# ON FUNCTIONS SUBHARMONIC IN A HALF-SPACE(¹)

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1. **Introduction.** The present article is devoted to the study of functions which are subharmonic in a half-space and whose upper limit is non-positive at the finite boundary points. Our method originated from a remark of Professor A. Weinstein. He pointed out to us that the following property of harmonic functions is implied by the correspondence principle [16; 17] for generalized axially symmetric potentials: *The function* $u(x_1, x_2, \cdots, x_n)$ *is harmonic in the half-space* $x_n > 0$ *if and only if the function* $v(\xi_1, \xi_2, \cdots, \xi_{n+2}) \equiv u(\xi_1, \xi_2, \cdots, \xi_{n-1}, (\xi_n^2+\xi_{n+1}^2+\xi_{n+2}^2)^{1/2})/(\xi_n^2+\xi_{n+1}^2+\xi_{n+2}^2)^{1/2}$ *is harmonic in the corresponding domain in* $(n+2)$*-dimensional space*(²).

2. **The method.** Let $R^n$ $(n \geqq 2)$ be an $n$-dimensional Euclidean space the points of which will be denoted by $x = (x_1, x_2, \cdots, x_n)$ or $y = (y_1, y_2, \cdots, y_n)$.

Definition. *The function* $u(x)$ *is said to be of class* A *if and only if the following conditions are fulfilled*: (1) $u$ *is defined and subharmonic in the half-space* $H = E[x \mid x_n > 0]$; (2) $u$ *satisfies*

$$\limsup_{x \to y} u(x) \leqq 0 \qquad (x \in H) \tag{2.1}$$

*at all points* $y$ *of the boundary* $D = E[x \mid x_n = 0]$.

It is convenient to introduce further an $(n+2)$-dimensional space $\mathrm{P}^{n+2}$ $(n \geqq 2)$ whose points will be denoted by $\xi = (\xi_1, \xi_2, \cdots, \xi_{n+2})$ or $\eta = (\eta_1, \eta_2, \cdots, \eta_{n+2})$.

Definition. *The function* $v(\xi)$ *is said to be of class* B *if and only if the following conditions are fulfilled*: (1) $v$ *is defined and subharmonic in the whole space* $\mathrm{P}^{n+2}$; (2) $v$ *is symmetric with respect to the subspace* $\Delta = E[\xi \mid \xi_n^2+\xi_{n+1}^2+\xi_{n+2}^2 = 0]$, *i.e. depending only on* $\xi_1, \xi_2, \cdots, \xi_{n-1}$ *and* $(\xi_n^2+\xi_{n+1}^2+\xi_{n+2}^2)^{1/2}$.

Our method is based on a simple one-to-one correspondence between these two classes which enables us to reduce problems concerning functions of class A to questions about functions of class B.

Lemma. *The transformation* $v(\xi) = S[u(x)]$, *defined by*

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(²) A direct calculation yields $\Delta u(x) = (\xi_n^2+\xi_{n+1}^2+\xi_{n+2}^2)^{1/2}\Delta v(\xi)$, where $\Delta$ denotes the Laplacian and $x_1 = \xi_1, x_2 = \xi_2, \cdots, x_{n+1} = \xi_{n-1}, x_n = (\xi_n^2+\xi_{n+1}^2+\xi_{n+2}^2)^{1/2}$.

$$(2.2)\quad v(\xi) \equiv u(\xi_1, \xi_2, \cdots, \xi_{n-1}, (\xi_n^2 + \xi_{n+1}^2 + \xi_{n+2}^2)^{1/2})/(\xi_n^2 + \xi_{n+1}^2 + \xi_{n+2}^2)^{1/2}$$

*for* $\xi \in \mathrm{P}^{n+2} - \Delta$, *and by*

$$(2.3)\qquad v(\xi) = \limsup_{\eta \to \xi} v(\eta) \qquad (\eta \in \mathrm{P}^{n+2} - \Delta)$$

*for* $\xi \in \Delta$, *establishes a one-to-one correspondence between the two classes of functions* A *and* B. *The inverse* $T$ *of* $S$ *is given by*

$$(2.4)\qquad u(x) = T[v(\xi)] \equiv x_n v(x_1, x_2, \cdots x_{n-1}, x_n, 0, 0).$$

REMARK. It is evident from the following proof that this lemma remains valid, if the class A is replaced by the set of functions which are defined and subharmonic in $H(R_1, R_2) = H \cap E[x \mid R_1 < |x| < R_2]$, where $|x| = (\sum_1^n x_i^2)^{1/2}$, and satisfy (2.1) for all points $y$ on $D(R_1, R_2) = D \cap E[x \mid R_1 < |x| < R_2]$. At the same time B must be replaced by the class of functions which are defined and subharmonic in $\Omega(R_1, R_2) = E[\xi \mid R_1 < |\xi| < R_2]$, where $|\xi| = (\sum_1^{n+2} \xi_i^2)^{1/2}$, and are symmetric with respect to $\Delta$. The values 0 for $R_1$ and $\infty$ for $R_2$ are admitted. This will be used in the proofs of Theorems 1 and 4.

