R_e\cap L_f$. Then in the group H_e the inverse of xy' is yx', and in the group H_f the inverse of $x'y$ is $y'x$.*

(iv) *The mappings λ_a: $x\rightarrow a'x$ and ρ_a: $x\rightarrow xa'$ are isomorphisms of the group A onto the groups H_f and H_e, respectively. If, instead of a, a different element b of $R_e\cap L_f$ be chosen, then $\lambda_a\neq\lambda_b$ and $\rho_a\neq\rho_b$.*

2. **A generalization of the Rees-Suschkewitsch Theorem.** We proceed now to give a theorem, concerning any regular $\mathfrak{D}$-class in a semigroup S, which reduces to the Rees-Suschkewitsch Theorem, [6] and [5], when S is completely simple (Theorem 2.93 in [5]). The method of proof is that of Rees, which Green carried over (Theorem 1 of [1]) to any $\mathfrak{D}$-class. Our only claim to novelty is that with any $\mathfrak{D}$-class D we associate a regular matrix semigroup $\mathfrak{M}$ such that D is partially isomorphic with $\mathfrak{M}$. It is not immediately evident from the construction of $\mathfrak{M}$ that it depends only on D. In order to prove the uniqueness of $\mathfrak{M}$, we construct directly from D a semigroup D^0,

which we call the *trace* of D, evidently depending only on D, and observe that $\mathfrak{M}$ is isomorphic with D^0.

LEMMA 5. *Let T be a union of $\mathfrak{D}$-classes of a semigroup S. Let 0 be a symbol not representing any element of S, and let $T^0 = T \cup 0$. Define a multiplication $\circ$ in T^0 as follows: for any $a, b \in T$,*

$$a \circ b = \begin{cases} ab & \text{if } ab \in R_a \cap L_b, \\ 0 & \text{otherwise}; \end{cases}$$

$$a \circ 0 = 0 \circ a = 0 \circ 0 = 0.$$

Then T^0 is a semigroup.

Proof. Clearly we need only show that

$$a \circ (b \circ c) = (a \circ b) \circ c$$

for all $a, b, c \in T$. Suppose first that $a \circ (b \circ c) \neq 0$. Then $b \circ c \neq 0$ also, and we have

$$b \circ c = bc \in R_b \cap L_c, \qquad a \circ (b \circ c) = a \circ (bc) = a(bc) \in R_a \cap L_{bc}.$$

From $b \mathfrak{R} bc$ and $a(bc) \mathfrak{R} a$ we have, since $\mathfrak{R}$ is a left congruence relation, $ab \mathfrak{R} a(bc) \mathfrak{R} a$, whence $ab \in R_a$. Now from Theorems 1 and 3 we see that a, b, c, ab, bc, and abc lie in a regular $\mathfrak{D}$-class of S. Hence, by Corollary 2 to Lemma 4, $abc \mathfrak{L} bc \mathfrak{L} c$ implies $Sabc = Sc$, and thus $yabc = c$ for some $y \in S$; and from $bc \in R_b$ we have $bcz = b$ for some $z \in S$. Hence

$$b = bcz = b(yabc)z = (bya)(bcz) = (bya)b = (by)(ab),$$

whence $b \mathfrak{L} ab$. Since we have already found that $ab \in R_a$, we now have $ab \in R_a \cap L_b$, and consequently $a \circ b = ab$. From $abcz = ab$ we have $abc \in R_{ab}$, and we have observed above that $abc \mathfrak{L} c$. Thus $abc \in R_{ab} \cap L_c$, whence $(ab) \circ c = abc$. Therefore $a \circ (b \circ c) = abc = (a \circ b) \circ c$. Similarly, we can show that if $(a \circ b) \circ c \neq 0$ then $(a \circ b) \circ c = a \circ (b \circ c)$. Thus the two expressions $a \circ (b \circ c)$ and $(a \circ b) \circ c$ are either both 0 or both abc.

We note that if a and b are not $\mathfrak{D}$-equivalent then $a \circ b = 0$. Our chief concern is the case in which the set T of Lemma 5 is a single $\mathfrak{D}$-class D. It will follow from the next theorem that in this case D^0 is completely simple.

Let G^0 be a group with zero (Rees' terminology), let I and Λ be any two sets, and, for each $i \in I$, $\lambda \in \Lambda$, let $p_{\lambda i}$ be an element of G^0. Let P denote the $\Lambda \times I$ matrix with elements $p_{\lambda i}$. Following Rees, we define the $I \times \Lambda$-matrix semigroup over G^0 with structure matrix P to be the set of all triples $(a; i, \lambda)$ ($a \in G^0$, $i \in I$, $\lambda \in \Lambda$) with multiplication defined by

$$(a; i, \lambda)(b; j, \mu) = (a p_{\lambda j} b; i, \mu)$$

($a, b \in G^0$; $i, j \in I$; $\lambda, \mu \in \Lambda$). Moreover, we identify all triples of the form $(0; i, \lambda)$ as a single element 0. (Actually, the set of all such triples is an ideal, and we take the Rees quotient.) We shall denote this semigroup by $\mathfrak{M}(G^0; I, \Lambda; P)$.