**Proof of the lemma.** Let $u(x)$ be an arbitrary function of class A. We first prove that $v(\xi) = S[u(x)]$ is subharmonic in $\mathrm{P}^{n+2} - \Delta$. In the case where $u$ is sufficiently regular, e.g. possesses continuous partial derivatives up to the second order, this can be verified by straightforward differentiation(2). A general function $u$ of class A can be represented (on any subcompact of $H$) as the limit of a nonincreasing sequence $\{u_k\}$ of regular subharmonic functions. The functions $v_k = S[u_k]$ are subharmonic. Obviously, $v_k \downarrow v$ for $k \to \infty$. Hence $v$ is subharmonic in $\mathrm{P}^{n+2} - \Delta$.

Furthermore, we demonstrate that $v$ is bounded from above in the neighborhood of any subcompact of $\Delta$. In order to prove this we consider, for any given $R > 0$, the least harmonic majorant $h_{2R}^+(x)$ of the function $u^+ = \max[u, 0]$ in the domain $H \cap [|x| < 2R] \cdot h_{2R}^+(x)$ assumes the boundary value 0 on $D \cap [|x| < 2R]$. From this follows easily that there exists a constant $C(R) < \infty$ such that $u(x) \leqq u^+(x) \leqq h_{2R}^+(x) \leqq C(R) x_n$ in a neighborhood of $D \cap [|x| \leqq R]$. In the corresponding neighborhood of $\Delta \cap [|\xi| \leqq R]$ we then have $v < C(R)$ as stated.

Therefore, $v$ is now known to be subharmonic in $\mathrm{P}^{n+2} - \Delta$ and bounded from above in the neighborhood of any subcompact of $\Delta$. Since $\Delta$ is of capacity zero, we can apply a theorem of M. Brelot [3, p. 31] and conclude that $v$ can be continued into $\Delta$ as a subharmonic function. (One of Brelot's hypotheses, namely the boundedness of the exceptional set, is not fulfilled. However, the reader will easily convince himself that one can get rid of this assumption by trivial modifications in Brelot's argument.) The continuation is uniquely determined and given by (2.3). Hence $v = S[u]$ is of class B.

Let $v(\xi)$ be an arbitrary function of class B. Then $u(x) = T[v]$ is subharmonic in $H$. If $v$ is sufficiently regular, this can be verified by differentiation.

The general case can again be treated by introducing an approximating sequence. Obviously (2.1) is fulfilled. Consequently, $u$ is of class A.

It is evident that the transformations $S$ and $T$ are inverse to each other. They establish a one-to-one correspondence between the classes A and B. Q.E.D.

We introduce the following notations:

$$\Sigma_r = E[\xi \mid |\xi| = r], \qquad S_r = E[x \mid |x| = r],$$

and $\omega_n = 2\pi^{n/2}/\Gamma(n/2)$ (area of the $(n-1)$-dimensional unit sphere).

3. **Properties of a mean value.** Let $u(x)$ be subharmonic in $H(R_1, R_2)$ and satisfy (2.1) for all $y \in D(R_1, R_2)$. We consider the corresponding function $v(\xi) = S[u]$, which is subharmonic in $\Omega(R_1, R_2)$.

A well known theorem of F. Riesz [12] states that the mean value

$$A(v; r) = \frac{1}{\omega_{n+2} r^{n+1}} \int_{\Sigma_r} v(\xi) d\Sigma_r(\xi) \tag{3.1}$$

is a convex function of $r^{-n}$ $(R_1 < r < R_2)$, i.e.

$$\begin{vmatrix} r_1^{-n} & A(r_1) & 1 \\ r^{-n} & A(r) & 1 \\ r_2^{-n} & A(r_2) & 1 \end{vmatrix} \geqq 0 \tag{3.2}$$

holds for any three radii $r_1$, $r$, and $r_2$ satisfying the inequality $R_1 < r_1 < r < r_2 < R_2$. It is easy to verify that

$$A(v; r) = \frac{1}{\omega_{n+2} r^{n+1}} \int_{S_r} \frac{u}{x_n} 4\pi x_n^2 dS_r(x) = \frac{2n}{\omega_n} \frac{m(r)}{r}. \tag{3.3}$$