Rees calls $\mathfrak{M}$ *regular* if no row or column of P consists of zeros. It is a fortunate accident of terminology that $\mathfrak{M}$ is regular in the sense of Rees if and only if it is regular in the sense of von Neumann. If $\mathfrak{M}$ is regular, it is easy to see that there are just two $\mathfrak{D}$-classes in $\mathfrak{M}$: one consists of 0 alone, and the other of all the nonzero elements of $\mathfrak{M}$. The $\mathfrak{L}$-classes [$\mathfrak{R}$-classes] are just the sets $L_\lambda[R_i]$ of all $(a; i, \lambda)$ with fixed $\lambda \in \Lambda[i \in I]$.

Now let D be a regular $\mathfrak{D}$-class of a semigroup S, and let $\{R_i; i \in I\}$ and $\{L_\lambda; \lambda \in \Lambda\}$ be the sets of $\mathfrak{R}$-classes and $\mathfrak{L}$-classes, respectively, contained in D. As before, we write $H_{i\lambda} = R_i \cap L_\lambda$, and without loss of generality we may choose the notation so that H_{11} contains an idempotent element; I and Λ are thereby regarded as having the element 1 in common. For each $i \in I$ and $\lambda \in \Lambda$, choose in any way elements $r_i \in H_{i1}$ and $q_\lambda \in H_{1\lambda}$. Let $H_{11}^0 = H_{11} \cup 0$, and define the $\Lambda \times I$-matrix P with elements $p_{\lambda i}$ in H_{11}^0 as follows:

$$p_{\lambda i} = \begin{cases} q_\lambda r_i & \text{if } q_\lambda r_i \in H_{11}, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathfrak{M}(H_{11}^0; I, \Lambda; P)$ be the $I \times \Lambda$-matrix semigroup over H_{11}^0 with structure matrix P. By Theorem 3, $q_\lambda r_i \in H_{11}$ if and only if $H_{i\lambda}$ contains an idempotent element; hence $\mathfrak{M}$ is regular.

THEOREM 6. *The regular $\mathfrak{D}$-class D is partially isomorphic with $\mathfrak{M}$ in the following sense. Every element of D is uniquely representable in the form $r_i x q_\lambda$, with $x \in H_{11}$, $i \in I$, $\lambda \in \Lambda$; and if ϕ is the one-to-one mapping of $\mathfrak{M} - \{0\}$ onto D defined by $\phi(x; i, \lambda) = r_i x q_\lambda$, then*

$$\phi[(x; i, \lambda)(y; j, \mu)] = \phi(x; i, \lambda)\phi(y; j, \mu)$$

if $(x; i, \lambda)(y; j, \mu) \neq 0$, i.e., if $p_{\lambda j} \neq 0$. Furthermore, $\mathfrak{M}$ is determined by D uniquely to within isomorphism; in fact $\mathfrak{M}$ is isomorphic with the trace D^0 of D.

Proof. By Lemma 3, any $\mathfrak{L}$-class in D contains at least one idempotent; for each $\lambda \in \Lambda$, let us select an idempotent e_λ in L_λ. By Theorem 2, each q_λ has a unique inverse q_λ' in $R_{e_\lambda} \cap L_1$. The mappings

$$x \to x q_\lambda \quad (x \in L_1), \qquad y \to y q_\lambda' \quad (y \in L_\lambda)$$

are mutually inverse mappings of L_1 and L_λ onto each other whereunder, for each $i \in I$, H_{i1} and $H_{i\lambda}$ are mapped onto each other. For $q_\lambda q_\lambda' = e$, where e is the idempotent in H_{11}, and $q_\lambda' q_\lambda = e_\lambda$; hence $(xq_\lambda)q_\lambda' = xe = x$ for every $x \in L_1$, and $(yq_\lambda')q_\lambda = ye_\lambda = y$ for every $y \in L_\lambda$. That $H_{i1} \leftrightarrow H_{i\lambda}$ follows from the fact that $xS = xq_\lambda S$, so that $x \mathfrak{R} x q_\lambda$, and similarly $y \mathfrak{R} y q_\lambda'$.

In the same way, for each r_i $(i \in I)$ we may select an inverse r_i' in R_1. Then

$$x \to r_i x \quad (x \in R_1) \qquad \text{and} \qquad y \to r_i' y \quad (y \in R_i)$$

are mutually inverse mappings of R_1 and R_i onto each other preserving $\mathfrak{L}$-equivalence. Continuing as in Green's Theorem 1, the mappings

$$x \to r_i x q_\lambda \quad (x \in H_{11}) \qquad \text{and} \qquad y \to r_i' y q_\lambda' \quad (y \in H_{i\lambda})$$

are mutually inverse mappings of H_{11} and $H_{i\lambda}$ onto each other. Consequently every element of D is uniquely representable in the form $r_i x q_\lambda$ with $x \in H_{11}$, $i \in I$, $\lambda \in \Lambda$.