Here we adopt the usual definition

$$m(r) = \int_{S_r} u(x) \sin \phi dS_1(x), \tag{3.4}$$

where $\sin \phi = x_n/r$ and $dS_1(x) = dS_r(x)/r^{n-1}$ is the central projection of the surface element $dS_r(x)$ on $S_1$. Inequality (3.2) is equivalent to each one of the following three relations:

$$\begin{vmatrix} m(r) r^{n/2-1} & r^{n/2} & r^{-n/2} \\ m(r_1) r_1^{n/2-1} & r_1^{n/2} & r_1^{-n/2} \\ m(r_2) r_2^{n/2-1} & r_2^{n/2} & r_2^{-n/2} \end{vmatrix} \geqq 0, \tag{3.5}$$

$$\begin{vmatrix} r_1^{-n} & m(r_1)/r_1 & 1 \\ r^{-n} & m(r)/r & 1 \\ r_2^{-n} & m(r_2)/r_2 & 1 \end{vmatrix} \geqq 0, \tag{3.6}$$

$$(3.7)\qquad \begin{vmatrix} r_1^n & r_1^{n-1}m(r_1) & 1 \\ r^n & r^{n-1}m(r) & 1 \\ r_2^n & r_2^{n-1}m(r_2) & 1 \end{vmatrix} \geqq 0.$$

We thus have proved:

THEOREM 1. *Let $u(x)$ be subharmonic in $H(R_1, R_2)$ and satisfy* (2.1) *on $D(R_1, R_2)$. Then the function $m(r)$, defined by* (3.4), *has the following properties:* (1) *$m(r)/r$ is a convex function of $r^{-n}$;* (2) *$r^{n-1}m(r)$ is a convex function of $r^n$*(3).

By comparison of formulation (3.5) with Theorem 7 in [1] we observe that this is a well known result of L. Ahlfors in somewhat extended form. The greater generality consists in the fact that we neither limit ourselves to nonnegative functions $u$, nor do we—and this would be equivalent—replace $u$ by $u^+ = \max [u, 0]$ in the definition of $m(r)$. For this reason results of A. Dinghas [5; 6] and M. Tsuji [14] are also contained in Theorem 1. These authors discussed the behavior of the mean value $m(r)$ of functions which are positive and harmonic throughout $H$. Their results follow from our statements if we restrict ourselves to functions $u$ which are negative and harmonic throughout $H$, and formulate the conclusions for $(-u)$. The same remark applies to Theorem 2 (below)(4).

The statements of Theorem 1 are the best possible in the following sense: given any convex function $\phi(t)$, $0 \leqq t_1 < t < t_2 \leqq \infty$, there exists a function $u(x)$ with the following properties: (a) $u$ is defined and subharmonic in $H(t_2^{-1/n}, t_1^{-1/n})$; (b) $u$ satisfies (2.1) on $D(t_2^{-1/n}, t_1^{-1/n})$; (c) $m(r)/r \equiv \phi(r^{-n})$. Indeed, the function $u(x) = T[v]$, where $v(\xi) = (2n/\omega_n)\phi(|\xi|^{-n})$, fulfills all of these conditions. It is even possible to construct examples which show that the postulate (a) may be replaced by: (a') $u$ is defined and harmonic in $H(t_2^{-1/n}, t_1^{-1/n})$. These functions illustrate the accuracy of statement (1) in Theorem 1. One can construct corresponding examples for statement (2) in an analogous way.

Suppose now that $u$ is of class A. Then $v = S[u]$ is of class B and, by a well known property of subharmonic functions, $A(v; r)$ is a nondecreasing function of $r$. We thus have:

THEOREM 2. *Let $u(x)$ be subharmonic in $H$ and satisfy* (2.1) *on $D$. Then $m(r)/r$, $m(r)$ being defined by* (3.4), *is a nondecreasing function of $r$.*

This is a classical theorem of L. Ahlfors [1]. He derived it for positive harmonic functions of class A as a consequence of the convexity property of

(3) Since the inequalities (3.6) and (3.7) follow from each other, the statements (1) and (2) are equivalent. Seemingly this fact has never been pointed out before.

(4) *Added February* 20, 1956. In a more recent publication (C.R. Acad. Sci. Paris vol. 237 (1953) pp. 594–595) A. Dinghas has announced further results in this field. These can also be derived by the method used in this article.

$m(r)/r$. Further proofs are contained in the articles of A. Dinghas [5] and M. Heins [8](5).

So far only elementary properties of subharmonic functions have been applied. We shall now make use of deeper results, such as the Riesz decomposition theorem.

4. **Representation theorems.** Let us first consider a function $u(x)$ which is defined in $H$, subharmonic and nonpositive. Obviously, $u$ is of class A. The corresponding function $v(\xi)=S[u]$ is nonpositive throughout $P^{n+2}$.

In any given sphere $|\xi|<r$ $(r>0)$ we have the Riesz decomposition [12, II, p. 357]

$$v(\xi) = h_r(\xi) - \int_{|\eta|<r} \gamma_r(\xi, \eta) d\mu(e_\eta). \tag{4.1}$$

Here $h_r(\xi)$ is the least harmonic majorant of $v(\xi)$ in $|\xi|<r$, $\gamma_r(\xi, \eta)$ denotes Green's function for this domain and $\mu(e_\eta)$ is a non-negative Borel measure.

Let $r\nearrow\infty$. Then, for any fixed pair of points $\xi$ and $\eta$, $\gamma_r(\xi, \eta)\nearrow|\xi-\eta|^{-n}$, where $\xi-\eta=(\xi_1-\eta_1, \cdots, \xi_{n+2}-\eta_{n+2})$. Furthermore, $\{h_r(\xi)\}$ is a nondecreasing sequence of nonpositive harmonic functions. It tends to a limit function which is nonpositive and harmonic throughout $P^{n+2}$, therefore a constant. Hence

$$v(\xi) = c - \int_{P^{n+2}} |\xi-\eta|^{-n} d\mu(e_\eta). \tag{4.2}$$

We observe that the measure $\mu$ is symmetric with respect to $\Delta$, i.e. $\mu(e_\eta)=\mu(e_\eta')$ holds whenever the set $e_\eta$ can be transformed into $e_\eta'$ by a rotation of the space $P^{n+2}$ around $O$ which leaves $\Delta$ invariant. This property of $\mu$ follows at once from the corresponding symmetry of $v$ and the well known fact that $\mu$ is uniquely determined by $v$.

We are now going to determine the representation of $u(x)$ which corresponds to (4.2). It is convenient for this purpose to decompose the measure $\mu$ into a sum $\mu=\mu_1+\mu_2$, $\mu_1$ and $\mu_2$ being of support $\Delta$ and $P^{n+2}-\Delta$, respectively. If we put

$$v_1 = \int_\Delta |\xi-\eta|^{-n} d\mu_1(e_\eta), \tag{4.3}$$

we obviously have

$$T[v_1] = x_n \int_D |x-y|^{-n} d\nu_1(e_y), \tag{4.4}$$

the measure $\nu_1$ being defined by

(5) Professor M. Riesz kindly drew our attention to another (unpublished) proof by L. Gårding and B. Kjellberg. Their method is different from the one used in this paper.

$$\nu_1(e_y) = \mu_1(E[\eta \mid \eta_1 = y_1, \cdots, \eta_{n-1} = y_{n-1}, \eta_n^2 + \eta_{n+1}^2 + \eta_{n+2}^2 = 0; y \in e_y]). \tag{4.5}$$

Moreover, defining

$$v_2 = \int_{\mathrm{P}^{n+2}-\Delta} |\xi - \eta|^{-n} d\mu_2(e_\eta), \tag{4.6}$$

we state that

$$T[v_2] = \int_H g(x, y) d\nu_2(e_y), \tag{4.7}$$

where $g(x, y)$ denotes Green's function for $H$ and $\nu_2$ is given by

$$\nu_2(e_y) = (\lambda_n/y_n)\mu_2(E[\eta \mid \eta_1 = y_1, \cdots, \eta_{n-1} = y_{n-1}, \eta_n^2 + \eta_{n+1}^2 + \eta_{n+2}^2 = y_n^2; y \in e_y]), \tag{4.8}$$

where $\lambda_2 = 1/2$ and $\lambda_n = 1/2(n-2)$ for $n \geqq 3$. In order to demonstrate (4.7), we first consider a fixed point $y_0$ and assume that $\mu_2$ consists of the unit mass, distributed uniformly over the set

$$\Gamma(y_0) = E[\eta \mid \eta_1 = y_{10}, \cdots, \eta_{n-1} = y_{n-1,0}, \eta_n^2 + \eta_{n+1}^2 + \eta_{n+2}^2 = y_{n0}^2].$$

In this case we have

$$v_2 = \frac{1}{2}\int_0^\pi \sin t dt \Big/ \left[ \sum_1^{n-1} (\xi_i - y_{i0})^2 + \xi_n^2 + \xi_{n+1}^2 + \xi_{n+2}^2 + y_{n0}^2 - 2y_{n0}(\xi_n^2 + \xi_{n+1}^2 + \xi_{n+2}^2)^{1/2} \cos t \right]^{n/2}.$$

This formula was obtained by direct calculation. It can be verified a posteriori by the following simple reasoning. The potential $v_2(\xi)$ is uniquely determined by the following properties: (1) $v_2$ is harmonic in $\mathrm{P}^{n+2} - \Gamma(y_0)$; (2) $v_2$ is symmetric with respect to $\Delta$; (3) $v_2(\xi) = [\sum_1^{n-1} (\xi_i - y_{i0})^2 + y_{n0}]^{-n/2}$ on $\Delta$. This is implied by the identification principle of A. Weinstein [16]. It is easy to see that the above expression satisfies all of these conditions. For $n \geqq 3$ we conclude that

$$\begin{aligned} T[v_2] &= \frac{x_n}{2}\int_0^\pi \frac{\sin t dt}{[|x - y_0|^2 + 2x_n y_{n0}(1 - \cos t)]^{n/2}} \\ &= \frac{1}{2(n-2)y_{n0}}\left[\frac{1}{|x - y_0|^{n-2}} - \frac{1}{[|x - y_0|^2 + 4x_n y_{n0}]^{(n-2)/2}}\right] \\ &= \frac{g(x, y_0)}{(2n-2)y_{n0}}. \end{aligned} \tag{4.9}$$

An analogous formula holds for $n=2$. (4.9) is a special case of (4.7) from which the general case can easily be deduced.