Suppose now that $p_{\lambda j} \neq 0$ ($\lambda \in \Lambda, j \in I$). By definition of P, this means that $q_\lambda r_j \in H_{11}$ and $p_{\lambda j} = q_\lambda r_j$. We then have $x p_{\lambda j} y \in H_{11}$ for any x, $y \in H_{11}$; and so for any $i \in I$, $\mu \in \Lambda$,

$$\begin{aligned} \phi[(x;\, i,\, \lambda)(y;\, j,\, \mu)] &= \phi(x p_{\lambda j} y;\, i,\, \mu) = r_i(x p_{\lambda j} y) q_\mu \\ &= r_i x q_\lambda r_j y q_\mu = \phi(x;\, i,\, \lambda)\phi(y;\, j,\, \mu). \end{aligned}$$

Finally to show the isomorphism between $\mathfrak{M}$ and the trace D^0 of D, we need only show that

$$(r_i x q_\lambda)(r_j y q_\mu) \in R_i \cap L_\mu$$

if and only if $q_\lambda r_j \in H_{11}$ (since $r_i x q_\lambda \in R_i$ and $r_j y q_\mu \in L_\mu$). By Theorem 3, the former is the case if and only if $R_j \cap L_\lambda$ contains an idempotent element, and, by the same theorem, this in turn is the case if and only if $q_\lambda r_j \in H_{11}$.

If $H_{i\lambda}$ ($i \neq 1$, $\lambda \neq 1$) is an $\mathfrak{H}$-class containing an idempotent f, then, in the proof of Theorem 6, we can choose for r_i the inverse of q_λ in H_{i1}. If we pick (as we always can) $q_1 = r_1 =$ identity element e of H_{11}, then we see that $p_{11} = p_{1\lambda} = p_{i1} = p_{i\lambda} = e$. From this (or else directly) we have the first part of the following corollary. The second part is immediate from Lemma 1. We call the set Q, defined in the corollary, a *quadrilateral set*. The point of the corollary is that the structure of any quadrilateral set is exactly half known. In addition to this, we know roughly where the other half of the products lie: in at most two other $\mathfrak{D}$-classes.

COROLLARY. *Let e and f be $\mathfrak{D}$-equivalent idempotent elements of a semigroup. Let*

$$\begin{aligned} H_{11} &= R_e \cap L_e, \qquad H_{12} = R_e \cap L_f, \\ H_{21} &= R_f \cap L_e, \qquad H_{22} = R_f \cap L_f, \\ Q &= H_{11} \cup H_{12} \cup H_{21} \cup H_{22}. \end{aligned}$$

Then every element of Q is uniquely expressible in the form $(x;\, i,\, j)$ (with $x \in H_{11}$; $i, j = 1, 2$) such that $(x;\, i,\, j) \in H_{ij}$ and such that $(x;\, i,\, j)(y;\, j,\, k) = (xy;\, i,\, k)$. Moreover, there exist $\mathfrak{D}$-classes D_{12} and D_{21} such that

$$(x;\, i,\, 1)(y;\, 2,\, k) \in D_{12} \quad \text{and} \quad (x;\, i,\, 2)(y;\, 1,\, k) \in D_{21}.$$

As an example, let S be the semigroup generated by two symbols p, q, subject to the generating relations

$$pqp = p, \qquad qpq = q.$$

S may be described as the free semigroup generated by a pair of inverse elements. We can show that every $\mathcal{R}$-class [$\mathcal{L}$-class] of S contains exactly two elements, and hence that every $\mathcal{D}$-class contains four elements, while every $\mathcal{H}$-class consists of a single element. In fact, if w is a word beginning with p and ending with q (say), the set of four elements

$$\begin{Bmatrix} w & wp \\ qw & qwp \end{Bmatrix}$$

is a $\mathcal{D}$-class, the two elements in each row [column] constituting an $\mathcal{R}$-class [$\mathcal{L}$-class], and similarly for the other three possibilities.

$$D = \begin{Bmatrix} pq & p \\ q & qp \end{Bmatrix}$$

is the only regular $\mathcal{D}$-class in S. The elements of D may be represented as follows:

$$pq \to (e; 1, 1), \qquad p \to (e; 1, 2),$$
$$q \to (e; 2, 1), \qquad qp \to (e; 2, 2).$$

We find that

$$D_{12} = \begin{Bmatrix} q^2 & q^2p \\ pq^2 & pq^2p \end{Bmatrix}, \qquad D_{21} = \begin{Bmatrix} p^2 & p^2q \\ qp^2 & qp^2q \end{Bmatrix}.$$

Here $Q = D$, and we see that D, D_{12}, and D_{21} are all distinct.

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