Since $v=c-v_1-v_2$, we have $u=cx_n-T[v_1]-T[v_2]$ and, by (4.4) and (4.7), we arrive at the following result:

THEOREM 3. *A function $u(x)$ which is subharmonic and nonpositive throughout $H$ admits the representation*

$$u(x) = cx_n - x_n \int_D |x-y|^{-n} d\nu_1(e_y) - \int_H g(x, y) d\nu_2(e_y), \tag{4.10}$$

*where $\nu_1$ and $\nu_2$ are non-negative Borel measures, of support $D$ and $H$, respectively, $g$ denotes Green's function for $H$, and $c$ is a nonpositive constant.*

This theorem is well known. In the case $n=2$ it has been demonstrated by M. Riesz [13, p. 10], J. L. Doob and B. O. Koopman [7], S. Verblunsky [15], A. Dinghas [5], M. Tsuji [14] and by L. H. Loomis and D. V. Widder [10](6). The extension to higher dimensions has been published by A. Dinghas [6] and by J. Lelong-Ferrand [9]. These articles contain proofs of the representation for harmonic functions $u(x)$ from which the general case can easily be inferred making use of a result of F. Riesz [12, II, p. 357].

The constant $c$ is closely connected with the mean value considered in §3. For, by the definition of $h_r(\xi)$ and the Gauss mean value theorem, we have $h_r(0)=A(v;r)$. Hence, by (3.2),

$$c = \lim_{r\to\infty} h_r(0) = \frac{2n}{\omega_n} \lim_{r\to\infty} \frac{m(r)}{r} \cdot \tag{4.11}$$

This is a result of A. Dinghas [5; 6].

It is easy to see that the proof of Theorem 3 can still be carried through if the hypothesis $u\leqq 0$ is replaced by $u^+(x)=O(|x|)$ for $|x|\to\infty$. In this case $c$ becomes an arbitrary real constant. (4.11) remains valid.

We shall now derive representation theorems of a more general type. Our method will be the same as the one employed above except that, instead of applying the Riesz decomposition theorem, we shall make use of a generalization due to M. Brelot [4] (see also A. Pfluger [11]). For the sake of simplicity we shall limit ourselves from now on to the case $n=2$.

Let us consider a function $u(x)$ which is defined and subharmonic in $H(R, \infty)$ and satisfies (2.1) on $D(R, \infty)$. We further assume that $u(x)$ is *of order less than $p$ at infinity*, where $p$ is a positive integer, not less than 2. By this we understand that there exists a number $q<p$ such that

$$u^+(x) = o(|x|^q) \qquad \text{for } |x| \to \infty, \tag{4.12}$$

where $u^+=\max[u, 0]$.

(6) We are indebted to Dr. K. C. Hsu for pointing out to us several of these references

The relations (3.2), (3.3), and (4.12) imply

$$(4.13) \qquad A(v^+; r) = o(r^{q-1}) \qquad \text{for } r \to \infty,$$

where $v^+ = \max [S[u], 0] = S[u^+]$. Let $R_1 > R$ be given arbitrarily. By a theorem of M. Brelot (Theorem 2′ in [4]), $v$ admits the representation

$$(4.14) \quad v(\xi) = h(\xi) - \int_{\Omega(R_1,\infty)} \left[ |\xi - \eta|^{-2} - \sum_{k=0}^{p-2} C_k^1(\cos \omega_{\xi\eta}) |\xi|^k |\eta|^{-k-2} \right] d\mu(e_\eta)$$

in $\Omega(R_1, \infty)$. Here $h(\xi)$ denotes a function which is harmonic in $\Omega(R_1, \infty)$, symmetric with respect to $\Delta$, and which satisfies $h(\xi) = O(|\xi|^{p-2})$ for $|\xi| \to \infty$. $\mu$ is a non-negative Borel measure, symmetric with respect to $\Delta$. The $C_k^1$'s are Gegenbauer polynomials (see e.g. [18, p. 329]), defined as the coefficients of the development

$$(1 - 2zt + t^2)^{-1} = \sum_{k=0}^{\infty} C_k^1(z) t^k.$$

Finally, $\cos \omega_{\xi\eta} = \sum_1^4 \xi_i \eta_i / |\xi| |\eta|$.

In order to obtain a representation for $u = T[v]$ we shall now apply the transformation $T$ to the right-hand side of (4.14). We find it convenient to consider the space $R^2$ as a complex number plane, putting $x = x_1 + ix_2$, $y = y_1 + iy_2$ etc.

It follows from the properties of $h$ that $T[h]$ is defined and harmonic in $H(R_1, \infty)$ and assumes the boundary value 0 on $D(R_1, \infty)$. If we continue $T[h]$ across $D(R_1, \infty)$ as an odd function of $x_2$, we obtain a function which is defined and harmonic in the region $|x| > R_1$. Since furthermore $T[h] = O(|x|^{p-1})$ for $x \to \infty$, we conclude that $T[h]$ can be represented in the form

$$(4.15) \qquad T[h] = \operatorname{Im} \sum_{k=-\infty}^{p-1} a_k x^k$$

with real-valued coefficients $a_k$ $(k = p-1, p-2, \cdots)$.

We decompose $\mu = \mu_1 + \mu_2$, $\mu_1$ and $\mu_2$ being of support $\Omega(R_1, \infty) \cap \Delta$ and $\Omega(R_1, \infty) - \Delta$, respectively. Let

$$(4.16) \qquad w_1 = \int_{\Omega(R_1,\infty)\cap\Delta} K(\xi, \eta) d\mu_1(e_\eta)$$

and

$$(4.17) \qquad w_2 = \int_{\Omega(R_1,\infty)-\Delta} K(\xi, \eta) d\mu_2(e_\eta),$$

where $K(\xi, \eta) = |\xi - \eta|^{-2} - \sum_{k=0}^{p-2} C_k^1(\cos \omega_{\xi\eta}) |\xi|^k |\eta|^{-k-2}$. We state that

$$T[w_1] = \operatorname{Im} \int_{D(R_1,\infty)} \left[ \frac{1}{y - x} - \sum_{k=1}^{p-1} \frac{x^k}{y^{k+1}} \right] d\nu_1(e_y) \tag{4.18}$$

and

$$T[w_2] = \int_{H(R_1,\infty)} \left[ g(x, y) - \operatorname{Re} \sum_{k=1}^{p-1} \frac{x^k(\bar{y}^k - y^k)}{k \mid y \mid^{2k}} \right] d\nu_2(e_y), \tag{4.19}$$

where $\nu_1$ and $\nu_2$ are given by (4.5) and (4.8), respectively, and $\bar{y} = y_1 - iy_2$ denotes the conjugate of $y$.

We begin the proof of (4.18) with the consideration of a special case. Let $\mu_1$ be the unit mass, concentrated in an arbitrarily chosen point $\eta_0 = (y_{10}, 0, 0, 0)$ on $\Omega(R_1, \infty) \cap \Delta$. Then $w_1 = K(\xi, \eta_0)$, i.e. $w_1$ is the function $|\xi - \eta_0|^{-2}$ diminished by the terms of order $\leqq p-2$ of its Taylor development around 0. Now, it is quite obvious that if we apply the transformation $T$ to a function which is analytic (in the variables $\xi_1$, $\xi_2$, $\xi_3$ and $\xi_4$) in the neighborhood of the origin in $P^4$, then we obtain a function which is analytic (in the variables $x_1$ and $x_2$) in the neighborhood of the origin in $R^2$. Furthermore, the Taylor series may be transformed term by term. Therefore, the sum of the terms of order $\leqq p-2$ in the development of $|\xi - \eta_0|^{-2}$ is transformed into the sum of the terms of order $\leqq p-1$ in the development of

$$T[\,|\,\xi - \eta_0\,|^{-2}] = x_2 \,|\, x - y_0 \,|^{-2},$$

where $y_0 = (y_{10}, 0)$. (The increase in order is caused by the factor $x_2$ arising from (2.4).) But since, in complex notation,

$$\frac{x_2}{|\,x - y_0\,|^2} = \operatorname{Im} \frac{1}{y_0 - x} = \operatorname{Im} \sum_{k=0}^{\infty} \frac{x^k}{y_0^{k+1}} = \operatorname{Im} \sum_{k=1}^{\infty} \frac{x^k}{y_0^{k+1}},$$

we conclude that

$$T[w_1] = \operatorname{Im} \left[ \frac{1}{y_0 - x} - \sum_{k=1}^{p-1} \frac{x^k}{y_0^{k+1}} \right].$$

This proves (4.18) for the considered special measure $\mu_1$. It is easy to extend the result to the case of a finite number of concentrated masses. Finally, a limiting process establishes (4.18) in its full generality.

The verification of (4.19) is quite analogous. We begin with a special case. Let $y_0 \in H(R_1, \infty)$ be chosen arbitrarily. Suppose then that $\mu_2$ consists of the unit mass, distributed uniformly over the set $\Gamma(y_0)$. The function $w_2$ is defined as the potential $\int_{\Gamma(y_0)} |\xi - \eta|^{-2} d\mu_2(e_\eta)$ diminished by the terms of order $\leqq p-2$ of its Taylor development around 0. Let $g$ denote Green's function for $H$. Making use of (4.7) we conclude that $T[w_2]$ is obtained by subtracting from the function $g(x, y_0)/2y_{20}$ those terms of its Taylor development around 0 which are of order $\leqq p-1$. Since

$$g(x, y_0) = \log |x - \bar{y}_0| - \log |x - y_0|$$
$$= \operatorname{Re} \left\{ \log\left(1 - \frac{x}{\bar{y}_0}\right) - \log\left(1 - \frac{x}{y_0}\right) \right\}$$
$$= \operatorname{Re} \left\{ \sum_{k=1}^{\infty} \frac{x^k}{k y_0^k} - \sum_{k=1}^{\infty} \frac{x^k}{k \bar{y}_0^k} \right\} = \operatorname{Re} \sum_{k=1}^{\infty} \frac{x^k(\bar{y}_0^k - y_0^k)}{k |y_0|^{2k}},$$

it follows that

$$T[w_2] = \frac{1}{2y_{20}} \left[ g(x, y_0) - \operatorname{Re} \sum_{k=1}^{p-1} \frac{x^k(\bar{y}_0^k - y_0^k)}{k |y_0|^{2k}} \right].$$

This is the desired result for the special case. It is easy to proceed now to the general formula (4.19).

Clearly $v = h - w_1 - w_2$. Hence $u = T[h] - T[w_1] - T[w_2]$ and, by (4.15), (4.18), and (4.19), we obtain

THEOREM 4. *Let $u(x_1 + ix_2)$ be defined and subharmonic in $H(R, \infty)$ and satisfy* (2.1) *on $D(R, \infty)$. Suppose further that $u$ is of order less than $p$ at infinity, where $p$ is a positive integer $\geqq 2$. Then, given any $R_1 > R$, the following representation holds in $H(R_1, \infty)$*

$$(4.20) \quad \begin{aligned} u(x) = {} & \operatorname{Im} \sum_{k=-\infty}^{p-1} a_k x^k - \operatorname{Im} \int_{D(R_1,\infty)} \left[ \frac{1}{y - x} - \sum_{k=1}^{p-1} \frac{x^k}{y^{k+1}} \right] d\nu_1(e_y) \\ & - \int_{H(R_1,\infty)} \left[ g(x, y) - \operatorname{Re} \sum_{k=1}^{p-1} \frac{x^k(\bar{y}^k - y^k)}{k |y|^{2k}} \right] d\nu_2(e_y). \end{aligned}$$

*Here $\nu_1$ and $\nu_2$ denote non-negative Borel measures, of support $D(R_1, \infty)$ and $H(R_1, \infty)$, respectively, and the coefficients $a_k$ $(k = p-1, p-2, \cdots)$ are real numbers. $g$ denotes Green's function for $H$.*

A similar theorem can be deduced for the neighborhood of 0. We introduce the following terminology: A function $u(x)$, defined and subharmonic in $H(0, R)$, is said to be *of order less than $p$ at the origin*, if there exists a number $q < p$ such that

$$u^+(x) = o(|x|^{-q}) \qquad \text{for } x \to 0.$$

With this definition the above mentioned result can be formulated as follows:

THEOREM 5. *Let $u(x_1 + ix_2)$ be defined and subharmonic in $H(0, R)$ and satisfy* (2.1) *on $D(0, R)$. Suppose further that $u$ is of order less than $p$ at the origin, where $p$ is a positive integer $\geqq 2$. Then, given any $R_1 < R$, the following representation holds in $H(0, R_1)$*

$$u(x) = \operatorname{Im} \sum_{k=1-p}^{\infty} a_k x^k + \operatorname{Im} \int_{D(0,R_1)} \left[ \frac{1}{x-y} - \sum_{k=1}^{p-1} \frac{y^{k-1}}{x^k} \right] d\nu_1(e_y)$$

(4.21)

$$- \int_{H(0,R_1)} \left[ g(x, y) - \operatorname{Re} \sum_{k=1}^{p-1} \frac{y^k - \bar{y}^k}{k x^k} \right] d\nu_2(e_y).$$

*Here $\nu_1$ and $\nu_2$ are non-negative Borel measures, of support $D(0, R_1)$ and $H(0, R_1)$, respectively, and the coefficients $a_k$ $(k=1-p, 2-p, \cdots)$ are real numbers. $g$ denotes Green's function for $H$.*

This result is an easy consequence of Theorem 4. It can also be derived directly making use of Brelot's Theorem 2 [4].

Suppose that $u(x)$ is of class A, being of order less than $p$ at infinity. In this case it might be of interest to have a representation for $u$ which is valid throughout $H$. Such a formula is obtained by means of the following argument: The reader will easily convince himself that if $v(\xi)=S[u]$ is harmonic in a neighborhood of 0, then $u$ can be represented throughout $H$ by (4.20), the domains of integration $D(R_1, \infty)$ and $H(R_1, \infty)$ being changed into $D$ and $H$, respectively. If $v$ does not fulfill this special assumption, then we consider $v^*(\xi)=v(\xi)-\int_{|\eta|<1}|\xi-\eta|^{-2}d\mu(e_\eta)$. Since $v^*$ is harmonic in $|\xi|<1$, we conclude that $T[v^*]$ is representable in the form (4.20). Making use, furthermore, of (4.4) and (4.7), we find that $u$ admits the representation (4.20), modified as follows: (1) $D(R_1, \infty)$ and $H(R_1, \infty)$ have to be replaced by $D$ and $H$, respectively; (2) in both integrands the subtracted terms (finite sums) have to be omitted for $|y|<1$; (3) $a_k=0$ for $k\leqq -1$.

The growth of the function $u$ and the behavior of the measures $\nu_1$ and $\nu_2$ are closely connected. By means of Brelot's Lemmas 2′ and 4′ we find the following properties. If $u$ fulfills the hypotheses of Theorem 4, then the measure $\mu$, associated with $v=S[u]$, satisfies the following condition: There exists a number $s<p$ such that

$$\int_{\Omega(R_1,\infty)} |\eta|^{-s-1} d\mu(e_\eta) < \infty. \tag{4.22}$$

Conversely, suppose that measures $\nu_1$ and $\nu_2$ are given such that the corresponding measure $\mu$, to be determined from (4.5) and (4.8), satisfies (4.22) for some $s<p$. Then (4.20) yields a function $u$ which is defined and subharmonic in $H(R_1, \infty)$, of order less than $p$ at infinity, and which satisfies (2.1) on $D(R_1, \infty)$. (Of course, at the same time the coefficients $a_k$ are supposed to have been chosen in such a way that the series $\sum_{-\infty}^{p-1} a_k x^k$ converges in $|x|>R_1$.)

It should be noticed that we did not make full use of Brelot's theorems. Our results could be refined by applying the precise statements in [4]. We shall not go into this.

We limited ourselves to the case $n=2$. But the method works in higher dimensions as well. This generalization is also left to the reader.

We conclude with two applications of Theorem 4.

COROLLARY 1. *Suppose that the function $f(x)$ is regular analytic in $H(R, \infty)$. If the imaginary part of $f$ satisfies* (2.1) *on $D(R, \infty)$ and is of order less than $p$ (positive integer $\geqq 2$) at infinity, then, for any given $R_1 > R$, the representation*

$$f(x) = \sum_{k=-\infty}^{p-1} a_k x^k - \int_{D(R_1,\infty)} \left[\frac{1}{y-x} - \sum_{k=1}^{p-1} \frac{x^k}{y^{k+1}}\right] d\nu(e_y)$$

*holds in $H(R_1, \infty)$. Here $\nu$ denotes a non-negative Borel measure and the coefficients $a_k$ $(k = p-1, p-2, \cdots)$ are real numbers.*

We define: An analytic function $f(x)$ is of order less than $p$ if and only if the subharmonic function $u(x) = \log |f(x)|$ has this property according to the above definition. This convention agrees with the usual terminology in the theory of entire functions.

COROLLARY 2. *Suppose that the function $f(x)$ is regular analytic in $H(R, \infty)$ and satisfies*

$$\limsup_{x \to y} |f(x)| \leqq 1 \qquad (x \in H(R, \infty))$$

*for all $y \in D(R, \infty)$. Assume further that $f$ is of order less than $p$ (positive integer $\geqq 2$) at infinity. Let the zeros of $f$ (enumerated in any order) be denoted by $x_m$ $(m = 1, 2, 3, \cdots)$. Then, given any $R_1 > R$, $f$ admits the representation*

$$f(x) = e^{i\Phi(x)} \prod_{|x_m| > R_1} \left[\frac{x - x_m}{x - \bar{x}_m} \exp\left\{\sum_{k=1}^{p-1} \frac{\overset{k}{x}(\overset{k}{\bar{x}}_m - \overset{k}{x}_m)}{k |x_m|^{2k}}\right\}\right],$$

*where*

$$\Phi(x) = \sum_{k=-\infty}^{p-1} a_k x^k + \int_{D(R_1,\infty)} \left[\frac{1}{y-x} - \sum_{k=1}^{p-1} \frac{x^k}{y^{k-1}}\right] d\nu(e_y),$$

*in $H(R_1, \infty)$. Here $\nu$ denotes a non-negative Borel measure and the coefficients $a_k$ $(k = p-1, p-2, \cdots)$ are real numbers.*

This representation can be considered as an analogue of Hadamard's factorization theorem for entire functions of finite order (see e.g. [2, p. 22]). It is obtained from Theorem 4 by letting $u = \log |f(x)|$.

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